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Exploring a new $SU(4)$ symmetry of meson interpolators

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In recent lattice calculations it has been discovered that mesons upon truncation of the quasizero modes of the Dirac operator obey a symmetry larger than the $SU(2)_L \times SU(2)_R \times U(1)_A$ symmetry of the QCD Lagrangian. This symmetry has been suggested to be $SU(4) \supset SU(2)_L \times SU(2)_R \times U(1)_A$ that mixes not only the u- and d-quarks of a given chirality, but also the left- and right-handed components. Here it is demonstrated that bilinear $\bar{q}q$ interpolating fields of a given spin $J \geq 1$ transform into each other according to irreducible representations of $SU(4)$ or, in general, $SU(2N_F)$. This fact together with the coincidence of the correlation functions establishes $SU(4)$ as a symmetry of the $J \geq 1$ mesons upon quasizero mode reduction. It is shown that this symmetry is a symmetry of the confining instantaneous charge-charge interaction in QCD. Different subgroups of $SU(4)$ as well as the $SU(4)$ algebra are explored.

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I. INTRODUCTION

In recent $N_F = 2$ dynamical lattice simulations with the manifestly chiral-invariant overlap Dirac operator, a new symmetry of mesons of given spin has been discovered upon truncation of the quasizero modes of the Dirac operator, Refs. [1,2] (A hint for this symmetry had been seen in a previous study, Ref. [3].) Namely, the $J = 1$ mesons $\rho, \rho', \omega, \omega', a_1, b_1, h_1, f_1$ get degenerate after removal of the lowest-lying Dirac eigenmodes.[1] A similar degeneracy is seen also in $J = 2$ mesons, Ref. [4]. This symmetry has been suggested to be $SU(4) \supset SU(2)_L \times SU(2)_R \times U(1)_A$ that mixes components of the fundamental vector (u_L, u_R, d_L, d_R), Ref. [5]. It is higher than the broken $SU(2)_L \times SU(2)_R \times U(1)_A$ symmetry of the QCD Lagrangian and should be considered as an emergent symmetry in $J \geq 1$ mesons that reflects the QCD dynamics once the quasizero modes of the Dirac operator have been removed. It has been proposed that this symmetry might be a symmetry of the dynamical QCD string because there is no color-magnetic interaction (field) in the system, Ref. [5].

In the present paper we extend findings of [5] and show that the composite $J \geq 1 \bar{q}q$ bilinear operators (interpolating fields) with nonexotic quantum numbers transform according to irreducible dim = 15 and dim = 1 representations of the $SU(4) \supset SU(2)_L \times SU(2)_R \times U(1)_A \times C_i$ group. This result holds irrespective of the observations made in Refs. [1,2] as well as possible physics interpretations in Ref. [5]. The correlation functions obtained with these operators get indistinguishable after truncation, Ref. [2]. This fact establishes consequently the proposed $SU(4)$ symmetry as the symmetry of the $J \geq 1$ spectra upon the quasizero mode reduction. We also study different subgroups of $SU(4)$, the corresponding algebras as well as transformation properties of the interpolators with respect to these subgroups.

The outline of the article is as follows: In Sec. II we review the classification of the spin-1 $\bar{q}q$-bilinears with respect to $SU(2)_L \times SU(2)_R$ and $U(1)_A$ transformations, Ref. [6]. In Sec. III we demonstrate that all these interpolators are connected with each other through the $SU(4)$ transformations that include not only the chiral rotations but also a mixing between the left- and right-handed components, specify interpolators that transform according to different subgroups of $SU(4)$ and construct the respective algebras. A generalization to $SU(2N_F)$ and to general spin is also discussed. In the last section we show that the observed $SU(4)$ symmetry implies the absence of the color-magnetic field in the confined system after the low-mode elimination and can be considered as a manifestation of the dynamical string in QCD.

II. CHIRAL CLASSIFICATION OF THE $J = 1$ BILINEAR OPERATORS

We work in Minkowski space with the chiral representation of the γ-matrices. In flavor space we use the Pauli matrices $\boldsymbol{\tau}$. The basic definitions are collected in the Appendix. With the notation

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{1}$$

we make explicit the two flavors in the quark field. The left- and right-handed quark fields for one flavor are defined via the projection operator $P_\pm = 1/2(\mathbb{1} \pm \gamma^5)$, which can be generalized for two quark flavors by defining the projectors as $\Gamma_\pm = (\mathbb{1}_F \otimes P_\pm)$:

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[1]It is not yet entirely clear from the lattice results whether the f_1 state is degenerate with other $J = 1$ mesons. While the quality of the effective mass plateau is excellent for the $\rho, \rho', \omega, \omega', a_1, b_1, h_1$ mesons it is not so for the f_1 state.

$$\Psi_{\rm L} = \Gamma_{-}\Psi, \qquad \Psi_{\rm R} = \Gamma_{+}\Psi. \tag{2}$$

All $\bar{q}q$-mesons and respective operators with nonexotic quantum numbers can be arranged into irreducible representations of the parity-chiral group $SU(2)_L \times SU(2)_R \times C_i$, Ref. [7]. We use the notation $(I_{\rm L}, I_{\rm R})$, with left-handed $(I_{\rm L})$ and right-handed $(I_{\rm R})$ isospin for each irreducible representation of $SU(2)_L \times SU(2)_R$. The classification of spin-1 meson operators is presented in Fig. 1. Below each meson a corresponding interpolator $J^r_{(I,J^{PC})}$ is given with r being the index of an irreducible representation of the parity-chiral group.

As an example we now compare the combination of left- and right-handed quarks within the interpolators of the two isovectors 1^{--}. We start with the interpolators $J^{(1,0)\oplus(0,1)}_{(1,1^{--})} = \bar{\Psi}(\tau^a \otimes \gamma^k)\Psi$ and write it in terms of left- and right-handed quarks:

$$J^{(1,0)\oplus(0,1)}_{(1,1^{--})} = \bar{\Psi}_{\rm L}(\tau^a \otimes \gamma^k)\Psi_{\rm L} + \bar{\Psi}_{\rm R}(\tau^a \otimes \gamma^k)\Psi_{\rm R}. \tag{3}$$

It has the chiral content $\bar{L}L + \bar{R}R$. The interpolator $J^{(1/2,1/2)_b}_{(1,1^{--})} = \bar{\Psi}(\tau^a \otimes \gamma^0\gamma^k)\Psi$ can be split up as

$$J^{(1/2,1/2)_b}_{(1,1^{--})} = \bar{\Psi}_{\rm L}(\tau^a \otimes \gamma^0\gamma^k)\Psi_{\rm R} + \bar{\Psi}_{\rm R}(\tau^a \otimes \gamma^0\gamma^k)\Psi_{\rm L}, \tag{4}$$

and has the chiral content $\bar{L}R + \bar{R}L$.

The axial part of the $SU(2)_L \times SU(2)_R$ transformations is defined by

$$\Psi \to \Psi' = e^{i\frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{\tau}}{2}\otimes\gamma^5}\Psi \equiv U\Psi. \tag{5}$$

These axial transformations do not form a closed group. However, we use $SU(2)_A$ as a shorthand notation for these transformations in the text below and in Fig. 1.

The matrix U has the property $U^\dagger(\mathbb{1}_{\rm F} \otimes \gamma^0) = (\mathbb{1}_{\rm F} \otimes \gamma^0)U$, from which $\bar{\Psi}' = \bar{\Psi}U$ follows. It can be expressed in closed form as

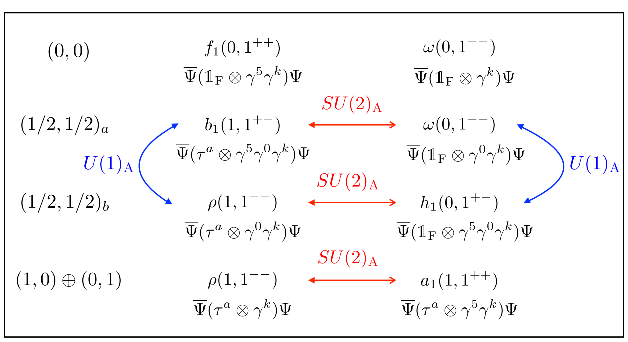

FIG. 1 (color online). On the left column the irreducible representations of the parity-chiral group are given. Each meson is denoted as (I, J^{PC}), with I isospin, J total spin, P parity and C charge conjugation. Below each state a current from which it can be generated is given. The $SU(2)_A$ and $U(1)_A$ connections are denoted by red and blue lines, respectively.

$$U = (\mathbb{1}_{\rm F} \otimes \mathbb{1}_{\rm D}) \cos\left[\frac{|\boldsymbol{\varepsilon}|}{2}\right] + i(\hat{\boldsymbol{\varepsilon}} \cdot \boldsymbol{\tau} \otimes \gamma^5) \sin\left[\frac{|\boldsymbol{\varepsilon}|}{2}\right], \tag{6}$$

with $\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}/|\boldsymbol{\varepsilon}|$. We now apply the $SU(2)_A$ transformation U on the individual interpolators of Fig. 1. For instance, the interpolator $J^{(1/2,1/2)_a}_{(0,1^{--})}$ transforms as

$$\bar{\Psi}'(\mathbb{1}_{\rm F} \otimes \gamma^0\gamma^k)\Psi' = \bar{\Psi}(\mathbb{1}_{\rm F} \otimes \gamma^0\gamma^k)\Psi \cdot \mathcal{E} + \bar{\Psi}(\tau^a \otimes \gamma^5\gamma^0\gamma^k)\Psi \cdot \mathcal{F}^a, \tag{7}$$

with $\mathcal{E} = \cos|\boldsymbol{\varepsilon}|$ and $\mathcal{F}^a = i\hat{\boldsymbol{\varepsilon}}^a \sin|\boldsymbol{\varepsilon}|$ being functions of the rotation vector $\boldsymbol{\varepsilon}$ only. We find that the following pairs become connected via the $SU(2)_A$ (see Fig. 1):

$$J^{(1/2,1/2)_a}_{(1,1^{+-})} \longleftrightarrow J^{(1/2,1/2)_a}_{(0,1^{--})}, \tag{8}$$

$$J^{(1/2,1/2)_b}_{(1,1^{--})} \longleftrightarrow J^{(1/2,1/2)_b}_{(0,1^{+-})}, \tag{9}$$

$$J^{(1,0)\oplus(0,1)}_{(1,1^{--})} \longleftrightarrow J^{(1,0)\oplus(0,1)}_{(1,1^{++})}. \tag{10}$$

Similarly, the $U(1)_A$ transformation

$$\Psi \to \Psi' = e^{i\alpha(\mathbb{1}_{\rm F}\otimes\gamma^5)}\Psi \tag{11}$$

connects interpolators from the $(1/2, 1/2)_a$ and $(1/2, 1/2)_b$ representations which have the same isospin but opposite spatial parity. These four interpolators form an irreducible representation of $SU(2)_L \times SU(2)_R \times U(1)_A$. The interpolators from the $(1, 0) \oplus (0, 1)$ representation are self-dual with respect to the $U(1)_A$ transformations. The singlet interpolators from the (0,0) representations are invariant with respect to both $U(1)_A$ and $SU(2)_A$ transformations.

III. EXTENDING $SU(2)_L \times SU(2)_R \times U(1)_A \times C_i$ TO $SU(4)$

A. Left-right mixing

Our task is to find transformations that mix different representations of the parity-chiral group. The representations $(1/2, 1/2)$ have the quark content $\bar{L}R \pm \bar{R}L$ and the representations $(0, 0), (1, 0) \oplus (0, 1)$ have the quark content $\bar{L}L \pm \bar{R}R$. Consequently, in order to connect these representations one needs to find a symmetry transformation, which mixes left- and right-handed quarks, Ref. [5].

Consider the fundamental doublets

$$\mathrm{U} = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

and

$$\mathrm{D} = \begin{pmatrix} d_L \\ d_R \end{pmatrix}$$

constructed from Weyl spinors. We can consider $SU(2)_{\mathrm{u}}$ and $SU(2)_{\mathrm{d}}$ rotations of these doublets in an imaginary three-dimensional space that mix the u_{L} and u_{R} as well as the d_{L} and d_{R} spinors. It is similar to the well familiar concept of the isospin space: The electric charges of particles are conserved quantities, but rotations in the isospin space mix particles with different electric charges. In our case the chirality of a massless quark is a conserved quantity but the $SU(2)_{\mathrm{u}}$ and $SU(2)_{\mathrm{d}}$ rotations mix quarks with different chiralities:

$$\mathrm{U} \to \mathrm{U}' = e^{i\frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}}{2}}\mathrm{U}, \qquad \mathrm{D} \to \mathrm{D}' = e^{i\frac{\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}}{2}}\mathrm{D}, \tag{12}$$

where $\boldsymbol{\sigma}$ are the standard Pauli matrices which obey the $\mathfrak{su}(2)$ algebra:

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k. \tag{13}$$

We refer to this imaginary three-dimensional space as the *chiralspin* space.

Instead of the Weyl spinors we can consider the left- and right-handed Dirac bispinors. Then, the required $\mathfrak{su}(2)$ algebra can be constructed with the 4×4 matrices

$$\Sigma = \{\gamma^0, i\gamma^5\gamma^0, -\gamma^5\}, \tag{14}$$

with the commutation relation

$$[\Sigma^i, \Sigma^j] = 2i\epsilon^{ijk}\Sigma^k. \tag{15}$$

These rotations act in Dirac space only and are diagonal in flavor space:

$$\Psi \to \Psi' = e^{i(\mathbb{1}_{\mathrm{F}} \otimes \frac{\boldsymbol{\varepsilon}\cdot\Sigma}{2})}\Psi \equiv V\Psi. \tag{16}$$

We denote this symmetry group as $SU(2)_{\mathrm{cs}}$. We note that in the compact notation of Eq. (16) two $SU(2)_{\mathrm{u}}$ and $SU(2)_{\mathrm{d}}$ symmetries for the individual quark flavors u and d are hidden.

In analogy to Eq. (6) we express V as

$$V = (\mathbb{1}_{\mathrm{F}} \otimes \mathbb{1}_{\mathrm{D}}) \cos\left[\frac{|\boldsymbol{\varepsilon}|}{2}\right] + i(\mathbb{1}_F \otimes \hat{\boldsymbol{\varepsilon}} \cdot \Sigma) \sin\left[\frac{|\boldsymbol{\varepsilon}|}{2}\right]. \tag{17}$$

Now we apply these chiralspin rotations on the interpolators in Fig. 1 and find the following triplets[2] of interpolators that are connected to each other[3]:

$$J^{(0,0)}_{(0,1^{--})} \leftrightarrow J^{(1/2,1/2)_a}_{(0,1^{--})} \leftrightarrow J^{(1/2,1/2)_b}_{(0,1^{+-})}, \tag{18}$$

$$J^{(1,0)\oplus(0,1)}_{(1,1^{--})} \leftrightarrow J^{(1/2,1/2)_b}_{(1,1^{--})} \leftrightarrow J^{(1/2,1/2)_a}_{(1,1^{+-})}. \tag{19}$$

This means that, transforming any of the interpolators in Eq. (18) with respect to V, the result can always be decomposed as

$$\bar{\Psi}(\mathbb{1}_{\mathrm{F}} \otimes \gamma^k)\Psi \cdot \mathcal{E}^{(i)} + \bar{\Psi}(\mathbb{1}_{\mathrm{F}} \otimes \gamma^5\gamma^0\gamma^k)\Psi \cdot \mathcal{F}^{(i)} + \bar{\Psi}(\mathbb{1}_{\mathrm{F}} \otimes \gamma^0\gamma^k)\Psi \cdot \mathcal{G}^{(i)}, \tag{20}$$

with $i = 1, 2, 3$ labeling the interpolators, and $\mathcal{E}^{(i)}, \mathcal{F}^{(i)}, \mathcal{G}^{(i)}$ being functions of the rotation vector $\boldsymbol{\varepsilon}$ only. Performing a transformation of the interpolating currents in Eq. (19) leads to the same decomposition with τ^a instead of $\mathbb{1}_{\mathrm{F}}$ in flavor space. It is clear why we get two triplets of states: in Eq. (16) two $SU(2)$ symmetries, namely for up and down quarks, appear.[4] The interpolators

$$J^{(0,0)}_{(0,1^{++})} = \bar{\Psi}(\mathbb{1}_{\mathrm{F}} \otimes \gamma^5\gamma^k)\Psi, \tag{21}$$

$$J^{(1,0)\oplus(0,1)}_{(1,1^{++})} = \bar{\Psi}(\tau^a \otimes \gamma^5\gamma^k)\Psi, \tag{22}$$

are invariant[5] with respect to $SU(2)_{\mathrm{cs}}$. In group-theoretical language, we have shown the multiplication rule $2 \otimes 2 = 3 \oplus 1$ for $SU(2)$.

The $SU(2)_{CS}$ triplets and singlets are shown in Fig. 2.

The $U(1)_A$ symmetry is contained in $SU(2)_{\mathrm{cs}}$ as a subgroup.

Let us, at the end of this section, emphasize that the $SU(2)_{\mathrm{cs}}$ symmetry is not a symmetry of the QCD Lagrangian. We apply a $SU(2)_{\mathrm{cs}}$ transformation on the fermion part of the QCD Lagrangian:

$$\bar{\Psi}'(\mathbb{1}_{\mathrm{F}} \otimes \gamma^\mu D_\mu)\Psi' = \bar{\Psi}(\mathbb{1}_{\mathrm{F}} \otimes \gamma^0 D_0)\Psi - \bar{\Psi}(\mathbb{1}_{\mathrm{F}} \otimes \gamma^0)V^\dagger(\mathbb{1}_{\mathrm{F}} \otimes \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{D})V\Psi. \tag{23}$$

The γ^0-part is invariant under this transformation. The spacial part would only be invariant if $\bar{\chi}^i = \chi^i (i = 1, 2, 3)$, see Eq. (A1), i.e., both the left- and right-handed fermions fulfilled the same Weyl equations, as intended by the symmetry.[6] An invariance can also be achieved by a spatial coupling in the Lagrangian of the form $\gamma^5\gamma^k$, see Eq. (21).

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[2]Consequently, the chiralspin 1 should be ascribed to these fields.

[3]When applying V on $\bar{\Psi}$ we have to be careful, since V and $(\mathbb{1}_{\mathrm{F}} \otimes \gamma_0)$ do not commute. We write $\bar{\Psi}' = \bar{\Psi}(\mathbb{1}_{\mathrm{F}} \otimes \gamma^0)V^\dagger(\mathbb{1}_{\mathrm{F}} \otimes \gamma^0)$.

[4]The symmetry $SU(2)_{CS}$ connects the interpolators with off-diagonal γ-structure to interpolators with diagonal γ-structure (in the chiral representation of the γ-matrices). This is how left- and right-handed quarks are mixed.

[5]I.e., their chiralspin is 0.

[6]For example, this symmetry is manifestly violated by instantons. Only the left-handed quark satisfies the Dirac equation with zero eigenvalue in the field of an instanton, while only the right-handed quark produces a zero mode in the field of an anti-instanton, Refs. [8,9].

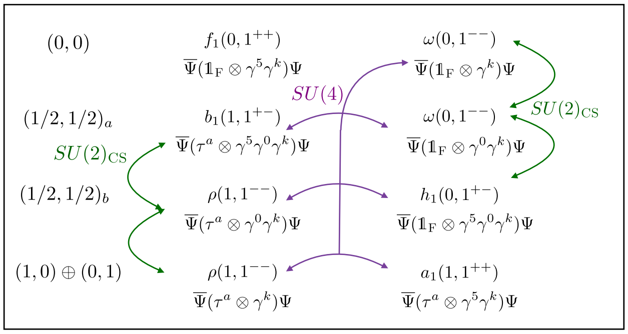

FIG. 2 (color online). The $SU(2)_{CS}$ triplets are denoted by green lines; f_1 and a_1 are the $SU(2)_{CS}$-singlets. The $SU(4)$ 15-plet is indicated by purple lines; f_1 is the $SU(4)$-singlet.

B. $SU(4)$

When we try to find a common algebra for the $SU(2)_L \times SU(2)_R$ and the $SU(2)_{\rm cs}$ symmetries, we immediately arrive at the $\mathfrak{su}(4)$ algebra. This is due to the commutator

$$[(\mathbb{1}_{\rm F} \otimes \Sigma^i), (\tau^a \otimes \Sigma^3)] = 2i\epsilon^{i3k}(\tau^a \otimes \Sigma^k), \tag{24}$$

with $a = 1, 2, 3$ and $i, k = 1, 2, 3$. The 15 matrices altogether,

$$\{(\tau^a \otimes \mathbb{1}_D), (\mathbb{1}_F \otimes \Sigma^i), (\tau^a \otimes \Sigma^i)\}, \tag{25}$$

form the generators T^l of the $\mathfrak{su}(4)$ algebra, satisfying the following commutation relations:

$$[T^l, T^m] = 2if^{lmn}T^n, \qquad f^{lmn} = \frac{1}{8i}{\rm Tr}[[T^l, T^m]T^n], \tag{26}$$

$$\{T^l, T^m\} = 2\delta^{lm}\mathbb{1} + 2d^{lmn}T^n,$$
$$d^{lmn} = \frac{1}{8}{\rm Tr}[\{T^l, T^m\}T^n], \tag{27}$$

with f^{lmn} denoting the totally antisymmetric structure constants and d^{lmn} a totally symmetric tensor, $l, m, n = 1, 2, \ldots, 15$. The formula

$$(\boldsymbol{\epsilon}\cdot\mathbf{T})(\boldsymbol{\epsilon}\cdot\mathbf{T}) = \boldsymbol{\epsilon}^2 + (if^{lmn} + d^{lmn})\epsilon^l\epsilon^m T^n \tag{28}$$

follows from the (anti)commutation relations.

We denote this symmetry as

$$\Psi \to \Psi' = e^{i\boldsymbol{\epsilon}\cdot\mathbf{T}/2}\Psi \equiv W\Psi, \tag{29}$$

with the fundamental vector Ψ being four-dimensional:

$$\Psi = \begin{pmatrix} u_{\rm L} \\ u_{\rm R} \\ d_{\rm L} \\ d_{\rm R} \end{pmatrix}. \tag{30}$$

The $SU(4)$ symmetry transformation mixes both quark flavors and left/right-handed components. For instance, a left-handed up quark now transforms as

$$u_{\rm L} \to a\cdot u_{\rm L} + b\cdot u_{\rm R} + c\cdot d_{\rm L} + e\cdot d_{\rm R}, \tag{31}$$

with a, b, c, d being functions of the rotation vector $\boldsymbol{\varepsilon}$. The new mixing, not present for $SU(2)_L \times SU(2)_R$ and $SU(2)_{\rm cs}$, is between $u_{\rm L}$ and $d_{\rm R}$ (and accordingly for the other quark flavors).

In principle the matrix W could be written in linearized form [according to Eqs. (6) and (17)]

$$W = a_0(\mathbb{1}_{\rm F} \otimes \mathbb{1}_{\rm D}) + a_1(i\boldsymbol{\epsilon}\cdot\mathbf{T}/2), \tag{32}$$

with the coefficients a_0, a_1 being expressions of f^{abc} and d^{abc}, see Ref. [10]. We perform an analytical evaluation with MATHEMATICA, where we express W by its spectral decomposition. We calculate which fields (mesons) are connected via $SU(4)$ by transforming each interpolator in Fig. 1 with respect to W, Eq. (29). We arrive that the following interpolators get mixed via W:

$$J^{(0,0)}_{(0,1^{--})} \leftrightarrow J^{(1/2,1/2)_a}_{(0,1^{--})} \leftrightarrow J^{(1/2,1/2)_b}_{(0,1^{+-})} \leftrightarrow J^{(1,0)\oplus(0,1)}_{(1,1^{--})}$$
$$\leftrightarrow J^{(1/2,1/2)_a}_{(1,1^{+-})} \leftrightarrow J^{(1/2,1/2)_b}_{(1,1^{--})} \leftrightarrow J^{(1,0)\oplus(0,1)}_{(1,1^{++})}. \tag{33}$$

They form basis vectors for a dim = 15 irreducible representation of $SU(4)$. Hence, any of the currents above, when transformed with respect to W, Eq. (29), can be decomposed as

$$\bar\Psi(\Xi^\alpha \otimes \gamma^k)\Psi\cdot\mathcal{E}^\alpha_{(i)} + \bar\Psi(\Xi^\alpha \otimes \gamma^0\gamma^k)\Psi\cdot\mathcal{F}^\alpha_{(i)} \tag{34}$$

$$+\bar\Psi(\Xi^\alpha \otimes \gamma^5\gamma^0\gamma^k)\Psi\cdot\mathcal{G}^\alpha_{(i)} + \bar\Psi(\tau^a \otimes \gamma^5\gamma^k)\Psi\cdot\mathcal{K}^a_{(i)}, \tag{35}$$

where we used the compact notation $\Xi^\alpha = (\mathbb{1}_{\rm F}, \tau^a)$, $(\alpha = 1, 2, 3, 4)$ and $\mathcal{E}^\alpha_{(i)}, \mathcal{F}^\alpha_{(i)}, \mathcal{G}^\alpha_{(i)}, \mathcal{K}^a_{(i)}$ are functions of the rotation parameter $\boldsymbol{\varepsilon}$ only. The index i labels the interpolators.

In this decomposition the interpolator

$$J^{(0,0)}_{(0,1^{++})} = \bar\Psi(\mathbb{1}_{\rm F} \otimes \gamma^5\gamma^k)\Psi \tag{36}$$

is missing, because it is invariant with respect to W, Eq. (29), i.e., represents a singlet representation of $SU(4)$. We have thus shown the following $SU(4)$ multiplication rule: $\bar 4 \otimes 4 = 15 \oplus 1$. The $SU(4)$ singlet and 15-plet are shown in Fig. 2.

C. Other transformations

The $SU(2)_L \times SU(2)_R$ and $SU(2)_{\rm cs}$ symmetries are two subgroups of $SU(4)$. The matrices

$$T^{a,i} = \{(\tau^a \otimes \mathbb{1}_D), (\tau^a \otimes \Sigma^i)\}, \qquad i = 1, 2, \tag{37}$$

with Σ^1 and Σ^2 given in Eq. (14), generate two additional subgroups of $SU(4)$. The transformations

$$\Psi \to \Psi' = e^{i(\frac{\epsilon\cdot\tau}{2}\otimes\Sigma_1)}\Psi \equiv X\Psi, \tag{38}$$

$$\Psi \to \Psi' = e^{i(\frac{\epsilon\cdot\tau}{2}\otimes\Sigma_2)}\Psi \equiv Y\Psi, \tag{39}$$

do not form closed subgroups but we denote them for shortness as $SU(2)_X$ and $SU(2)_Y$. They can be expressed in closed form according to Eq. (6) with γ^0 ($i\gamma^5\gamma^0$) instead of γ^5 in Dirac space.

The following left/right-handed quark flavors mix with u_L via these symmetries:

$$u_\mathrm{L} \to a\cdot u_\mathrm{L} + b\cdot u_\mathrm{R} + c\cdot d_\mathrm{R}, \tag{40}$$

which means that u_L mixes with all L/R-quark flavors except d_L. The same is true for u_R, which mixes with all flavors except d_R. So the mixings of the chiral $SU(2)_L \times SU(2)_R$ symmetry, namely $u_\mathrm{L} \leftrightarrow d_\mathrm{L}, u_\mathrm{R} \leftrightarrow d_\mathrm{R}$, do not occur for these two X and Y transformations.

We now identify which interpolators become connected. We start with the transformation X, Eq. (38), for which the following mixings occur:

$$J^{(1/2,1/2)_a}_{(0,1^{--})} \longleftrightarrow J^{(1,0)\oplus(0,1)}_{(1,1^{--})}, \tag{41}$$

$$J^{(1/2,1/2)_a}_{(1,1^{+-})} \longleftrightarrow J^{(1,0)\oplus(0,1)}_{(1,1^{++})}, \tag{42}$$

$$J^{(1/2,1/2)_b}_{(1,1^{--})} \longleftrightarrow J^{(0,0)}_{(0,1^{--})}. \tag{43}$$

The interpolators

$$J^{(0,0)}_{(0,1^{++})} = \bar{\Psi}(\mathbb{1}_\mathrm{F} \otimes \gamma^5\gamma^k)\Psi, \tag{44}$$

$$J^{(1/2,1/2)_b}_{(0,1^{+-})} = \bar{\Psi}(\mathbb{1}_\mathrm{F} \otimes \gamma^5\gamma^0\gamma^k)\Psi, \tag{45}$$

are invariant.

Now we turn to the transformation Y, Eq. (39). Here the particles with interpolators

$$J^{(1/2,1/2)_b}_{(0,1^{+-})} \longleftrightarrow J^{(1,0)\oplus(0,1)}_{(1,1^{--})}, \tag{46}$$

$$J^{(1/2,1/2)_b}_{(1,1^{--})} \longleftrightarrow J^{(1,0)\oplus(0,1)}_{(1,1^{++})}, \tag{47}$$

$$J^{(1/2,1/2)_a}_{(1,1^{+-})} \longleftrightarrow J^{(0,0)}_{(0,1^{--})}, \tag{48}$$

form doublets. The interpolators

$$J^{(0,0)}_{(0,1^{++})} = (\mathbb{1}_\mathrm{F} \otimes \gamma^5\gamma^k), \tag{49}$$

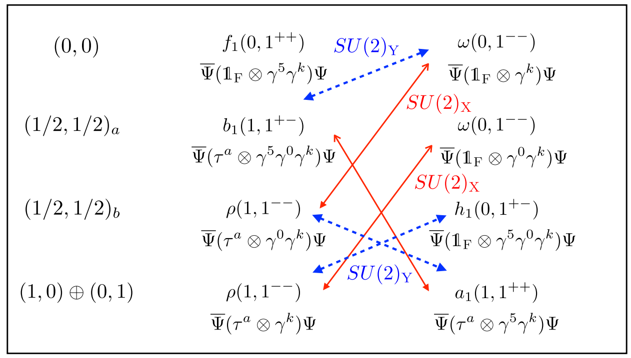

FIG. 3 (color online). Symmetry transformations $SU(2)_X$, Eq. (38) (red), $SU(2)_Y$, Eq. (39) (dotted blue) connecting spin-1 meson fields.

$$J^{(1/2,1/2)_a}_{(0,1^{--})} = (\mathbb{1}_\mathrm{F} \otimes \gamma^0\gamma^k), \tag{50}$$

are invariant.

To make our findings more transparent, in Fig. 3 we show how the transformations $SU(2)_X$ (red), $SU(2)_Y$ (dotted blue) connect the different mesons of spin-1.

D. Generalization to arbitrary spin

The $SU(4)$ symmetry holds also for arbitrary spin $J \geq 1$, because for any $J \geq 1$ one can construct interpolators with derivatives that have exactly the same chiral transformation properties as those in Fig. 1, see for details Ref. [7].

For even spins, $J = 2n, n = 1, 2, \ldots$ we have the following 15-plets:

$$J^{(0,0)}_{(0,J^{++})} \leftrightarrow J^{(1/2,1/2)_a}_{(0,J^{++})} \leftrightarrow J^{(1/2,1/2)_b}_{(0,J^{-+})} \leftrightarrow J^{(1,0)\oplus(0,1)}_{(1,J^{++})} \leftrightarrow J^{(1/2,1/2)_a}_{(1,J^{-+})} \leftrightarrow J^{(1/2,1/2)_b}_{(1,J^{++})} \leftrightarrow J^{(1,0)\oplus(0,1)}_{(1,J^{--})}, \tag{51}$$

and for mesons with spin $J = 2n - 1$ we have

$$J^{(0,0)}_{(0,J^{--})} \leftrightarrow J^{(1/2,1/2)_a}_{(0,J^{--})} \leftrightarrow J^{(1/2,1/2)_b}_{(0,J^{+-})} \leftrightarrow J^{(1,0)\oplus(0,1)}_{(1,J^{--})} \leftrightarrow J^{(1/2,1/2)_a}_{(1,J^{+-})} \leftrightarrow J^{(1/2,1/2)_b}_{(1,J^{--})} \leftrightarrow J^{(1,0)\oplus(0,1)}_{(1,J^{++})}. \tag{52}$$

The $SU(4)$-singlets are $J^{(0,0)}_{(0,J^{--})}$ for even spin and $J^{(0,0)}_{(0,J^{++})}$ for odd spin.

For $J = 0$ only the $(1/2, 1/2)_a$ and $(1/2, 1/2)_b$ chiral representations are possible and the symmetry group is $SU(2)_L \otimes SU(2)_R \times U(1)_A$.

E. Generalization to three and N_F flavors

The three flavor mesons are classified according to $SU(3)_L \otimes SU(3)_R$, and fall into the irreducible representations $(1, 1), (\bar{3}, 3) \oplus (3, \bar{3}), (8, 1) \oplus (1, 8)$. The symmetry connecting the interpolators in these distinct irreducible representations is $SU(6)$ with the 35 generators

$$T^l = \{(\lambda^a \otimes \mathbb{1}_{\rm D}), (\mathbb{1}_{\rm F} \otimes \Sigma^i), (\lambda^a \otimes \Sigma^i)\}, \quad (53)$$

and λ^a the Gell-Mann matrices $(a = 1, \ldots, 8)$, $l = 1, 2, \ldots, 35$. The fundamental vector Ψ is six dimensional and we have for fixed spin J the multiplication rule: $\bar{6} \otimes 6 = 35 \oplus 1$. This can be further generalized to $N_{\rm F}$ flavors, by simply replacing the Gell-Mann λ^a matrices in T^l with any other $\mathfrak{su}(N_{\rm F})$-generators in flavor space. We then arrive at the $(2N_{\rm F})^2 - 1$ generators of the $SU(2N_{\rm F})$ symmetry. All symmetry patterns derived in the above sections for two flavors, apply for three and $N_{\rm F}$ flavors as well.

IV. IMPLICATIONS

We have mentioned in the Introduction that upon subtraction of the lowest-lying Dirac modes from the valence quark propagators a degeneracy of all mesons from the 15-plet is observed. Clearly, this is not accidental and reflects some inherent in QCD dynamics.

A priori one expects that elimination of the quasizero modes should restore the chiral $SU(2)_L \times SU(2)_R$ symmetry in hadrons since the quark condensate of the vacuum is connected with the density of the quasizero modes via the Banks-Casher relation [11]. However, not only degeneracy patterns from the groups $SU(2)_L \times SU(2)_R$ and $U(1)_A$ are seen, but a larger symmetry $SU(4) \supset SU(2)_L \times SU(2)_R \times U(1)_A$.

Naively one could assume that all the interesting nonperturbative physics is removed with the low-lying modes and what remains is some output from perturbative interactions. Such a simple assumption can be ruled out, however. In Ref. [1] we have proven that a system of free or weakly interacting quarks in a box is not compatible with the degeneracy of the ground states with opposite spatial parity seen in our lattice results. Such a degeneracy necessarily implies that we deal with the bound (confined) system of quarks. Second, within a perturbative approach only the $SU(2)_L \times SU(2)_R$ symmetry can be obtained, since it is a symmetry of the QCD Lagrangian and consequently of perturbation theory. Third, the energy of the quark-antiquark system with perturbative interactions should be around two bare quark masses, and not of the order 1 GeV.

The very fact that we observe a higher symmetry than the symmetry of the QCD Lagrangian does imply that we deal with a highly nontrivial nonperturbative system. In Ref. [5] it has been proven that there is no color-magnetic field in this bound (confined) system. There is only a color-electric field that binds the quarks. Here we present an alternative way to support this statement.

From Eq. (23) it follows that the color-Coulomb part of the QCD Hamiltonian in Coulomb gauge (J denotes the Faddeev-Popov determinant),

$$H_C = \frac{g^2}{2} \int J^{-1} \rho^a(\boldsymbol{x}) F^{ab}(\boldsymbol{x}, \mathbf{y}) J \rho^b(\mathbf{y}), \quad (54)$$

which describes the interaction between color charge densities $\rho^a(\mathbf{x})$ mediated by the color-Coulomb kernel $F^{ab}(\mathbf{x}, \mathbf{y})$ is invariant with respect to both $SU(2)_{CS}$ and $SU(4)$ transformations. Therefore such a term survives in an $SU(4)$-symmetric hadron. It is the term which arises from the longitudinal part of the color-electric Yang-Mills Hamiltonian after resolving Gauss law.

However, the coupling of quarks to spatial gluons,

$$H_T = -g \int d^3x \Psi^\dagger(\boldsymbol{x}) \boldsymbol{\alpha} \cdot \boldsymbol{A}(\boldsymbol{x}) \Psi(\boldsymbol{x}), \quad (55)$$

is not $SU(2)_{CS}$- and $SU(4)$-symmetric and therefore its expectation value must vanish in the $SU(4)$-symmetric hadron wave function. The only interaction left is the color-Coulomb part H_C, Eq. (54).[7]

We conclude that after reduction of the quasizero modes the only interaction left in the system arises from the color-electric field components and there is no color-magnetic field contribution. In the untruncated case the color-magnetic field contribution is still there, but only through the quasizero modes. Indeed, e.g., the instanton fluctuations, which do lead the quasizero modes [12,13], do contain the magnetic field.

The instantaneous color-Coulomb potential, given as the expectation value of F^{ab} in the gluon sector, is confining [14] and can be approximately obtained from the variational approach [15].

The linear color-Coulomb potential implies that lines of the color-electric field are squeezed into a flux tube. A flux tube between static quark sources has been observed on the lattice, see Ref. [16] and references therein. The observed $SU(4)$ symmetry of hadrons after the low-mode elimination can be consequently connected to the existence of the *dynamical* QCD string and its energy [5].

V. SUMMARY

We have found a new $SU(4)$ symmetry of the bilinear quark-antiquark fields of any spin $J \geq 1$ with nonexotic quantum numbers. This symmetry contains not only chiral transformations, but also the left-right rotations of massless quarks. We have classified interpolating fields according to different irreducible representations of $SU(4)$ and its subgroups. These results are straightforwardly generalized to $N_{\rm F}$ massless flavors and the respective group is $SU(2N_{\rm F})$.

[7]It could be also seen directly from (23): the γ^0 part of the interaction Lagrangian represents the Coulombic interaction coming from Gauss law, while the spatial part is due to the interaction of the quark current with spatial gluons $\mathbf{j} \cdot \mathbf{A}$, where color-magnetic contributions can occur.

The very fact that the correlation functions calculated with all operators from the 15-plet of $SU(4)$ in Ref. [2] upon subtraction of the quasizero modes of the Dirac operator get indistinguishable, establishes the new $SU(4)$ symmetry of mesons after removal of the quasizero modes. This symmetry is higher than the symmetry of the QCD Lagrangian and should be consequently considered as an emergent symmetry. This symmetry implies the absence of magnetic interactions (of the color-magnetic field) in the system and might be interpreted as a manifestation of the dynamical QCD string, Ref. [5].

An interesting question is whether the f_1 meson, which belongs to the singlet representation of $SU(4)$, is degenerate or not with other $J = 1$ mesons. If yes, then there should exist a higher symmetry, which contains $SU(4)$ as a subgroup and that combines both the 15-plet and the singlet representation into a higher representation. It cannot be $U(4)$, because a transition from $U(4)$ to its subgroup $SU(4)$ does not reduce the irreducible representations of $U(4)$ into a sum of irreducible representations of $SU(4)$.

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APPENDIX: BASIC DEFINITIONS AND CONVENTIONS

The chiral representation of γ-matrices enables us to write γ^μ in a compact notation:

$$\gamma^\mu = \begin{pmatrix} 0 & \chi^\mu \\ \bar{\chi}^\mu & 0 \end{pmatrix}, \qquad \chi^\mu = (1, \boldsymbol{\chi}), \qquad \bar{\chi}^\mu = (1, -\boldsymbol{\chi}), \tag{A1}$$

and the chirality matrix γ^5 is given as

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A2}$$

so that for a single flavor the quark field in L/R-components is given as

$$\psi = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}. \tag{A3}$$

Important for the construction of the meson symmetries are matrices of the form $M_{\mathrm{F}} \otimes N_{\mathrm{D}}$ with M and N matrices in flavor and Dirac space, respectively. As an example, we construct

$$(\mathbb{1}_{\mathrm{F}} \otimes \gamma^k) = \begin{pmatrix} 0 & \chi^k & 0 & 0 \\ -\chi^k & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi^k \\ 0 & 0 & -\chi^k & 0 \end{pmatrix}, \tag{A4}$$

$$(\tau^1 \otimes \gamma^k) = \begin{pmatrix} 0 & 0 & 0 & \chi^k \\ 0 & 0 & -\chi^k & 0 \\ 0 & \chi^k & 0 & 0 \\ -\chi^k & 0 & 0 & 0 \end{pmatrix}, \tag{A5}$$

$$(\tau^2 \otimes \gamma^k) = i \begin{pmatrix} 0 & 0 & 0 & -\chi^k \\ 0 & 0 & \chi^k & 0 \\ 0 & \chi^k & 0 & 0 \\ -\chi^k & 0 & 0 & 0 \end{pmatrix}, \tag{A6}$$

$$(\tau^3 \otimes \gamma^k) = \begin{pmatrix} 0 & \chi^k & 0 & 0 \\ -\chi^k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\chi^k \\ 0 & 0 & \chi^k & 0 \end{pmatrix}. \tag{A7}$$

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