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A hidden symmetry of AdS resonances

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Recent investigations have revealed powerful selection rules for resonant energy transfer between modes of nonlinear perturbations in global anti–de Sitter (AdS) space-time. It is likely that these selection rules are due to the highly symmetric nature of the underlying AdS background, though the precise relation has remained unclear. In this article, we demonstrate that the equation satisfied by the scalar field mode functions in AdS_{d+1} has a hidden $SU(d)$ symmetry, and explicitly specify the multiplets of this $SU(d)$ symmetry furnished by the mode functions. We also comment on the role this structure might play in explaining the selection rules.

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I. INTRODUCTION AND OVERVIEW

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction. (attributed to Sydney Coleman)

The stability of AdS space-times [1] against small amplitude (but nonlinear) perturbations is an interesting question in its own right, as well as in light of the AdS/CFT correspondence. On the gravity side, a turbulent instability would be significant in the formation of black holes (possibly after repeated reflections of the perturbations from the boundary), and on the gauge theory side it would indicate thermalization in a strongly coupled quantum field theory at low energies.

Investigations of this question have resulted in various surprises. In the pioneering article [2] it was found that weak amplitude Gaussian initial data in global AdS lead to black hole formation after multiple reflections from the boundary. The expectation there was that resonant energy transfer between the modes might be the reason behind the instability. Apparently noncollapsing initial data have been subsequently found in [3,4]. It was further claimed in [5] that when only low-lying modes are turned on and the amplitude is small enough (parametrized by a dimensionless number ϵ), the system does not collapse, at least for a very long time. The numerical results for the specific initial data considered in [5] have been challenged in [6], but it might still be possible that for sufficiently low-amplitude initial data, collapse can be delayed by much longer than $\sim 1/\epsilon^2$. Various other papers also suggest that the instability (if it exists) of low-lying low-amplitude modes might not be as virulent as it was originally thought [7–13].

Some understanding of the situation was gained in [7], where using renormalization group resummation techniques, effective equations were derived, describing slow energy transfer between the modes. It was shown that the resonant energy transfer between modes in AdS is highly restricted, due to the fact that a majority of the possible growing (secular) terms in perturbation theory (signaling resonant energy transfer) do not in fact arise. In [8] it was argued that a probe quartic scalar field theory in AdS could be used as an instructive model for the full gravitational instability question—among other things, it was noticed there as well that there are restrictions on secular terms and resonant energy transfer. The description in [8] was couched in the language of the two-time formalism (TTF) of [5], which is equivalent to the renormalization group (RG) resummation of [7] at relevant order. These results were further extended in [12] where the spherical symmetry assumption that was inherent in all previous discussions was dropped, and it was shown by direct manipulations with the mode functions that the scalar field still exhibits highly restrictive selection rules in its resonances. Another related observation is that the RG/TTF-resummed version of the theory possesses a number of conserved charges—this structure was discovered for the scalar theory in [8] and for the full gravity theory in [10].

Our main motivation in this paper will be to gain a more concrete understanding of this vanishing of the various RG/TTF coefficients. We believe that the existence of resonance selection rules as well as the conservation laws is strongly indicative of some underlying symmetry principle. Note that explicit symmetries (and related conservation laws) can be identified in the RG/TTF effective

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equations [8,10], but this is not what concerns us here. Rather, we are looking for a symmetry of the underlying AdS background that leads to these conservation laws in the resummed effective theory.

The fact that symmetries of the background can restrict the energy flow due to nonlinearities among the perturbation modes is not unfamiliar in the context of nonlinear system studies. For example, some symmetry-based selection rules have been discussed for vibrations of nonlinear atomic lattices, in particular, in relation to the so-called "bushes" of modes—these are subsets of modes only transferring energy to each other, but not to other modes, due to symmetry restrictions [14].

Our starting point in identifying the relevant symmetry will be that the frequencies of the scalar field mode functions in AdS background display a characteristic $2n + l$ degeneracy. Here, n is a "radial quantum number" governing the radial dependence of the mode functions, and l is the angular momentum governing their angular dependence. Typically, one expects degeneracies to arise when there are symmetries in the problem. Note however that this present degeneracy *cannot* be understood in terms of spherical symmetry explicitly present in the equations. The spherical symmetry leads merely to the familiar fact that there is degeneracy in "azimuthal quantum numbers" for a given l. The fact that the frequencies only depend on $2n + l$, such that different choices of n and l produce the same frequency, clearly implies a bigger symmetry group.

In the rest of the paper, we will argue that this degeneracy can be explicitly understood in terms of a hidden symmetry in the mode function equation. We study the degenerate multiplets of AdS_{d+1} mode functions, reveal that they form representations of (a rather nonobvious) $SU(d)$ group and explicitly specify the representation in which each mode function resides. The way we identify the hidden symmetry is by first relating the scalar wave equation in AdS to that in an Einstein static universe via a conformal transformation. Then we employ arguments similar in spirit to those associated to the $SO(4)$ enhancement of the symmetry group of the hydrogen atom to claim that we have a hidden $SU(d)$ symmetry in the problem. The mode function equation turns out to be essentially the Schrödinger equation for a quantum particle on a sphere with a spherical analog of the harmonic oscillator potential. This problem has been previously treated in [15,16] (see also [17]), and is known to possess a hidden $SU(d)$ symmetry. We will conclude by making some comments about how this symmetry might be responsible for the selection rules noticed in [7,8,10,12].

II. PERTURBATION THEORY, RESONANCES AND SELECTION RULES

We will start with a self-interacting scalar field in AdS_{d+1},

$$S = \int d^{d+1}x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right), \tag{2.1}$$

with

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{(N+1)!}\phi^{N+1}. \tag{2.2}$$

The selection rule problem that drives our interest is only meaningful for some discrete values of the mass, most prominently including $m = 0$, the specific value on which the previous investigations focused [8,12,13]. However, the hidden symmetries we want to display are present for any values of the mass, hence we shall for now treat it as arbitrary. The application of these hidden symmetries to selection rules should of course be discussed in the context of appropriate mass values.

The global AdS metric (after setting the AdS radius to unity) is

$$\begin{aligned} ds^2_{\mathrm{AdS}_{d+1}} &= \sec^2x(-dt^2 + dx^2 + \sin^2x d\Omega^2_{d-1}) \\ &\equiv \sec^2x(ds^2_{\mathrm{ES}}), \end{aligned} \tag{2.3}$$

where we have identified the metric on the Einstein static universe ds^2_{ES} for future convenience. (More specifically, only half of each spherical spatial slice of the Einstein static universe is included, since x varies between 0 and $\pi/2$, rather than 0 and π. The resulting boundary is just a conformal image of the boundary of the AdS.) We will use Ω to collectively denote the $(d-1)$-sphere coordinates appearing in $d\Omega_{d-1}$. The equations of motion for the scalar field are given by

$$\Box_{\mathrm{AdS}_{d+1}}\phi - m^2\phi \equiv \cos^2x(-\partial_t^2\phi + \Delta_s^{(d)}\phi) - m^2\phi = \frac{\lambda}{N!}\phi^N \tag{2.4}$$

where

$$\Delta_s^{(d)} \equiv \frac{1}{\tan^{d-1}x}\partial_x(\tan^{d-1}x\partial_x) + \frac{1}{\sin^2x}\Delta_{\Omega_{d-1}}. \tag{2.5}$$

Here $\Delta_{\Omega_{d-1}}$ is the Laplacian on the $(d-1)$-sphere. The solution to the free theory [i.e., $\lambda = 0$ in (2.4)], which we shall denote $\phi^{(0)}$, can be found by separating variables, as presented (for example) in [18]:

$$\phi^{(0)}(t,x,\Omega) = \sum_{n=0}^{\infty}\sum_{l,k}(A_{nlk}e^{-i\omega_{nlk}t} + \bar{A}_{nlk}e^{i\omega_{nlk}t})e_{nlk}(x,\Omega), \tag{2.6}$$

where A_{nlk} are arbitrary complex amplitudes and

$$\omega_{nlk} = 2n + l + \Delta, \tag{2.7}$$

with $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$. The mode functions are

$$e_{nlk}(x,\Omega) = \cos^{\Delta} x \sin^{l} x P_n^{(\Delta-\frac{d}{2}, l+\frac{d}{2}-1)}(-\cos 2x) Y_{lk}(\Omega). \tag{2.8}$$

Y_{lk} are the $(d-1)$-dimensional spherical harmonics, and the set of "azimuthal numbers" on a general $(d-1)$-sphere is collectively indicated by the label k. Its details will not affect the discussion below. $P_n^{(a,b)}(y)$ are the Jacobi polynomials. The mode functions satisfy the equation

$$\left(\Delta_s^{(d)} - \frac{m^2}{\cos^2 x}\right) e_{nlk}(x,\Omega) = -\omega_{nlk}^2 e_{nlk}(x,\Omega), \tag{2.9}$$

and they are orthogonal with respect to a scalar product defined by

$$(f,g) = \int dx d\Omega \tan^{d-1} x f(x,\Omega) g(x,\Omega). \tag{2.10}$$

One can then take the nonlinearities into account perturbatively by expanding solutions to (2.4) as

$$\phi = \phi^{(0)} + \lambda \phi^{(1)} + \cdots. \tag{2.11}$$

For $\phi^{(1)}$, one gets

$$-\partial_t^2 \phi^{(1)} + \left(\Delta_s^{(d)} - \frac{m^2}{\cos^2 x}\right)\phi^{(1)} = \frac{\lambda}{N!} \frac{(\phi^{(0)})^N}{\cos^2 x}. \tag{2.12}$$

It is convenient to expand $\phi^{(1)}$ in the basis of e_{nlk}:

$$\phi^{(1)}(t,x,\Omega) = \sum_{nlk} c_{nlk}^{(1)}(t) e_{nlk}(x,\Omega). \tag{2.13}$$

Substituting (2.6), (2.9) and (2.13) in (2.12), and projecting on the eigenmodes e_{nlk} using (2.10) one gets

$$\ddot{c}_{nlk}^{(1)} + \omega_{nlk}^2 c_{nlk}^{(1)} \sim \frac{\lambda}{N!} \sum_{n_1 l_1 k_1} \cdots \sum_{n_N l_N k_N} C_{nlk|n_1 l_1 k_1|\cdots|n_N l_N k_N} \times (A_{n_1 l_1 k_1}^{(0)} e^{-i\omega_{n_1 l_1 k_1} t} + \bar{A}_{n_1 l_1 k_1}^{(0)} e^{i\omega_{n_1 l_1 k_1} t}) \ldots (A_{n_N l_N k_N}^{(0)} e^{-i\omega_{n_N l_N k_N} t} + \bar{A}_{n_N l_N k_N}^{(0)} e^{i\omega_{n_N l_N k_N} t}), \tag{2.14}$$

where we have used a proportionality (rather than equality) sign since we have not kept track of the normalization of the mode functions, which is inessential for our purposes. The coefficients C are given by

$$C_{nlk|n_1 l_1 k_1|\ldots|n_N l_N k_N} = \int dx d\Omega \tan^{d-1} x \sec^2 x e_{nlk} e_{n_1 l_1 k_1} \ldots e_{n_N l_N k_N}. \tag{2.15}$$

The right-hand side of (2.14) consists of a sum of simple oscillating terms of the form $e^{i\omega t}$ with

$$\omega = \pm\omega_{n_1 l_1 k_1} \pm \cdots \pm \omega_{n_N l_N k_N}, \tag{2.16}$$

where all the plus-minus sign choices are independent. If ω is different from $\pm\omega_{nlk}$, the corresponding term produces an innocuous oscillating contribution to $c^{(1)}$, and the corresponding contribution to ϕ remains bounded, with an amplitude proportional to λ for all times. However, if $\omega = \pm\omega_{nlk}$, the corresponding term is in *resonance* with the left-hand side of (2.14), producing a *secular* term in $c^{(1)}$ and ϕ, a term that will grow indefinitely with time and invalidate the perturbation theory at times of order $1/\lambda$.

In order to make the perturbation theory usable for large times (which is the regime of physical interest), different resummation schemes can be employed, such as the two-time formalism [5,8,13], renormalization group resummation [7,10] and averaging [8,10]. Some systematic discussion of resummation techniques in the context of AdS dynamics can be found in [7,10]. All the resummation schemes mentioned are equivalent at the lowest nontrivial order, which is our present setting. The result of resummation procedures is an improved perturbation theory valid at times of order $1/\lambda$ in which secular terms become replaced with (resummed) slow changes of the mode amplitudes. The picture of nonlinearities inducing energy transfer between linear modes (this transfer being slow in the weakly nonlinear regime) is physically very intuitive.

Each resonant term in the sum on the right-hand side of (2.14) produces the corresponding term in c_1 and, after resummation, a corresponding term in the flow equations describing the slow variations of the complex amplitudes, cf. (2.6), due to the energy transfer between the modes. The general resonance condition reads

$$\omega_{nlk} = \pm\omega_{n_1 l_1 k_1} \pm \cdots \pm \omega_{n_N l_N k_N}. \tag{2.17}$$

It has been noted, however, that in AdS settings, many of the plus-minus choices in the above expression do not in fact result in secular terms, because the corresponding C coefficients in (2.14) vanish. In [7], this phenomenon was proved for spherically symmetric perturbations of dynamical gravity coupled to a scalar field. In [8], it was noted that similar vanishing occurs for spherically symmetric configurations of a self-interacting scalar field in a fixed AdS geometry, which is our present setting. In [12], the assumption of spherical symmetry was relaxed and a powerful set of *selection rules* was given for the C coefficients as defined by (2.15).

More specifically, the considerations of [12] established that, if $d(N+1)$ is even and $m=0$, the resonances corresponding to choosing all plus signs in (2.17) always drop out from the dynamics due to the vanishing of the corresponding C coefficients. Using (2.7), this particular resonance condition can be rewritten as

$$2n + l = (N-1)\Delta + 2n_1 + l_1 + \cdots + 2n_N + l_N. \tag{2.18}$$

Note that the resonance condition itself cannot be satisfied for general values of the mass m, since Δ is in general a noninteger. However, $m=0$ implies $\Delta = d$, in which case there are always some modes satisfying the above condition. Despite the fact that the resonance condition is satisfied, however, the vanishing of the corresponding C coefficients results in the absence of the corresponding secular terms and energy transfer channels. This, in turn, finds expression in extra conservation laws restricting the slow energy transfer analyzed in [8,10–12].

III. HIDDEN $SU(d)$ SYMMETRY AND MODE FUNCTION MULTIPLETS

In the previous section, we have reviewed nonlinear perturbation theory for scalar fields in AdS space-time and the emergence of resonances generating significant (slow) energy transfer between the modes for arbitrarily small nonlinearities. We have displayed a set of *selection rules* forcing some of these resonances to vanish despite they could be present on general grounds. In all the previous analytic considerations [7,12], the selection rules were proved using brute force manipulations involving the properties of orthogonal polynomials contained in (2.8). It is natural to believe that there exists a more qualitative explanation for the selection rules, most likely based on the high degree of symmetry and other special properties of the underlying AdS background. The concept of the ground state symmetries restricting the energy flow between the perturbation modes due to nonlinearities is familiar in more conventional settings, such as vibrations of crystalline lattices [14]. Similar suspicions of the symmetry origins of the AdS selection rules have been voiced in [12].

One is thus confronted with the question of the symmetry properties of the mode functions (2.8) appearing in the integrals (2.15). Here, one immediately observes an intriguing structure. The mode functions are eigenfunctions defined by (2.9), and each set of eigenfunctions with the same eigenvalue ω^2_{nlk} must form an irreducible representation of the symmetry group of the operator on the left-hand side of (2.9), whose eigenvalue problem is studied. The only obvious symmetry this operator ($\Delta_s^{(d)} - m^2/\cos^2 x$) has is the $SO(d)$ rotations of the $(d-1)$-sphere parametrized by Ω. This symmetry explains why the eigenvalues (2.7) do not depend on the azimuthal numbers k. The degeneracy is much higher however, since the eigenvalues are not only independent of k, but also only depend on l and n through the combination $2n+l$. Different representations of the rotational $SO(d)$ labeled by l are bundled together to form much bigger representations of what must be a bigger symmetry group. What can this group be?

One might have been tempted to look for isometries of AdS as the source of degeneracies. Since the mode functions are defined on a single spatial slice, rather than in the whole space-time, one might have tried to talk of the isometry group of a single spatial slice of AdS_{d+1}, which is $SO(d,1)$. This is a wrong perspective, however, since the Laplacian on a single spatial slice of AdS_{d+1} is $\cos^2 x\Delta_s^{(d)}$, which is different from the operator on the left-hand side of Eq. (2.9) defining the mode functions, even for $m=0$. Not surprisingly, representations of $SO(d,1)$ do not decompose into representations of its rotational subgroup $SO(d)$ in a way compatible with the AdS frequency degeneracies. The eigenvalues in (2.9) depend only on $2n+l$, which implies that the values of the angular momentum l entering each "level" are either all even or all odd. This property is not shared by the $SO(d)$ decomposition of $SO(d,1)$ representations.

To reveal the actual enhanced symmetry group of (2.9) it is convenient to first recall the conformal relation (2.3) between the AdS metric and the Einstein static metric. Using the standard conformal transformation formulas displayed, for instance, in (3.5) of [19], one can obtain the following identity for arbitrary $\phi(t,x,\Omega)$:

$$\cos^{(d+3)/2}x\left(\Box_{\mathrm{ES}} - \frac{(d-1)^2}{4}\right)\frac{\phi(t,x,\Omega)}{\cos^{(d-1)/2}x} = \left(\Box_{\mathrm{AdS}_{d+1}} + \frac{d^2-1}{4}\right)\phi(t,x,\Omega). \tag{3.1}$$

Let us emphasize for clarity that on the left-hand side the operator is acting on $\phi(t,x,\Omega)/\cos^{(d-1)/2}x$. Now,

$$\Box_{\mathrm{ES}} = -\partial_t^2 + \Delta_{\Omega_d}, \tag{3.2}$$

where the d-sphere Laplacian is explicitly

$$\Delta_{\Omega_d} \equiv \frac{1}{\sin^{d-1}x}\partial_x(\sin^{d-1}x\partial_x) + \frac{1}{\sin^2 x}\Delta_{\Omega_{d-1}}. \tag{3.3}$$

Correspondingly, by relating the AdS wave-function equation to the ES wave equation using (3.1), one can rewrite the mode function equation (2.9) in the form of a Schrödinger equation

$$(-\Delta_{\Omega_d} + V(x))\tilde{e}_{nlk} = E_{nlk}\tilde{e}_{nlk}, \tag{3.4}$$

with

$$V(x) = \frac{1}{\cos^2 x}\left(m^2 + \frac{d^2-1}{4}\right) \quad \text{and}$$
$$E_{nlk} = \omega_{nlk}^2 - \frac{(d-1)^2}{4}. \tag{3.5}$$

We have defined $\tilde{e}_{nlk} \equiv e_{nlk}/\cos^{(d-1)/2} x$ for convenience. Note that the range of x is only a hemisphere, $x \in [0, \pi/2)$, since the potential $V(x)$ is unbounded and confines the "particle" to this hemisphere.

Enhanced symmetries of the Schrödinger equation for a particle on a d-sphere, with a potential $V(x) \sim 1/\cos^2 x$ have been studied in [15,16] from a purely quantum-mechanical perspective. (It may also be useful to consult [20], which uses notation more similar to ours.) Historically, the equations were solved in [17] and the observed abnormal energy degeneracies of the sort we described above (with energies depending only on $2n + l$) prompted an investigation into enhanced symmetries.

The easiest way to notice the presence of enhanced symmetries in (3.4) is by looking at the corresponding classical problem. Solving the equations of motion in centrally symmetric potentials is standard and we shall not review the details here. It is easy to show that if the orbital shape of a trajectory in a central potential $V(r)$ in the ordinary d-dimensional flat space is $r = r(\varphi)$, then the orbital shape of the motion on a d-sphere in the potential $V(\tan x)$ is $x = \arctan(r(\varphi))$, with x being the polar angle on the sphere, as in (3.4). It is a simple corollary that if the orbits close for a central potential $V(r)$ in the ordinary d-dimensional flat space, then they will close as well for the central potential $V(\tan x)$ on a d-sphere.

It is known that the orbits close in flat space only for two potentials: the Coulomb potential $V(r) \sim 1/r$ and the isotropic harmonic oscillator potential $V(r) \sim r^2$. This is the so-called Bertrand's theorem [21]. The corresponding potentials on a d-sphere, for which the closure of orbits is guaranteed by the above consideration, are the sphere Coulomb potential $V(x) \sim \cot x$ and the sphere "harmonic oscillator" potential $V(x) \sim 1/\cos^2 x$. It is the latter potential that appears in the "Schrödinger" equation (3.4).

In flat space, the closure of orbits is explained by enhanced symmetries and the corresponding conserved quantities. For the Coulomb potential $V(r) \sim 1/r$, the conserved quantity is the Laplace-Runge-Lenz vector, which, together with the angular momentum, forms an $so(d+1)$ Lie algebra with respect to taking the Poisson brackets. For the isotropic harmonic oscillator potential $V(r) \sim r^2$ the corresponding conserved quantity is a traceless symmetric second-rank tensor known as Elliott's quadrupole after [22], or the Fradkin tensor after [23]. Together with the angular momentum (antisymmetric second-rank tensor), it forms an $su(d)$ Lie algebra.

The situation on a d-sphere forms a close parallel to the one described above for flat space. The d-sphere Coulomb problem first appeared in [24] and reveals an $SO(d+1)$ symmetry and the corresponding degeneracy pattern, exactly identical to the flat space case. The sphere harmonic oscillator, which is our main interest here, has been analyzed in [15,16]. The classical version of the corresponding $SU(d)$ symmetry has been displayed and the corresponding conserved quantities have been constructed. A quantum version of this symmetry, leaving equation (3.4) invariant, has only been constructed [15] for $d = 2$, which corresponds to AdS_3 in our setting, because of the problems with resolving the ordering ambiguities. Its explicit construction remains an outstanding technical problem, to the best of our knowledge. Nevertheless, the fact that the degeneracies of the energy levels of (3.4) and their properties under $SO(d)$ spatial rotations fit representations of $SU(d)$ demonstrates that the classical $SU(d)$ symmetry is in no way upset by quantization.

We would like to emphasize that the symmetry, the degeneracies and the multiplets furnished by the eigenfunctions are exactly the same for our sphere harmonic oscillator as for the usual straightforward isotropic harmonic oscillator in flat space. Of course, the symmetries are realized in a much more nontrivial way in the nonlinear case of motion on a sphere. For flat space, the $SU(d)$ symmetry can be seen immediately by simply writing the Hamiltonian $H \sim \sum(p_i^2 + x_i^2)$ in terms of the creation-annihilation operators $H \sim \sum a_i^\dagger a_i$. Transforming a_i to $\tilde{a}_i = S_{ik} a_k$, with any $SU(d)$ matrix S_{ik}, obviously leaves the Hamiltonian invariant. It is furthermore straightforward, due to the linear nature of the flat space harmonic oscillator, to implement any such transformation as a unitary operator on the Hilbert space, $\tilde{a}_i = U a_i U^\dagger$.

We finally identify the $SU(d)$ transformation properties of the multiplets corresponding to each "energy" level in (3.4). These can be easily reconstructed from the transformation properties of the mode functions under the obvious $SO(d)$ subgroup of $SU(d)$ representing the spatial rotations in (3.4). For each given n and l, the rotational properties of the mode functions are determined by the spherical harmonics $Y_{lk}(\Omega)$, which transform according to the traceless symmetric rank l tensor representation of $SO(d)$. A given representation of $SU(d)$ is formed by all such functions with the same value of energy, i.e., with the same value of $2n + l$. Since both n and l are positive, there will be a maximal possible value of l in each multiplet, which we shall call L. Each level will then be composed of the following $SO(d)$ multiplets: $(n = 0, l = L)$, $(n = 1, l = L - 2)$, etc. This is precisely the $SO(d)$ content of the fully symmetrized Lth power of the fundamental representation of $SU(d)$. Indeed, to separate irreducible representations of $SU(d)$ into irreducible representations of its $SO(d)$ subgroup, one must separate each tensor into its trace and traceless parts [25]. Applied to a fully symmetric tensor of rank L, this will generate traceless fully symmetric tensors of ranks L, $L - 2$, etc. (since two indices get contracted to produce each trace). These are precisely the rotational representations appearing for mode functions at each given level, with the angular momentum varying in steps of 2.

IV. COMMENTS ON THE SYMMETRY ORIGIN OF THE SELECTION RULES

Having established that the mode functions (2.8) of frequency $\Delta + L$ form multiplets transforming as the fully symmetrized Lth power of the fundamental representation of $SU(d)$, the hidden symmetry group of the mode function equation (2.9), one might wonder what repercussions this observation has on the selection rules for the energy flow coefficients (2.15).

One may first recall how selection rules arise in more conventional settings, for example, for cases with ordinary spherical symmetry $SO(d)$. Consider an integral of spherical harmonics

$$\int d\Omega Y_{l_1 k_1}(\Omega)\ldots Y_{l_N k_N}(\Omega). \tag{4.1}$$

One can use the standard angular momentum addition theory to decompose the product of spherical harmonics into a sum of irreducible representations of $SO(d)$. Integrated over all angles, any nontrivial (nonscalar) irreducible representation will produce a zero result. The only way the integral can be nonzero is if the trivial (scalar) representation is contained in the direct product of the representations corresponding to $Y_{l_1 k_1}, \ldots, Y_{l_N k_N}$. By the usual addition of angular momenta, this can only happen if each l_i is less than or equal to the sum of all the other l_i. Hence the angular momentum selection rules.

The application of $SU(d)$ symmetry to the integral (2.15) is considerably less straightforward for the following reasons. First, the $SU(d)$ is not made of purely spatial transformations. In the Schrödinger equation language of (3.4) it is a quantum symmetry originating from classical canonical transformations mixing coordinates and momenta.[1] It therefore does not act on the integral (2.15) as straightforwardly as spatial rotations act on (4.1). Even more frustrating, explicit construction of the symmetry generators for (3.4) has evaded dedicated effort in [15,16], except for the relatively simple case $d = 2$. This is despite the fact that the symmetry is certainly there, as evidenced by the symmetries of the corresponding classical problem, and the level degeneracies and wave-function rotational multiplets of the quantum problem matching the decomposition of multiplets of the $SU(d)$ in terms of its rotational subgroup $SO(d)$. It is these technical complications that encouraged us to present our understanding in its current form, postponing more detailed investigations to future work.

Despite the above technical complications, one might envisage the possible algebraic patterns responsible for the selection rules in (2.15) in the presence of the $SU(d)$ symmetry. The symmetry transformations connect the mode functions e_{nlk} in the same multiplet, i.e., with the same values of $2n + l$. There must exist the corresponding conjugate raising and lowering operators $\hat{A}_{\pm}$, increasing (decreasing) l by 2 and decreasing (increasing) n by 1. Analogous operators have been constructed for the simple flat space isotropic harmonic oscillator in [27]. Imagine then that $2n + l = L$ in (2.15) is large. One can write e_{nlk} as a certain number of lowering operators $\hat{A}_-$ acting on a mode function in the same multiplet with the highest value of l, i.e., $e_{0L\tilde{k}}$. It should be possible to use the conjugation properties of $\hat{A}_{\pm}$ to turn the $\hat{A}_-$ acting on $e_{0L\tilde{k}}$ into $\hat{A}_+$ acting on the remaining mode functions under the integral, which will raise their angular momentum values somewhat. The result will be an integral of a product of $e_{0L\tilde{k}}$, carrying a very high angular momentum L, with transformed $e_{n_i l_i k_i}$. Note that acting with $SU(d)$ symmetry transformations on $e_{n_i l_i k_i}$ can never increase their angular momentum beyond $2n_i + l_i$, because there are no such states in the multiplets. In the end, after applying the $SU(d)$ transformation to the integral (2.15) one will end up with an integral of a product of a mode function with a very large angular momentum $2n + l$ and other mode functions with angular momenta of at most $2n_i + l_i$. This product is strongly constrained by the *ordinary* angular momentum selection rules of the sort described under (4.1). Note that, after the $SU(d)$ transformation has been applied, restrictive angular momentum constraints will arise even if all the values of l and l_i in the original integral (2.15) were zero, in which case a direct application of angular momentum selection rules would have been completely vacuous.

At a practical level, there is a number of technical obstructions to implementing the above program in a detailed fashion. As we have already mentioned, we are not aware of an explicit construction of the $SU(d)$ symmetry generators for (3.4), and the attempts of [15,16] have been plagued by algebraic difficulties. This precludes a straightforward specification of the raising and lowering operators, unlike the much more obvious flat space case of [27]. One has to worry, furthermore, about the trigonometric insertions in (2.15). Part of those insertions will be absorbed by the integration measure necessary to make the raising and lowering operators conjugate. What remains will be acted upon by the raising operators, if one attempts the construction in the previous paragraph, and its $SU(d)$ transformation properties will have to be discussed explicitly before the detailed form of the resulting selection

[1]The $SU(d)$ hidden symmetry group we find is closely related to the isometry group of the complex projective space $\mathbb{CP}^{d-1}$. Indeed, the corresponding geometric structure can be revealed in the phase space of the ordinary d-dimensional flat space isotropic harmonic oscillator, as in Sec. 5.4.5.3 of [26]. If an analogous representation is found for the d-sphere case, it may turn out useful for our pursuits.

rules can be exposed. All of this would require a more direct understanding of how the $SU(d)$ symmetry acts on the Hilbert space of (3.4).

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