Quark deconfinement and gluon condensate in a weak magnetic field from QCD sum rules

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We study QCD finite energy sum rules (FESR) for the axial-vector current correlator in the presence of a magnetic field, in the weak field limit and at zero temperature. We find that the perturbative QCD and the hadronic contribution to the sum rules get explicit magnetic-field-dependent corrections and that these in turn induce a magnetic field dependence on the deconfinement phenomenological parameter s_0 and on the gluon condensate. The leading corrections turn out to be quadratic in the field strength. We find from the dimension $d = 2$ first FESR that the magnetic field dependence of s_0 is proportional to the absolute value of the light-quark condensate. Hence, it increases with increasing field strength. This implies that the parameters describing chiral symmetry restoration and deconfinement behave similarly as functions of the magnetic filed. Thus, at zero temperature the magnetic field is a catalyzing agent of both chiral symmetry breaking and confinement. From the dimension $d = 4$ second FESR we obtain the behavior of the gluon condensate in the presence of the external magnetic field. This condensate also increases with increasing field strength.

I. INTRODUCTION

Lately, the properties of strongly interacting matter in the presence of external magnetic fields has become a very active research field. One of the driving motivations behind this interest is the possibility to study experimentally such properties in peripheral collisions of heavy nuclei at high energy. In addition, recent lattice QCD (LQCD) results show that the critical temperature for deconfinement/chiral symmetry restoration decreases with increasing field strength [1]. This behavior is dubbed *inverse magnetic catalysis*, and it reveals an unexpected, nontrivial phenomenon: in a thermal environment, near the transition temperature, the presence of a magnetic field acting on strongly interacting matter hinders the formation of a quark-antiquark condensate. LQCD calculations [2] show

that the quark condensate does increase with increasing magnetic field at low temperatures. This behavior corresponds to *magnetic catalysis*. However, as the temperature increases approaching the crossover region $T \simeq 150$ MeV, the quark condensate reaches a maximum value smaller than at $T = 0$ (for the same field strength). Subsequently, the condensate decreases as a function of the field strength. Finally, for temperatures above the crossover values the condensate decreases monotonically as a function of the magnetic field. Some of the possible scenarios aiming to understand this behavior include (i) invoking a fermion paramagnetic contribution to the pressure with a sufficiently large magnetization [3]; (ii) the competition between the valence and sea contributions at the phase transition [4] produced by a backreaction of the Polyakov loop, which depends on the magnetic field [5]; (iii) magnetic inhibition

due to neutral meson fluctuations in a strong magnetic field [6]; (iv) accounting for nonperturbative effects by means of Schwinger-Dyson equations and renormalization group analyses [7–9]; (v) a decreasing magnetic field and temperature-dependent coupling with [10–12] and without [13,14] plasma screening effects; (vi) the proper account of the gluon contribution during the phase transition [15]; and (vii) quark antiscreening due to the anomalous magnetic moment of quarks in strong [16] and weak [17] fields. On the other hand, this behavior is not obtained in mean field approaches describing the thermal environment [18–23], nor when calculations beyond the mean field do not include magnetic effects on the coupling constants [24]. For recent reviews see Refs. [25,26].

Given the dual nature of the QCD phase transition, a pertinent question is to what extent inverse magnetic catalysis is due to the mechanisms of either chiral symmetry restoration and/or of deconfinement. One way to address this question is to find a relation between deconfinement and chiral symmetry restoration parameters as a function of the magnetic field. Since the transition happens for temperatures in the realm of nonperturbative phenomena, the relation searched for needs to carry nonperturbative information. An extensively used tool in the context of effective models at finite temperature and zero [27] and finite [22] magnetic field is the Polyakov loop [28]. When coupled to quark degrees of freedom this loop sheds light on how chiral symmetry and deconfinement behave during the QCD transition as a function of the field intensity. Another nonperturbative tool that does not rely on effective models is that of QCD finite energy sum rules (FESR). This approach has been successfully applied both at zero [29] and at finite temperature [30] to understand hadronic properties. Of particular mention are (i) the prediction, in this framework, of the survival of charmonium and bottomonium above the critical temperature [31]; (ii) the temperature behavior of the hadronic width of charmonium and bottonium [31] in qualitative agreement with bottonium results from LQCD [32]; and (iii) the description of the dimuon spectrum in heavy-ion collisions [33], in excellent agreement with data [34] in the region of the rho-meson peak. A key parameter that emerges from this analysis signaling quark-gluon deconfinement is the squared energy threshold, s_0, above which the hadronic spectral function is well approximated by perturbative QCD (pQCD). An interesting relation between s_0 and the quark condensate $\langle \bar{q}q \rangle$, whereby the former is proportional to the latter, has been found, in the absence of a magnetic field and at finite temperature, in [35], and at finite temperature and density in [36].

In this paper we use FESR in the axial-vector channel, and in the presence of an external magnetic field, to explore the relation between (i) the deconfinement and chiral symmetry restoration parameters, s_0 and $\langle \bar{q}q \rangle$; and (ii) obtain the behavior of the gluon condensate as a function of the magnetic field intensity at zero temperature.

Other formulations of the QCD sum rules, also at $T = 0$, have been used to explore the dependence of heavy-quark meson masses and mixing as a function of the magnetic field strength [37]. The paper is organized as follows: In Sec. II we set up the FESR. We show that since the pQCD contribution to the current correlator receives corrections that can be expressed as powers of the magnetic field strength divided by powers of the squared energy s, there appear additional terms that contribute to higher order FESR. In Sec. III we explicitly compute the pQCD corrections to the current correlator and solve the first two FESR to find the dependence of s_0 and the gluon condensate on the magnetic field strength. We show that for magnetic field strengths eB smaller than s_0, the former follows the magnetic field dependence of the quark condensate, which we parametrize from LQCD results [2]. We also show that the magnetic field dependence of the gluon condensate receives nontrivial corrections from the pQCD sector and that overall it is a monotonically increasing function of the field strength. We summarize and conclude in Sec. IV, leaving for the appendixes the explicit computation of the imaginary part of the hadronic contribution and the vanishing of the first order correction in the field strength of the pQCD contribution to the axial-vector current correlator.

II. FINITE ENERGY QCD SUM RULES IN THE PRESENCE OF A MAGNETIC FIELD

The charged axial-vector current correlator in the absence of a magnetic field and at $T = 0$ can be written as

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iqx} \langle 0|T(A_\mu(x), A_\nu^\dagger(0))|0\rangle$$
$$= (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi_A(q^2) + q_\mu q_\nu \Pi_0(q^2), \quad (1)$$

where $A_\mu(x) =: \bar{d}(x)\gamma_\mu\gamma_5 u(x):$ is the (charged) axial-vector current, q_μ is the four-momentum carried by the current, with $s \equiv q^2 > 0$ the squared energy. The functions $\Pi_{A,0}(q^2)$ are free of kinematical singularities, an important property needed in writing dispersion relations and sum rules. Concentrating on e.g. $\Pi_0(q^2)$ and invoking the operator product expansion (OPE) of current correlators at short distances beyond perturbation theory, one of the two pillars of the QCD sum rule method, one has

$$\Pi_0(Q^2)|_{\text{QCD}} = C_0\hat{I} + \sum_{N=1} \frac{C_{2N}(Q^2, \mu^2)}{Q^{2N}} \langle \hat{O}_{2N}(\mu^2)\rangle, \quad (2)$$

where $Q^2 \equiv -q^2$, $\langle \hat{O}_{2N}(\mu^2)\rangle \equiv \langle 0|\hat{O}_{2N}(\mu^2)|0\rangle$, and μ^2 is a renormalization scale. The Wilson coefficients C_N depend on the Lorentz indexes and quantum numbers of the currents, and on the local gauge invariant operators $\hat{O}_{2N}$ built from the quark and gluon fields in the QCD Lagrangian. These operators are ordered by increasing

dimensionality and the Wilson coefficients are calculable in pQCD. The unit operator above has dimension $d = 0$ and $C_0 \hat{I}$ stands for the purely perturbative contribution normalized according to

$$C_0 \hat{I} = \frac{1}{4\pi} \ln \left(\frac{-s}{\mu^2} \right) [1 + \mathcal{O}(\alpha_s(s))]. \quad (3)$$

Since there are no dimension $d = 2$ operators built from the QCD fields, it is generally assumed that the OPE starts at dimension $d = 4$. This is fully confirmed by determinations of condensates from experimental data [38]. The dimension $d = 4$ in the chiral limit is proportional to the renormalization group invariant gluon condensate

$$C_4 \langle \hat{O}_4 \rangle = \frac{\pi}{3} \langle \alpha_s G^2 \rangle. \quad (4)$$

The second pillar of the QCD sum rule method is to consider an integration contour in the complex square energy plane, as in Fig. 1, and invoke Cauchy's theorem assuming that QCD can be used on the circle of radius $|s_0|$, provided $|s_0|$ is large enough (quark-hadron duality). On the real axis there is a discontinuity associated with the hadronic states entering the spectral function. Since there are no further singularities this leads to the FESR

$$-\frac{1}{2\pi i} \oint_{C(|s_0|)} ds \, s^{N-1} \Pi_0^{\text{QCD}}(s) = \frac{1}{\pi} \int_0^{s_0} ds \, s^{N-1} \text{Im} \Pi_0^{\text{HAD}}(s), \quad (5)$$

with $N \geq 1$, and $\Pi_0^{\text{QCD}}(s)$ given by the OPE, Eq. (2). It will be shown later that in the presence of a magnetic field, and in the weak field limit $eB < s_0$, the Wilson coefficients acquire themselves a B-field dependence. In this work we shall compute the corrections to the FESR due to a weak
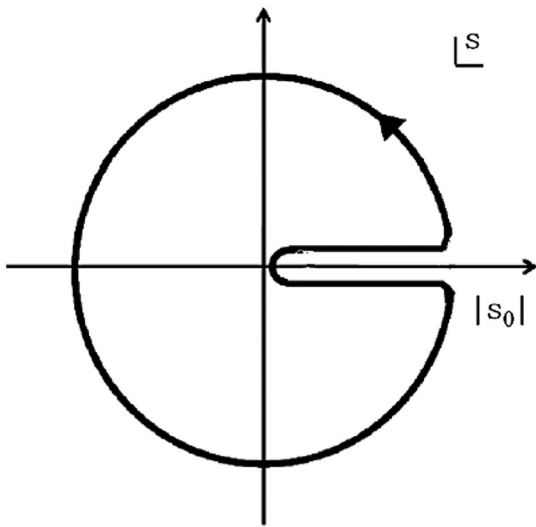

FIG. 1. Cauchy integration contour in the complex squared energy s-plane used to obtain QCD FESR. The radius of the circle, s_0, is the threshold for pQCD.

magnetic field, which can be expressed as a series in powers of eB. Since the magnetic field carries dimension of energy squared, on dimensional grounds one finds the replacements

$$C_0 \ln \left(\frac{-s}{\mu^2} \right) \rightarrow C_0 \ln \left(\frac{-s}{\mu^2} \right) + \sum_{n=1}^{\infty} C_0^{(n)} \frac{(eB)^n}{s^n}$$

$$C_{2N} \rightarrow \sum_{m=0}^{\infty} C_{2N}^{(m)} \frac{(eB)^m}{s^m} \quad (6)$$

where $C_{2N}^{(m)}$ are dimensionless quantities that can be computed in pQCD at a given order in eB. Substituting Eqs. (6) and (2) into Eq. (5), one obtains

$$-\sum_{m=0}^{N-1} (-1)^{N-m} C_{2(N-m)}^{(m)} (eB)^m \langle O_{2(N-m)} \rangle$$

$$= \frac{1}{\pi} \int_0^{s_0} ds \, s^{N-1} \text{Im} \Pi_0^{\text{HAD}}(s) - \frac{C_0}{N} s_0^N + C_0^{(N)}(eB)^N. \quad (7)$$

Note that in general the presence of the magnetic field mixes operators of different dimension in the FESR. For instance, the first two sum rules ($N = 1, 2$) become

$$0 = \frac{1}{\pi} \int_0^{s_0} ds \, \text{Im} \Pi_0^{\text{HAD}}(s) - C_0 s_0 + C_0^{(1)}(eB), \quad (8)$$

$$-C_4^{(0)} \langle O_4 \rangle + C_2^{(1)}(eB) \langle O_2 \rangle = \frac{1}{\pi} \int_0^{s_0} ds \, s \, \text{Im} \Pi_0^{\text{HAD}}(s)$$

$$- \frac{C_0}{2} s_0^2 + C_0^{(2)}(eB)^2. \quad (9)$$

In order to set up explicitly the relevant FESR we start by computing the hadronic contribution. The axial-vector current in the presence of a magnetic field can be interpolated by the charged pion current

$$A_\mu = -f_\pi D_\mu \pi^+ = -f_\pi (\partial_\mu - ie\mathcal{A}_\mu) \pi^+, \quad (10)$$

where $f_\pi = 130.28(14)$ MeV [39] is the pion decay constant, π^+ the pion field, and $\mathcal{A}_\mu = (B/2)(0, -y, x, 0)$ the vector potential in the symmetric gauge, which gives rise to a constant magnetic field along the $\hat{z}$ direction. Therefore, the axial-vector correlator in the hadronic sector can be written as

$$\Pi_{\mu\nu}^{\text{HAD}}(x, y) \equiv \langle 0 | T(A_\mu(x), A_\nu^\dagger(y)) | 0 \rangle$$

$$= if_\pi^2 \langle 0 | T[D_\mu \pi^+(x) D_\nu^* \pi^-(y)] | 0 \rangle$$

$$= if_\pi^2 D_\mu(x) D_\nu^*(y) G_\pi(x, y)$$

$$= ie^{ie\Phi(x,y)} f_\pi^2 (\partial/\partial x^\mu)(\partial/\partial y^\nu) \tilde{G}_\pi(x - y)$$

$$\equiv e^{ie\Phi(x,y)} \tilde{\Pi}_{\mu\nu}^{\text{HAD}}(x - y), \quad (11)$$

where we have used the fact that the charged pion propagator $G_\pi(x, y)$ in the presence of a magnetic field can be written as a product of a translationally invariant piece $\tilde{G}_\pi(x - y)$ and a *phase* factor $e^{ie\Phi(x,y)}$. The phase factor does not depend on the integration path so that choosing a straight line trajectory it can be written as

$$\Phi(x, y) = \int_x^y \mathcal{A}(\xi)d\xi, \qquad (12)$$

where $\xi = yt + x(1 - t)$, $t \in [0, 1]$. It is easy to show that the above phase factor can be *gauged away* by a suitable gauge transformation of the vector potential. Hence, we keep only the translational invariant part of the hadronic correlator whose Fourier transform is

$$\Pi_0^{HAD}(q^2) = if_\pi^2 \tilde{G}_\pi(q^2), \qquad (13)$$

where $\tilde{G}_\pi(q^2)$ stands for the Fourier transform of $\tilde{G}_\pi(x - y)$. Using Schwinger's proper time method this quantity can be written as

$$\tilde{G}_\pi(q^2) = \int_0^\infty \frac{d\tau}{\cos(eB\tau)} e^{i\tau[q_\parallel^2 - q_\perp^2 \tan(eB\tau)/eB\tau + i\epsilon]}, \qquad (14)$$

where $m_\pi = 0$ as we consider the chiral limit. Hereafter we shall use the notation

$$g_{\mu\nu} = g_{\mu\nu}^\parallel - g_{\mu\nu}^\perp$$
$$g_{\mu\nu}^\parallel = \mathrm{diag}(1, 0, 0, -1)$$
$$g_{\mu\nu}^\perp = \mathrm{diag}(0, 1, 1, 0). \qquad (15)$$

Consequently

$$a \cdot b = (a \cdot b)_\parallel - (a \cdot b)_\perp$$
$$(a \cdot b)_\parallel = g_{\mu\nu}^\parallel a^\mu b^\nu$$
$$= a_0 b_0 - a_3 b_3$$
$$(a \cdot b)_\perp = g_{\mu\nu}^\perp a^\mu b^\nu$$
$$= a_1 b_1 + a_2 b_2$$
$$g_{\mu\nu}^\parallel g^{\mu\nu} = 2$$
$$g_{\mu\nu}^\perp g^{\mu\nu} = -2. \qquad (16)$$

Extending the integration in Eq. (14) to the lower complex plane, it can be shown [40] that the charged pion propagator is expressed in terms of a sum over Landau levels

$$\tilde{G}_\pi(q^2) = 2i \sum_{l=0}^\infty \frac{(-1)^l L_l(2q_\perp^2/eB)e^{-q_\perp^2/eB}}{q_\parallel^2 - (2l + 1)eB}. \qquad (17)$$

As it is customary in problems where Lorentz invariance is lost and the correlators depend separately on the time and space parts of the four-momentum, to simplify the analysis hereafter we study the behavior of the correlator in the *static* limit where the space part of the four momentum vanishes. However, note that since the magnetic field separates space-time into longitudinal and transverse directions with respect to the direction of the field, q_0 comes together with q_3 to form $q_\parallel^2 = q_0^2 - q_3^2$. It is therefore equally simple to set $q_\perp^2 = 0$, perform the analysis in terms of $q_\parallel^2$ and then at the end, if needed, take $q_3 \to 0$. In this limit, Eq. (17) becomes

$$\tilde{G}_\pi(q^2) = 2i \sum_{l=0}^\infty \frac{(-1)^l}{q_\parallel^2 - (2l + 1)eB}. \qquad (18)$$

Therefore the hadronic contribution to the correlator becomes explicitly

$$\Pi_0^{HAD}(q_\parallel^2 = s) = -2f_\pi^2 \sum_{l=0}^\infty \frac{(-1)^l}{s - (2l + 1)eB}. \qquad (19)$$

As we show in Appendix A, the imaginary part of Eq. (19) in the weak field limit, $eB < s_0$, is given by

$$\mathrm{Im}\Pi_0^{HAD}(s) = f_\pi^2 \pi \delta(s - eB). \qquad (20)$$

It is important to mention that Eq. (20) holds provided s_0 satisfies also the condition $s_0 < 3eB$, as explicitly shown in Appendix A. This means that the weak field condition ($eB < s_0$) has to be supplemented with a further restriction for Eq. (20) to remain valid. Recall that s_0 represents the onset for the pQCD description for the axial-vector spectral density and that this quantity is a decreasing function of temperature [33]. For cold nuclear matter, as in the case of a neutron star, the condition $eB < s_0 < 3eB$ may be difficult to meet, especially for a weak field where not only the situation $eB < s_0$, but even $neB < s_0$, $(n \geq 1)$, can happen. However, for a heavy-ion collision, around the deconfinement/chiral symmetry restoration transition, when s_0 has dropped off to small values, the weak field condition can also be made compatible with $eB < s_0 < 3eB$. Hereafter we keep in mind this last observation as the working scenario, aiming to eventually incorporate thermal effects to describe the behavior of a magnetized medium near the phase transition.

Using Eq. (20), the hadronic line integral in the FESR is given explicitly by

$$\frac{1}{\pi} \int_0^{s_0} ds\, s^{N-1} \mathrm{Im}\Pi_0^{HAD}(s) = f_\pi^2(eB)^{N-1}. \qquad (21)$$

Substituting Eq. (21) into the QCD sum rules Eqs. (8)–(9) gives

$$0 = f_\pi^2 - C_0 s_0 + C_0^{(1)}(eB)$$

$$-C_4\langle O_4\rangle = f_\pi^2(eB) - \frac{C_0}{2}s_0^2 + C_0^{(2)}(eB)^2. \quad (22)$$

In order to solve these equations, we now proceed to compute explicitly the coefficients $C_0^{(1)}$ and $C_0^{(2)}$.

III. SOLUTIONS FOR s_0 AND $C_4\langle O_4\rangle$

To perform the perturbative calculation of the coefficients $C_0^{(1)}$ and $C_0^{(2)}$ in the weak field limit we make use of the weak field expansion of the quark propagator in the presence of a constant magnetic field [41], and in the chiral limit, up to order $\mathcal{O}(B^2)$

$$iS_B(k) = i\frac{k}{k^2} - (e_qB)\frac{\gamma_1\gamma_2(\gamma\cdot k)_\parallel}{k^4}$$
$$- 2i(e_qB)^2\frac{[k_\perp^2(\gamma\cdot k)_\parallel - k_\parallel^2(\gamma\cdot k)_\perp]}{k^8}, \quad (23)$$

where e_q is the absolute value of the quark's charge. We emphasize that the weak field expansion of the fermion propagator in the chiral limit is a well-defined object. This happens because the weak field limit of the propagator can be thought of as a series representation in powers of eB of the full propagator independent of any relation between the strength of the magnetic field and the fermion mass. One can then use this series representation and take the chiral limit as much as one can do it for the usual fermion propagator in the absence of the field; depending on the process considered, this limit could lead to infrared divergences but it does not preclude the validity of taking $m \to 0$ inside the propagator. In the present context, the field can be considered as weak when compared to s_0, which is the only other energy (squared) present in the problem, in such a way that the expansion in Eq. (6) converges.

The pQCD contribution to the axial-vector current correlator in the presence of a magnetic field is depicted in Fig. 2, where we also define the kinematics. The thick internal lines represent the full quark propagators in the magnetic field background. To first order in e_qB only one

of the two quark propagators carries the magnetic effects. This is depicted in Fig. 3 where the wavy line starting from a cross represents the external magnetic field. The two diagrams in Fig. 3 that determine the coefficient $C_0^{(1)}$ vanish identically when contracted with the momenta carried by the axial-vector currents. This is due to a straightforward application of Furry's theorem, and to the fact that the vector and axial-vector correlators are chiral symmetric. However, we show explicitly in Appendix B that each of the contributions in the diagrams, Fig. 3, vanishes.

The first nontrivial magnetic contribution to the pQCD axial-vector current correlator is of order $(e_qB)^2$. The relevant diagrams are shown in Fig. 4. First, we compute the diagram where one magnetic field line is attached to each one of the two quark propagators. For these, we use Eq. (23) to first order in e_qB. We call this contribution $\Pi_{\mu\nu}^{(11)}(q^2)$, and its explicit expression is

$$\Pi_{\mu\nu}^{(11)}(q^2) = -iN_c(q_uq_dB^2)$$
$$\times \int \frac{d^4k}{(2\pi)^4}\frac{\text{Tr}[\gamma_\mu\gamma_1\gamma_2[\gamma\cdot(k-q)_\parallel]\gamma_\nu\gamma_1\gamma_2(\gamma\cdot k)_\parallel]}{(k-q)^4k^4}. \quad (24)$$

Since according to Eq. (5), we are interested in the magnetic corrections to the coefficient of the longitudinal structure, $\Pi_0(q^2)$, we project $\Pi_{\mu\nu}^{(11)}(q^2)$ with $q^\mu q^\nu$ and
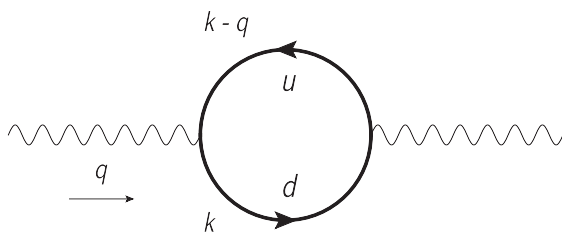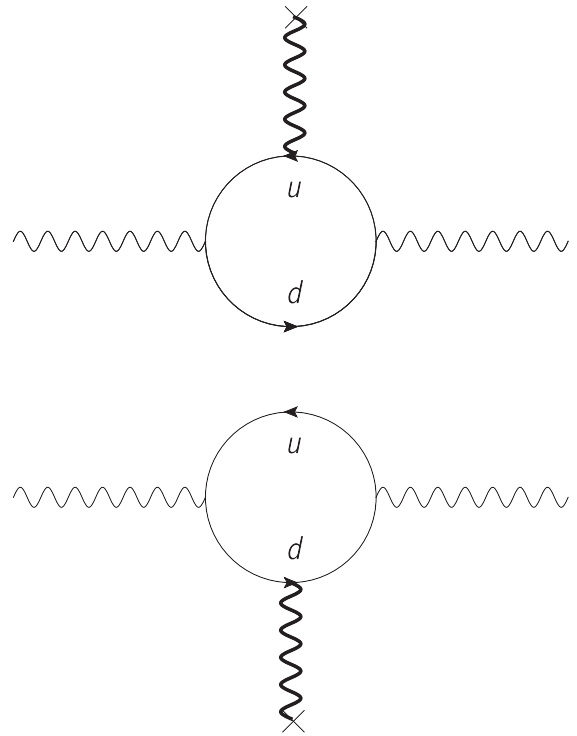

FIG. 2. pQCD contribution to the axial-vector current correlator in the presence of a magnetic field. The thick internal lines represent the full quark propagators in the magnetic field background.

FIG. 3. pQCD contribution to the axial-vector current correlator in the presence of a magnetic field to first order in e_qB. The thick wavy lines ending in a cross represent the external magnetic field.
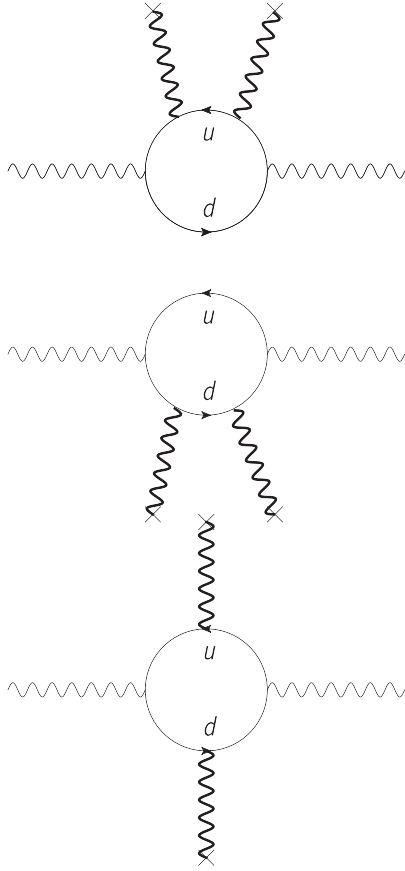
FIG. 4. pQCD contribution to the axial-vector current correlator in the presence of a magnetic field to second order in $e_q B$. The thick wavy lines ending in a cross represent the external magnetic field.

define

$$\tilde{\Pi}_0^{(11)}(q^2) \equiv q^\mu q^\nu \Pi_{\mu\nu}^{(11)}(q^2). \tag{25}$$

Using

$$\gamma_1 \gamma_2 [\gamma \cdot (k-q)_\parallel] = [\gamma \cdot (k-q)_\parallel] \gamma_1 \gamma_2, \tag{26}$$

together with

$$\gamma_1 \gamma_2 \not{q} \gamma_1 \gamma_2 = 2(\gamma \cdot q)_\perp - \not{q}, \tag{27}$$

gives

$$\tilde{\Pi}_0^{(11)} = i N_c (q_u q_d B^2) \int \frac{d^4 k}{(2\pi)^4}$$
$$\times \frac{\mathrm{Tr}[\not{q}[\gamma \cdot (k-q)]_\parallel (-2(\gamma \cdot q)_\perp + \not{q})(\gamma \cdot k)_\parallel]}{(k-q)^4 k^4}. \tag{28}$$

The evaluation of the trace yields

$$\mathrm{Tr}[\not{q}[\gamma \cdot (k-q)]_\parallel (-2(\gamma \cdot q)_\perp + \not{q})(\gamma \cdot k)_\parallel]$$
$$= -4(q_\parallel^2 + q_\perp^2)[(k-q) \cdot k]_\parallel$$
$$+ 8(q \cdot k)_\parallel [(k-q) \cdot q]_\parallel. \tag{29}$$

We now use the Feynman parametrization

$$\frac{1}{(k-q)^4 k^4} = 3! \int_0^1 dx \frac{x(x-1)}{[(k-xq)^2 - x(x-1)q^2]^4}, \tag{30}$$

and the change of variable

$$k \to l = k - xq, \tag{31}$$

to obtain

$$\tilde{\Pi}_0^{(11)}(q^2) = 4 i N_c (q_u q_d B^2) 3!$$
$$\times \int_0^1 x(x-1) \int \frac{d^4 l}{(2\pi)^4} \frac{1}{[l^2 - \Delta]^4}$$
$$\times [q^2 q_\parallel^2 x(x-1) - (q_\parallel^2 + q_\perp^2) l_\parallel^2 + 2(q \cdot l)_\parallel^2], \tag{32}$$

where we have discarded terms with odd powers of l and defined $\Delta = x(x-1)q^2$. The integrals over l are computed by means of

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2 - \Delta]^n} = i \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{[l^2 - \Delta]^n} = i \frac{(-1)^{n-1}}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-d/2-1)}{\Gamma(n)}$$
$$\times \left(\frac{1}{\Delta}\right)^{n-d/2-1}, \tag{33}$$

with $n = 4$ and $d = 4$. Using Eq. (33) in Eq. (32), and after integrating over x, we find

$$\tilde{\Pi}_0^{(11)} = -\frac{N_c}{4\pi^2} (q_u q_d B^2) \frac{[q_\parallel^2 + q_\perp^2]}{q^2}. \tag{34}$$

In the limit $q_\perp^2 \to 0$ Eq. (34) becomes

$$\tilde{\Pi}_0^{(11)} \overset{q_\perp^2 \to 0}{\to} -\frac{N_c}{4\pi^2} (q_u q_d B^2). \tag{35}$$

In a similar fashion we compute the diagrams in Fig. 4 to second order in eB in the u-quark and in the d-quark propagator. Calling the longitudinal projections $\tilde{\Pi}_0^{(20)}$ and $\tilde{\Pi}_0^{(02)}$, respectively, the result is

$$\tilde{\Pi}_0^{(20)}(q^2) = -\frac{N_c}{24\pi^2}(q_u B)^2 \left[\frac{(q_\parallel^2 + q_\perp^2)}{q^2} + 2\frac{q_\parallel^2 q_\perp^2}{q^4}\right]$$

$$\xrightarrow{q_\perp^2 \to 0} -\frac{N_c}{24\pi^2}(q_u B)^2$$

$$\tilde{\Pi}_0^{(02)}(q^2) = -\frac{N_c}{24\pi^2}(q_d B)^2 \left[\frac{(q_\parallel^2 + q_\perp^2)}{q^2} + 2\frac{q_\parallel^2 q_\perp^2}{q^4}\right]$$

$$\xrightarrow{q_\perp^2 \to 0} -\frac{N_c}{24\pi^2}(q_d B)^2. \tag{36}$$

Adding all three contributions, and using the absolute values $q_u = 2/3e$, $q_d = 1/3e$, and $N_c = 3$, we obtain the coefficient of the longitudinal structure of the axial-vector current correlator to second order in the magnetic field

$$\Pi_0^{B^2} = -\left(\frac{17}{18}\right)\frac{(eB)^2}{4\pi^2}. \tag{37}$$

Using this result together with the first equation in Eq. (6), we obtain the Wilson coefficient of the pQCD contribution to second order in the magnetic field

$$C_0^{(2)} = -\left(\frac{17}{18}\right)\frac{1}{4\pi^2}. \tag{38}$$

The last ingredient needed to find s_0 and $C_4\langle O_4 \rangle$ is the magnetic field dependence of f_π. This may be obtained from the Gell-Mann-Oakes-Renner (GMOR) relation, which relates f_π with the light-quark condensate, and the pion mass with the light-quark masses. The GMOR relation is a low energy theorem involving the pseudoscalar current correlator

$$\psi_5(q^2) = i\int d^4x \, e^{iqx}\langle 0|T(\partial^\mu A_\mu(x)\partial^\nu A_\nu^\dagger(0))|0\rangle, \tag{39}$$

where the divergence of the axial-vector current, $\partial^\mu A_\mu(x)$, is proportional to the sum of the up- and down-quark masses. At zero momentum the GMOR relation reads

$$m_\pi^2 f_\pi^2 = -2(m_u + m_d)\langle\bar{q}q\rangle. \tag{40}$$

In the standard chiral interpretation of this relation f_π is related to the light quark condensate

$$f_\pi^2 = -2\mathcal{B}\langle\bar{q}q\rangle, \tag{41}$$

and the pion mass is related to the light-quark masses

$$m_\pi^2 = \frac{1}{\mathcal{B}}(m_u + m_d), \tag{42}$$

where $\mathcal{B}$ is a chiral perturbation theory constant. In order to find the constant $\mathcal{B}$ one needs to use the physical pion and light-quark masses. In the hadronic sector the pion pole

contribution to the pseudoscalar current correlator, $\psi_5(q^2)$, is given by

$$\psi_5^{\text{HAD}}(q^2) = if_\pi^2 m_\pi^4 \tilde{G}_\pi(q^2), \tag{43}$$

where $\tilde{G}_\pi(q^2)$ is the pion propagator. In the QCD sector the light-quark condensate contribution to $\psi_5(0)$ is given by

$$\psi_5^{\text{QCD}}(0) = -2(m_u + m_d)\langle\bar{q}q\rangle. \tag{44}$$

According to Eq. (18) [see also the discussion leading to Eq. (A7) for the normalization] the pion propagator for $q^2 \geq 0$ in the presence of a weak magnetic field can be approximated as

$$\tilde{G}_\pi(q^2) \simeq \frac{i}{q^2 - m_\pi^2 - eB}, \tag{45}$$

so that the GMOR relation becomes

$$\frac{f_\pi^2 m_\pi^4}{m_\pi^2 + eB} \simeq -2(m_u + m_d)\langle\bar{q}q\rangle. \tag{46}$$

For weak fields, we use the expansion

$$\frac{1}{m_\pi^2 + eB} \sim \frac{1}{m_\pi^2}\left(1 - \frac{eB}{m_\pi^2}\right). \tag{47}$$

Inserting Eq. (47) into Eq. (46), and neglecting a magnetic field dependence of the pion and quark masses, the term $\frac{eB}{m_\pi^2}$ introduces a correction to the GMOR relation of higher order than the one being considered in this work. Therefore, to a good approximation for small fields the magnetic field dependence of f_π^2 is determined by that of $\langle\bar{q}q\rangle$ and the constant $\mathcal{B}$ from its vacuum value. We point out that the validity of the GMOR relation in the presence of a magnetic field was first shown in the first of Refs. [42].
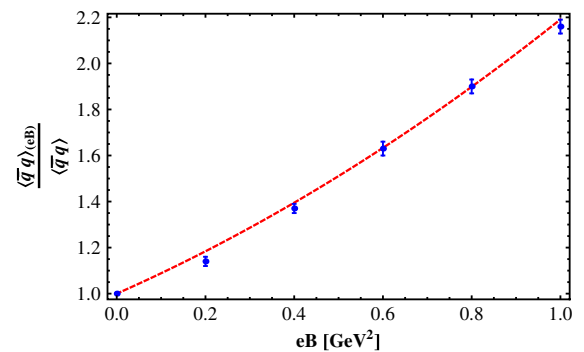

FIG. 5 (color online). Light-quark condensate normalized to its vacuum value as a function of the magnetic field strength eB in units of GeV^2. The data points are from Ref. [2] and the dotted line corresponds to the fit $\langle\bar{q}q\rangle_{(eB)}/\langle\bar{q}q\rangle = 1 + a(eB) + b(eB)^2$, with $a = 0.85 \text{ GeV}^{-2}$, $b = 0.34 \text{ GeV}^{-4}$.
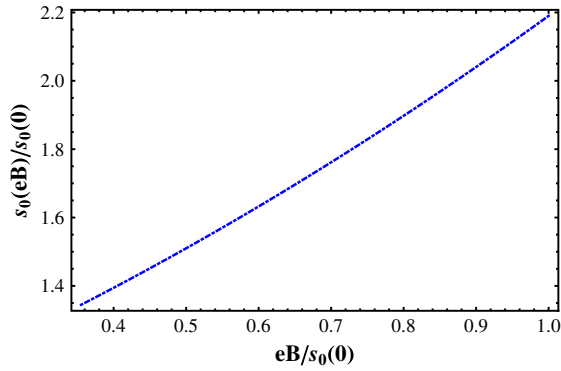
FIG. 6 (color online). Squared energy pQCD threshold, s_0, normalized to its $eB = 0$ value, as a function of the field strength eB. Note that the plotted range is consistent with the requirement $eB < s_0 < 3eB$.
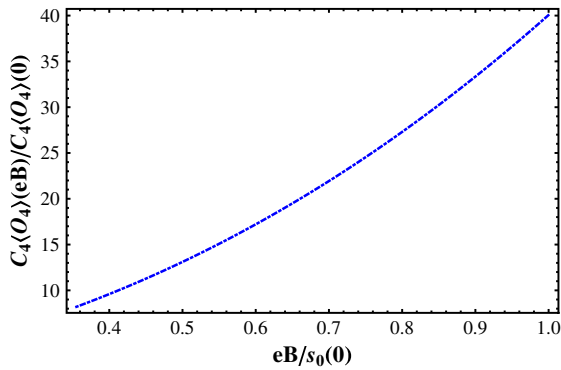

FIG. 7 (color online). The gluon condensate, $C_4\langle O_4\rangle$, normalized to its $eB = 0$ value, as a function of eB. Note that the plotted range is consistent with the requirement $eB < s_0 < 3eB$.

The light-quark condensate in the presence of the magnetic field has been computed in Ref. [2]. We make use of this result, and parametrize the magnetic field dependence of the light-quark condensate with a quadratic fit

$$\langle \bar{q}q\rangle_{(eB)}/\langle \bar{q}q\rangle = 1 + a(eB) + b(eB)^2, \quad (48)$$

where $a = 0.85$ GeV^{-2}, $b = 0.34$ GeV^{-4} and (eB) is given in GeV^2. The data from Ref. [2] together with the fit are shown in Fig. 5. Using this information we finally write the explicit solutions for s_0 and $C_4\langle O_4\rangle$ from Eq. (22):

$$s_0 = -8\pi\mathcal{B}\langle \bar{q}q\rangle_{(eB)}$$

$$C_4\langle O_4\rangle = -2(eB)\mathcal{B}\langle \bar{q}q\rangle_{(eB)} + 8\pi(\mathcal{B}\langle \bar{q}q\rangle_{(eB)})^2$$

$$+ \left(\frac{17}{18}\right)\frac{(eB)^2}{4\pi^2}. \quad (49)$$

The solutions for s_0 and for $C_4\langle O_4\rangle$ as functions of eB are plotted in Figs. 6 and 7, respectively. Note that s_0 is

proportional to the absolute value of the light-quark condensate, and that together with $C_4\langle O_4\rangle$ it increases with increasing magnetic field.

IV. SUMMARY AND CONCLUSIONS

In this paper we studied QCD FESR for the axial-vector current correlator in the presence of a magnetic field in the weak field limit $eB < s_0$, where s_0 is the squared energy threshold for the onset of pQCD. Given that we work in the massless quark limit, the weak field limit can be understood as an expansion in the small parameter eB/s_0. We have shown that the presence of the field modifies both the pQCD and the hadronic sectors of the FESR. The direct modification to the hadronic contribution for the equation that governs the behavior of s_0 vanishes nontrivially. The only change in this sector comes from the dependence of the pion decay constant on the magnetic field. In turn, this pion decay constant is proportional to the quark condensate through the GMOR relation. We have shown that in the limit where the magnetic field is small compared to the physical pion mass squared, the proportionality constant between f_π and $\langle \bar{q}q\rangle$ can be approximated to be the same as in vacuum. Use of lattice QCD data on the quark condensate as a function of the magnetic field allows us then to extract the magnetic field dependence of f_π. We should point out that since lattice QCD results are valid for physical pion masses, the calculation in this work cannot be directly compared to the results in Refs. [42] which are performed using chiral perturbation theory in the limit where $qB \gg m_\pi^2$. The magnetic field dependence of s_0 is proportional to the magnetic field dependence of the absolute value of the light-quark condensate. This means that the behavior of these two parameters as a function of the magnetic field is similar. Therefore the magnetic field both helps the formation of the condensate and acts against deconfinement. Next, we computed explicitly the magnetic corrections to the pQCD contribution which affect the behavior of the gluon condensate as a function of field strength. In the weak field limit the first correction is quadratic in the field. The gluon condensate also grows as a function of the field strength which goes hand in hand with the behavior of the magnetic field, both as a catalyst of chiral symmetry breaking and confinement.

The results obtained here should serve as a basis for studies at finite temperature in an external magnetic field, thus providing clues on the relation between chiral symmetry restoration and deconfinement at the transition temperature. This research is in progress and will be reported elsewhere.

APPENDIX A: IMAGINARY PART OF THE HADRONIC CONTRIBUTION TO THE AXIAL-VECTOR CURRENT CORRELATOR

The charged axial-vector current correlator in the presence of a constant magnetic field, and in the limit $q_\perp \to 0$, is written as

$$\Pi_0^{\mathrm{HAD}}(s) = -2f_\pi^2 \sum_{l=0}^{\infty} \frac{(-1)^l}{s - (2l+1)eB}. \qquad (A1)$$

The above expression can be split into two sums, one for the even and the other for the odd values of l, namely

$$\Pi_0^{\mathrm{HAD}}(s) = -2f_\pi^2 \left\{ \sum_{l=0,\mathrm{even}} \frac{1}{s - (2l+1)eB} - \sum_{l=\mathrm{odd}} \frac{1}{s - (2l+1)eB} \right\}. \qquad (A2)$$

Pulling out a factor $-1/4eB$ from both sums and adding and subtracting the element with $l = -1$ we obtain from Eq. (A2)

$$\Pi_0^{\mathrm{HAD}}(s) = 2f_\pi^2 \left\{ \frac{1}{4eB} \sum_{l'=0}^{\infty} \frac{1}{l' - \frac{s/eB-1}{4}} - \frac{1}{4eB} \sum_{l'=0}^{\infty} \frac{1}{l' - \frac{s/eB+1}{4}} - \frac{1}{s+eB} \right\}, \qquad (A3)$$

where we defined $l' = \frac{l}{2}$ for the sum with even l and $l' = \frac{l+1}{2}$ for the sum with odd l. The sums in Eq. (A3) are divergent. In order to extract the finite piece we regularize them as

$$\Pi_0^{\mathrm{HAD}}(s) = 2f_\pi^2 \lim_{\epsilon \to 1} \left\{ \frac{1}{4eB} \sum_{l'=0}^{\infty} \frac{1}{(l' - \frac{s/eB-1}{4})^\epsilon} - \frac{1}{4eB} \sum_{l'=0}^{\infty} \frac{1}{(l' - \frac{s/eB+1}{4})^\epsilon} - \frac{1}{s+eB} \right\}$$

$$= 2f_\pi^2 \lim_{\epsilon \to 1} \left\{ \frac{1}{4eB} \zeta(\epsilon, (s/eB - 1)/4) - \frac{1}{4eB} \zeta(\epsilon, (s/eB + 1)/4) - \frac{1}{s+eB} \right\}, \qquad (A4)$$

where $\zeta(a, z)$ is the Hurwitz zeta function. Expanding around $\epsilon = 1$ we find

$$\Pi_0^{\mathrm{HAD}}(s) = 2f_\pi^2 \left\{ \frac{1}{4eB} \left[\frac{1}{\epsilon - 1} - \psi\left(\frac{-(s/eB - 1)}{4} \right) \right] - \frac{1}{4eB} \left[\frac{1}{\epsilon - 1} - \psi\left(\frac{-(s/eB + 1)}{4} \right) \right] - \frac{1}{s+eB} \right\}$$

$$= -2f_\pi^2 \left\{ \frac{1}{4eB} \psi\left(\frac{-(s/eB - 1)}{4} \right) + \frac{1}{4eB} \psi\left(\frac{-(s/eB + 1)}{4} \right) + \frac{1}{s+eB} \right\}, \qquad (A5)$$

where $\psi(x)$ is the digamma function. We note that the divergent pieces cancel when $\epsilon \to 1$. Recall that $\psi(x)$ is singular for $x = 0, -1, -2, \dots$ In the region $0 \le eB < s_0$, neither of the digamma functions in Eq. (A5) becomes singular. The first singularity for $\psi(-(s/eB - 1)/4)$ happens at $s = eB$ and for $\psi(-(s/eB + 1)/4)$ at $s = 3B$. Therefore, by restricting the analysis to the region $eB \le s_0 < 3eB$ we can compute the discontinuity, or imaginary part of Eq. (A5), with the result

$$\mathrm{Im}\Pi_0^{\mathrm{HAD}}(s) = f_\pi^2 \pi \delta(s - eB), \qquad (A6)$$

where since s is strictly larger than or equal to 0, one has

$$\lim_{\epsilon \to 0} \frac{\epsilon}{(s - eB) + \epsilon^2} = \frac{\pi}{2} \delta(s - eB). \qquad (A7)$$

Finally, in the limit $eB \to 0$ the imaginary part of the correlator becomes

$$\mathrm{Im}\Pi_0^{\mathrm{HAD}}(s) = f_\pi^2 \pi \delta(s), \qquad (A8)$$

which coincides with the known value in the absence of a magnetic field.

APPENDIX B: FIRST ORDER MAGNETIC CORRECTION TO THE PQCD CONTRIBUTION TO THE AXIAL-VECTOR CURRENT CORRELATOR

The contribution to the pQCD axial-vector current correlator of order (eB) is given by

$$\Pi_{\mu\nu}^B(q^2) = \Pi_{\mu\nu}^{(10)}(q^2) + \Pi_{\mu\nu}^{(01)}(q^2) \qquad (B1)$$

where

$$\Pi_{\mu\nu}^{(10)}(q^2) = N_c(q_u B) \int \frac{d^4 k}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_1 \gamma_2 [\gamma \cdot (k-q)_\parallel] \gamma_\mu k \gamma_\nu]}{k^2 (k-q)^4}$$

$$\Pi_{\mu\nu}^{(01)}(q^2) = N_c(q_d B) \int \frac{d^4 k}{(2\pi)^4} \frac{\mathrm{Tr}[(k - \slashed{q}) \gamma_\mu \gamma_1 \gamma_2 [\gamma \cdot q]_\parallel \gamma_\nu]}{(k-q)^2 k^4}. \qquad (B2)$$

Note that the traces in Eqs. (B2) are equal except for an overall sign, due to the ordering of elements inside the trace. Therefore, we evaluate one of the traces,

$$
\begin{aligned}
\text{Tr}[\gamma_1\gamma_2[\gamma\cdot(k-q)_\parallel]\gamma_\mu k\gamma_\nu] \\
&= k^\alpha(k-q)_\parallel^\beta = \text{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_1\gamma_2\gamma_\beta^\parallel] \\
&= -ik^\alpha\text{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\rho\gamma_5 \\
&\quad \times \{b^\rho[(k-q)\cdot u] \\
&\quad - u^\rho[(k-q)\cdot b]\}],
\end{aligned}
\tag{B3}
$$

where u^ρ and b^ρ are four-vectors describing the particle's rest frame and the direction of the magnetic field, respectively. In the rest frame, these are given by

$$
\begin{aligned}
u &= (1,0,0,0) \\
b &= (0,0,0,1).
\end{aligned}
\tag{B4}
$$

Using that

$$
\text{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\rho\gamma_5] = -4i\epsilon_{\mu\alpha\nu\rho},
\tag{B5}
$$

we can write Eq. (B3) as

$$
\begin{aligned}
\text{Tr}[\gamma_1\gamma_2[\gamma\cdot(k-q)_\parallel]\gamma_\mu k\gamma_\nu] \\
&= -4k^\alpha\epsilon_{\mu\alpha\nu\rho}\{b^\rho[(k-q)\cdot u] - u^\rho[(k-q)\cdot b]\}.
\end{aligned}
\tag{B6}
$$

Notice that when Eq. (B6) is contracted with $q^\mu q^\nu$ the coefficient of the longitudinal structure vanishes.

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