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Embedding qubits into fermionic Fock space: Peculiarities of the four-qubit case

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We give a fermionic Fock space description of embedded entangled qubits. Within this framework the problem of classification of pure state entanglement boils down to the problem of classifying spinors. The usual notion of separable states turns out to be just a special case of the one of pure spinors. By using the notion of single, double and mixed occupancy representation with intertwiners relating them a natural physical interpretation of embedded qubits is found. As an application of these ideas one can make a physical sound meaning of *some* of the direct sum structures showing up in the context of the so-called black-hole/qubit correspondence. We discuss how the usual invariants for qubits serving as measures of entanglement can be obtained from invariants for spinors in an elegant manner. In particular a detailed case study for recovering the invariants for four-qubits within a spinorial framework is presented. We also observe that reality conditions on complex spinors defining Majorana spinors for embedded qubits boil down to self-conjugate states under the Wootters spin flip operation. Finally we conduct a study on the explicit structure of $\mathrm{Spin}(16,\mathbb{C})$ invariant polynomials related to the structure of possible measures of entanglement for fermionic systems with eight modes. Here we find an algebraically independent generating set of the generalized stochastic local operations and classical communication invariants and calculate their restriction to the dense orbit. We point out the special role the largest exceptional group E_8 is playing in these considerations.

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I. INTRODUCTION

It is well known that a system of n distinguishable qubits can naturally be embedded into a system of n fermions with $2n$ modes. This idea has been widely used with applications in quantum chemistry [1,2], in studies concerning the relationship between spin systems and fermionic ones [3], the quantum Merlin–Arthur completeness of the N-representability problem [4], entanglement classification and canonical forms [5–8], ground state properties of fermionic systems [9] and the so-called black-hole/qubit correspondence (BHQC) [10].

In a recent study it has been shown [11] that in order to gain further insight into the structure of such entangled systems it is rewarding to regard them as embedded ones into the *full* fermionic Fock space. Physically this means that apart from the usual protocols of preserving the number of fermions we should also allow ones for manipulating such systems via changing the fermion number. This idea leads us to the notion of generalized Bogoliubov transformations [11]. Though the physical significance of this idea is yet to be explored even at this stage it makes it possible to regard the classification problem of entanglement types under stochastic local operations and classical communication (SLOCC) [12] as a special case of a problem well known to mathematicians as the problem of classifying *spinors* [13–17]. This observation makes a step towards establishing a unified framework for understanding entanglement properties of quantum systems consisting of subsystems with both distinguishable and indistinguishable constituents. The aim of the present paper is to take a further step in this direction and present a systematic study of n-qubit systems as ones living inside the *full* fermionic Fock space with particular emphasis put on four-qubit systems.

The motivation for embarking in this investigation is twofold. The first is to provide further clues for understanding the structure of pure state multipartite entanglement measures as the ones arising from invariants for spinors. These are homogeneous polynomials in the complex amplitudes of the pure fermionic states, capable of identifying certain types of entanglement in Fock space. It has been observed [11] that when considering in the broader fermionic context some of the multiqubit polynomial invariants have a more transparent geometric and algebraic structure than in the original multiqubit one. Making use of the full Fock space these structures are easy to identify and straightforward to do calculations with. With our investigations we would like to shed some further light on such issues by working out explicitly the embedded four-qubit case. Our treatise can also be regarded as an elaboration on some of the ideas presented in Ref. [7]

in connection with four-qubit states regarded as fermionic ones.

The second source of motivation is coming from the BHQC [10]. In this context it was observed [18,19] that it is useful to reinterpret some of the irreducible representation spaces of groups like $E_7(\mathbb{C})$ and $SO(12,\mathbb{C})$ and many others [20] (whose real forms are showing up in black hole entropy formulas of certain supergravity theories) as ones composed of a certain number of qubits. Since the invariants associated to these representation spaces have the physical meaning as the black hole entropy this qubit picture lends itself naturally to an interpretation of black hole entropy as a manifestation of some sort of entanglement. In simple special cases this interpretation has turned out to be a useful one for obtaining further insight into the structure of black hole solutions in supergravity [10]. However, for the more complicated cases no conventional entanglement based reinterpretation have been found.

In order to illustrate this problem arising in BHQC let us consider the so-called R-R subsector of $N = 8$ supergravity [10]. In this case we have 32 charges describing the winding configurations of certain extended objects like membranes and strings on noncontractible cycles of extra dimensions. These charges are transforming according to the spinor representation of the group $SO(6,6)$ a real form of the group $SO(12,\mathbb{C})$. Then the representation space is $\mathcal{H} = \mathbb{R}^{32}$. Naively one would think that since $32 = 2^5$ this representation space is amenable to a 5 real qubit (or rebit) interpretation, i.e., $\mathcal{H} = V_A \otimes V_B \otimes V_C \otimes V_D \otimes V_E$, where $V_{A,B,C,D,E}$ are five copies of $\mathbb{R}^2$s. Of course this interpretation is wrong since it cannot accommodate the $SO(6,6)$ action. However, one can accommodate this action via employing six real qubits[1] to build up the spinor representation space $\mathcal{H}$ in the form [18,19]

$$\mathcal{H} = V_{ACF} \oplus V_{ADE} \oplus V_{BCE} \oplus V_{BDF},$$
$$V_{ACF} \equiv V_A \otimes V_C \otimes V_F \text{ etc.} \tag{1}$$

However, the problem with this structure is that at first sight it is not amenable to any conventional quantum information theoretic interpretation as an entangled system. The reason for this is simple: the presence of the direct sums. In the language of representation theory this problem can be rephrased as the decomposition of $SO(6,6)$ under the subgroup $SL(2)_A \times SL(2)_B \times SL(2)_C \times SL(2)_D \times SL(2)_E \times SL(2)_F$, namely,

$$32 \to (2,1,2,1,1,2) \oplus (2,1,1,2,2,1)$$
$$\oplus (1,2,2,1,2,1) \oplus (1,2,1,2,1,2). \tag{2}$$

In this language the problem is that unlike the doublets (qubits) in the conventional theory of quantum entanglement we cannot make sense of the singlets. In Appendix C of the review paper of Borsten *et al.* [20] many more examples of that kind have been discussed. Some of them are related to representation spaces of exceptional groups, and having direct relevance to string theory and supergravity. According to Ref. [20] the unusual "tripartite entanglement of six qubits" of Eq. (1) can be given the conventional quantum information theoretic interpretation by regarding it as a subspace of six entangled *qutrits*. The weak point of this suggestion is that there is no physically sound reason why we should restrict our attention to this particular 32-dimensional subspace inside the 3^6 dimensional space rather than to any other one. For this proposal to make sense one should somehow specify the physical protocols which are represented by those transformations of this qutrit space that leave this 32-dimensional subspace invariant. Similar criticism should be applied to the remaining systems of Appendix C of Ref. [20]. In this paper we show that at least in the special case of systems amenable to a fermionic Fock space description structures like the one of Eq. (1) can be made a natural interpretation as embedded qubit systems which avoids the problem posed above.

The plan of this paper is as follows. In Sec. II we summarize the basic material on fermionic Fock space and identify the generalized SLOCC group as $\mathbb{C}^\times \times \mathrm{Spin}(2N,\mathbb{C})$. Here $\mathrm{Spin}(2N,\mathbb{C})$ is accommodating generalized Bogoliubov transformations. In Sec. III we reconsider some of the ideas of Ref. [11] on entanglement in fermionic Fock space. We show how ordinary SLOCC transformations [12] are accommodated within the formalism and how the SLOCC classification problem is recovered within the framework of a more general mathematical problem, namely the classification of spinors. Here we also comment on the special role of *pure* spinors giving a natural generalization of the notion of separable states. In Sec. IV we begin our investigation of embedded n-qubit systems by studying the single occupancy representation. We observe that the largest subgroup of the fermionic SLOCC group $GL(2n,\mathbb{C})$ leaving invariant the single occupancy subspace is the group $\tilde{G} = S_n \ltimes G$ where $G = GL(2,\mathbb{C})^{\times n}$ and S_n is the symmetric group. We will make use of this in our analysis of four-qubit states when searching for permutation invariant combinations of the usual SLOCC invariants within a spinorial formalism. In Sec. V we embark on an elementary discussion on different ways for embedding qubits. We go through in detail the basic structures we come across up to four qubits. Here we introduce intertwiners relating the double, single and mixed occupancy representations occurring in these descriptions of qubits. An interesting observation on the role of these intertwiners as maps related to the mirror map of string theory relating the IIA and IIB duality frames is presented. These

[1]Clearly one can permute the labels A, B, C, D, E, F for convenience provided we leave the incidence structure of Eq. (1) intact. This incidence structure is that of a terahedron. For alternative labeling of these qubits see Refs. [18,19].

investigations make it clear how one should make sense of *some* of the direct sum structures (the occurrence of singlets) in the BHQC. Section VI is devoted to a detailed study of invariants and covariants paying special attention to the spinorial case yielding embedded four-qubit states. For the four-qubit case the structure of the basic spinorial covariant is of block diagonal form of the blocks related to the two-partite reduced density matrices via a conjugation operation. This is the spinorial generalization [21] of the well-known Wootters spin flip operation [22]. We point out that the notion of Wootters self-conjugate spinor is just the notion of a *Majorana* spinor. In closing this section we recover the basic four-qubit $\tilde{G}$ invariants in a spinorial framework. In Sec. VII we conduct a study on the explicit structure of $\mathrm{spin}(16,\mathbb{C})$ invariant polynomials, i.e., on the structure of generalized SLOCC invariants for fermionic systems with eight modes. We point out the special role the largest exceptional group E_8 is playing in this respect. We introduce a fermionic state which is a representative of the semisimple orbit, depending on eight complex parameters. Then in terms of these parameters for this particular state we calculate the values of eight polynomial invariants, which form a basis of generators of the ring of $\mathrm{Spin}(16,\mathbb{C})$ invariant polynomials. We show that the resulting polynomials in eight variables can neatly be expressed in terms of eight algebraically independent polynomials which are invariant under the Weyl group of E_8. Our conclusion and comments are left for Sec. VIII.

II. FERMIONIC FOCK SPACE

In this section we summarize results concerning spinors in a fermionic Fock space language [11,13–16]. Let V be an N-dimensional complex vector space and V^* its dual. We regard $V \simeq \mathbb{C}^N$ with $\{e_i\}$, $i = 1,2,\ldots N$ the canonical basis and $\{e^i\}$ is the dual basis. Elements of V will be called *one particle states*. We tacitly assume that V is a finite dimensional Hilbert space also equipped with a Hermitian inner product, but at first we will not make use of this extra structure until Sec. VI F. We also introduce the $2N$ dimensional vector space

$$\mathcal{V} \equiv V \oplus V^* \tag{3}$$

with basis $\{e^I\} \equiv \{e_i, e^j\}$, $I = 1,\ldots N, N+1,\ldots 2N$. An element of $\mathcal{V}$ is of the form $x = v + \alpha$ where v is a vector and α is a linear form with $v = v^i e_i$ and $\alpha = \alpha_j e^j$. According to the method of second quantization to any element $x \equiv x_I e^I \in \mathcal{V}$ one can associate a linear operator $\hat{x}$ acting on a 2^N dimensional complex vector space $\mathcal{F}$ called the *fermionic Fock space* $\mathcal{F}$ as follows. Take the exterior (Grassmann) algebra $\wedge^\bullet V^*$ where

$$\wedge^\bullet V^* = \mathbb{C} \oplus V^* \oplus \wedge^2 V^* \oplus \cdots \oplus \wedge^N V^*. \tag{4}$$

Then the Fock space is defined as

$$\mathcal{F} \equiv \wedge^\bullet V^* \otimes (\wedge^N V)^{-1/2}. \tag{5}$$

The origin of the last factor will be explained later [see Eq. (31)]. Temporarily the reader should regard $\mathcal{F}$ merely as the Grassmann algebra (4) based on V^*. Now the operator $\hat{x} = x_I \hat{e}^I = v^i \hat{e}_i + \alpha_j \hat{e}^j$ acting on $\mathcal{F}$ is obtained by the assignment

$$e^i \mapsto \hat{e}^i \equiv e^i \wedge, \qquad e_i \mapsto \hat{e}_i \equiv \iota_{e_i} \tag{6}$$

i.e., the basis vectors are mapped to the operators of exterior and interior multiplication. Defining $\{\hat{x},\hat{y}\} \equiv \hat{x}\hat{y} + \hat{y}\hat{x}$ we have

$$\{\hat{e}^i, \hat{e}_j\} = \delta^i{}_j \hat{1}, \qquad \{\hat{e}^i, \hat{e}^j\} = \{\hat{e}_i, \hat{e}_j\} = 0 \tag{7}$$

which are the usual fermionic *anticommutation* relations.

The one-dimensional subspace $\wedge^0 V^* = \mathbb{C}$ corresponds to the ray of the *vacuum state* denoted by $|0\rangle$. The operators $\hat{e}^i$ and $\hat{e}_j$ are the *creation* and *annihilation* operators. For later convenience we redefine these as

$$\hat{e}^i \equiv \hat{p}^i, \qquad \hat{e}_j \equiv \hat{n}_j \tag{8}$$

with

$$\{\hat{p}^i, \hat{n}_j\} = \delta^i{}_j \hat{1}, \qquad \{\hat{p}^i, \hat{p}^j\} = \{\hat{n}_i, \hat{n}_j\} = 0. \tag{9}$$

The algebra above will be called the canonical anticommutation relations (CAR) algebra (the algebra of canonical anticommutation relations). This algebra can compactly be expressed as

$$\{\hat{e}_I, \hat{e}_J\} = g_{IJ}\hat{1}, \qquad g_{IJ} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{10}$$

where g_{IJ} is a $2N \times 2N$ matrix with $N \times N$ blocks and I is the $N \times N$ identity matrix.

Clearly

$$\hat{n}_j|0\rangle = 0 \tag{11}$$

encapsulates the defining property of the vacuum, namely that it contains no "particles" or "excitations" at all. On the other hand the state

$$\hat{p}^i|0\rangle \tag{12}$$

represents a single "particle" which is in the ith "mode." Similarly states of the form

$$\hat{p}^i\hat{p}^j|0\rangle, \qquad \hat{p}^i\hat{p}^j\hat{p}^k|0\rangle, \qquad \ldots,$$
$$\hat{p}^1\hat{p}^2\cdots\hat{p}^N|0\rangle, \qquad i < j < k \quad \text{etc.} \tag{13}$$

are the two, three $\ldots N$ particle states. Generally the $\binom{N}{k}$-dimensional k-particle subspace is spanned by the basis vectors $\hat{p}^{i_1}\hat{p}^{i_2}\cdots\hat{p}^{i_k}|0\rangle$ with $1 \le i_1 < i_2 \cdots < i_k \le N$. It then follows that an arbitrary state of $\mathcal{F}$ can be written in the form

$$|\psi\rangle \equiv \hat{\Psi}|0\rangle, \qquad \hat{\Psi} \equiv \sum_{k=0}^{N} \sum_{i_1 i_2 \cdots i_k = 1}^{N} \frac{1}{k!} \psi^{(k)}_{i_1 i_2 \cdots i_k} \hat{p}^{i_1}\hat{p}^{i_2}\cdots\hat{p}^{i_k}. \tag{14}$$

Here the kth order totally antisymmetric tensors $\psi^{(k)}_{i_1 i_2 \cdots i_k}$ encapsulate the complex amplitudes of the k-particle subspace. An element $|\psi\rangle \in \mathcal{F}$ is called a *spinor*.

One can alternatively define the linear combinations

$$\hat{\Gamma}_i = \hat{p}^i + \hat{n}_i, \qquad \hat{\Gamma}_{i+N} = \hat{p}^i - \hat{n}_i, \qquad i = 1, 2, \ldots, N \tag{15}$$

satisfying

$$\{\hat{\Gamma}_I, \hat{\Gamma}_J\} = 2\eta_{IJ}\hat{1}, \qquad \eta_{IJ} = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & -\delta_{ij} \end{pmatrix}, \qquad I, J = 1, \ldots, 2N, \qquad i, j = 1, \ldots, N. \tag{16}$$

The matrix representatives of the $\hat{\Gamma}_I$ operators correspond to the usual gamma matrices in physics. Let us now define the operator

$$\hat{\Gamma} \equiv [\hat{n}_1, \hat{p}^1][\hat{n}_2, \hat{p}^2][\hat{n}_3, \hat{p}^3]\cdots[\hat{n}_N, \hat{p}^N] = (-1)^{N(N-1)/2}\hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3\cdots\hat{\Gamma}_{2N}. \tag{17}$$

It is easy to check that $\hat{\Gamma}^2 = \hat{1}$ hence the eigenvalues of this operator are ± 1. Spinors $|\psi_\pm\rangle$ which are eigenvectors of $\hat{\Gamma}$ corresponding to the eigenvalues ± 1 are called *Weyl spinors* of positive and negative *helicity* or *chirality*. One can check that in Eq. (14) spinors of positive chirality have terms with an *even*, and negative chirality have terms with an *odd* number of creation operators. Hence we have the decomposition

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-. \tag{18}$$

Let us now take the operators $\hat{x} = x^I\hat{e}_I = \alpha_i\hat{p}^i + v^j\hat{n}_j$ and $\hat{y} = y^J\hat{e}_J = \beta_i\hat{p}^i + w^j\hat{n}_j$ answering the corresponding vectors x and y having the same expansions with hats removed. Then

$$\{\hat{x}, \hat{y}\} = g(x, y)\hat{1} = g_{IJ}x^I y^J\hat{1} = (\alpha_i w^i + v^j\beta_j)\hat{1}. \tag{19}$$

Here $g\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is a nondegenerate symmetric bilinear form with matrix g_{IJ} known from Eq. (10). The group of transformations which leave this form invariant is the orthogonal group $O(\mathcal{V}, g) \equiv O(2N, \mathbb{C})$. We take its connected component to the identity which is $SO(2N, \mathbb{C})$. We have

$$g(\mathcal{S}(x), \mathcal{S}(y)) = g(x, y), \qquad \mathcal{S} \in SO(2N, \mathbb{C}). \tag{20}$$

Using matrices this equation yields

$$\mathcal{S}^t g \mathcal{S} = g. \tag{21}$$

Writing $\mathcal{S} = e^s$ where $s \in \mathfrak{so}(2N)$ and using Eq. (20) by taking the infinitesimal version of Eq. (21) one can see that s can be parametrized as

$$s = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \qquad C^t = -C, \qquad B^t = -B, \tag{22}$$

where A, B, C are $N \times N$ matrices with B^t refers to the transposed matrix of B. One can also regard an $\mathcal{S}$ as a transformation acting on the operators $\hat{x}$. Then combining Eqs. (19) and (20) one can see that elements of $SO(2N, \mathbb{C})$ also leave the CAR algebra invariant. These transformations will be called generalized Bogoliubov transformations.

We would also like to have an action of these generalized Bogoliubov transformations on our Fock space $\mathcal{F}$. The usual way to define this action is via introducing operators $\hat{S}$ that are mapped to the transformations $\mathcal{S}$ via the relation

$$\hat{S}\,\hat{x}\,\hat{S}^{-1} = \mathcal{S}(\hat{x}), \qquad x \in \mathcal{V}, \qquad \mathcal{S} \in SO(2N, \mathbb{C}). \tag{23}$$

Here $\mathcal{S}(\hat{x}) = x^I\mathcal{S}(\hat{e}_I) = x^I\hat{e}_J\mathcal{S}^J{}_I$ where the matrix $\mathcal{S}^J{}_I$ is the exponential of the matrix given by Eq. (22). It is well known that the set of such transformations $\hat{S}$ gives the double cover of $SO(2N, \mathbb{C})$ which is the group $\mathrm{Spin}(2N, \mathbb{C})$. Let us write the infinitesimal version of Eq. (23) in the form

$$[\hat{s}, \hat{x}] = s(\hat{x}), \quad \hat{s} \in \mathrm{spin}(2N), \quad s \in \mathfrak{spin}(2N) \simeq \mathfrak{so}(2N). \tag{24}$$

Now a calculation shows that

$$\hat{s} = \frac{1}{2}A_i{}^j[\hat{p}^i, \hat{n}_j] + \frac{1}{2}B_{ij}\hat{p}^i\hat{p}^j + \frac{1}{2}C^{ij}\hat{n}_i\hat{n}_j. \tag{25}$$

Clearly for the action $\hat{S} = e^{\hat{s}}$ the subspaces $\mathcal{F}_\pm$ are invariant ones. In the following transformations of the form

$$|\psi_\pm\rangle \mapsto \lambda e^{\hat{s}}|\psi_\pm\rangle, \qquad (\lambda, e^{\hat{s}}) \in \mathbb{C}^\times \times \mathrm{Spin}(2N, \mathbb{C}), \qquad |\psi_\pm\rangle \in \mathcal{F}_\pm \tag{26}$$

will be called *generalized SLOCC transformations*. The rationale for also including the group $\mathbb{C}^\times$ of nonzero complex numbers will be given in Eq. (39). Notice that the set of generalized SLOCC transformations is respecting the chirality of Weyl spinors.

III. ENTANGLEMENT

In this section we summarize results on entanglement in fermionic Fock space [11].

The main advantage of the subspaces $\mathcal{F}_\pm$ is that we can embed into them the state spaces of a large variety of multipartite entangled systems taken together with the action of their respective SLOCC groups. In order to see this one just has to realize that the particle number conserving subgroup of the generalized SLOCC group is obtained by setting $B = C = 0$ in Eq. (25). Now we have

$$\hat{s} = \frac{1}{2} A_i{}^j [\hat{p}^i, \hat{n}_j] = A_i{}^j \hat{p}^i \hat{n}_j - \frac{1}{2} \mathrm{Tr}(A) \hat{1}. \tag{27}$$

The exponential of this is

$$\hat{S} = e^{\hat{s}} = e^{-\mathrm{Tr}A/2} e^{A_i{}^j \hat{p}^i \hat{n}_j}. \tag{28}$$

Let us now consider the action of $\hat{S}$ on a k-particle subspace

$$\hat{S}|\psi^{(k)}\rangle = \frac{1}{k!} \psi^{(k)}{}_{i_1 i_2 \ldots i_k} (\hat{S}\hat{p}^{i_1}\hat{S}^{-1})(\hat{S}\hat{p}^{i_2}\hat{S}^{-1}) \cdots (\hat{S}\hat{p}^{i_k}\hat{S}^{-1}) \hat{S}|0\rangle. \tag{29}$$

According to Eq. (23) we have

$$\hat{S}\hat{p}^i\hat{S}^{-1} = \hat{p}^j \mathcal{S}_j{}^i, \qquad \mathcal{S} = e^A \in GL(N, \mathbb{C}) \tag{30}$$

hence

$$\hat{S}|\psi^{(k)}\rangle = \frac{1}{k!} \psi'^{(k)}_{j_1 \ldots j_k} \hat{p}^{j_1} \cdots \hat{p}^{j_k} |0\rangle,$$
$$\psi'^{(k)}_{j_1 \cdots j_k} = (\mathrm{Det}\mathcal{S})^{-1/2} \mathcal{S}_{j_1}{}^{i_1} \cdots \mathcal{S}_{j_k}{}^{i_k} \psi^{(k)}{}_{i_1 \cdots i_k}. \tag{31}$$

Here we have used the identity $e^{\mathrm{Tr}A} = \mathrm{Det}\mathcal{S}$ where $\mathcal{S} = e^A$. Equation (31) shows that apart from the extra term $(\mathrm{Det}\mathcal{S})^{-1/2}$ the totally antisymmetric tensor $\psi^{(k)}_{i_1 i_2 \cdots i_k}$ incorporating the $\binom{N}{k}$ complex amplitudes transforms via N identical copies of the usual fermionic SLOCC group i.e., $GL(N, \mathbb{C})$ well known from the theory of fermionic entanglement. Notice also that the presence of the extra term clearly shows that the fermionic Fock space should be the one of Eq. (5) we started our considerations with.

Let us fix a spinor $|\psi\rangle \in \mathcal{F}$ and define its *annihilator subspace* $\mathcal{M}_\psi$ of $\mathcal{V}$ as the set of vectors $x \in \mathcal{V}$ such that their corresponding operators $\hat{x}$ annihilate $|\psi\rangle$:

$$\mathcal{M}_\psi \equiv \{x \in \mathcal{V} | \hat{x}|\psi\rangle = 0\}. \tag{32}$$

From Eq. (19) it follows that if $x, y \in \mathcal{M}_\psi$ then $g(x, y) = 0$. Hence $\mathcal{M}_\psi$ is a *totally isotropic subspace* of $\mathcal{V}$. Clearly due to the structure of our bilinear form g the maximal dimension of a totally isotropic subspace is N. A spinor $|\psi\rangle$ such that $\mathcal{M}_\psi$ is a *maximal* totally isotropic subspace of $\mathcal{V}$ is called a *pure spinor*. (Cartan calls them *simple spinors*, Chevalley calls them *pure spinors*. Here we follow the conventions based on the English literature and will call them pure spinors.)

First of all notice that all of the spinors showing up in the sequence of Eqs. (11)–(13) are pure. Indeed, take for instance $|\psi\rangle = \hat{p}^{k+1}\hat{p}^{k+2} \cdots \hat{p}^N |0\rangle$ where $k = 0, 1, \ldots N$. Then

$$\mathcal{M}_\psi = \mathrm{span}\{n_1, n_2, \ldots n_k, p^{k+1}, p^{k+2}, \ldots p^N\}. \tag{33}$$

Since any pair of operators corresponding to vectors taken from this set is pairwise anticommuting, according to Eq. (19) this is a totally isotropic subspace, with maximal dimension N. From this it follows that all the pure spinors of the form

$$\hat{p}^{i_1}\hat{p}^{i_2} \cdots \hat{p}^{i_k}|0\rangle \leftrightarrow e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k} \tag{34}$$

are *Slater determinants*. Since in the usual theory of fermionic entanglement (with fixed particle number) the states corresponding to the rays of Slater determinants are called *separable* we conclude that in the realm of generalized SLOCC transformations the separable states should be identified with the pure spinors. It is important to note that *pure spinors are Weyl* and there is a one-to-one correspondence between the rays of pure spinors and the set of maximally totally isotropic subspaces [15]. An arbitrary pure spinor can always be represented in the form

$$|\psi^{\mathrm{pure}}\rangle = \lambda e^{\hat{B}} \hat{p}^{i_1}\hat{p}^{i_2} \cdots \hat{p}^{i_k}|0\rangle, \qquad \hat{B} = \frac{1}{2} B_{ij} \hat{p}^i \hat{p}^j,$$
$$B_{ij} = -B_{ji} \tag{35}$$

for some $k = 0, 1, \ldots N$ and $\lambda \in \mathbb{C}^\times$. We will refer to the content of this equation as the fact that a pure spinor is the so-called *B-transform* of a Slater determinant. Spinors that are not pure will be called *entangled*.

Classification of entanglement types in fermionic Fock space amounts to finding the generalized SLOCC classes, i.e., finding the orbit structure under the group action of Eq. (26). Since the nontrivial subgroup of this group is $\mathrm{Spin}(2N, \mathbb{C})$ and this group respects the chirality of the spinors one can obtain generalized SLOCC orbits for Weyl spinors of either type i.e., $\mathcal{F}_+$ or $\mathcal{F}_-$. In the mathematics literature finding the generalized SLOCC classes via determining a representative state from each orbit and its stabilizer is called the *classification problem of spinors*. It is known that for $N = 1, 2, 3$ every spinor is pure [16,17]. It means that the action of the group $\mathrm{Spin}(2N, \mathbb{C})$ on the space of Weyl spinors of say positive chirality is transitive. From the physical point of view it means that there are no entangled states in the fermionic Fock space for 1, 2, or 3 single particle states. The classification problem of spinors was solved by Igusa [17] for $N = 4, 5, 6$, by Popov [23] for $N = 7$, and by Antonyan and Elashvili [24] for $N = 8$. These results give the full orbit structure of entangled states. For $N > 8$ coarse classification schemes have been proposed based on the notion of the nullity of a spinor [16]. The *nullity* is just the dimension of the subspace of $\mathcal{V}$ characterized by vectors giving rise to operators annihilating an entangled state $|\psi\rangle$.

IV. EMBEDDED QUBITS

We have seen that the generalized SLOCC group contains naturally the ordinary fermionic SLOCC group. These groups are acting on state spaces which are representing quantum systems with *indistinguishable constituents*. One can however, relax this restriction. Our formalism based on fermionic Fock spaces is also capable of incorporating systems with *distinguishable constituents*. In this paper we consider the possibility of incorporating n-qubit systems with particular emphasis put on the four-qubit case.

Let $N = 2n$, hence our Hilbert space of one-particle states is now *even* dimensional. In this case it is convenient to introduce a new labeling for the single particle basis states:

$$\{e_1, e_2, \ldots e_n, e_{n+1}, e_{n+2}, \ldots e_{2n}\} = \{e_1, e_2, \ldots e_n, e_{\bar{1}}, e_{\bar{2}}, \ldots e_{\bar{n}}\}. \tag{36}$$

Let us now consider $\mathcal{F}^{(n)}$, the $\binom{2n}{n}$ dimensional n-particle subspace of $\mathcal{F}$. An n-fermion state with $2n$ one-particles states can be written in the form

$$|Z\rangle = \frac{1}{n!} Z_{i_1 i_2 \cdots i_n} \hat{p}^{i_1} \hat{p}^{i_2} \cdots \hat{p}^{i_n} |0\rangle. \tag{37}$$

Under the SLOCC subgroup of Eq. (28) the amplitudes of this state transform as

$$Z_{i_1 \cdots i_n} \mapsto (\mathrm{Det}\mathcal{S})^{-1/2} \mathcal{S}_{j_1}{}^{i_1} \cdots \mathcal{S}_{j_n}{}^{i_n} Z_{i_1 \cdots i_n} \equiv S_{j_1}{}^{i_1} \cdots S_{j_n}{}^{i_n} Z_{i_1 \cdots i_n}, \tag{38}$$

where

$$S_j{}^i \equiv (\mathrm{Det}\mathcal{S})^{-\frac{1}{2n}} \mathcal{S}_j{}^i \in SL(2n, \mathbb{C}). \tag{39}$$

Hence in this special case the (28) subgroup of transformations coming from the group $\hat{S} \in \mathrm{Spin}(4n, \mathbb{C})$ with $B_{ij} = C^{ij} = 0$ will *not* produce the full SLOCC group $GL(2n, \mathbb{C})$ only an $SL(2n, \mathbb{C})$ subgroup. Luckily, in Eq. (26) we defined the *generalized SLOCC group* as $\mathbb{C}^\times \times \mathrm{Spin}(4n, \mathbb{C})$. Thanks to this extra $\mathbb{C}^\times$ even in this special case our generalized SLOCC group will contain the ordinary SLOCC one, namely $GL(2n, \mathbb{C})$. Notice however, that for k-fermion states with $k \neq n$ this subtlety for obtaining the full SLOCC group is not needed.[2]

From the set of basis vectors of $\mathcal{F}^{(n)}$ we choose a special subset containing merely 2^n elements as follows:

$$\hat{p}^1 \hat{p}^2 \cdots \hat{p}^n |0\rangle, \qquad \hat{p}^1 \hat{p}^2 \cdots \hat{p}^{\bar{n}} |0\rangle, \qquad \ldots, \\ \hat{p}^{\bar{1}} \hat{p}^{\bar{2}} \cdots \hat{p}^n |0\rangle, \qquad \hat{p}^{\bar{1}} \hat{p}^{\bar{2}} \cdots \hat{p}^{\bar{n}} |0\rangle. \tag{40}$$

These basis vectors will be spanning the state space of embedded n-qubit states. Indeed, let

$$|\psi\rangle = \sum_{\mu_1, \ldots, \mu_n = 0,1} \psi_{\mu_1 \cdots \mu_n} |\mu_1 \ldots \mu_n\rangle \tag{41}$$

be an n-qubit state, i.e., an element of $\mathbb{C}^{2^n}$. Let us now define a map

$$f\colon \mathbb{C}^{2^n} \to \wedge^n \mathbb{C}^{2n} \simeq \mathcal{F}^{(n)} \tag{42}$$

as follows:

$$|\psi\rangle \mapsto |Z_\psi\rangle = (\psi_{00\cdots 0} \hat{p}^1 \hat{p}^2 \cdots \hat{p}^n + \psi_{00\cdots 1} \hat{p}^1 \hat{p}^2 \cdots \hat{p}^{\bar{n}} + \cdots + \psi_{11\cdots 1} \hat{p}^{\bar{1}} \hat{p}^{\bar{2}} \cdots \hat{p}^{\bar{n}}) |0\rangle. \tag{43}$$

In this way we have embedded an n-qubit state to $\mathcal{F}^{(n)}$.

Now we consider the fermionic SLOCC transformations of Eqs. (28) and (31). These are transformations, characterized by a $2n \times 2n$ matrix $\mathcal{S} = e^A$, which leave the n-particle subspace of the Fock space invariant. Our aim is to restrict $\mathcal{S}$ in such a way that the resulting matrix also leaves the n-qubit subspace, spanned by the basis vectors of Eq. (40), invariant and at the same time this new matrix also gives rise to the usual $SL(2, \mathbb{C})^{\times n}$ part of the SLOCC action on $|Z_\psi\rangle$.

Looking at Eq. (38) it is easy to see that such transformations can be organized to a matrix of the form

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2n, \mathbb{C}), \tag{44}$$

where $a = \mathrm{diag}(a_1, \ldots a_n)$, $b = \mathrm{diag}(b_1, \ldots b_n)$, $c = \mathrm{diag}(c_1, \ldots c_n)$, $\mathrm{diag}(d_1, \ldots d_n)$, i.e., the $n \times n$ blocks of S are *diagonal matrices*. One can also place these complex numbers into an n element set of 2×2 matrices

[2]This factor of $\mathbb{C}^\times$ is also needed to be in accord with the classification of spinors for $N = 6$, i.e., $n = 3$. As it is well known from the classification theory of prehomogeneous vector spaces in this case we have a dense orbit of the group $GL(6, \mathbb{C}) \simeq \mathbb{C}^\times \times SL(6, \mathbb{C})$ on the 3-fermion state space $\wedge^3 \mathbb{C}^6$. This orbit is just the fermionic generalization of the GHZ orbit known for three-qubits.

$$S^{(l)} \equiv \begin{pmatrix} a_l & b_l \\ c_l & d_l \end{pmatrix} \in SL(2,\mathbb{C}), \qquad l = 1,2,\ldots n. \tag{45}$$

Taken together with the extra factor $\mathbb{C}^\times$ known from Eq. (26) the transformation $|Z_\psi\rangle \mapsto \lambda\hat{S}|Z_\psi\rangle$ gives rise to the one

$$\psi_{\mu_1\ldots\mu_n} \mapsto \mathcal{A}^{(1)}{}_{\mu_1}{}^{\nu_1}\cdots\mathcal{A}^{(n)}{}_{\mu_n}{}^{\nu_n}\psi_{\nu_1\ldots\nu_n}, \\ \mathcal{A}^{(l)} \in GL(2,\mathbb{C}), \qquad l = 1,2\ldots n \tag{46}$$

which is just the usual SLOCC action for n-qubits.

In the case of SLOCC classification one generally obtains different *families* of entangled states. The families can contain inequivalent orbits under the SLOCC group. It can also happen that under permutations of the qubits one particular orbit in a family is mapped to another orbit in another family. Then in the case of qubits it is rewarding to explore the orbits of the group $\tilde{G} = S_n \ltimes G$ where $G = GL(2,\mathbb{C})^{\times n}$ and S_n is the symmetric group. One can then ask what is the relationship between the *largest subgroup* G' of the fermionic SLOCC group $GL(2n,\mathbb{C})$ which leaves invariant the n-qubit subspace spanned by the vectors of Eq. (40), and $\tilde{G}$. Surprisingly according to Lemma III.8 of Ref. [25] the answer to this question is $G' = \tilde{G}$. For this to make sense one should embed S_n into $GL(2n,\mathbb{C})$ such that for an element $\sigma \in S_n$ we have

$$(1,2,\ldots,n,\bar{1},\bar{2},\ldots,\bar{n}) \\ \mapsto (\sigma(1),\sigma(2),\ldots,\sigma(n),\sigma(\bar{1}),\sigma(\bar{2}),\ldots,\sigma(\bar{n})), \tag{47}$$

meaning that the basis vectors of Eq. (36) should be transformed accordingly. The group $\tilde{G}$ will be used in Sec. VI G when studying four-qubit invariants.

For illustrative purposes it is useful to invoke the following physical interpretation [7]. Our Hilbert space of one-particle states is $\mathcal{H} = \mathbb{C}^{2n} \simeq \mathbb{C}^n \otimes \mathbb{C}^2 \equiv \mathcal{H}_{\rm site} \otimes \mathcal{H}_{\rm spin}$. In this picture the fermions can be localized to n sites (boxes), and each site (box) can be filled with a spin which is either up or down. This way of representing the 2^n basis states of Eq. (40) will be called *single occupancy* representation. The remaining $\binom{2n}{n} - 2^n$ basis states contain *double and mixed occupancy states* as well. In this case some of the boxes can also be empty or filled with two spins, one is up the other is down. The single and double occupancy representations of qubits are illustrated in Figs. 1 and 2.

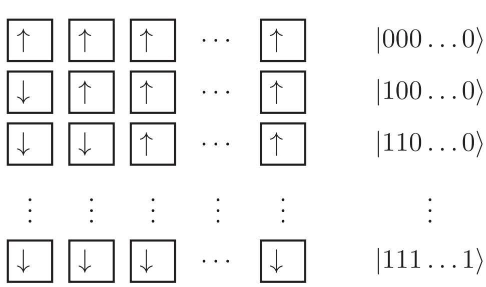

FIG. 1. Single occupancy embedding of the n-qubit Hilbert space (2^n basis vectors) inside $\mathcal{F}_+$ (for $n = 2k$ boxes and $N = 2n$ single particle states) or $\mathcal{F}_-$ (for $n = 2k+1$ boxes and $N = 2n$ single particle states).

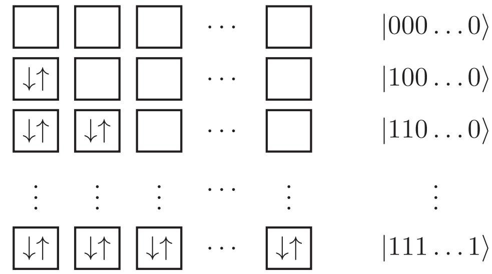

FIG. 2. Double occupancy embedding of the n-qubit Hilbert space (2^n basis vectors) inside $\mathcal{F}_+$ (n boxes and $N = 2n$ single particle states).

V. WAYS OF EMBEDDING QUBITS

A. Embedding one qubit

Apart from the "canonical" way discussed in the previous section, there exist actually many more ways for obtaining embedded n-qubit systems. Since these different types of embedding will be of importance for us, here we start to clarify the technique of embedding. We start with the elementary case of a single qubit.

In the case of a single qubit we have $n = 1$ and $V = \mathbb{C}^2$. We have states from the even chirality sector such as

$$|\psi_+\rangle = (\eta\hat{\mathbf{1}} + \xi\hat{p}^1\hat{p}^{\bar{1}})|0\rangle \in \mathcal{F}_+, \qquad \eta,\xi \in \mathbb{C} \tag{48}$$

and for the odd chirality sector such as

$$|\psi_-\rangle = (Z_1\hat{p}^1 + Z_{\bar{1}}\hat{p}^{\bar{1}})|0\rangle \in \mathcal{F}_-, \qquad Z_1, Z_{\bar{1}} \in \mathbb{C}. \tag{49}$$

The nontrivial part of the generalized SLOCC group comprises the group Spin$(4,\mathbb{C})$. According to Eq. (25) an element of this group can be written in the form $\hat{S} = e^{\hat{s}}$, where

$$\hat{s} = A_1{}^1\hat{p}^1\hat{n}_1 + A_{\bar{1}}{}^{\bar{1}}\hat{p}^{\bar{1}}\hat{n}_{\bar{1}} + A_{\bar{1}}{}^1\hat{p}^{\bar{1}}\hat{n}_1 + A_1{}^{\bar{1}}\hat{p}^1\hat{n}_{\bar{1}} \\ + B_{1\bar{1}}\hat{p}^1\hat{p}^{\bar{1}} + C^{1\bar{1}}\hat{n}_1\hat{n}_{\bar{1}} - \frac{1}{2}(A_1{}^1 + A_{\bar{1}}{}^{\bar{1}})\hat{\mathbf{1}}. \tag{50}$$

Under $|\psi\rangle \mapsto \hat{s}|\psi\rangle$ where $|\psi\rangle = |\psi_+\rangle + |\psi_-\rangle$ we have $(\eta,\xi) \mapsto (\eta',\xi')$ and $(Z_1, Z_{\bar{1}}) \mapsto (Z'_1, Z'_{\bar{1}})$ where

$$\begin{pmatrix} \eta' \\ \xi' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(A_1{}^1 + A_{\bar{1}}{}^{\bar{1}}) & -C^{1\bar{1}} \\ B_{1\bar{1}} & \frac{1}{2}(A_1{}^1 + A_{\bar{1}}{}^{\bar{1}}) \end{pmatrix}\begin{pmatrix} \eta \\ \xi \end{pmatrix}, \tag{51}$$

$$\begin{pmatrix} Z'_1 \\ Z'_{\bar{1}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(A_1{}^1 - A_{\bar{1}}{}^{\bar{1}}) & A_1{}^{\bar{1}} \\ A_{\bar{1}}{}^1 & -\frac{1}{2}(A_1{}^1 - A_{\bar{1}}{}^{\bar{1}}) \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_{\bar{1}} \end{pmatrix}. \tag{52}$$

It is obvious that the state $|\psi_-\rangle$ is an embedded qubit. Its embedding is described by the usual process based on Eq. (43). Looking at (52) we see that in this case the one parameter subgroups of $SL(2,\mathbb{C})$ are

$$\begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ \beta & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix},$$
$$\log\alpha \equiv \frac{1}{2}(A_1{}^1 - A_{\bar{1}}{}^{\bar{1}}), \qquad \beta \equiv A_{\bar{1}}{}^1, \qquad \gamma \equiv A_1{}^{\bar{1}}. \tag{53}$$

On the other hand due to its transformation properties under $SL(2,\mathbb{C})$ the state $|\psi_+\rangle$ is also a qubit. However, it is an unusual one. Its state space is a subspace of $\mathcal{F}$ where the particle number is *not conserved*. Indeed, according to Eq. (25) the SLOCC transformations also contain the transformations $e^{\hat{B}}$, the so-called B-transforms that are creating two particles from the vacuum. Similarly, we have $e^{\hat{C}}$, the C-transforms that are annihilating two particles from a two particle state. The corresponding one parameter subgroups are

$$\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$
$$\log a \equiv -\frac{1}{2}(A_1{}^1 + A_{\bar{1}}{}^{\bar{1}}), \qquad b \equiv B_{1\bar{1}}, \qquad c \equiv -C^{1\bar{1}}. \tag{54}$$

The four-dimensional space $\mathcal{F}$ is a *direct sum*: $\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-$. Let us call the restriction of $\hat{s}$ of Eq. (50) with $A_1{}^1 = -A_{\bar{1}}{}^{\bar{1}}$ and $B_{1\bar{1}} = C^{1\bar{1}} = 0$ the operator $\hat{s}_-$:

$$\hat{s}_- = \log\alpha(\hat{p}^1\hat{n}_1 - \hat{p}^{\bar{1}}\hat{n}_{\bar{1}}) + \beta\hat{p}^{\bar{1}}n_1 + \gamma\hat{p}^1\hat{n}_{\bar{1}}. \tag{55}$$

Similarly the restriction with $A_1{}^1 = A_{\bar{1}}{}^{\bar{1}}$ and $A_1{}^{\bar{1}} = A_{\bar{1}}{}^1 = 0$ will be called $\hat{s}_+$:

$$\hat{s}_+ = \log a(\hat{n}_1\hat{p}^1 - \hat{p}^{\bar{1}}\hat{n}_{\bar{1}}) + b\hat{p}^1\hat{p}^{\bar{1}} - c\hat{n}_1\hat{n}_{\bar{1}}. \tag{56}$$

Then $\hat{s} = \hat{s}_+ + \hat{s}_-$. This corresponds to the well-known fact that $\mathfrak{spin}(4) \equiv \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Notice that due to $\hat{s}_\pm|\psi_\mp\rangle = 0$ we have $\hat{s}|\psi\rangle = \hat{s}_+|\psi_+\rangle + \hat{s}_-|\psi_-\rangle$.

Let us relate the two different realizations of qubits on $\mathcal{F}$. Recall the operators of Eq. (15). Clearly

$$\hat{\Gamma}_1|0\rangle = \hat{p}^1|0\rangle, \qquad \hat{\Gamma}_1\hat{p}^1\hat{p}^{\bar{1}}|0\rangle = \hat{p}^{\bar{1}}|0\rangle. \tag{57}$$

Since $\hat{\Gamma}_{\bar{1}}^2 = \hat{\mathbf{1}}$ one can move between the basis vectors of the realizations of Eqs. (48) and (49) back and forth. Moreover, if $(\alpha,\beta,\gamma) \mapsto (a,b,c)$ then

$$\hat{\Gamma}_1\hat{s}_-\hat{\Gamma}_1 = \hat{s}_+. \tag{58}$$

B. Embedding two qubits

Though it was useful for setting the stage, the previous case was physically uninteresting. This case was lacking the phenomenon of entanglement our main concern. Now for the problem of embedding entangled qubits in different ways we consider our first nontrivial example, the case of two qubits. We have $n = 2$ and $N = 4$ hence the generalized SLOCC group is $\mathbb{C}^\times \times \mathrm{Spin}(8,\mathbb{C})$. In this case we have the range of indices $i,j = 1,2,\bar{1},\bar{2}$ and the parametrizations

$$|\psi_+\rangle = \left(\eta\hat{\mathbf{1}} + \frac{1}{2!}Z_{ij}\hat{p}^i\hat{p}^j + \xi\hat{p}^1\hat{p}^{\bar{1}}\hat{p}^2\hat{p}^{\bar{2}}\right)|0\rangle,$$
$$Z_{ij} = -Z_{ji}, \tag{59}$$

$$|\psi_-\rangle = \left(X_i\hat{p}^i + \frac{1}{3!}\epsilon_{ijkl}Y^i\hat{p}^j\hat{p}^k\hat{p}^l\right)|0\rangle. \tag{60}$$

Our aim is to identify two-qubit systems inside the eight-dimensional Fock spaces $\mathcal{F}_\pm$. On each space $\mathcal{F}_\pm$ four copies of $SL(2,\mathbb{C})$ act. Their $3 \times 4 = 12$ complex parameters can be accommodated in a generator of the form Eq. (25) with parameters placed inside the matrices A, B, C as

$$A_i{}^j = \begin{pmatrix} \log\alpha_1 - \log a_1 & 0 & \gamma_1 & 0 \\ 0 & \log\alpha_2 - \log a_2 & 0 & \gamma_2 \\ \beta_1 & 0 & -\log\alpha_1 - \log a_1 & 0 \\ 0 & \beta_2 & 0 & -\log\alpha_2 - \log a_2 \end{pmatrix}, \tag{61}$$

$$B_{ij} = \begin{pmatrix} 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_2 \\ -b_1 & 0 & 0 & 0 \\ 0 & -b_2 & 0 & 0 \end{pmatrix}, \qquad C^{ij} = \begin{pmatrix} 0 & 0 & -c_1 & 0 \\ 0 & 0 & 0 & -c_2 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \end{pmatrix}. \tag{62}$$

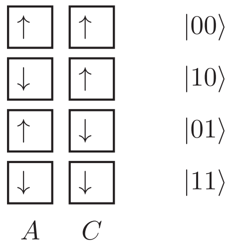

FIG. 3. Single occupancy embedding of the two-qubit Hilbert space inside $\mathcal{F}_+$.

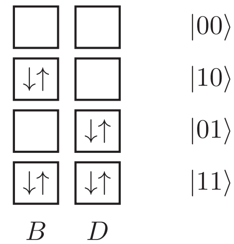

FIG. 4. Double occupancy embedding of the two-qubit Hilbert space inside $\mathcal{F}_+$.

Clearly inside $\mathcal{F}_+$ we have a subsystem arising from the mapping of Eq. (43). This is the subsystem of single occupancy states. States of this subsystem are of the form

$$|\psi_+^{\text{single}}\rangle \equiv (Z_{12}\hat{p}^1\hat{p}^2 + Z_{1\bar{2}}\hat{p}^1\hat{p}^{\bar{2}} + Z_{\bar{1}2}\hat{p}^{\bar{1}}\hat{p}^2 + Z_{\bar{1}\bar{2}}\hat{p}^{\bar{1}}\hat{p}^{\bar{2}})|0\rangle \in \mathcal{F}_+. \tag{63}$$

On this state two copies of $SL(2,\mathbb{C})$s act nontrivially. Call them $SL(2,\mathbb{C})_A$ and $SL(2,\mathbb{C})_C$. The generators of these groups are of the same form as the right-hand side of Eq. (55), where the range of indices is either $1,\bar{1}$ or $2,\bar{2}$. The parameters are $(\alpha_1,\beta_1,\gamma_1)$ and $(\alpha_2,\beta_2,\gamma_2)$ respectively. We will call the four-dimensional subspace that the state of Eq. (63) belongs to as V_{AC}. It is easy to check that the remaining two copies of $SL(2,\mathbb{C})$s, to be called $SL(2,\mathbb{C})_B$ and $SL(2,\mathbb{C})_D$ having the same form as the right-hand side of Eq. (56) and characterized by the parameters (a_1,b_1,c_1) and (a_2,b_2,c_2), act on V_{AC} trivially. This means that the corresponding generators with the (56) form annihilate $|\psi_+^{\text{single}}\rangle$. The single occupancy embedding of two qubits is illustrated in Fig. 3.

Due to the product nature of the action of $SL(2,\mathbb{C})_A \times SL(2,\mathbb{C})_C$ one can regard the space V_{AC} as one having a tensor product structure corresponding to two qubits, i.e., $V_{AC} = V_A \otimes V_C$. Similar reasoning shows that the double occupancy subspace V_{BD} with the representative

$$|\psi_+^{\text{double}}\rangle = (\eta\hat{\mathbf{1}} + Z_{1\bar{1}}\hat{p}^1\hat{p}^{\bar{1}} + Z_{2\bar{2}}\hat{p}^2\hat{p}^{\bar{2}} + \xi\hat{p}^1\hat{p}^{\bar{1}}\hat{p}^2\hat{p}^{\bar{2}})|0\rangle \tag{64}$$

is annihilated by $SL(2,\mathbb{C})_A \times SL(2,\mathbb{C})_C$, but having a usual $SL(2,\mathbb{C})_B \times SL(2,\mathbb{C})_D$ action. Hence one can write $V_{BD} = V_B \otimes V_D$. The double occupancy embedding of two qubits is illustrated in Fig. 4.

The result of these considerations is that one can write

$$\mathcal{F}_+ = (V_A \otimes V_C) \oplus (V_B \otimes V_D). \tag{65}$$

Of course this is just the well-known fact that the spinor representation 8_s of $\text{Spin}(8,\mathbb{C})$ under the subgroup $SL(2)_A \times SL(2)_B \times SL(2)_C \times SL(2)_D$ decomposes as

$$8_s = (2,1,2,1) \oplus (1,2,1,2). \tag{66}$$

It is important to realize at this point that the (65) structure of $\mathcal{F}_+$ is induced by our choice of the physically relevant subset of SLOCC transformations. In particular the tensor product structures $V_A \otimes V_C$ and $V_B \otimes V_D$ are induced by the input coming from physics, namely our identification of a subset of transformations playing a special role. Under this process we distinguished the four generators $\hat{s}_A, \hat{s}_B, \hat{s}_C$ and $\hat{s}_D$ as the ones representing a special set of physical protocols to be performed on the physical states represented by elements of our fermionic Fock space $\mathcal{F}_+$. In order to shed some light on what we mean by a "special set of physical protocols" let us write out explicitly $\hat{s}_A, \hat{s}_B, \hat{s}_C$ and $\hat{s}_D$:

$$\hat{s}_A = \log\alpha_1(\hat{p}^1\hat{n}_1 - \hat{p}^{\bar{1}}\hat{n}_{\bar{1}}) + \beta_1\hat{p}^{\bar{1}}n_1 + \gamma_1\hat{p}^1\hat{n}_{\bar{1}}, \tag{67}$$

$$\hat{s}_B = \log a_1(\hat{n}_1\hat{p}^1 - \hat{p}^{\bar{1}}\hat{n}_{\bar{1}}) + b_1\hat{p}^1\hat{p}^{\bar{1}} - c_1\hat{n}_1\hat{n}_{\bar{1}}, \tag{68}$$

$$\hat{s}_C = \log\alpha_2(\hat{p}^2\hat{n}_2 - \hat{p}^{\bar{2}}\hat{n}_{\bar{2}}) + \beta_2\hat{p}^{\bar{2}}n_2 + \gamma_2\hat{p}^2\hat{n}_{\bar{2}}, \tag{69}$$

$$\hat{s}_D = \log a_2(\hat{n}_2\hat{p}^2 - \hat{p}^{\bar{2}}\hat{n}_{\bar{2}}) + b_2\hat{p}^2\hat{p}^{\bar{2}} - c_2\hat{n}_2\hat{n}_{\bar{2}}. \tag{70}$$

From these expressions it is clear that transformations $\hat{s}_{A,B}$ act on the modes $\{1,\bar{1}\}$, and $\hat{s}_{C,D}$ act on the ones $\{2,\bar{2}\}$ of the Hilbert space of single-particle states. In the box picture these operations act on the states of the first and second box respectively. Moreover, the difference between $\hat{s}_{A,C}$ and $\hat{s}_{B,D}$ is the one of single or double occupancy of the corresponding box. When we think of the boxes as sites of a lattice with two state systems (e.g., $1/2$ spins) attached to them, the physical protocols are just the ones of addressing only *one* of the sites and at the *same time also* deciding on the (single or double occupancy) type of manipulations to be performed on their spins. Clearly these types of manipulations will provide different types of access to the resources available in this simple lattice system characterized by the spinor $|\psi_+\rangle$.

Notice also that apart from the tensor product structures Eq. (65) is also featuring a *direct sum*. The two parts of this direct sum correspond to the physical sectors of single or double occupancy. These sectors are reminiscent of some superselection sectors used in quantum theory. Namely, if for some physical reason we have no access to physical manipulations represented by generalized SLOCC transformations intertwining between these sectors, then we say that a superselection rule forbids us to go from single to double occupancy or vice versa.

A comment here is in order. It is important to realize that had we immediately started with four qubits and the corresponding spaces V_A, V_B, V_C and V_D, *physically* we would have had no *a priori* reason for using a *mathematical* construct such as $V_{AC} \oplus V_{BD}$ for quantum information processing. The reason is that in this case this construct is not representing any physically sound entangled system.

Now thanks to our constructions based on fermionic Fock space the status of *certain*[3] direct sums combined with tensor products has changed. Indeed, for fermionic systems we have a sound generalization of the notion of SLOCC transformations hence in this special case it is easy to make *physical* sense of their embedded subsystems.

Closing this section let us also comment on a dual construction based on the odd chirality sector $\mathcal{F}_-$. Let us write Eq. (60) in the form

$$|\psi_-\rangle = |\psi_-^{sd}\rangle + |\psi_-^{ds}\rangle, \tag{71}$$

where

$$|\psi_-^{sd}\rangle = (X_1\hat{p}^1 + X_{\bar{1}}\hat{p}^{\bar{1}} + Y^{\bar{1}}\hat{p}^1\hat{p}^{2\bar{2}} - Y^1\hat{p}^{\bar{1}}\hat{p}^{2\bar{2}})|0\rangle, \tag{72}$$

$$|\psi_-^{ds}\rangle = (X_2\hat{p}^2 + X_{\bar{2}}\hat{p}^{\bar{2}} + Y^{\bar{2}}\hat{p}^{1\bar{1}}\hat{p}^2 - Y^2\hat{p}^{1\bar{1}}\hat{p}^{\bar{2}})|0\rangle. \tag{73}$$

Here we have employed the shorthand notation $\hat{p}^{1\bar{1}} \equiv \hat{p}^1\hat{p}^{\bar{1}}$ etc., moreover we have used the combinations of letters sd and ds to indicate the hybrid nature of these states, i.e., they are combinations like "single-double" or "double-single." It means that $|\psi_-^{sd}\rangle$ and $|\psi_-^{ds}\rangle$ represent two-qubit systems when one of the qubits is taken in single and the other in double occupancy representation. The mixed occupancy embedding of two qubits corresponding to $|\psi_-^{sd}\rangle$ is illustrated in Fig. 5.

Now we have a decomposition

$$\mathcal{F}_- = (V_A \otimes V_D) \oplus (V_B \otimes V_C) \tag{74}$$

which corresponds to the decomposition of the conjugate spinor representation 8_c of $\mathrm{Spin}(8,\mathbb{C})$ as

[3]Of course we are not expecting that *any* direct sum structure can be embedded into *some* fermionic Fock space. For example in the conclusions we will see that though the 56-dimensional fundamental representation space of the exceptional group $E_7(\mathbb{C})$ is arising as a special direct sum of seven three-qubit sectors however, this structure cannot be embedded into a single fermionic Fock space.

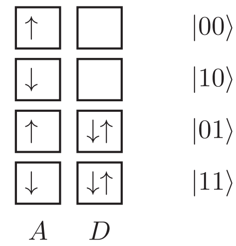

FIG. 5. Mixed occupancy embedding of the two-qubit Hilbert space inside $\mathcal{F}_-$.

$$8_c = (2,1,1,2) \oplus (1,2,2,1). \tag{75}$$

Let us write

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_- = (\mathcal{F}^{00} \oplus \mathcal{F}^{11}) \oplus (\mathcal{F}^{01} \oplus \mathcal{F}^{10})$$
$$= (V_{AC} \oplus V_{BD}) \oplus (V_{AD} \oplus V_{BC}). \tag{76}$$

This way of decomposing $\mathcal{F}$ displays that qubits A and C are in the single occupancy (0) and qubits B and D are in the double occupancy (1) representation. Notice that between the basis states of V_{AC} and V_{BD} (i.e., $\mathcal{F}^{00}$ and $\mathcal{F}^{11}$) and the action of the corresponding SLOCC groups $SL(2,\mathbb{C})_A \times SL(2,\mathbb{C})_C$ and $SL(2,\mathbb{C})_B \times SL(2,\mathbb{C})_D$ the operator $\hat{\Gamma}_1\hat{\Gamma}_2$ intertwine. Using similar intertwining properties one can write

$$\mathcal{F}^{00} = \hat{\mathbf{1}}\mathcal{F}^{00}, \qquad \mathcal{F}^{10} = \hat{\Gamma}_1\mathcal{F}^{00},$$
$$\mathcal{F}^{01} = \hat{\Gamma}_2\mathcal{F}^{00}, \qquad \mathcal{F}^{11} = \hat{\Gamma}_1\hat{\Gamma}_2\mathcal{F}^{00}. \tag{77}$$

Hence the state spaces of the embedded two-qubit systems are generated from the one of the canonical two-qubit system of Eq. (63) via the action of suitable intertwining operators.

C. Embedding three qubits

Many aspects of the three-qubit case have already been addressed within a fermionic Fock space context [11]. However, related to the system of Eq. (1) mentioned in the Introduction it is important to revisit this case from the viewpoint of embedded systems. Here we have $n=3$ and $N=6$ with the generalized SLOCC group $\mathbb{C}^\times \times \mathrm{Spin}(12,\mathbb{C})$. Let us parametrize in this case the Weyl spinors of positive and negative chirality as

$$|\psi_-\rangle = \left(U_i\hat{p}^i + \frac{1}{3!}Z_{ijk}\hat{p}^{ijk} + \frac{1}{5!}W^i\varepsilon_{ijklmn}\hat{p}^{ijklm}\right)|0\rangle, \tag{78}$$

$$|\psi_+\rangle = \left(\eta\hat{\mathbf{1}} + \frac{1}{2!}Y_{ij}\hat{p}^{ij} + \frac{1}{2!4!}X^{ij}\varepsilon_{ijklmn}\hat{p}^{klmn} + \xi\hat{p}^{123456}\right)|0\rangle, \tag{79}$$

where $\hat{p}^{ijk} = \hat{p}^i\hat{p}^j\hat{p}^k$ etc.

Now we have $(1,2,3,4,5,6) \equiv (1,2,3,\overline{1},\overline{2},\overline{3})$. As usual in the box picture we have three boxes or sites with two possible spin projections: up or down. Now on each space $\mathcal{F}_\pm$ six copies of $SL(2,\mathbb{C})$ act. Their $3 \times 6 = 18$ parameters are placed inside the 6×6 matrices A, B, C similar to the pattern we already know from Eqs. (61) and (62).

In the case of fermions with six single particle states the canonical three-qubit system connected to single occupancy is living inside $\mathcal{F}_-$. It is related to the general pattern of embedding known from Eq. (43). Using the notation familiar from the end of the previous subsection we denote this subspace as $\mathcal{F}^{000}$. Hence we have

$$|\psi_-^{sss}\rangle \equiv (Z_{123}\hat{p}^{123} + Z_{12\overline{3}}\hat{p}^{12\overline{3}} + \cdots + Z_{\overline{123}}\hat{p}^{\overline{123}})|0\rangle \in \mathcal{F}^{000}. \tag{80}$$

The notation "sss" or 000 refers to the three $SL(2,\mathbb{C})$ generators that act nontrivially on this state. They are all in the single occupancy representation. This means that we have to use three copies of generators of the form of Eqs. (67), (69) and a third one with labels featuring 3 and $\overline{3}$. The remaining three copies of $SL(2,\mathbb{C})$s with generators having the form of Eqs. (68), (70) and again a third one act trivially on $\mathcal{F}^{000}$. The result of these considerations is that now we have the decomposition of $\mathcal{F}$ to the 32 and 32′ representations corresponding to $\mathcal{F}_\pm$ as follows:

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_- = (\mathcal{F}^{001} \oplus \mathcal{F}^{010} \oplus \mathcal{F}^{100} \oplus \mathcal{F}^{111}) \oplus (\mathcal{F}^{000} \oplus \mathcal{F}^{011} \oplus \mathcal{F}^{101} \oplus \mathcal{F}^{110}). \tag{81}$$

Clearly one can identify eight copies of three-qubit systems living inside $\mathcal{F}$. Unlike in the two-qubit case, now the single and double occupancy subspaces, namely $\mathcal{F}^{000}$ and $\mathcal{F}^{111}$ are living inside subspaces of different chirality. The double occupancy state is of the form

$$|\psi_+^{ddd}\rangle = (\eta\hat{\mathbf{1}} + Y_{1\overline{1}}\hat{p}^{1\overline{1}} + \cdots - X^{1\overline{1}}\hat{p}^{2\overline{2}3\overline{3}} - \cdots - \xi\hat{p}^{1\overline{1}2\overline{2}3\overline{3}})|0\rangle \in \mathcal{F}^{111}. \tag{82}$$

The subspaces like $\mathcal{F}^{001}$ are in a mixed representation, meaning that two of the qubits are in the single and one of the qubits is in the double occupancy representation of the $SL(2,\mathbb{C})^{\times 6}$ subgroup.

As in the two-qubit case one can see that

$$\mathcal{F}^{000} = \hat{\mathbf{1}}\mathcal{F}^{000}, \qquad \mathcal{F}^{100} = \hat{\Gamma}_1\mathcal{F}^{000}, \qquad \ldots$$
$$\mathcal{F}^{110} = \hat{\Gamma}_1\hat{\Gamma}_2\mathcal{F}^{000}, \qquad \mathcal{F}^{111} = \hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3\mathcal{F}^{000}. \tag{83}$$

Hence the state spaces of the embedded three-qubit systems are again generated from the one of the canonical three-qubit subspaces via the action of suitable intertwining operators.

Let us associate to the first box (site) qubits A, and B. Qubit A is in single and qubit B in double occupancy. Similarly to the second box we associate C and D, for the third box E and F. Then the decomposition of Eq. (81) takes the form

$$\mathcal{F} = (V_{ACF} \oplus V_{ADE} \oplus V_{BCE} \oplus V_{BDF}) \oplus (V_{ACE} \oplus V_{ADF} \oplus V_{BCF} \oplus V_{BDE}). \tag{84}$$

Notice the structure of either $\mathcal{F}_+$ or $\mathcal{F}_-$. One can arrange the four summands to the vertices of the tetrahedron. Then the six edges will correspond to the six common qubits. Alternatively one can consider an incidence geometry consisting of four points labeled by triples like ACF, ADE, BCE, BDF and the lines by A, B, C, D, E, F. Then e.g., points ACF and ADE are connected by line A, etc. This incidence structure coincides with the one of the complement of a line of the Fano plane. We also note that the decomposition of Eq. (76) is precisely the one of Eq. (1) familiar from the Introduction mentioned in connection with the BHQC. However, unlike in previous attempts now to such constructs a quantum information theoretic meaning was given.

It is instructive to calculate $|\tilde{\psi}_+\rangle = \hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3|\psi_-\rangle$. This gives the 32 complex amplitudes of the positive chirality part parametrized by the 32 complex amplitudes of the negative chirality one. The result is

$$\tilde{X}^{ij} = \begin{pmatrix} 0 & U_3 & -U_2 & Z_{1\overline{23}} & Z_{1\overline{31}} & Z_{1\overline{12}} \\ -U_3 & 0 & U_1 & Z_{2\overline{23}} & Z_{2\overline{31}} & Z_{2\overline{12}} \\ U_2 & -U_1 & 0 & Z_{3\overline{23}} & Z_{3\overline{31}} & Z_{3\overline{12}} \\ -Z_{1\overline{23}} & -Z_{2\overline{23}} & -Z_{3\overline{23}} & 0 & W^{\overline{3}} & -W^{\overline{2}} \\ -Z_{1\overline{31}} & -Z_{2\overline{31}} & -Z_{3\overline{31}} & -W^{\overline{3}} & 0 & W^{\overline{1}} \\ -Z_{1\overline{12}} & -Z_{2\overline{12}} & -Z_{3\overline{12}} & W^{\overline{2}} & -W^{\overline{1}} & 0 \end{pmatrix},$$
$$\tilde{\xi} = Z_{\overline{123}}, \tag{85}$$

$$\tilde{Y}_{ij} = \begin{pmatrix} 0 & -W_3 & W_2 & -Z_{\overline{1}23} & -Z_{\overline{2}23} & -Z_{\overline{3}23} \\ W_3 & 0 & -W_1 & -Z_{\overline{1}31} & -Z_{\overline{2}31} & -Z_{\overline{3}31} \\ -W_2 & W_1 & 0 & -Z_{\overline{1}12} & -Z_{\overline{2}12} & -Z_{\overline{3}12} \\ Z_{\overline{1}23} & Z_{\overline{1}31} & Z_{\overline{1}12} & 0 & -U_{\overline{3}} & U_{\overline{2}} \\ Z_{\overline{2}23} & Z_{\overline{2}31} & Z_{\overline{2}12} & U_{\overline{3}} & 0 & -U_{\overline{1}} \\ Z_{\overline{3}23} & Z_{\overline{3}31} & Z_{\overline{3}12} & -U_{\overline{2}} & U_{\overline{1}} & 0 \end{pmatrix},$$
$$\tilde{\eta} = -Z_{123}. \tag{86}$$

This dictionary provides an explicit form for $|\tilde{\psi}_+^{ddd}\rangle = \hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3|\psi_-^{sss}\rangle$ hence for the intertwining map between $\mathcal{F}^{000}$

and $\mathcal{F}^{111}$, i.e., the map between the single and double occupancy representation of three-qubits. Note that the intertwiner above has a special significance in string theory. It is related to the so-called mirror map which for toroidal compactifications is relating via T-duality the IIA and IIB duality frames of the relevant string theories. Restricting attention to the subset of the eight amplitudes $(Z_{123}, Z_{12\overline{3}}, \ldots, Z_{\overline{123}})$ describing three-qubit states we obtain a new labeling for three-qubits. Originally this unusual representation of three-qubits equivalent to our double occupancy representation was the first to appear within the context of the BHQC [10].

D. Embedding four qubits

We have $n = 4$ and $N = 8$ and the generalized SLOCC group is $\mathbb{C}^{\times} \times \mathrm{Spin}(16, \mathbb{C})$. A Weyl spinor of positive chirality is now of the form

$$|\psi_+\rangle = \left(\eta\hat{\mathbf{1}} + \frac{1}{2!}X_{ij}\hat{p}^{ij} + \frac{1}{4!}Z_{ijkl}\hat{p}^{ijkl} + \frac{1}{6!}\varepsilon_{ijklmnrs}Y^{ij}\hat{p}^{klmnrs} + \xi\hat{p}^{12345678}\right)|0\rangle. \quad (87)$$

As usual the most natural way of embedding four qubits into $\mathcal{F}_+$ is via single occupancy

$$|\psi_+^{ssss}\rangle = (Z_{1234}\hat{p}^{1234} + Z_{123\overline{4}}\hat{p}^{123\overline{4}} + \cdots + Z_{\overline{1234}}p^{\overline{1234}} + Z_{\overline{1234}}\hat{p}^{\overline{1234}})|0\rangle \in \mathcal{F}^{0000}. \quad (88)$$

There are eight different embedded four-qubit subspaces in $\mathcal{F}_+$. These are

$$\mathcal{F}^{\mu_1\mu_2\mu_3\mu_4} \equiv \hat{\Gamma}_1^{\mu_1}\hat{\Gamma}_2^{\mu_2}\hat{\Gamma}_3^{\mu_3}\hat{\Gamma}_4^{\mu_4}\mathcal{F}^{0000},$$
$$\mu_1 + \mu_2 + \mu_3 + \mu_4 \equiv 0,$$
$$\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{Z}_2. \quad (89)$$

Similarly, we have eight further embeddings into $\mathcal{F}_-$ with $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1$. Hence one can write

$$\mathcal{F} = \bigoplus_{(\mu_1\mu_2\mu_3\mu_4)\in(\mathbb{Z}_2)^4} \mathcal{F}^{\mu_1\mu_2\mu_3\mu_4}. \quad (90)$$

VI. INVARIANTS AND COVARIANTS

A. The invariant bilinear form

We start by recapitulating some of the results of Ref. [11]. Let us consider a collection of k elements $\{x_1, x_2, \ldots, x_k\}$ of the vector space $\mathcal{V}$. These give rise to a set $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k\}$ of operators. We multiply these operators and define a map, called the *transposed* map as follows:

$$(\hat{x}_1\hat{x}_2\ldots\hat{x}_k)^T = \hat{x}_k\ldots\hat{x}_2\hat{x}_1. \quad (91)$$

Consider now the operator $\hat{n}_1\hat{n}_2\ldots\hat{n}_N$. This operator annihilates all the terms from the expansion of Eq. (14) except the term from the one-dimensional subspace of $\mathcal{F}$ corresponding to the $\wedge^N V^*$ part of the (4) Grassmann algebra. It is just the subspace spanned by the basis vector

$$|\mathrm{top}\rangle \equiv \hat{p}^1\hat{p}^2\ldots\hat{p}^N|0\rangle. \quad (92)$$

For this vector we have

$$\hat{n}_1\hat{n}_2\ldots\hat{n}_N|\mathrm{top}\rangle = (-1)^{\frac{N(N-1)}{2}}\hat{n}_N\ldots\hat{n}_2\hat{n}_1\hat{p}^1\hat{p}^2\ldots\hat{p}^N|0\rangle = (-1)^{\frac{N(N-1)}{2}}|0\rangle. \quad (93)$$

Let us now consider two elements of the fermionic Fock space

$$|\psi\rangle = \hat{\Psi}|0\rangle \in \mathcal{F}, \qquad |\phi\rangle = \hat{\Phi}|0\rangle \in \mathcal{F}. \quad (94)$$

Our aim is to define a nondegenerate bilinear form

$$(\cdot,\cdot)\colon\ \mathcal{F}\times\mathcal{F}\to\mathbb{C} \quad (95)$$

invariant under the nontrivial subgroup of the generalized SLOCC group i.e., $\mathrm{Spin}(2N, \mathbb{C})$. We define

$$(\psi, \phi)|0\rangle \equiv (-1)^{\frac{N(N-1)}{2}}(\hat{n}_1\hat{n}_2\ldots\hat{n}_N)\hat{\Psi}^T\hat{\Phi}|0\rangle. \quad (96)$$

By virtue of Eq. (93) the meaning of the bilinear form is clear: it picks out the complex coefficient of the "top" part of the state $\hat{\Psi}^T\hat{\Phi}|0\rangle$. From the definition it is obvious that for any operator $\hat{O}$ we have

$$(\psi, \hat{O}\phi) = (\hat{O}^T\psi, \phi). \quad (97)$$

One can also check that

$$(\psi, \phi) = (-1)^{\frac{N(N-1)}{2}}(\phi, \psi). \quad (98)$$

Hence this bilinear form is symmetric for $N = 0, 1 \pmod 4$ and antisymmetric for $N = 2, 3 \pmod 4$.

Now a look at Eq. (25) shows that the generators of $\mathrm{Spin}(2N, \mathbb{C})$ satisfy $\hat{s}^T = -\hat{s}$. Combining this with Eq. (97) shows that

$$(\psi, \hat{s}\phi) + (\hat{s}\psi, \phi) = 0 \quad (99)$$

which demonstrates the invariance of our bilinear form under the generalized SLOCC transformations of the form $\hat{S} = e^{\hat{s}}$.

B. Invariants and covariants for $N = 4,6$

For $N = 2n$ we have embedded n-qubit systems. Our aim is to construct the basic covariants and invariants of fermionic systems and relate these quantities to the corresponding ones of qubits.

Let us construct covariants using our bilinear form. The simplest choices are

$$K_I \equiv (\psi, \hat{e}_I \psi), \qquad K^I = g^{IJ} K_J = (\psi, \hat{e}^I \psi). \tag{100}$$

Using Eqs. (97), (23) and $\hat{s}^T = -\hat{s}$ we see that under the generalized SLOCC group K_I transforms as

$$\begin{aligned}(\psi, \hat{e}_I \psi) \mapsto (\hat{S}\psi, \hat{e}_I \hat{S}\psi) &= (\psi, \hat{S}^{-1} \hat{e}_I \hat{S}\psi) \\ &= (\psi, \hat{e}_J \psi)(\mathcal{S}^{-1})^J_I.\end{aligned} \tag{101}$$

As a result of this and Eq. (21) one has

$$K_I \mapsto K_J (\mathcal{S}^{-1})^J{}_I, \qquad K^I \mapsto \mathcal{S}^I{}_J K^J. \tag{102}$$

In order to build invariants one can try to experiment with further covariants. A natural choice is a covariant

$$\mathcal{K}^I{}_J \equiv (\psi, \hat{e}^I \hat{e}_J \psi). \tag{103}$$

Since it transforms as $\mathcal{K} \mapsto \mathcal{S}K\mathcal{S}^{-1}$ one can form the invariants

$$I_{2n} \equiv \mathrm{Tr}(\mathcal{K}^n). \tag{104}$$

In order to explore the structure of these invariants we write $\mathcal{K}$ as

$$\begin{aligned}\mathcal{K}^I{}_J &= \frac{1}{2} g^{IL}(\psi, [\hat{e}_L, \hat{e}_J]\psi) + \frac{1}{2} g^{IL}(\psi, \{\hat{e}_L, \hat{e}_J\}\psi) \\ &= \frac{1}{2} g^{IL}(\psi, [\hat{e}_L, \hat{e}_J]\psi) + \frac{1}{2}\delta^I{}_J(\psi, \psi),\end{aligned} \tag{105}$$

where we have used $\{\hat{e}_I, \hat{e}_J\} = g_{IJ}\hat{1}$.

For $N = 0, 1 \pmod 4$ the bilinear form is symmetric. In this case by virtue of the fact that the $[\hat{e}_I, \hat{e}_J]$ are just the generators of $\mathrm{Spin}(2N, \mathbb{C})$ and Eqs. (98) the first term gives zero. Hence in this case

$$\mathcal{K}^I{}_J = \frac{1}{2}\delta^I{}_J(\psi, \psi). \tag{106}$$

Using (104) we get

$$I_{2r} = 2^{1-r} N(\psi, \psi)^r. \tag{107}$$

Hence in this case apart from the quadratic invariant (ψ, ψ) no new invariant of this kind is obtained. We note that for the $r = 1$, $N = 4$ case a restriction of Eq. (59) to two fermions with four modes gives for I_2 four times a quadratic form which corresponds to the usual Plücker relations. Its square is just the determinant of the 4×4 antisymmetric matrix Z_{ij}. The magnitude of this quadratic form up to constant factors is just the usual measure of entanglement introduced in [26], which for embedded two qubits boils down to the well-known concurrence.

On the other hand for $N = 2, 3 \pmod 4$ the bilinear form is antisymmetric hence the last term of (105) vanishes giving the result

$$\mathcal{K}^I{}_J = \frac{1}{2} g^{IL}(\psi, [\hat{e}_L, \hat{e}_J]\psi). \tag{108}$$

Clearly since g^{IJ} is symmetric and the commutator is antisymmetric in this case $I_2 = 0$. So the first new nontrivial nonzero invariant should be a quartic one, I_4. Indeed, a calculation in the special case $N = 6$ shows [11,27] that when restricted to the subspace of positive chirality this invariant is just the quartic invariant introduced by Igusa [17] for his classification of spinors up to $N = 6$. This invariant is also related to the so-called generalized Hitchin functional [28].

Notice also that in the case $N = 4m + 2$ using the matrices of Eqs. (10) and (22) one can form the new matrix $\Lambda \equiv sg$ satisfying $\Lambda^t = -\Lambda^t$. Then using Eq. (108) one gets

$$\begin{aligned}&\frac{1}{2}\mathrm{Tr}(s\mathcal{K}_\psi) = (\psi, \hat{s}\psi), \qquad \hat{s} = \frac{1}{2}\Lambda^{IJ}\hat{e}_I\hat{e}_J \in \mathfrak{spin}(2N), \\ &s \in \mathfrak{so}(2N),\end{aligned} \tag{109}$$

where one can check that this expression for $\hat{s}$ coincides with the usual one of Eq. (25). Then we have a mapping $\mathcal{F}_\pm \to \mathfrak{so}(2N)$ of the form $|\psi\rangle \mapsto \mathcal{K}_\psi$. Since in this case the (96) bilinear form is antisymmetric, we can regard it as a symplectic form on $\mathcal{F}_\pm$, hence we can think of the spaces $\mathcal{F}_\pm$ as phase spaces of a classical mechanical system with the generalized SLOCC transformations defining a group action on it. It can then be shown that in this case the association $|\psi\rangle \mapsto \mathcal{K}_\psi$ described by Eq. (109) is the so-called moment map [11,28,29].

C. Invariants and covariants for $N = 8$

Let us consider the case when $N \equiv 0 \pmod 4$. Here an important special case is the $N = 8$ one which contains four fermions with eight single particle states. This is the setting where one can embed naturally four-qubit systems our main concern here. The corresponding state is living inside a Weyl spinor of the (87) form. As described in Eq. (90) this case incorporates eight different classes of embedded four-qubit states. The most natural embedding is the (88) one based on single occupancy.

The basic covariant we should consider here is the one

$$\mathcal{K}^{IJ}{}_{KL} \equiv (\psi, \hat{e}^I \hat{e}^J \hat{e}_K \hat{e}_L \psi). \tag{110}$$

Writing $\hat{e}^I\hat{e}^J = \frac{1}{2}[\hat{e}^I, \hat{e}^J] + \frac{1}{2}\{\hat{e}^I, \hat{e}^J\}$ and noting that in the $N \equiv 0 \,(\mathrm{mod}\, 4)$ case $(\psi, [\hat{e}_I, \hat{e}_J]\psi) = 0$ and $(\psi, \phi) = (\phi, \psi)$ we get

$$\mathcal{K}^{IJ}{}_{KL} \equiv g^{II'}g^{JJ'}(\psi, \hat{e}_{I'}\hat{e}_{J'}\hat{e}_K\hat{e}_L\psi) = \mathcal{R}^{IJ}{}_{KL} + \frac{1}{4}g^{IJ}g_{KL}(\psi,\psi), \tag{111}$$

where

$$\mathcal{R}^{IJ}{}_{KL} = \frac{1}{4}g^{II'}g^{JJ'}(\psi, [\hat{e}_{I'}, \hat{e}_{J'}][\hat{e}_K, \hat{e}_L]\psi). \tag{112}$$

For fermionic systems described by Weyl spinors of the form (87) the basic invariants under the generalized SLOCC group $\mathrm{Spin}(16, \mathbb{C})$ are the ones

$$\mathcal{J}_{2p} = \mathcal{K}^{I_1J_1}{}_{I_2J_2}\mathcal{K}^{I_2J_2}{}_{I_3J_3}\ldots\mathcal{K}^{I_pJ_p}{}_{I_1J_1}. \tag{113}$$

However, due to Eq. (111) it is enough to consider the invariants

$$\mathcal{I}_{2p} = \mathcal{R}^{I_1J_1}{}_{I_2J_2}\mathcal{R}^{I_2J_2}{}_{I_3J_3}\ldots\mathcal{R}^{I_pJ_p}{}_{I_1J_1}. \tag{114}$$

D. Four fermions with eight modes

In the following we consider the positive chirality sector of the $N = 8$ case with $|\psi\rangle \equiv |\psi_+\rangle$ [see Eq. (87)] with the constraint

$$|\psi\rangle = \frac{1}{4!}Z_{ijkl}\hat{p}^{ijkl}|0\rangle. \tag{115}$$

Now the quadratic SLOCC invariant is

$$(\psi, \psi) = \frac{1}{4!4!}\varepsilon^{ijklmnrs}Z_{ijkl}Z_{mnrs}. \tag{116}$$

Recall now that $\hat{e}_I = (\hat{e}_i, \hat{e}_{i+8}) = (\hat{n}_i, \hat{p}^i)$ to show that the only nonzero independent components of $\mathcal{R}^{IJ}{}_{KL}$ are $\mathcal{R}^{ij+8}{}_{k+8l}$ and $\mathcal{R}^{ij}{}_{kl}$. For example we have

$$\mathcal{R}^{ij}{}_{kl} = (\psi, \hat{p}^{ij}\hat{n}_{kl}\psi) = (\psi, \hat{n}_{kl}\hat{p}^{ij}\psi) = \mathcal{R}^{k+8l+8}{}_{i+8j+8}. \tag{117}$$

Similarly we have

$$\mathcal{R}^{ij+8}{}_{k+8l} = \frac{1}{4}(\psi, [\hat{p}^i, \hat{n}_j][\hat{p}^k, \hat{n}_l]\psi) = \frac{1}{4}(\psi, [\hat{p}^k, \hat{n}_l][\hat{p}^i, \hat{n}_j]\psi) = \mathcal{R}^{kl+8}{}_{i+8j}. \tag{118}$$

For the explicit form of $\mathcal{R}^{ij}{}_{kl}$ one gets

$$\mathcal{R}^{ij}{}_{kl} = \frac{1}{2!4!}\varepsilon^{ijabcdef}Z_{lkab}Z_{cdef}. \tag{119}$$

On the other hand,

$$\mathcal{R}^{ij+8}{}_{k+8l} = (\psi, \hat{p}^i\hat{n}_j\hat{p}^k\hat{n}_l\psi) + \frac{1}{4}\delta^i_j\delta^k_l(\psi,\psi) - \frac{1}{2}\delta^k_l(\psi, \hat{p}^i\hat{n}_j\psi) - \frac{1}{2}\delta^i_j(\psi, \hat{p}^k\hat{n}_l\psi). \tag{120}$$

Now

$$(\psi, \hat{p}^i\hat{n}_j\psi) = \frac{1}{2}(\psi, [\hat{p}^i, \hat{n}_j]\psi) + \frac{1}{2}(\psi, \{\hat{p}^i, \hat{n}_j\}\psi) = 0 + \frac{1}{2}\delta^i_j(\psi,\psi), \tag{121}$$

hence

$$\mathcal{R}^{ij+8}{}_{k+8l} = (\psi, \hat{p}^i\hat{n}_j\hat{p}^k\hat{n}_l\psi) - \frac{1}{4}\delta^i_j\delta^k_l(\psi,\psi) \tag{122}$$

yielding the result

$$\mathcal{R}^{ij+8}{}_{k+8l} = \left(\frac{1}{2}\delta^k_j\delta^i_l - \frac{1}{4}\delta^i_j\delta^k_l\right)(\psi,\psi) - \mathcal{R}^{ik}{}_{jl}. \tag{123}$$

Then according to Eq. (111) the net result is that all the components of the covariant $\mathcal{K}^{IJ}{}_{KL}$ can entirely be expressed in terms of the invariant (ψ,ψ) and the quantity $\mathcal{R}^{ij}{}_{kl}$:

$$\mathcal{K}^{ij}{}_{kl} = \mathcal{R}^{ij}{}_{kl}, \qquad \mathcal{K}^{ij+8}{}_{k+8l} = \frac{1}{2}(\psi,\psi)\delta^i_l\delta^k_j - \mathcal{R}^{ik}{}_{jl}. \tag{124}$$

$\mathcal{R}^{ij}{}_{kl}$ with the (119) explicit form is a covariant with respect to the ordinary SLOCC subgroup $SL(8,\mathbb{C})$ of $\mathrm{Spin}(16,\mathbb{C})$. It is just the covariant introduced by Katanova [30]. Hence for the construction of invariants for embedded four fermionic systems with eight single particle states it is enough to consider the invariants formed by the matrix $\mathcal{R}^{ij}{}_{kl}$ of Eq. (119). These invariants are of the form

$$I_{2p} = \mathcal{R}^{i_1j_1}{}_{i_2j_2}\mathcal{R}^{i_2j_2}{}_{i_3j_3}\ldots\mathcal{R}^{i_pj_p}{}_{i_1j_1}. \tag{125}$$

It is known [7,30] that

$$\{I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}\} \tag{126}$$

gives an algebraically independent set of generators.

E. Embedded four-qubits, covariants and density matrices

In this section via restricting the spinor of Eq. (115) we start deriving the usual set of four-qubit invariants in a spinorial language. Our starting point is the embedded four-qubit state in the single occupancy representation

$$|\psi\rangle = (\psi_{0000}\hat{p}^{1234} + \psi_{0001}\hat{p}^{123\overline{4}} + \ldots + \psi_{1110}\hat{p}^{\overline{1234}} + \psi_{1111}\hat{p}^{\overline{1234}})|0\rangle \in \mathcal{F}_+ \tag{127}$$

hence $\psi_{0000} = Z_{1234}$, $\psi_{0001} = Z_{123\overline{4}} = Z_{1238}$ etc. This is to be compared with the conventional way of writing this state as

$$|\psi\rangle = \sum_{\mu_1\mu_2\mu_3\mu_4\in 0,1} \psi_{\mu_1\mu_2\mu_3\mu_4}|\mu_1\mu_2\mu_3\mu_4\rangle \in \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2. \tag{128}$$

Using the 16 complex amplitudes of our state one can define three basic 4×4 matrices containing four four-component vectors U, V, W and Z:

$$\mathcal{L} \equiv \begin{pmatrix} \psi_{0000} & \psi_{0001} & \psi_{0010} & \psi_{0011} \\ \psi_{0100} & \psi_{0101} & \psi_{0110} & \psi_{0111} \\ \psi_{1000} & \psi_{1001} & \psi_{1010} & \psi_{1011} \\ \psi_{1100} & \psi_{1101} & \psi_{1110} & \psi_{1111} \end{pmatrix} \equiv \begin{pmatrix} U^0 & U^1 & U^2 & U^3 \\ V^0 & V^1 & V^2 & V^3 \\ W^0 & W^1 & W^2 & W^3 \\ Z^0 & Z^1 & Z^2 & Z^3 \end{pmatrix}, \tag{129}$$

$$\mathcal{M} = \begin{pmatrix} U^0 & U^1 & V^0 & V^1 \\ W^0 & W^1 & Z^0 & Z^1 \\ U^2 & U^3 & V^2 & V^3 \\ W^2 & W^3 & Z^2 & Z^3 \end{pmatrix},$$

$$\mathcal{N} = \begin{pmatrix} U^0 & U^2 & V^0 & V^2 \\ U^1 & U^3 & V^1 & V^3 \\ W^0 & W^2 & Z^0 & Z^2 \\ W^1 & W^3 & Z^1 & Z^3 \end{pmatrix}. \tag{130}$$

Notice that the matrices $\mathcal{M}$ and $\mathcal{N}$ are obtained from the matrix $\mathcal{L}$ via the permutations. The index structure of these matrices is

$$\mathcal{L} \leftrightarrow \psi_{\mu_1\mu_2\mu_3\mu_4} \qquad \mathcal{M} \leftrightarrow \psi_{\mu_3\mu_1\mu_2\mu_4} \qquad \mathcal{N} \leftrightarrow \psi_{\mu_1\mu_4\mu_2\mu_3}. \tag{131}$$

If we use the (128) representation these matrices appear in the reduced density matrices

$$\varrho_{12} = \mathcal{L}\mathcal{L}^\dagger, \qquad \overline{\varrho}_{34} = \mathcal{L}^\dagger\mathcal{L}, \tag{132}$$

$$\varrho_{13} = \mathcal{M}\mathcal{M}^\dagger, \qquad \overline{\varrho}_{24} = \mathcal{M}^\dagger\mathcal{M}, \tag{133}$$

$$\varrho_{14} = \mathcal{N}\mathcal{N}^\dagger, \qquad \overline{\varrho}_{23} = \mathcal{N}^\dagger\mathcal{N}, \tag{134}$$

coming from the corresponding operators like $\hat{\varrho}_{12} \equiv \mathrm{Tr}_{34}|\psi\rangle\langle\psi|$.

We can embed these density matrices inside of a 28×28 one as follows. Write $|\psi\rangle = \frac{1}{4!}Z_{ijkl}\hat{p}^{ijkl}|0\rangle$ with only the relevant 16 nonzero complex amplitudes ($ijkl \in \{1, 2, \ldots, \overline{4}\}$). Define the two-partite reduced density matrix as

$$\varrho^{ij}{}_{kl} \equiv \frac{1}{2}\langle\psi|\hat{p}^{ij}\hat{n}_{kl}\psi\rangle. \tag{135}$$

Explicitly one has

$$\varrho^{ij}{}_{kl} = \frac{1}{4}\overline{Z}^{ijmn}Z_{lkmn}. \tag{136}$$

By virtue of this it can be checked that if $\langle\psi|\psi\rangle = 1$ then $\varrho^{ij}{}_{ij} = 6$, i.e., $\varrho^{ij}{}_{kl}$ satisfies the usual Löwdin normalization adopted by quantum chemists [31]. From the 28 independent index pairs ij only 24 give nonzero contribution (pairs like $1\overline{1}, 2\overline{2}, 3\overline{3}, 4\overline{4}$ give zero), and similarly for the index pairs kl one only has to take into consideration 24 ones. It is easy to see that using for this 24×24 block the somewhat unusual labeling for the rows and columns as $(12, 1\overline{2}, \overline{1}2, \overline{12}, \ldots, 34, 3\overline{4}, \overline{3}4, \overline{34})$ we are left with a block diagonal matrix consisting of six 4×4 blocks. These are precisely the six reduced density matrices of Eqs. (132)–(134). Since these are all having trace equals to one, the trace of $\varrho^{ij}{}_{kl} = 6$ as it has to be.

Consider now one-half of the covariant $\mathcal{R}^{ij}{}_{kl}$ of Eq. (117) i.e.,

$$\frac{1}{2}\mathcal{R}^{ij}{}_{kl} = \frac{1}{2}(\psi, \hat{p}^{ij}\hat{n}_{kl}\psi) \tag{137}$$

with explicit form given by Eq. (119) i.e.,

$$\frac{1}{2}\mathcal{R}^{ij}{}_{kl} = \frac{1}{4} * Z^{ijmn}Z_{lkmn}, \tag{138}$$

where $*\psi$ denotes the Hodge dual of the four-form ψ.

F. Majorana fermions

Comparing Eqs. (135) and (137) we see that our covariant and the two-partite reduced density matrix is of the same structure up to the important difference that the former features the bilinear pairing and the latter the usual Hermitian scalar product. Now it is known that this structural similarity is related to the generalization [21] of the usual Wootters spin flip operation [22] as follows. For a spinor ψ define its *spined flipped spinor* $\tilde{\psi}$ via the formula

$$\langle\tilde{\psi}|\phi\rangle \equiv (\psi, \phi). \tag{139}$$

Then writing $|\psi\rangle$ and $|\phi\rangle$ as in Eq. (94) we obtain the result

$$|\tilde{\psi}\rangle = (\hat{\Psi}^T)^\dagger|\mathrm{top}\rangle, \tag{140}$$

where $|\mathrm{top}\rangle$ is defined in Eq. (92) and clearly $\hat{n}_i^\dagger = \hat{p}^i$. In our special case

$$|\tilde{\psi}\rangle = \frac{1}{4!}\tilde{Z}_{ijkl}\hat{p}^{ijkl}|0\rangle = \frac{1}{4!} * \overline{Z}_{ijkl}\hat{p}^{ijkl}|0\rangle,$$

$$\tilde{Z}_{ijkl} = *\overline{Z}_{ijkl} = \frac{1}{4!}\varepsilon_{ijklabcd}\overline{Z}^{abcd}. \qquad (141)$$

Now notice that the condition

$$|\tilde{\psi}\rangle = |\psi\rangle \qquad (142)$$

is a *reality condition* imposed on our originally *complex* spinor. It can be checked that this condition is precisely the *Majorana condition* for spinors [32]. In the following a spinor satisfying Eq. (142) will be referred to a *Majorana spinor*. Hence we obtained the nice result that the generalized Wootters spin flip operation of Ref. [21] is naturally related to the notion of Majorana spinors. As an extra bonus of embedding qubits into Fock space we have managed to understand the meaning of the well-known Wootters spin flip operation of quantum information as a special case of an operation related to a natural reality condition for spinors. In our special case after comparing Eqs. (136) and (138) we see that for a Majorana spinor (Wootters self-conjugate spinor) our basic covariant $\mathcal{R}^{ij}{}_{kl}$ (the covariant of Katanova [30]) is up to a factor of 2 just the two-partite reduced density matrix $\varrho^{ij}{}_{kl}$.

Let us see how these findings manifest themselves in the special case of embedded four-qubits. Now the independent components of $\mathcal{R}^{ij}{}_{lk}$ form a 28×28 matrix[4] $\mathcal{R}$ whose structure is similar to the one of $\varrho^{ij}{}_{lk}$. Namely this matrix is block diagonal and consists of seven 4×4 blocks. One block is containing merely zeros, the remaining six ones are

$$\mathcal{R}_{12} \equiv \epsilon\otimes\epsilon\mathcal{L}\epsilon\otimes\epsilon\mathcal{L}^t, \qquad \mathcal{R}_{34} \equiv \epsilon\otimes\epsilon\mathcal{L}^t\epsilon\otimes\epsilon\mathcal{L}, \qquad (143)$$

$$\mathcal{R}_{13} \equiv \epsilon\otimes\epsilon\mathcal{M}\epsilon\otimes\epsilon\mathcal{M}^t, \qquad \mathcal{R}_{24} \equiv \epsilon\otimes\epsilon\mathcal{M}^t\epsilon\otimes\epsilon\mathcal{M}, \qquad (144)$$

$$\mathcal{R}_{14} \equiv \epsilon\otimes\epsilon\mathcal{N}\epsilon\otimes\epsilon\mathcal{N}^t, \qquad \mathcal{R}_{23} \equiv \epsilon\otimes\epsilon\mathcal{N}^t\epsilon\otimes\epsilon\mathcal{N}, \qquad (145)$$

where

$$\epsilon\otimes\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad (146)$$

[4]Notice that the matrix $\mathcal{R}$ is defined with a swap of the indices kl hence a sign change.

and $\mathcal{L}^t$ refers to the transposed matrix of $\mathcal{L}$ [not to be confused with the transposition used for spinors in expressions like $\hat{\Psi}^T$ in (140), see also Eq. (91)]. Explicitly we have

$$\mathcal{R} = \frac{1}{2}\begin{pmatrix} \mathcal{R}_{12} & & & & & & \\ & \mathcal{R}_{34} & & & & & \\ & & \mathcal{R}_{13} & & & & \\ & & & \mathcal{R}_{24} & & & \\ & & & & \mathcal{R}_{14} & & \\ & & & & & \mathcal{R}_{23} & \\ & & & & & & 0 \end{pmatrix}. \qquad (147)$$

Now for a four-qubit state of the form (128) the amplitudes of the usual Wootters spin flipped state spin flipped state are defined as

$$\tilde{\psi}_{\mu_1\mu_2\mu_3\mu_4} = \epsilon^{\mu_1\mu_1'}\epsilon^{\mu_2\mu_2'}\epsilon^{\mu_3\mu_3'}\epsilon^{\mu_4\mu_4'}\overline{\psi}_{\mu_1'\mu_2'\mu_3'\mu_4'}. \qquad (148)$$

After representing the 16 amplitudes in terms of 4×4 matrices the Wotters spin flip corresponds to transformations like $\mathcal{L}\mapsto\epsilon\otimes\epsilon\overline{\mathcal{L}}\epsilon\otimes\epsilon$ terms showing up in Eqs. (143)–(145).

G. Recovering the basic four-qubit invariants

Let us now try to recover the basic four-qubit invariants in our spinorial framework. From the work of Luque and Thibon [33] we know that for four qubits we have four algebraically independent invariants. One of them is of order 2 (H), two of order 4 (L and M), and one of order 6 (D). For a pair of arbitrary vectors $U, V \in \mathbb{C}^4$ let us define an inner product by the formula

$$U\cdot V = U^0V^3 - U^1V^2 - U^2V^1 + U^3V^0 \qquad (149)$$

i.e., the inner product has the matrix of Eq. (146). Then in the notation of Eq. (129) we have the formula for H

$$H = U\cdot Z - V\cdot W = \epsilon^{\mu_1\mu_1'}\epsilon^{\mu_2\mu_2'}\epsilon^{\mu_3\mu_3'}\epsilon^{\mu_4\mu_4'}\psi_{\mu_1\mu_2\mu_3\mu_4}\psi_{\mu_1'\mu_2'\mu_3'\mu_4'}$$
$$= \frac{1}{2}(\psi,\psi). \qquad (150)$$

Notice that the first expression for H is hiding its permutation invariance, clearly displayed by the second. The third expression relates this important invariant to our bilinear form defined for spinors, see Eq. (116), where in this case for the labeling of the amplitudes Eq. (127) should be used.

For the definitions of the fourth order invariants we have

$$L = \mathrm{Det}\mathcal{L}, \qquad M = \mathrm{Det}\mathcal{M}, \qquad N = \mathrm{Det}\mathcal{N}, \qquad (151)$$

and the important identity [33]

$$L + M + N = 0. \tag{152}$$

In order to define the sixth order invariant D we use the characteristic polynomial of the matrix $R \equiv \mathcal{R}_{12}$ as a generating polynomial for the algebraically independent invariants [34]

$$\mathcal{P}(R,t) \equiv \mathrm{Det}(tI - R) = t^4 - s_1 t^3 + s_2 t^2 - s_3 t + s_4, \tag{153}$$

where

$$s_1 = \mathrm{Tr}R = 2H, \tag{154}$$

$$2s_2 = (\mathrm{Tr}R)^2 - \mathrm{Tr}R^2 = H^2 + 4M + 2L, \tag{155}$$

$$3!s_3 = (\mathrm{Tr}R)^3 - 3\mathrm{Tr}R\mathrm{Tr}R^2 + 2\mathrm{Tr}R^3 = 4D + 2HL, \tag{156}$$

$$4!s_4 = (\mathrm{Tr}R)^4 + 8\mathrm{Tr}R\mathrm{Tr}R^3 + 3(\mathrm{Tr}R^2)^2 - 6(\mathrm{Tr}R)^2\mathrm{Tr}R^2 - 6\mathrm{Tr}R^4 = 4!\mathrm{Det}R = 4!L^2. \tag{157}$$

An explicit computation shows that [34]

$$s_3 = 2\mathrm{Det}\begin{pmatrix} U\cdot U & U\cdot V & U\cdot Z \\ U\cdot W & V\cdot W & W\cdot Z \\ U\cdot Z & V\cdot Z & Z\cdot Z \end{pmatrix} - 2\mathrm{Det}\begin{pmatrix} U\cdot V & V\cdot V & V\cdot W \\ U\cdot W & V\cdot W & W\cdot W \\ U\cdot Z & V\cdot Z & W\cdot Z \end{pmatrix} \tag{158}$$

which implicitly defines D. For the algebraically independent set of $SL(2,\mathbb{C})^{\times 4}$ invariants either the set s_1, s_2, s_3, s_4 or the one H, L, M, D can be used. There is yet another way of looking at the sixth order invariants which will be useful. Using H, L, M and D one can define new sixth order combinations [33] E and F as follows:

$$D = E - HL, \qquad E = F - HN, \qquad F = D - HM. \tag{159}$$

Clearly these combinations are related by permutation symmetry of the qubits in a cyclic manner. This has the important corollary that if in the characteristic polynomial we plug in for R either of the matrices in Eqs. (143)–(145) based on $\mathcal{L}, \mathcal{N}$ or $\mathcal{M}$ then the invariants s_1, s_2, s_3, s_4 showing up will always have a similar form with the letters L, M, N and D, E, F cyclically permuted. Explicitly

$$s_1 = \begin{cases} 2H & \text{using } \mathcal{L} \\ 2H, & \text{using } \mathcal{N} \\ 2H, & \text{using } \mathcal{M} \end{cases} \tag{160}$$

$$s_2 = \begin{cases} H^2 + 2(M-N) & \text{using } \mathcal{L} \\ H^2 + 2(L-M), & \text{using } \mathcal{N} \\ H^2 + 2(N-L), & \text{using } \mathcal{M} \end{cases} \tag{161}$$

$$s_3 = \begin{cases} 2(D+E) & \text{using } \mathcal{L} \\ 2(E+F), & \text{using } \mathcal{N} \\ 2(F+D), & \text{using } \mathcal{M} \end{cases} \tag{162}$$

$$s_4 = \begin{cases} L^2 & \text{using } \mathcal{L} \\ N^2, & \text{using } \mathcal{N} \\ M^2, & \text{using } \mathcal{M}, \end{cases} \tag{163}$$

where for arriving at this form with permutation symmetry displayed we used the identities (152) and (159). Recall now that from the terms showing up in these expressions one can form four algebraically independent combinations which apart from $SL(2,\mathbb{C})^{\times 4}$ invariance displaying permutation invariance as well. These form the set [33,35] $\{H, \Sigma, \Gamma, \Pi\}$ where

$$\Sigma = L^2 + M^2 + N^2, \qquad \Gamma = D + E + F, \qquad \Pi = (L-M)(M-N)(N-L). \tag{164}$$

In order to reveal the spinorial origin of these invariants one calculates the traces of the relevant 4×4 blocks of the basic covariant $\mathcal{R}^{ij}{}_{kl}$ associated to the 28×28 matrix $\mathcal{R}$ of Eq. (147) with explicit structure given by Eqs. (143)–(145):

$$\frac{1}{2}\mathrm{Tr}R = \begin{cases} H & \text{using } \mathcal{L} \\ H, & \text{using } \mathcal{N} \\ H, & \text{using } \mathcal{M} \end{cases} \tag{165a}$$

$$\frac{1}{2}\mathrm{Tr}R^2 = \begin{cases} H^2 + 2(N-M) & \text{using } \mathcal{L} \\ H^2 + 2(M-L), & \text{using } \mathcal{N} \\ H^2 + 2(L-N), & \text{using } \mathcal{M} \end{cases} \tag{165b}$$

$$\frac{1}{2}\mathrm{Tr}R^3 = \begin{cases} H^3 + 6H(N-M) + 3(D+E) & \text{using } \mathcal{L} \\ H^3 + 6H(M-L) + 3(E+F), & \text{using } \mathcal{N} \\ H^3 + 6H(L-N) + 3(F+D), & \text{using } \mathcal{M} \end{cases} \tag{165c}$$

$$\frac{1}{2}\mathrm{Tr}R^4 = \begin{cases} H^4 + 12H^2(N-M) + 8H(D+E) + 4(N-M)^2 - 2L^2 & \text{using } \mathcal{L} \\ H^4 + 12H^2(M-L) + 8H(E+F) + 4(M-L)^2 - 2N^2, & \text{using } \mathcal{N} \\ H^4 + 12H^2(L-N) + 8H(F+D) + 4(L-N)^2 - 2M^2. & \text{using } \mathcal{M}. \end{cases} \tag{165d}$$

Further traces of powers can be calculated, here we merely give the expressions for the fifth and sixth powers we need later:

$$\frac{1}{2}\mathrm{Tr}R^5 = H^5 + 20H^3(N-M) + 15H^2(D+E) - 5HL^2 + 20H(N-M)^2 + 10(D+E)(N-M) \tag{165e}$$

$$\frac{1}{2}\mathrm{Tr}R^6 = H^6 + 30H^4(N-M) + 24H^3(D+E) - 9H^2L^2 + 60H^2(N-M)^2 + 48H(D+E)(N-M) + 6(E+D)^2 + 8(N-M)^2 - 6L^2(N-M), \tag{165f}$$

where in these expressions we have merely displayed the $R = \mathcal{R}_{12}$ choice.

Let us now define $SL(2,\mathbb{C})^{\times 4} \rtimes S_4$ invariants g_{2p}, $p = 1, 2, \ldots$ as follows:

$$g_{2p} = \frac{1}{2}\sum_{a<b} \mathrm{Tr}\mathcal{R}^p_{ab}, \tag{166}$$

i.e., in order to form these invariants we have to add the traces of the powers of the six nontrivial matrices showing up in Eq. (147). Then a straightforward calculation shows that

$$g_2 = 6H, \qquad g_4 = 6H^2, \qquad g_6 = 6H^3 + 12\Gamma \tag{167}$$

$$g_8 = 6H^4 + 32H\Gamma + 20\Sigma \tag{168}$$

$$g_{10} = 6H^5 + 90H\Sigma + 60H^2\Gamma \tag{169}$$

$$g_{12} = 6H^6 + 96H^3\Gamma + 250H^2\Sigma + 16\Gamma^2 - 60\Pi. \tag{170}$$

From this it follows that as an independent set of generators the set

$$\{g_2, g_6, g_8, g_{12}\} \tag{171}$$

can be used. In order to relate our set of generators to the one of Chen *et al.* [7] we express the invariant g_{10} in terms of the independent ones. The result is

$$2^5 \cdot 3^4 g_{10} = 7g_2^5 + 2^3 \cdot 3^5 g_2 g_8 - 2^3 \cdot 7 \cdot 9 g_2^2 g_6. \tag{172}$$

Now comparing this equation with Eq. (11) of Ref. [7] one concludes that the set of independent generators used by Chen *et al.* namely $\{f'_2, f'_6, f'_8, f'_{12}\}$ is related to ours simply,

$$f'_{2p} = 2^{1-p} g_{2p} = \mathrm{Tr}\mathcal{R}^p, \tag{173}$$

where $\mathcal{R}$ is the matrix of Eq. (147). Now the invariants I_{2p} of Eq. (125) are just trivial multiples of f'_{2p} namely

$$I_{2p} = (-1)^p 2^p f'_{2p}, \tag{174}$$

and the fermionic invariants J_{2p} can also be calculated using Eq. (124). Clearly J_{2p} will be again a polynomial of the set $\{f'_2, f'_6, f'_8, f'_{12}\}$ which we will not give here.

VII. THE ALGEBRA OF Spin(16,ℂ) INVARIANT POLYNOMIAL FUNCTIONS

Let $\mathcal{A}$ be the algebra of complex polynomial functions on either $\mathcal{F}_+$ or $\mathcal{F}_-$ i.e., on the 128-dimensional complex vector space of Weyl spinors of definite chirality which are invariant under $G_0 = \mathrm{Spin}(16,\mathbb{C})$. One can define the affine variety $\mathcal{F}_+/\mathrm{Spin}(16,\mathbb{C})$ associated to $\mathcal{A}$. One can then show that this variety is isomorphic [24] to $\mathbb{C}^8$. In this section we would like to elaborate on the structure of the eight algebraically independent generators for $\mathcal{A}$. From the physical point of view the magnitudes of these generators will give possible measures of entanglement, which are invariant under the generalized SLOCC subgroup G_0.

Note that such an investigation can be regarded as a natural generalization of the one initiated in Ref. [7] where four fermions with eight modes were considered. In this case the corresponding algebra of invariants $\mathcal{B}$ is the one of complex polynomial functions on the 70-dimensional complex vector space $\wedge^4 V^*$ with $V = \mathbb{C}^8$ invariant under $SL(8,\mathbb{C})$. The latter group is the nontrivial subgroup of the SLOCC group and the affine variety $\wedge^4 V^*/SL(8,\mathbb{C})$ is isomorphic to the affine space $\mathbb{C}^7$. In our fermionic formalism the seven generators of $\mathcal{B}$ are of the form [7,30] $f_{2p} = \mathrm{Tr}\mathcal{R}^p$ with $p = 1,3,4,5,6,7,9$ where $\mathcal{R}$ is a 28×28 matrix not subject to the restrictions displayed in Eq. (147). Furthermore, for embedded four-qubit systems our detailed calculations based on the special form of Eq. (147) show how the algebra $\mathcal{C}$ of complex polynomial functions on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ invariant under $SL(2,\mathbb{C})^{\times 4} \rtimes S_4$ are derived from the basic fermionic invariants. In this case the corresponding affine variety is isomorphic to $\mathbb{C}^4$ with generators $\{g_2, g_6, g_8, g_{12}\}$ of Eqs. (167)–(170). According to Eq. (173) these generators also correspond to the set [7] $\{f'_2, f'_6, f'_8, f'_{12}\}$. Clearly we have a sequence of embedded algebras

$$\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}. \tag{175}$$

In Ref. [7] the restriction map between the algebras $\mathcal{C} \subset \mathcal{B}$ has been studied. An aim of this section is to initiate a study concerning the algebra $\mathcal{A}$, as an object naturally incorporating all cases.

Now the results for the embedding $\mathcal{C} \subset \mathcal{B}$ follow from the decompositions based on the symmetric spaces

$$\mathfrak{so}_8 = (\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2) \oplus \mathfrak{m}_{16}, \tag{176}$$

$$\mathfrak{e}_7 = \mathfrak{sl}_8 \oplus \mathfrak{m}_{70}. \tag{177}$$

Note that results for the symmetric space decomposition

$$\mathfrak{e}_8 = \mathfrak{so}_{16} \oplus \mathfrak{m}_{128} \tag{178}$$

are also available in the literature [24]. Since the subspace $\mathfrak{m}_{128}$ in this approach is just the space of Weyl spinors i.e., $\mathcal{F}_+$ this observation enables an explicit exploration of the structure of the restriction map for the generators of the algebra $\mathcal{A}$ an issue which is the subject of the next subsections.

A. The semisimple orbit for four qubits

In order to gain some insight into the structure of $\mathcal{A}$ we reformulate some results already discussed in the literature. Take the representative of the semisimple orbit of four-qubit states in the form [36]

$$|\mathcal{G}(x)\rangle \equiv \sum_{\alpha=1}^{4} x_\alpha |\phi_\alpha\rangle, \qquad |\phi_\alpha\rangle \equiv |\varphi_\alpha\rangle \otimes |\varphi_\alpha\rangle,$$
$$x \equiv (x_1, x_2, x_3, x_4) \in \mathbb{C}^4, \tag{179}$$

where

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\varphi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{180}$$

$$|\varphi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |\varphi_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \tag{181}$$

Alternatively one can write

$$|\mathcal{G}(y)\rangle = y_1(|0000\rangle + |1111\rangle) + y_2(|0011\rangle + |1100\rangle) + y_3(|0101\rangle + |1010\rangle) + y_4(|0110\rangle + |1001\rangle), \tag{182}$$

where

$$x_1 = y_1 + y_4, \qquad x_2 = y_3 - y_2,$$
$$x_3 = y_3 + y_2, \qquad x_4 = y_1 - y_4. \tag{183}$$

Take the following 24 element set of elementary polynomials in x

$$\pm 2x_1, \qquad \pm 2x_2, \qquad \pm 2x_3, \qquad \pm 2x_4,$$
$$\pm x_1 \pm x_2 \pm x_3 \pm x_4, \tag{184}$$

where the last item in the list refers to all of the 16 possible sign combinations. Call these 24 elementary polynomials $e_s(x)$, $s = 1, 2, \ldots 24$. Let us define the new polynomials

$$\pi_{2p}(x) \equiv \sum_{s=1}^{24} [e_s(x)]^{2p}. \tag{185}$$

Then for example one has

$$\pi_2(x) = 24 \sum_{\alpha=1}^{4} x_\alpha^2 \tag{186}$$

$$\pi_6(x) = 48\left[3\sum_{\alpha=1}^{4} x_\alpha^6 + 5\sum_{\alpha\neq\beta} x_\alpha^2 x_\beta^4 + 30\sum_{\alpha>\beta>\gamma} x_\alpha^2 x_\beta^2 x_\gamma^2\right]. \tag{187}$$

Now a calculation of the simplest two invariants H and Γ for the state Eq. (179) shows that [37]

$$2H = \sum_{\alpha=1}^{4} x_\alpha^2,$$
$$2^5\Gamma = \sum_{\alpha=1}^{4} x_\alpha^6 - \sum_{\alpha\neq\beta} x_\alpha^2 x_\beta^4 + 18\sum_{\alpha>\beta>\gamma} x_\alpha^2 x_\beta^2 x_\gamma^2. \tag{188}$$

Comparing now π_{2p} and g_{2p} of Eq. (167) for $p = 1, 3$ one gets that

$$\pi_2(x) = 2^3 g_2(x), \qquad \pi_6(x) = 2^7 g_6(x). \tag{189}$$

A computer check shows that this simple pattern survives hence

$$\pi_{2p}(x) = 2^{2p+1} g_{2p}(x), \qquad p = 1, 3, 4, 6, \tag{190}$$

or alternatively

$$\pi_{2p}(x) = (-1)^p 2^{2p} I_{2p}, \tag{191}$$

where I_{2p} are the fermionic invariants defined in Eq. (125).

Notice that using the new parametrization of Eq. (183) for the e_r we immediately get the simple expressions

$$g_{2p} = \sum_{\alpha<\beta}(y_\alpha + y_\beta)^{2p} + \sum_{\alpha<\beta}(y_\alpha - y_\beta)^{2p},$$
$$\alpha, \beta = 1, 2, 3, 4. \qquad (192)$$

Notice that apart from a factor of $\frac{1}{6}$ and a different labeling convention used this expression for g_{2p} is of the same form as the invariants $\mathcal{F}_{2p}$ of Eq. (1) of Gour and Wallach [38]. It is also important to realize that these polynomials for $p = 1, 3, 4, 6$ constitute a set of algebraically independent polynomials [39,40] invariant under the Weyl group of the exceptional group F_4. This interesting connection between the dense orbit of four-qubit states and F_4 was emphasized in Refs. [36,38].

However, the algebraically independent sets $\{\mathcal{F}_2, \mathcal{F}_6, \mathcal{F}_8, \mathcal{F}_{12}\}$ and $\{g_2, g_6, g_8, g_{12}\}$ are not the same. Indeed in Ref. [38] instead of our parameters x_α the parametrization

$$(z_0, z_1, z_2, z_3) = (x_1, x_4, x_3, x_2) \qquad (193)$$

was used. In this parametrization [38]

$$\mathcal{F}_{2p} = \frac{1}{6}\sum_{\alpha<\beta}(z_\alpha - z_\beta)^{2p} - \frac{1}{6}\sum_{\alpha<\beta}(z_\alpha - z_\beta)^{2p},$$
$$\alpha, \beta = 0, 1, 2, 63. \qquad (194)$$

Hence according to Eq. (193) our polynomials g_{2p} expressed in terms of the parameters $y_\alpha, \alpha = 1, 2, 3, 4$ are of the same form as the polynomials $\mathcal{F}_{2p}$ expressed in terms of the parameters $x_\alpha, \alpha = 1, 2, 3, 4$.

As a result the explicit forms of the polynomials $\mathcal{F}_{2p}$ expressed in terms of the set $\{H, \Gamma, \Sigma, \Pi\}$ should be some different combinations than the ones shown in Eqs. (167)–(170). For example a quick calculation shows that $\mathcal{F}_6 = 12H^3 - 16\Gamma$ on the other hand according to Eq. (167) $g_6 = 6H^3 + 12\Gamma$. This is in accord with the result found in Sec. IV of Ref. [41]:

$$\mathcal{F}_2 = 2H, \qquad \mathcal{F}_6 = 4(3H^3 - 4\Gamma), \qquad (195)$$

$$\mathcal{F}_8 = \frac{4}{3}(33H^4 - 104H\Gamma + 40\Sigma) \qquad (196)$$

$$\mathcal{F}_{12} = \frac{4}{3}(513H^6 - 3012H^3\Gamma + 2180H^2\Sigma + 488\Gamma^2 + 480\Pi). \qquad (197)$$

Comparing these expressions with the ones of Eqs. (167)–(170) we see that our generating set $\{g_2, g_6, g_8, g_{12}\}$ is more elegant as the expansion coefficients are much simpler and they can be seen as the ones derived from a more general procedure based on fermionic systems as spinors. Notice, however, that according to Eq. (173) up to 2^{1-p} this generator system is the same as the one $\{f'_2, f'_6, f'_8, f'_{12}\}$ which already appeared in Ref. [7] as the one coming from the invariants of Katanova [30]. Here we added to these results a further twist by also displaying their explicit form in terms of the usual set $\{H, \Gamma, \Sigma, \Pi\}$ originally due to Schläfli [35]. We also note that apart from a factor of 6 the set $\{\mathcal{F}_2, \mathcal{F}_6, \mathcal{F}_8, \mathcal{F}_{12}\}$ is the same as the Saito-Sekiguchi set [42] of generators a point emphasized in the Appendix of Ref. [37].

B. Representing the dense orbit under $\mathbb{C}^\times \times \mathrm{Spin}(16, \mathbb{C})$

Our aim here is to use an eight parameter representative of the generic orbit in $\mathcal{F}_+$ under the action of the generalized SLOCC group $\mathbb{C}^\times \times \mathrm{Spin}(16, \mathbb{C})$ for obtaining explicit forms for the invariants. In Ref. [7] it was shown how the four parameter family of states of Eq. (179) can be embedded into a seven parameter family belonging to $\wedge^4 \mathbb{C}^8$. This means that this family can be regarded as restrictions of a more general one for four fermions with eight modes. This chain of generalizations is based on the (176)–(178) sequence of Lie algebras. Here we give a spinorial entanglement based generalization of the E_8 case. Note that our representative of the relevant entanglement class is equivalent to the representative of the standard semisimple orbit already known in the mathematics literature [24].

Let us consider the Fock space version of the state of Eq. (182):

$$|\mathcal{G}\rangle = y_1(\hat{p}^{1234} + \hat{p}^{\overline{1234}}) + y_2(\hat{p}^{12\overline{34}} + \hat{p}^{\overline{12}34}) + y_3(\hat{p}^{1\overline{2}3\overline{4}} + \hat{p}^{\overline{1}2\overline{3}4}) + y_4(\hat{p}^{1\overline{23}4} + \hat{p}^{\overline{1}23\overline{4}})|0\rangle = \sum_{\alpha=1}^{4} y_\alpha|E_\alpha\rangle, \qquad (198)$$

where

$$|E_1\rangle = (\hat{p}^{1234} + \hat{p}^{\overline{1234}})|0\rangle, \qquad |E_2\rangle = (\hat{p}^{12\overline{34}} + \hat{p}^{\overline{12}34})|0\rangle \qquad (199)$$

$$|E_3\rangle = (\hat{p}^{1\overline{2}3\overline{4}} + \hat{p}^{\overline{1}2\overline{3}4})|0\rangle, \qquad |E_4\rangle = (\hat{p}^{1\overline{23}4} + \hat{p}^{\overline{1}23\overline{4}})|0\rangle. \qquad (200)$$

Now if we make the identification

$$\{1, 2, 3, 4, \overline{1}, \overline{2}, \overline{3}, \overline{4}\} \equiv \{13572468\} \qquad (201)$$

then the basis vectors p_2, p_4, p_5 and $-p_6$ of Refs. [7,43] will correspond to the ones $|E_\alpha\rangle$. If according to (177) we identify these states as four generators belonging to the $\mathfrak{m}_{70}$ part of the Lie algebra $\mathfrak{e}_7$ we see that they define four from the seven of the basis states of a seven-dimensional Cartan subspace $\mathfrak{c}$. In this notation the remaining three basis vectors (denoted by p_1, p_3 and $-p_7$ in Ref. [7,43]) have the form

$$|E_5\rangle = (\hat{p}^{1\bar{1}4\bar{4}} + \hat{p}^{2\bar{2}3\bar{3}})|0\rangle, \quad |E_6\rangle = (\hat{p}^{1\bar{1}3\bar{3}} + \hat{p}^{2\bar{2}4\bar{4}})|0\rangle,$$
$$|E_7\rangle = (\hat{p}^{1\bar{1}2\bar{2}} + \hat{p}^{3\bar{3}4\bar{4}})|0\rangle. \tag{202}$$

Notice now that in the notation of Sec. IV D the basis vectors of the first kind $(p_2, p_4, p_5, -p_6)$ are spanning a subspace of the *single occupancy subspace* $\mathcal{F}^{0000} \subset \mathcal{F}_+$. On the other hand the ones of the second kind $(p_1, p_3, -p_7)$ are spanning a subspace of the *double occupancy subspace* $\mathcal{F}^{1111} \subset \mathcal{F}_+$. According to Eq. (89) these subspaces are related by the action of the operator

$$\hat{\Omega} \equiv \hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3\hat{\Gamma}_4. \tag{203}$$

Under the action of $\hat{\Omega}$ the basis vectors $|E_j\rangle, j = 1, \ldots 8$ are mapped to each other as

$$|E_\alpha\rangle \mapsto |E_{9-\alpha}\rangle, \qquad \alpha = 1, 2, 3, 4, \tag{204}$$

where we have introduced a new basis state $|E_8\rangle$ which is of the form

$$|E_8\rangle = (\hat{\mathbf{1}} + \hat{p}^{1234\overline{1234}})|0\rangle. \tag{205}$$

Hence the double occupancy version of the state of Eq. (198) is

$$|\mathcal{G}'\rangle = y_5(\hat{p}^{1\bar{1}4\bar{4}} + \hat{p}^{2\bar{2}3\bar{3}}|0\rangle + y_6(\hat{p}^{1\bar{1}3\bar{3}} + \hat{p}^{2\bar{2}4\bar{4}})|0\rangle$$
$$+ y_7(\hat{p}^{1\bar{1}2\bar{2}} + \hat{p}^{3\bar{3}4\bar{4}})|0\rangle + y_8(\hat{\mathbf{1}} + \hat{p}^{1234\overline{1234}})|0\rangle$$
$$= \sum_{\alpha=1}^{4} y_{9-\alpha}\hat{\Omega}|E_\alpha\rangle. \tag{206}$$

Now the eight parameter family of states we would like to propose is of the form

$$|G(y)\rangle \equiv \sum_{\alpha=1}^{4} (y_\alpha + y_{9-\alpha}\hat{\Omega})|E_\alpha\rangle. \tag{207}$$

An alternative form of this state is

$$|G(y)\rangle = \left(y_8\hat{\mathbf{1}} + \frac{1}{4!}Z_{ijkl}\hat{p}^{ijkl} + y_8\hat{p}^{1234\overline{1234}}\right)|0\rangle, \tag{208}$$

where

$$Z_{1234} = Z_{\overline{1234}} = y_1, \qquad Z_{12\overline{34}} = Z_{\overline{12}34} = y_2, \tag{209}$$

$$Z_{1\bar{2}3\bar{4}} = Z_{\bar{1}2\bar{3}4} = y_3, \qquad Z_{1\overline{23}4} = Z_{\bar{1}23\bar{4}} = y_4, \tag{210}$$

$$Z_{1\bar{1}4\bar{4}} = Z_{2\bar{2}3\bar{3}} = y_5, \qquad Z_{1\bar{1}3\bar{3}} = Z_{2\bar{2}4\bar{4}} = y_6,$$
$$Z_{1\bar{1}2\bar{2}} = Z_{3\bar{3}4\bar{4}} = y_7. \tag{211}$$

Notice that if we define the *dual tensor* $*Z^{ijkl}$ as

$$*Z^{ijkl} \equiv \frac{1}{4!}\varepsilon^{ijklabcd}Z_{abcd} \tag{212}$$

then we have

$$*Z^{1234} = Z_{\overline{1234}}, \qquad *Z^{\overline{1234}} = Z_{1234},$$
$$*Z^{12\overline{34}} = Z_{\overline{12}34}, \qquad \ldots \qquad *Z^{1\bar{1}2\bar{2}} = Z_{3\bar{3}4\bar{4}} \tag{213}$$

hence the tensor Z_{ijkl} is *self-dual*.

We will need the matrix elements

$$(G(y), G(y)) = 2y_8^2 + \frac{1}{4!} * Z^{ijkl}Z_{ijkl} = 2\sum_{n=1}^{8} y_n^2, \tag{214}$$

$$(G(y), \hat{p}^{ijkl}G(y)) = 2y_8 * Z^{ijkl},$$
$$(G(y), \hat{n}_{ijkl}G(y)) = 2y_8 Z_{lkji}, \tag{215}$$

$$(G(y), \hat{p}^{ij}\hat{n}_{kl}G(y)) = (\delta^i_l\delta^j_k - \delta^i_k\delta^j_l)y_8^2 + \frac{1}{2} * Z^{ijab}Z_{lkab}, \tag{216}$$

$$(G(y), \hat{p}^i\hat{n}_j\hat{p}^k\hat{n}_l G(y)) = \delta^i_l\delta^k_j \sum_{n=1}^{8} y_n^2 - (G(y), \hat{p}^{ik}\hat{n}_{jl}G(y)). \tag{217}$$

After using the results of Sec. V C and implementing self-duality for the matrix elements of the basic covariant $\mathcal{R}^{IJ}{}_{KL}$ we get

$$\mathcal{R}^{ij}{}_{kl} = (\delta^i_l\delta^j_k - \delta^i_k\delta^j_l)y_8^2 + \sum_{a<b} Z^{ijab}Z_{ablk}, \tag{218}$$

$$\mathcal{R}^{k+8l+8}{}_{i+8j+8} = \mathcal{R}^{ij}{}_{kl}, \tag{219}$$

$$\mathcal{R}^{ij}{}_{k+8l+8} = \mathcal{R}^{i+8j+8}{}_{kl} = 2y_8 Z_{ijkl}, \tag{220}$$

$$\mathcal{R}^{ij+8}{}_{k+8l} = \left(\delta^i_l\delta^k_j - \frac{1}{2}\delta^i_j\delta^k_l\right)\sum_{n=1}^{8} y_n^2 - \mathcal{R}^{ik}{}_{jl}. \tag{221}$$

C. Polynomial invariants for the generalized SLOCC group

Using the matrix elements in Eq. (114) one can calculate the invariants $\mathcal{I}_{2p}$. For the special case of the state of Eq. (207) these will be polynomials in the complex amplitudes $y_j, j = 1, \ldots 8$. As an algebraically independent set of these polynomials that are invariant under the nontrivial part of the generalized SLOCC group $G_0 = \mathrm{Spin}(16, \mathbb{C})$ we would like to propose

$$\{\mathcal{I}_2, \mathcal{I}_8, \mathcal{I}_{12}, \mathcal{I}_{14}, \mathcal{I}_{18}, \mathcal{I}_{20}, \mathcal{I}_{24}, \mathcal{I}_{30}\}. \tag{222}$$

Note that the order of the algebraically independent polynomials has been known for a long time [24,44–46]. Indeed carrying out the calculations a computer check shows that the set of Eq. (222) polynomials is algebraically independent.

In order to motivate our choice and also understand the meaning of these polynomials let us first consider another, 240 element set of elementary polynomials $e_s(x_1, \ldots, x_8), s = 1, 2, \ldots 240$, of the form

$$\pm x_i \pm x_j, \qquad \frac{1}{2}(x_i \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm x_6 \pm x_7 \pm x_8), \tag{223}$$

where in the second set only an *even* number of minus signs are allowed. In accord with the (178) decomposition this $240 = 112 + 128$ split of polynomials corresponds to the root system of the group E_8. Let us now define the polynomials

$$\Pi_{2p}(x) = \sum_s^{240} [e_s(x)]^{2p}, \qquad 2p = 2, 8, 12, 14, 18, 20, 24, 30. \tag{224}$$

They form an alternative set to our polynomials $\mathcal{I}_{2p}(y)$ coming from the set of Eq. (222). A computer check shows that they are algebraically independent as well.

Observe now that the set of vectors defined by (199)–(200), (202) and (205) defines a Cartan subspace $\mathfrak{c}$ (i.e., a maximal commutative subspace) of $\mathfrak{m} \equiv \mathfrak{m}_{128}$ of Eq. (178). (For the explicit form of the commutators of the $\mathfrak{e}_8$ Lie-algebra based on the decomposition of (178) see the paper of Antonyan and Elashvili [24].) Let $W \equiv W(\mathfrak{c}, \mathfrak{e}_8)$ be the Weyl group of $\mathfrak{e}_8$ regarded as a graded algebra. Then it is known [46] that the restriction of polynomial functions $\mathbb{C}[\mathfrak{m}] \to \mathbb{C}[\mathfrak{c}]$ induces an isomorphism $\mathbb{C}[\mathfrak{m}]^{G_0} \to \mathbb{C}[\mathfrak{c}]^W$. The upshot of these considerations is that if we restrict the (222) generating set taken from the space $\mathbb{C}[\mathfrak{m}]^{G_0}$ of generalized SLOCC invariant polynomials to the generic class represented by our state of Eq. (207) one obtains some combinations of an algebraically independent set taken from the space $\mathbb{C}[\mathfrak{c}]^W$ of polynomials that are invariant under the action of the Weyl group of E_8. Now it is known [39] (for alternative choices see [40,47]) that as an algebraically independent set of $\mathbb{C}[\mathfrak{c}]^W$ one can take our new polynomials of Eq. (224) constructed from the roots of $\mathfrak{e}_8$. In order to find the relationship between the Weyl invariant polynomials of Eq. (224) and our set of Eq. (222) restricted to the eight parameter family of (207) we have to relate the complex variables y_j and x_j $j = 1, \ldots 8$. We choose

$$y_1 = \frac{1}{2}(x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8), \qquad y_2 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4 - x_5 - x_6 + x_7 + x_8) \tag{225}$$

$$y_3 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4 - x_5 + x_6 - x_7 + x_8), \qquad y_4 = \frac{1}{2}(x_1 - x_2 - x_3 + x_4 - x_5 + x_6 + x_7 - x_8) \tag{226}$$

$$y_5 = \frac{1}{2}(x_1 - x_2 - x_3 + x_4 + x_5 - x_6 - x_7 + x_8), \qquad y_6 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + x_7 - x_8) \tag{227}$$

$$y_7 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4 + x_5 + x_6 - x_7 - x_8), \qquad y_8 = \frac{1}{2}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8). \tag{228}$$

Now let us solve this system of equations to obtain $x(y)$. Our aim is to find a relation between $\mathcal{I}_{2p}(y)$ and $\Pi_{2p}(x(y))$ for $2p = 2, 8, 12, 14, 18, 20, 24, 30.$ Using the results obtained in Eqs. (218)–(221) a computer calculation shows that

$$\mathcal{I}_{2p}(y) = (-1)^p 2^{2p-1} \Pi_{2p}(x(y)). \tag{229}$$

Let $\mathfrak{c}$ be the Cartan subspace of $\mathcal{F}_+$. Then $G_0\mathfrak{c}$ contains an open subset of $\mathcal{F}_+$ and is dense. From this it follows that any G_0 invariant polynomial on $\mathcal{F}_+$ is determined by its restriction to $\mathfrak{c}$. Hence our set of Eq. (222) can really be regarded as an algebraically independent set of G_0 invariant homogeneous polynomials. Clearly the extension of these polynomials to $\mathcal{F}_+$ can be given. The magnitudes of the polynomials showing up in this set we would like to propose for the characterization of the entanglement properties of systems of fermions with eight modes.

A comment on the structure of these invariants is in order. One can take for example the more general 71 parameter subset of states as given by Eq. (208). [Now we refrain from applying the restrictions of Eqs. (209)–(211).] Then using the explicit form of the matrix elements given by Eqs. (216)–(217) our invariants can explicitly be calculated. They can be expressed as polynomials in y_8 with expansion coefficients given by traces of powers of the 28×28 matrices Z and $*Z$. These expressions are even simpler for the 36 parameter family of self-dual states $*Z = Z$. Notice, however, that for the general case featuring all 128 amplitudes even for the simplest nontrivial invariant, i.e., the octic one $\mathcal{I}_8$, one would obtain a rather complicated formula. We mention that $\mathcal{I}_8$ is living inside the octic E_8 invariant calculated in the paper of Cederwall [48].

VIII. CONCLUSIONS

In this paper following the ideas of Ref. [11] we have been considering the problem of embedding qubits into fermionic Fock space based on an underlying Hilbert space of dimension N. Unlike previous studies making use of a subspace representing systems with the number of fermions fixed here we also allowed the possibility of creating and annihilating fermions via changing their total number. For this construction to make sense we made use of the full Fock space. Mathematically this corresponded to representing pure states of our quantum systems by spinors. This construction naturally leads to the idea of extending the fermionic SLOCC group $GL(N,\mathbb{C})$ to the one of $G=\mathbb{C}^\times\times\mathrm{Spin}(2N,\mathbb{C})$. In this picture classification of entanglement types boils down to the classification of spinors, i.e., the determinations of orbits under G finding their representatives and their stabilizers. As emphasized in Ref. [11] separable states in this formalism are represented by pure spinors a notion that dates back to Cartan and Chevalley. Hence entanglement in our new formalism corresponds to some sort of deviation from purity of spinors. Though our spinors serving as entangled states are inherently complex for obtaining real states one can also consider certain reality conditions. We have shown that a natural reality condition to be imposed on complex spinors is the one defining Majorana spinors. The naturality of this condition stems from the fact that for embedded qubits this condition boils down [21] to the one of self-conjugate states under the Wootters spin flip operation [22]. This operation is of utmost importance for defining physically well established measures such as the entanglement of formation for two qubits [22], and is a standard ingredient for defining multiqubit measures of entanglement. It is amusing to see this operation coming out easily from our Fock space considerations.

Looking at the phenomenon of pure state multipartite entanglement from our point of view is rewarding from many respects. Here we elucidated the usefulness of our approach by concentrating on special entangled systems made of few qubits embedded into Fock space. We clarified the structure of different types of embedding via applying the notions of single, double and mixed occupancy. These notions have transparent physical meaning. We have shown that the different types of embedding help us to clarify the physical meaning of structures showing up in the BHQC. The main problem there was the occurrence of direct sums combined with tensor products, or the occurrence of singlets apart from doublets. Though doublets (qubits) have a natural physical interpretation singlets have no clear cut interpretation within a conventional framework of entanglement theory. Embedding entanglement theory to the theory of spinors enables a natural physical interpretation of singlets.

It is important to note however that we are not pretending that our ideas solve the problem of singlets showing up in all contexts featuring the BHQC. Let us consider for instance the problem of the *tripartite entanglement of seven qubits* based on the 56-dimensional fundamental irreducible representation of the exceptional group E_7 of Refs. [18,19] related to the work of Manivel [49]. Since $8\times 7=56$ there the authors constructed this representation space as the sevenfold direct sum of eight-dimensional three-qubit spaces, namely,

$$V_{ABC}\oplus V_{ADE}\oplus V_{AFG}\oplus V_{BDF}\oplus V_{BEG}\oplus V_{CDG}\oplus V_{CEF}. \tag{230}$$

However, again without giving a physically sound recipe for what the seven *superselection sectors* in this case mean, this system is left in a state which is lacking any quantum information theoretic meaning. Although as demonstrated in Sec. V C. this system contains seven sectors of 32-dimensional subspaces amenable to a femionic interpretation based on the tripartite entanglement of six qubits the full 56-dimensional representation space cannot be embedded into fermionic Fock space. In order to see this just recall the decomposition of the 56 of $E_7(\mathbb{C})$ under $SL(2,\mathbb{C})\times SO(12,\mathbb{C})$:

$$56=(2,12)\oplus(1,32). \tag{231}$$

Here the extra $SL(2,\mathbb{C})$ factor corresponds to the seventh qubit (say qubit G). The second part (1,32) of this decomposition is featuring the 32-dimensional spinor representation amenable to a Fock space reinterpretation. Its meaning is clearly related to the tripartite entanglement of six qubits (say A,B,C,D,E,F) a picture coming from the embedded qubits[5] of Sec. V C. However, the first term is featuring the *vector* representation of $SO(12,\mathbb{C})$ for which no spinorial characterization is possible. Hence in order to make sense of these constructs from an entanglement point of view other ideas are needed.

As another application of our ideas we conducted a study on n-qubit invariants reinterpreted as spinorial structures. We have seen that it is rewarding to enlarge the n-qubit SLOCC group $GL(2,\mathbb{C})^{\times n}$ to $S_n\ltimes GL(2,\mathbb{C})^{\times n}$ by also taking into consideration permutations of qubits. Being the largest subgroup of the fermionic SLOCC group $GL(2n,\mathbb{C})$ which leaves invariant the n-qubit subspace spanned by the basis vectors of single occupancy it serves as a natural group directly related to the chain

$$S_n\ltimes GL(2,\mathbb{C})^{\times n}\subset GL(2n,\mathbb{C})\subset\mathbb{C}^\times\times\mathrm{Spin}(4n,\mathbb{C}). \tag{232}$$

[5]There a different labeling of qubits was used. However, the incidence structure of the decomposition is the same.

The rightmost member of this chain is our generalized SLOCC group of Eq. (26) taken for the special case of $N = 2n$. An important corollary of this observation is that n-qubit invariants featuring also permutation symmetry should be regarded as ones coming from the basic spinorial invariants and covariants. In other words for investigating these invariants we should consider a suitable restriction of the algebra $\mathcal{A}$ of complex polynomial functions on either $\mathcal{F}_+$ or $\mathcal{F}_-$. Indeed from a mathematical point of view the chain of algebras $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ answering the chain of groups of Eq. (232) should be regarded as the natural object of study. This idea first appeared in Ref. [7] for the inclusion $\mathcal{C} \subset \mathcal{B}$. In this paper we proposed to enlarge this inclusion to also include the algebra $\mathcal{A}$. As an illustration of these ideas we worked out the $n = 4$ case. Here we had the chance to compare our findings with numerous results already existing in the literature [7,30,33,34,36–38,41]. The $n = 4$ case is also highly special revealing an intriguing relationship to exceptional groups. Indeed considerations of Refs. [7,36,38] have already revealed that the algebras $\mathcal{C}$ and $\mathcal{B}$ are related to the structure of exceptional groups F_4 and E_7. Via the structure of the algebra $\mathcal{A}$ our considerations managed to add the largest exceptional group E_8 to the list. In particular we constructed an algebraically independent set of $\mathrm{Spin}(16, \mathbb{C})$ invariant polynomials. The magnitudes of these polynomials can serve as measures of entanglement in our fermionic Fock space context. With an explicit computation we have shown that when restricting these polynomials to the dense orbit the resulting polynomials on eight variables are invariant ones under $W(E_8)$ i.e., the Weyl group of E_8.

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