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θ-exact Seiberg-Witten maps at order e^3

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We study two distinct θ-exact Seiberg-Witten (SW) map expansions, (I) and (II), respectively, up to order e^3 for the gauge parameter, gauge field, and gauge field strengths of the noncommutative $\mathrm{U}_\star(1)$ gauge theory on the Moyal space. We derive explicitly the closed-form expression for the SW map ambiguity between the two and observe the emergence of several new totally commutative generalized star products. We also identify the additional gauge freedoms within each of the e^3-order field-strength expansions and define corresponding sets of deformation/ratio/weight parameters, (κ, κ_i) and (κ, κ_i'), for these two SW maps, respectively.

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I. INTRODUCTION

The θ-exact Seiberg-Witten (SW) map is an old and new subject in the noncommutative (NC) gauge field theory on the Moyal space. Some results emerged immediately after the map itself [1] was discovered [2–27]. Applications to the perturbative noncommutative quantum field theories started several years later. Until now it has been shown to be of great value for developing nontrivial variants of the noncommutative quantum field theory from both theoretical and phenomenological perspectives [28–43]. Yet most of them are restricted to the first/e^2 order of the θ-exact expansion only due to the complicated nature of (obtaining) the second/e^3-order expansion.[1] Recently a systematic construction of the θ-exact SW map expansion with respect to the powers of the coupling constant e for arbitrary gauge field theories on Moyal space was proposed in Ref. [44]. (See also Ref. [45] for its latest hybrid SW map extension.) This could trigger many further applications in the near future.

The main aim of this paper is to continue the aforementioned important progress on the θ-exact SW map expansion [44]. Here we focus on two topics. First, the e^3-order SW map expansion for the $\mathrm{U}_\star(1)$ gauge field obtained via the method in Ref. [44] [denoted as SW map (I)] appears to be different from the early results [7,34] [SW map (II)], which is not really surprising since has long been recognized that the SW map is far from unique when defined as the map between noncommutative and commutative fields that preserves the smooth commutative limit and satisfies the consistency condition relations [12–16,22,23,46]:

$$\delta_\Lambda A_\mu \equiv \partial_\mu\Lambda + i[\Lambda \stackrel{\star}{,} A_\mu] = \delta_\lambda A_\mu[a_\mu], \tag{1}$$

$$\delta_\Lambda F_{\mu\nu} \equiv i[\Lambda \stackrel{\star}{,} F_{\mu\nu}] = \delta_\lambda F_{\mu\nu}[a_\mu], \tag{2}$$

$$\Lambda[[\lambda_1,\lambda_2],a_\mu] = [\Lambda[\lambda_1,a_\mu] \stackrel{\star}{,} \Lambda[\lambda_2,a_\mu]] + i\delta_{\lambda_1}\Lambda[\lambda_2,a_\mu] - i\delta_{\lambda_2}\Lambda[\lambda_1,a_\mu]. \tag{3}$$

The Moyal star($\star$) product used in the above is defined as usual. Besides a possible connection by gauge transformation, there can be plenty of generic ambiguity/freedom/redundancy between two different gauge field SW maps. Still, it gives rise to the question of to exactly what extent are the two maps (I) and (II) different from each other. The ambiguity between the NC gauge field expansions can be characterized by looking at the composition of the first SW map expansion $\{\Lambda_1(a_\mu,\lambda), A_{1_\mu}(a_\mu)\ldots\}$ and by the inverse $\{\lambda_2(A_\mu,\Lambda), a_{2_\mu}(A_\mu)\ldots\}$ of the second SW map expansion $\{\Lambda_2(a_\mu,\lambda), A_{2_\mu}(a_\mu)\ldots\}$ [23]. Consistency conditions then lead to the following equality in the case of the NC $\mathrm{U}_\star(1)$ gauge theory:

$$\partial_\mu\lambda_2(A_{1_\mu}(a_\mu),\Lambda_1(a_\mu,\lambda)) = \delta_\lambda a_{2_\mu}(A_{1_\mu}(a_\mu)). \tag{4}$$

It was further pointed out [22,23] that such a composition bears the following general form:

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[1]Generically the ordering employed in this article is the formal power of field operators or the homogeneity in the fields [23]. The equivalent notation of coupling constant ordering is a consequence of the so-called $\mathrm{U}_\star(1)$ charge quantization issue and its resolution within the SW map approach [5,27]. In this resolution [5,27] the commutative field in the SW map expansion for a $\mathrm{U}_\star(1)$ theory is bundled with the commutative charge Qe in order to normalize the NC field to the quantized charges $0, \pm 1$, which in turn induces the equivalence between the field operator and coupling constant power ordering. We use the name coupling constant ordering for its easier visibility in (front of) the equations.

$$a_{2_\mu}(A_{1_\mu}(a_\mu)) = a_\mu + X_\mu(a_\mu) + \partial_\mu Y(a_\mu), \tag{5}$$

$$\lambda_2(A_{1_\mu}(a_\mu), \Lambda_1(a_\mu, \lambda)) = \lambda + \delta_\lambda Y(a_\mu), \tag{6}$$

where $\delta_\lambda X_\mu(a_\mu) = 0$. Thus, the procedure to thoroughly distinguish these two SW maps would be to determine $X_\mu(a_\mu)$ and $Y(a_\mu)$ explicitly.

Second, a large class of SW map ambiguities (field redefinitions) was constructed in the past by iteratively adding commutative gauge-covariant terms with free coefficients at the nth power of $\theta^{\mu\nu}$ to the known solution $A_\mu^{\theta^n}$ up to the same order [29,46]. It was found that such redefinitions contribute to the field strength in the action which can help to cancel certain divergences in the perturbative quantum loop computations [32,47–51]. These findings made such a method highly favorable in the θ-expanded studies, yet it remains an open question how to generalize this procedure to the θ-exact approach since the iteration is based solely on the powers of θ^{ij} and consequently gives no hint about how to resum over all orders of θ. Recently we observed that the e^2-order θ-exact expansion of the field strength for $\mathrm{U}_\star(1)$ gauge theory possesses a freedom by itself [52]: it contains a term invariant under the commutative gauge transformation. For this reason this term does not contribute to the consistency relation at e^2 order. One can then freely vary the ratio of this term with respect to the other term. The resulting deformed field-strength operator still allows the usual quadratic action to be gauge invariant up to e^2 order. The e^2-order ratio coefficient can be shown to be the inverse of the θ^1-order iteration induced coefficient reported in the early works. This fact motivates us to consider such ratio parameter(s) a possible substitute for the iteration-induced coefficient(s) in the θ-exact approach.

In this article we study both topics mentioned above at the e^3 order. We first compare—up to the cubic order of the coupling constant e—two distinct θ-exact SW map expansions for the NC $\mathrm{U}_\star(1)$ gauge theory: one obtained from the SW differential equation (I), and the other by inverting an early θ-exact inverted SW map solution (II). We give a closed-form expression for the SW map ambiguity between these two maps and show that this ambiguity/freedom could contribute to the field strength. We then extend the procedure in Ref. [52] to each of the e^3-order field-strength SW map expansions, identify the gauge-invariant parts inside each of the expansions, and assign the corresponding ratio parameters. With help from the new generalized star products found when studying the gauge field ambiguity, we are able to explicitly express each of the gauge-invariant parts in terms of the commutative field strength too.

The paper is structured as follows. In the second section we describe both θ-exact SW map expansion solutions up to the e^3 order. The SW map ambiguity between these two SW maps is given explicitly in Sec. III, where we also demonstrate the emergence of several new totally commutative generalized star products within the expressions. Section IV is devoted to the freedoms within each of the e^3-order θ-exact gauge field strength expansions, respectively. Discussions and conclusions then follow. In this article the capital letters denote NC objects, and the small letters denote commutative objects.

II. TWO DIFFERENT θ-EXACT SEIBERG-WITTEN EXPANSIONS UP TO THE e^3 ORDER

A. Seiberg-Witten map (I)

The first powerful method (I) to obtain the θ-exact SW map expansion for NC gauge theories on Moyal space is performed by solving the SW differential equations [1,14–16,22,23,44,45]. For the NC gauge parameter (Λ), the NC gauge field (A_μ), and the NC gauge field strength $(F_{\mu\nu})$ of the $\mathrm{U}_\star(1)$ gauge theory, these equations read

$$\frac{d}{dt}\Lambda(x) = -\frac{1}{4}\theta^{ij}\{A_i \overset{\star_t}{,} \partial_j \Lambda\}, \tag{7}$$

$$\frac{d}{dt}A_\mu(x) = \frac{1}{4}\theta^{ij}\{A_i \overset{\star_t}{,} \partial_j A_\mu + F_{j\mu}\}, \tag{8}$$

$$\frac{d}{dt}F_{\mu\nu}(x) = \frac{1}{4}\theta^{ij}[\{F_{\mu i} \overset{\star_t}{,} F_{\nu j}\} - \{A_i \overset{\star_t}{,} (D_j^{\star_t} + \partial_j)F_{\mu\nu}\}], \tag{9}$$

where the Moyal $\star$ product with an additional parameter t is defined as

$$(\phi \star_t \psi)(x) = e^{\frac{i}{2}t\theta^{ij}\partial_i^\eta \partial_j^\xi}\phi(x+\eta)\psi(y+\xi)|_{\eta,\xi\to 0}$$
$$\equiv \phi(x)e^{\frac{i}{2}\overleftarrow{\partial_i}t\theta^{ij}\overrightarrow{\partial_j}}\psi(x). \tag{10}$$

Note that in the rest of the article this parameter t will be absorbed into the definition of θ^{ij} when not needed. The NC covariant derivative is defined in the following way: $D_j^{\star_t} = \partial_j - i[A_j \overset{\star_t}{,}]$. By imposing the initial conditions [39]

$$\Lambda_{(\mathrm{I})}(x) = e\lambda + \mathcal{O}(e^2), \quad A_{\mu_{(\mathrm{I})}}(x) = ea_\mu + \mathcal{O}(e^2), \tag{11}$$

one can easily solve Eqs. (7) and (8) at the e^2 order and obtain the following solutions:

$$\Lambda_{(\mathrm{I})}(x) = e\lambda - \frac{e^2}{2}\theta^{ij}a_i \star_{2_t} \partial_j\lambda + \mathcal{O}(e^3),$$
$$A_{\mu_{(\mathrm{I})}}(x) = ea_\mu - \frac{e^2}{2}\theta^{ij}a_i \star_{2_t} (\partial_j a_\mu + f_{j\mu}) + \mathcal{O}(e^3). \tag{12}$$

Then, the next order of the SW differential equations can be written down recursively,

$$\frac{d}{dt}\Lambda^{e^3}(x) = \frac{e^3}{8}\theta^{ij}\theta^{kl}[\{a_i \overset{\star_t}{,} \partial_j(a_k \star_{2_t} \partial_l\lambda)\}$$
$$+ \{a_k \star_{2_t} (\partial_l a_i + f_{li}) \overset{\star_t}{,} \partial_j\lambda\}], \tag{13}$$

$$\frac{d}{dt}A_\mu^{e^3}(x) = \frac{e^3}{8}\theta^{ij}\theta^{kl}[\{a_i \overset{\star_{t'}}{,} (\partial_j(a_k\star_{2_t}(\partial_l a_\mu + f_{l\mu})) - 2(f_{jk}\star_{2_t}f_{\mu l} - a_i\star_{2_t}\partial_l f_{j\mu}))\} + \{(a_k\star_{2_t}(\partial_l a_i + f_{li}))\overset{\star_{t'}}{,}(\partial_j a_\mu + f_{j\mu})\}]. \tag{14}$$

The $\star_{2_t}$ product is defined analogously to the $\star_t$ product, i.e.,

$$\phi(x)\star_{2_t}\psi(x) = \frac{\sin(t\frac{\partial_1\theta\partial_2}{2})}{(\frac{\partial_1\theta\partial_2}{2})}\phi(x_1)\psi(x_2)\bigg|_{x_1=x_2=x}. \tag{15}$$

The fact that denominator is not scaled by t is crucial in solving the higher-order SW differential equations [44]. On the other hand, the t^n ordering still remains the same as the θ^n ordering because of the extra θ^{ij} outside the generalized star product(s).

Since Eqs. (13) and (14) involve only $\{f \overset{\star_{t'}}{,} (g\star_{2_t}h)\}$-type terms, in accord with the technique from Ref. [44] one can immediately introduce a new generalized $\star_{3'}$ product,

$$[fgh]_{\star_{3'}} = \int_0^t dt'\{f \overset{\star_{t'}}{,} (g\star_{2_{t'}}h)\} = \cdot\left(\frac{\cos[t(\frac{\partial_f\theta\partial_g}{2} + \frac{\partial_f\theta\partial_h}{2} - \frac{\partial_g\theta\partial_h}{2})] - 1}{(\frac{\partial_f\theta\partial_g}{2} + \frac{\partial_f\theta\partial_h}{2} - \frac{\partial_g\theta\partial_h}{2})(\frac{\partial_g\theta\partial_h}{2})} - \frac{\cos[t(\frac{\partial_f\theta\partial_g}{2} + \frac{\partial_f\theta\partial_h}{2} + \frac{\partial_g\theta\partial_h}{2})] - 1}{(\frac{\partial_f\theta\partial_g}{2} + \frac{\partial_f\theta\partial_h}{2} + \frac{\partial_g\theta\partial_h}{2})(\frac{\partial_g\theta\partial_h}{2})}\right) f\otimes g\otimes h, \tag{16}$$

for a universal expression of the e^3-order expansion. In this notation we find the following θ-exact solutions for the SW differential equations up to the e^3 order:

$$\Lambda_{(\mathrm{I})}(x) = e\lambda - \frac{e^2}{2}\theta^{ij}a_i\star_2\partial_j\lambda + \frac{e^3}{8}\theta^{ij}\theta^{kl}[a_i\partial_j(a_k\partial_l\lambda) - \partial_i\lambda a_k(\partial_l a_j + f_{lj})]_{\star_{3'}} + \mathcal{O}(e^4), \tag{17}$$

$$\begin{aligned} A_{\mu_{(\mathrm{I})}}(x) = {} & ea_\mu - \frac{e^2}{2}\theta^{ij}a_i\star_2(\partial_j a_\mu + f_{j\mu}) + \frac{e^3}{8}\theta^{ij}\theta^{kl}([a_i\partial_j(a_k(\partial_l a_\mu + f_{l\mu}))]_{\star_{3'}} - 2[a_i(f_{jk}f_{l\mu} - a_k\partial_l f_{j\mu})]_{\star_{3'}} \\ & + [(\partial_j a_\mu + f_{j\mu})a_k(\partial_l a_i + f_{li})]_{\star_{3'}}) + \mathcal{O}(e^4). \end{aligned} \tag{18}$$

B. Seiberg-Witten map (II)

Another type (II) of θ-exact SW map expansion [34] was obtained by inverting the solutions from Ref. [7]; the explicit expansion for A_μ and Λ up to e^3 order is as follows:

$$\begin{aligned} \Lambda_{(\mathrm{II})}(x) = {} & e\lambda - \frac{e^2}{2}\theta^{ij}a_i\star_2\partial_j\lambda + \frac{e^3}{2}\theta^{ij}\theta^{kl}\left[\frac{1}{2}(a_k\star_2(\partial_l a_i + f_{li}))\star_2\partial_j\lambda + \frac{1}{2}a_i\star_2\partial_j(a_k\star_2\partial_l\lambda)\right] \\ & - \frac{e^3}{2}\theta^{ij}\theta^{kl}[\partial_k\partial_i\lambda a_j a_l + \partial_k\lambda a_i\partial_l a_j]_{\star_3} + \mathcal{O}(e^4), \end{aligned} \tag{19}$$

$$\begin{aligned} A_{\mu_{(\mathrm{II})}}(x) = {} & ea_\mu - \frac{e^2}{2}\theta^{ij}a_i\star_2(\partial_j a_\mu + f_{j\mu}) + \frac{e^3}{2}\theta^{ij}\theta^{kl}\left[\frac{1}{2}(a_k\star_2(\partial_l a_i + f_{li}))\star_2(\partial_j a_\mu + f_{j\mu})\right. \\ & \left. + a_i\star_2\left(\partial_j(a_k\star_2(\partial_l a_\mu + f_{l\mu})) - \frac{1}{2}\partial_\mu(a_k\star_2(\partial_l a_j + f_{lj}))\right) - \frac{1}{2}a_i\star_2(\partial_k a_j\star_2\partial_l a_\mu)\right] \\ & + \frac{e^3}{2}\theta^{ij}\theta^{kl}[a_i\partial_k a_\mu(\partial_j a_l + f_{jl}) - \partial_k\partial_i a_\mu a_j a_l - 2\partial_k a_i\partial_\mu a_j a_l]_{\star_3} + \mathcal{O}(e^4). \end{aligned} \tag{20}$$

Clearly this gives the same e^2-order solution as in Eqs. (17) and (18) of the SW map (I); however, the e^3 order starts to show a difference. The totally commutative $\star_3$ product [7] is defined as follows:

$$[f(x)g(x)h(x)]_{\star_3} = \left(\frac{\sin(\frac{\partial_2\theta\partial_3}{2})\sin(\frac{\partial_1\theta(\partial_2+\partial_3)}{2})}{\frac{(\partial_1+\partial_2)\theta\partial_3}{2}\frac{\partial_1\theta(\partial_2+\partial_3)}{2}} + \{1\leftrightarrow 2\}\right)f(x_1)g(x_2)h(x_3)\bigg|_{x_i=x}. \tag{21}$$

The generalized star products $\star_2$, $\star_3$, and $\star_{3'}$ are connected with star commutators by the following relations:

$$[f(x)\stackrel{\star}{,}g(x)] = i\theta^{ij}\partial_i f(x)\star_2\partial_j g(x), \tag{22}$$

$$f(x)\star_2[g(x)\stackrel{\star}{,}h(x)] + g(x)\star_2[f(x)\stackrel{\star}{,}h(x)] = i\theta^{ij}[\partial_i f(x)g(x)\partial_j h(x) + f(x)\partial_i g(x)\partial_j h(x)]_{\star_3}, \tag{23}$$

$$[f(x)\stackrel{\star}{,}g(x)\star_2 h(x)] = \frac{i}{2}[h(x)\partial_i f(x)\partial_j g(x) + g(x)\partial_i f(x)\partial_j h(x) + \partial_i f(x)\partial_j g(x)h(x) + \partial_i f(x)g(x)\partial_j h(x)]_{\star_{3'}}. \tag{24}$$

Here we see that there are extra derivatives in the generalized star product formula for the star commutators, which provides the opportunity to "integral over" the infinitesimal transformation $\partial_i\lambda \to a_i$. Note also the difference between the $\star_3$ and $\star_{3'}$ products: $\star_{3'}$ helps to realize the $[f(x)\stackrel{\star}{,}g(x)\star_2 h(x)]$ structure, which takes place in the infinitesimal commutative gauge transformation of the SW map expansion of the NC $\mathrm{U}_\star(1)$ gauge field in terms of the commutative $\mathrm{U}(1)$ gauge field, while $\star_3$ realizes the $f(x)\star_2[g(x)\stackrel{\star}{,}h(x)]$, which is typical in the inverse SW map expansion of the commutative $\mathrm{U}(1)$ gauge field in terms of the NC $\mathrm{U}_\star(1)$ gauge field [7].

III. THE θ-EXACT SEIBERG-WITTEN MAP AMBIGUITY AT THE e^3 ORDER

The two e^3-order SW map expansions presented in the last section look quite different, although they bear similarities to a certain degree: they both start at the order θ^2 and bear a similar tensor structure. It is not hard to show that they are indeed not equal to each other by, for example, inspecting the θ^2- and θ^4-order expansions of each solution. Therefore, a certain ambiguity structure should exist between these two solutions. In this section we compare these two θ-exact SW maps up to the e^3 order given in Sec. II in detail. Following the arguments in Refs. [22,23], we consider the composition of one of the SW maps and the inverse of the other. Now, since SW map (II) was derived from a θ-exact inverse SW map expansion in Ref. [7], we choose the original inverse map of (II) for the ambiguity analysis outlined in the Introduction. This inverse SW map expansion is as follows:

$$\lambda_{(\mathrm{II})}(A_\mu,\Lambda) = \Lambda + \frac{1}{2}\theta^{ij}(A_i\star_2\partial_j\Lambda + \theta^{kl}[\partial_i\partial_k\Lambda A_j A_l + \partial_i\Lambda A_k\partial_j A_l]_{\star_3}) + O(A^3)\Lambda, \tag{25}$$

$$a_{\mu_{(\mathrm{II})}}(A_\mu) = A_\mu + \frac{1}{2}\theta^{ij}(A_i\star_2(\partial_j A_\mu + F_{j\mu}) + \theta^{kl}[-A_i\partial_k A_\mu(\partial_j A_l + F_{jl}) + \partial_i\partial_k A_\mu A_j A_l + \partial_k A_i\partial_\mu A_j A_l]_{\star_3}) + O(A^4). \tag{26}$$

Now, by expanding the compositions $\lambda_{(\mathrm{II})}(A_{\mu_{(\mathrm{I})}}(a_\mu),\Lambda_{(\mathrm{I})}(a_\mu,\lambda))$ and $a_{\mu_{(\mathrm{II})}}(A_{\mu_{(\mathrm{I})}}(a_\mu))$ up to the e^3 order, we find

$$\lambda_{(\mathrm{II})}(A_{\mu_{(\mathrm{I})}}(a_\mu),\Lambda_{(\mathrm{I})}(a_\mu,\lambda)) = e\lambda(x) + \Lambda^{e^3}_{(\mathrm{I})}(x) - \Lambda^{e^3}_{(\mathrm{II})}(x) + \mathcal{O}(e^4), \tag{27}$$

$$a_{\mu_{(\mathrm{II})}}(A_{\mu_{(\mathrm{I})}}(a_\mu)) = ea_\mu(x) + A^{e^3}_{\mu_{(\mathrm{I})}}(x) - A^{e^3}_{\mu_{(\mathrm{II})}}(x) + \mathcal{O}(e^4). \tag{28}$$

Note that the e^2 order vanishes as expected. Equations (5) and (6) in the Introduction then indicate that the (I) minus (II) differences at the e^3 order should bear the following expressions:

$$A^{e^3}_{\mu_{(\mathrm{I})}}(x) - A^{e^3}_{\mu_{(\mathrm{II})}}(x) = X^{e^3}_\mu(x) + \partial_\mu Y^{e^3}(x),\ \Lambda^{e^3}_{(\mathrm{I})}(x) - \Lambda^{e^3}_{(\mathrm{II})}(x) = \delta_\lambda Y^{e^3}(x). \tag{29}$$

In order to find a solution for the explicit forms of $X^{e^3}_\mu(x)$ and $Y^{e^3}(x)$, we first Fourier transform $A^{e^3}_\mu(x)$ into a momentum-space quantity $\tilde{A}^{e^3}_\mu(p,q,k)$:

$$\tilde{A}^{e^3}_\mu(p,q,k) = -\frac{e^3}{8}[\tilde{a}_\mu(k)((\tilde{a}(p)\theta q)(\tilde{a}(q)\theta k)M_1 + (\tilde{a}(p)\theta\tilde{a}(q))(q\theta k)M_2 + (\tilde{a}(p)\theta k)(\tilde{a}(q)\theta k)M_3) + k_\mu((\tilde{a}(p)\theta q)(\tilde{a}(q)\theta\tilde{a}(k))M_4 + (\tilde{a}(p)\theta\tilde{a}(q))(q\theta\tilde{a}(k))M_5 + (\tilde{a}(p)\theta\tilde{a}(k))(\tilde{a}(q)\theta k)M_6)]. \tag{30}$$

Then from Eqs. (17) to (20) we read out the coefficients M_is for the SW maps (I) and (II), respectively:

$$
\begin{aligned}
M_{1_{(\mathrm{I})}} &= 4f_{\star_{3'}}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right) + 4f_{\star_{3'}}\left(-\frac{q\theta k}{2},-\frac{p\theta k}{2},-\frac{p\theta q}{2}\right) = -2M_{2_{(\mathrm{I})}},\\
M_{3_{(\mathrm{I})}} &= 4f_{\star_{3'}}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right),\\
M_{4_{(\mathrm{I})}} &= -3f_{\star_{3'}}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right) - 2f_{\star_{3'}}\left(-\frac{q\theta k}{2},-\frac{p\theta k}{2},-\frac{p\theta q}{2}\right),\\
M_{5_{(\mathrm{I})}} &= -2f_{\star_{3'}}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right) - f_{\star_{3'}}\left(-\frac{q\theta k}{2},-\frac{p\theta k}{2},-\frac{p\theta q}{2}\right),\\
M_{6_{(\mathrm{I})}} &= 2f_{\star_{3'}}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right) + f_{\star_{3'}}\left(\frac{q\theta k}{2},-\frac{p\theta q}{2},-\frac{p\theta k}{2}\right),
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
M_{1_{(\mathrm{II})}} &= 8f_{\star_2}\left(\frac{p\theta q}{2}\right)f_{\star_2}\left(\frac{(p+q)\theta k}{2}\right) + 8f_{\star_2}\left(\frac{q\theta k}{2}\right)f_{\star_2}\left(\frac{p\theta(q+k)}{2}\right) - 8f_{\star_3}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right) = -2M_{2_{(\mathrm{II})}},\\
M_{3_{(\mathrm{II})}} &= 8f_{\star_2}\left(\frac{q\theta k}{2}\right)f_{\star_2}\left(\frac{p\theta(q+k)}{2}\right) - 4f_{\star_3}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right),\\
M_{4_{(\mathrm{II})}} &= -4f_{\star_2}\left(\frac{p\theta q}{2}\right)f_{\star_2}\left(\frac{(p+q)\theta k}{2}\right) - 6f_{\star_2}\left(\frac{q\theta k}{2}\right)f_{\star_2}\left(\frac{p\theta(q+k)}{2}\right) + 8f_{\star_3}\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right),\\
M_{5_{(\mathrm{II})}} &= 2f_{\star_2}\left(\frac{p\theta q}{2}\right)f_{\star_2}\left(\frac{(p+q)\theta k}{2}\right) + 4f_{\star_2}\left(\frac{q\theta k}{2}\right)f_{\star_2}\left(\frac{p\theta(q+k)}{2}\right),\\
M_{6_{(\mathrm{II})}} &= -4f_{\star_2}\left(\frac{q\theta k}{2}\right)f_{\star_2}\left(\frac{p\theta(q+k)}{2}\right) - 2f_{\star_2}\left(\frac{p\theta k}{2}\right)f_{\star_2}\left(\frac{q\theta(p+k)}{2}\right).
\end{aligned}
\tag{32}
$$

The functions $f_{\star_2}(a)$, $f_{\star_3}(a,b,c)$, and $f_{\star_{3'}}(a,b,c)$ are defined as follows:

$$f_{\star_2}(a) = \frac{\sin a}{a}, \tag{33}$$

$$f_{\star_3}(a,b,c) = \frac{\sin b \sin(a+b)}{(a+b)(b+c)} + \frac{\sin c \sin(a-c)}{(a-c)(b+c)}, \tag{34}$$

$$f_{\star_{3'}}(a,b,c) = \frac{\cos(a+b-c)-1}{(a+b-c)c} - \frac{\cos(a+b+c)-1}{(a+b+c)c}. \tag{35}$$

From Eqs. (31) and (32) we observe that under the permutation $q \leftrightarrow k$,

$$M_{5_{(\mathrm{I,II})}} = -M_{6_{(\mathrm{I,II})}}(q \leftrightarrow k), \tag{36}$$

which indicates that the $M_{5_{(\mathrm{I,II})}}$ and $M_{6_{(\mathrm{I,II})}}$ (and part of the $M_{4_{(\mathrm{I,II})}}$) contributions could be made equivalent to two appropriate infinitesimal NC gauge transformation(s), respectively,

$$
\begin{aligned}
A'^{e^3}_{\mu_{(\mathrm{I,II})}}(x) &= A^{e^3}_{\mu_{(\mathrm{I,II})}}(x) + \delta_{\Xi^{e^3}_{(\mathrm{I,II})}} A^{e^3}_{\mu_{(\mathrm{I,II})}}(x)\\
&= A^{e^3}_{\mu_{(\mathrm{I,II})}}(x) + \partial_\mu \Xi^{e^3}_{(\mathrm{I,II})}(x) + (O)(e^4),
\end{aligned}
\tag{37}
$$

with the rest of the gauge field $A'^{e^3}_{\mu_{(\mathrm{I,II})}}(x)$ bearing the following form in momentum space:

$$
\begin{aligned}
\tilde{A}'^{e^3}_{\mu_{(\mathrm{I,II})}} = -\frac{e^3}{8}[&\tilde{a}_\mu(k)((\tilde{a}(p)\theta q)(\tilde{a}(q)\theta k)M_{1_{(\mathrm{I,II})}} + (\tilde{a}(p)\theta\tilde{a}(q))(q\theta k)M_{2_{(\mathrm{I,II})}} + (\tilde{a}(p)\theta k)(\tilde{a}(q)\theta k)M'_{3_{(\mathrm{I,II})}})\\
&+ k_\mu((\tilde{a}(p)\theta\tilde{a}(q))(q\theta\tilde{a}(k))M'_{4_{(\mathrm{I,II})}})].
\end{aligned}
\tag{38}
$$

The above gauge parameters $\Xi^{e^3}_{(\mathrm{I,II})}(x)$ have been found explicitly for both cases (I) and (II),

$$\Xi^{e^3}_{(\mathrm{I})}(x) = -\frac{e^3}{8}\theta^{ij}\theta^{kl}(2[a_l\partial_j a_k a_i]_{\star_{3'}} + [a_i\partial_j a_k a_l]_{\star_{3'}}) + \mathcal{O}(e^4), \tag{39}$$

$$\Xi^{e^3}_{(\mathrm{II})}(x) = -\frac{e^3}{4}\theta^{ij}\theta^{kl}(2(a_i\star_2\partial_j a_k)\star_2 a_l + a_i\star_2(\partial_j a_k\star_2 a_l)) + \mathcal{O}(e^4), \tag{40}$$

respectively. It also turns out that the coefficients $M'_{4_{(\mathrm{I,II})}}$ satisfy

$$M'_{4_{(\mathrm{I,II})}} = -M_{1_{(\mathrm{I,II})}}. \tag{41}$$

Meanwhile, the $p\leftrightarrow q$ permutation symmetry of the $(\tilde a(p)\theta k)(\tilde a(q)\theta k)$ term, leads to $M'_{3_{(\mathrm{I,II})}}$,

$$2M'_{3_{(\mathrm{I,II})}} = M_{3_{(\mathrm{I,II})}}[p,q,k] + M_{3_{(\mathrm{I,II})}}[q,p,k], \tag{42}$$

which further simplifies Eq. (38).

Finishing all of the above transformations, we now examine the remaining difference $W_\mu(x) = A'^{e^3}_{\mu_{(\mathrm{I})}}(x) - A'^{e^3}_{\mu_{(\mathrm{II})}}(x)$ in momentum space:

$$\tilde W_\mu = \tilde A'^{e^3}_{\mu_{(\mathrm{I})}} - \tilde A'^{e^3}_{\mu_{(\mathrm{II})}} = e^3\bigg[\tilde a_\mu(k)\bigg((\tilde a(p)\theta q)(\tilde a(q)\theta k)\tilde W_1 - \frac{1}{2}(\tilde a(p)\theta\tilde a(q))(q\theta k)\tilde W_1 + (\tilde a(p)\theta k)(\tilde a(q)\theta k)\tilde W_3\bigg) - k_\mu((\tilde a(p)\theta\tilde a(q))(q\theta\tilde a(k)))\tilde W_1\bigg], \tag{43}$$

where

$$\tilde W_1 = \frac{p\theta k}{2}\left[\left(\frac{p\theta q}{2} - \frac{p\theta k}{2} + \frac{q\theta k}{2}\right) f_1\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right) + \frac{p\theta q}{2}\frac{p\theta k}{2}\frac{q\theta k}{2} f_2\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2};0\right)\right] = 2\frac{p\theta k}{q\theta p}\tilde W_3, \tag{44}$$

where the functions $f_1(a,b,c)$ and $f_2(a,b,c;n)$ have relatively complicated structures,

$$f_1(a,b,c) = \frac{1}{4}\bigg(\frac{\cos(a+b+c)}{(a+b)(b+c)(a+c)(a+b+c)} + \frac{\cos(a+b-c)}{(a+b)(b+c)(a-c)(a+b-c)} - \frac{\cos(a-b+c)}{(a-b)(b-c)(a+c)(a-b+c)} - \frac{\cos(a-b-c)}{(a-b)(b+c)(a-c)(a-b-c)} - \frac{8}{(a+b+c)(a+b-c)(a-b+c)(a-b-c)}\bigg), \tag{45}$$

$$f_2(a,b,c;n) = \frac{a^{2n}\cos a\sin b\sin c}{bc(a^2-b^2)(c^2-a^2)} + \frac{b^{2n}\sin a\cos b\sin c}{ac(a^2-b^2)(b^2-c^2)} + \frac{c^{2n}\sin a\sin b\cos c}{ab(b^2-c^2)(c^2-a^2)}. \tag{46}$$

One can immediately observe that the functions f_1 and f_2 are both completely symmetric under any permutation over a,b,c. This enables us to express the relevant part $W_\mu(x)$ of the difference between two θ-exact SW maps (I) and (II) via two new generalized entirely symmetric 3-products $\diamond_1$ and $\diamond_2(n)$:

$$[fgh]_{\diamond_1}(x) = \int e^{-i(p+q+k)x}\tilde f(p)\tilde g(q)\tilde h(k) f_1\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2}\right), \tag{47}$$

$$[fgh]_{\diamond_2}(x) = \int e^{-i(p+q+k)x}\tilde f(p)\tilde g(q)\tilde h(k) f_2\left(\frac{p\theta q}{2},\frac{p\theta k}{2},\frac{q\theta k}{2};n\right), \tag{48}$$

After reformulating the inverse fourier transformation of Eq. (43) in terms of $\diamond_1$ and $\diamond_{2(0)}$ products and some lengthy rearrangement of the fields and (the rest of) the indices, we find the following expression for $W_\mu(x)$ in terms of the $\diamond_1$ and $\diamond_{2(0)}$ 3-products:

$$W_\mu(x) = -\frac{e^3}{8}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}\bigg([\partial_r f_{ip} f_{jk}\partial_s\partial_q f_{l\mu}]_{\diamond_1} + [\partial_r f_{ip} f_{jk}\partial_s\partial_l f_{q\mu}]_{\diamond_1} + [\partial_p f_{ri}\partial_q f_{jk}\partial_s f_{l\mu}]_{\diamond_1} + \frac{1}{4}\theta^{ab}\theta^{cd}[\partial_p\partial_a f_{ri}\partial_q\partial_c f_{jk}\partial_s\partial_b\partial_d f_{l\mu}]_{\diamond_2(0)} + \partial_\mu\bigg([\partial_p f_{ir}\partial_q f_{jk}\partial_s a_l]_{\diamond_1} + 2[\partial_p\partial_r a_i\partial_q\partial_j a_k\partial_s a_l]_{\diamond_1} - \frac{1}{12}\theta^{ab}\theta^{cd}(3[\partial_p\partial_a\partial_r a_i\partial_q\partial_c\partial_j a_k\partial_s\partial_b\partial_d a_l]_{\diamond_2(0)} - [\partial_p\partial_a\partial_i a_r\partial_q\partial_c\partial_k a_j\partial_s\partial_b\partial_d a_l]_{\diamond_2(0)})\bigg)\bigg). \tag{49}$$

Note that both $f_1(a,b,c)$ and $f_2(a,b,c;n)$ have a smooth $\theta\to 0$ limit, as expected. Consequently, $W_\mu(x)$ will end at the θ^6 order if we expand the 3-products $\diamond_1$ and $\diamond_{2(0)}$ to their lowest order only. Finally, we add $(\partial_\mu\Xi^{e^3}_{(\mathrm{I})}(x)-\partial_\mu\Xi^{e^3}_{(\mathrm{II})}(x))$ back to the $W_\mu(x)$ and obtain explicit solutions to the equations (29):

$$X^{e^3}_\mu(x)=-\frac{e^3}{8}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}([\partial_r f_{ip}f_{jk}\partial_s\partial_q f_{l\mu}]_{\diamond_1}+[\partial_r f_{ip}f_{jk}\partial_s\partial_l f_{q\mu}]_{\diamond_1}+[\partial_p f_{ri}\partial_q f_{jk}\partial_s f_{l\mu}]_{\diamond_1}$$
$$+\frac{1}{4}\theta^{ab}\theta^{cd}[\partial_p\partial_a f_{ri}\partial_q\partial_c f_{jk}\partial_s\partial_b\partial_d f_{l\mu}]_{\diamond_{2(0)}}), \tag{50}$$

$$Y^{e^3}(x)=\frac{e^3}{8}\theta^{ij}\theta^{kl}(2[a_l\partial_j a_k a_i]_{\star_{3'}}+[a_i\partial_j a_k a_l]_{\star_{3'}}-4(a_i\star_2\partial_j a_k)\star_2 a_l-2a_i\star_2(\partial_j a_k\star_2 a_l))$$
$$-\frac{e^3}{8}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}([\partial_p f_{ir}\partial_q f_{jk}\partial_s a_l]_{\diamond_1}+2[\partial_p\partial_r a_i\partial_q\partial_j a_k\partial_s a_l]_{\diamond_1}$$
$$-\frac{1}{12}\theta^{ab}\theta^{cd}(3[\partial_p\partial_a\partial_r a_i\partial_q\partial_c\partial_j a_k\partial_s\partial_b\partial_d a_l]_{\diamond_{2(0)}}-[\partial_p\partial_a\partial_i a_r\partial_q\partial_c\partial_k a_j\partial_s\partial_b\partial_d a_l]_{\diamond_{2(0)}})). \tag{51}$$

The simple structure for the ambiguity between two gauge field SW maps at the e^3 order [Eq. (29)] leads to the following result for the gauge field strength difference/comparison:

$$F^{e^3}_{\mu\nu_{(\mathrm{I})}}(x)-F^{e^3}_{\mu\nu_{(\mathrm{II})}}(x)=\partial_\mu X^{e^3}_\nu(x)-\partial_\nu X^{e^3}_\mu(x)$$
$$=\frac{e^3}{4}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}\Big([\partial_r\partial_p f_{\mu i}\partial_q f_{jk}\partial_s f_{\nu l}]_{\diamond_1}-[\partial_r\partial_p f_{\mu i}f_{jk}\partial_s\partial_q f_{\nu l}]_{\diamond_1}+[\partial_p f_{\mu i}\partial_r f_{jk}\partial_q\partial_s f_{\nu l}]_{\diamond_1}$$
$$+[\partial_r f_{pi}\partial_s\partial_j f_{\mu k}\partial_q f_{\nu l}]_{\diamond_1}-[\partial_r f_{pi}\partial_j f_{\mu k}\partial_s\partial_q f_{\nu l}]_{\diamond_1}+[f_{pi}\partial_r\partial_j f_{\mu k}\partial_s\partial_q f_{\nu l}]_{\diamond_1}$$
$$+\frac{1}{4}\theta^{ab}\theta^{cd}([\partial_p\partial_r\partial_a f_{\mu i}\partial_q\partial_c f_{jk}\partial_s\partial_b\partial_d f_{\nu l}]_{\diamond_{2(0)}}+[\partial_p\partial_a f_{ri}\partial_q\partial_c\partial_j f_{\mu k}\partial_s\partial_b\partial_d f_{\nu l}]_{\diamond_{2(0)}})\Big)$$
$$+\frac{e^3}{8}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}\Big(2[\partial_r f_{pi}f_{jk}\partial_s\partial_q\partial_l f_{\mu\nu}]_{\diamond_1}-[\partial_r f_{pi}\partial_s f_{jk}\partial_q\partial_l f_{\mu\nu}]_{\diamond_1}$$
$$-\frac{1}{4}\theta^{ab}\theta^{cd}[\partial_r\partial_a f_{pi}\partial_s\partial_c f_{jk}\partial_q\partial_b\partial_d\partial_l f_{\mu\nu}]_{\diamond_{2(0)}}\Big). \tag{52}$$

Clearly the $Y^{e^3}(x)$-related terms drop out, leaving Eq. (52) with only the U(1) gauge field strengths.

IV. THE θ-EXACT GAUGE FIELD STRENGTH UP TO THE e^3 ORDER

In the last sections we examined the SW map ambiguities between two known gauge field expansions. This section focuses on another type of freedom within each of the two field-strength SW map expansions. Our motivation is to find a certain θ-exact alternative of the earlier θ-iterative freedom parameters. In the past the majority of studies on SW map ambiguities followed the θ-iterative field redefinition procedure in Refs. [29,46]. At θ^1 order this procedure introduces the following correction to the gauge field expansion [32]:

$$\varphi_\mu=\frac{b}{4}e\theta^{ij}D_\mu f_{ij}, \tag{53}$$

which then gives the gauge field strength correction

$$\Phi_{\mu\nu}=D_\mu\varphi_\nu-D_\nu\varphi_\mu=\frac{b}{4}e\theta^{ij}f_{\mu\nu}f_{ij}, \tag{54}$$

and consequently modifies the action into

$$S^{e^2\theta^1}_b=-\int e^2\theta^{ij}f^{\mu\nu}\Big(f_{\mu i}f_{\nu j}-\frac{1+b}{4}f_{ij}f_{\mu\nu}\Big). \tag{55}$$

To find a θ-exact alternative of Eq. (55) we first study the e^2-order field-strength SW map expansion. The gauge field expansions $A_{\mu_{(\mathrm{I})}}(x)$ and $A_{\mu_{(\mathrm{II})}}(x)$ [from Eqs. (18) and (20), respectively] lead to the same noncommutative $\mathrm{U}_\star(1)$ gauge field strength expansion up to the e^2 order,

$$F_{\mu\nu}(x)=ef_{\mu\nu}+e^2\theta^{ij}(f_{\mu i}\star_2 f_{\nu j}-a_i\star_2\partial_j f_{\mu\nu})+\mathcal{O}(e^3). \tag{56}$$

On the other hand, at the e^2 order, the general consistency condition for the gauge field strength

$$\delta_\lambda F_{\mu\nu} = i[\Lambda \stackrel{\star}{,} F_{\mu\nu}] \tag{57}$$

becomes

$$\delta_\lambda F_{\mu\nu}^{e^2} = ie^2[\lambda \stackrel{\star}{,} f_{\mu\nu}]. \tag{58}$$

Examining the gauge field strength (56), we find that the variation of the first term at e^2 order, $e^2\theta^{ij} f_{\mu i}\star_2 f_{\nu j}$, vanishes; therefore, the consistency condition (58) is fulfilled solely through the second term $-e^2\theta^{ij}a_i\star_2\partial_j f_{\mu\nu}$ as

$$\delta_\lambda(-e^2\theta^{ij}a_i\star_2\partial_j f_{\mu\nu}) = -e^2\theta^{ij}\partial_i\lambda\star_2\partial_j f_{\mu\nu} = ie^2[\lambda \stackrel{\star}{,} f_{\mu\nu}], \tag{59}$$

thanks to the relation between the $\star_2$ product and the $\star$ commutator (22). This observation encourages us to put an arbitrary parameter κ in front of the term $e^2\theta^{ij} f_{\mu i}\star_2 f_{\nu j}$ in Eq. (56) since this does not break the e^2-order consistency condition (58). Such a procedure leads to the κ-deformed gauge field strength up to the e^2 order,

$$F_{\mu\nu}(x)_\kappa = ef_{\mu\nu} + e^2\theta^{ij}(\kappa f_{\mu i}\star_2 f_{\nu j} - a_i\star_2\partial_j f_{\mu\nu}) + \mathcal{O}(e^3). \tag{60}$$

The restriction of the $F_{\mu\nu}(x)_\kappa$ to the θ^1 order gives

$$F_{\mu\nu}(x)_\kappa^{\theta^1} = e^2\theta^{ij}(\kappa f_{\mu i} f_{\nu j} - a_i\partial_j f_{\mu\nu}). \tag{61}$$

Also, the deformed action at the θ^1 order reads

$$S_\kappa^{e^2\theta^1} = -\int e^2\theta^{ij} f^{\mu\nu}\left(\kappa f_{\mu i} f_{\nu j} - \frac{1}{4} f_{ij} f_{\mu\nu}\right). \tag{62}$$

Here we see that the b correction (54) to the gauge field strength does not match the κ correction in Eq. (61). However, the b and κ corrections are instead connected by the (inter)actions since the κ and/or $a = 1 + b$ are present the ratio between two gauge-invariant terms in an inverted fashion. For this reason we consider the κ deformation as a possible substitute for the b (a in the literature) modification in the θ-exact approach.

To extend the κ deformation to e^3 order we must handle the effect of κ in the e^3-order consistency relation as well as the identification of possible new gauge-invariant terms. This can be done by solving the consistency relation

$$\begin{aligned}\delta_\lambda F_{\mu\nu}^{e^3}(x)_\kappa &= ie([\Lambda^{e^2} \stackrel{\star}{,} f_{\mu\nu}] + [\lambda \stackrel{\star}{,} F_{\mu\nu}^{e^2}(x)_\kappa]) \\ &= ie([\Lambda^{e^2} \stackrel{\star}{,} f_{\mu\nu}] + [\lambda \stackrel{\star}{,} e^2\theta^{ij}(\kappa f_{\mu i}\star_2 f_{\nu j} - a_i\star_2\partial_j f_{\mu\nu})]).\end{aligned} \tag{63}$$

We start by observing that both SW maps in Sec. II satisfy the e^3-order consistency relation (63) when $\kappa = 1$, i.e., without the κ deformation; then, within the undeformed NC field strength $F_{\mu\nu_{\rm (I,II)}}(x)$, we identify those terms relevant to the to-be-deformed term $i\theta^{ij}[\lambda \stackrel{\star}{,} f_{\mu i}\star_2 f_{\nu j}]$ and make them κ proportional. We also search for a possible κ-unrelated freedom/ambiguity in the undeformed NC field strength.

A. Gauge field strength from the Seiberg-Witten map (I)

The easiest way to determine the gauge field strength corresponding to the gauge field $A_{\mu_{\rm (I)}}(x)$ is by solving directly the SW differential equation for the gauge field strength [1],

$$\begin{aligned}\frac{d}{dt}F_{\mu\nu_{\rm (I)}}(x) = \frac{1}{4}\theta^{ij}[2\{F_{\mu i} \stackrel{\star_t}{,} F_{\nu j}\} \\ - \{A_i \stackrel{\star_t}{,} (2\partial_j F_{\mu\nu} - i[A_j \stackrel{\star_t}{,} F_{\mu\nu}])\}],\end{aligned} \tag{64}$$

which at the e^3 order yields

$$\begin{aligned}F_{\mu\nu_{\rm (I)}}^{e^3}(x) = \frac{e^3}{2}\theta^{ij}\theta^{kl}\Big[&([f_{\mu k} f_{\nu i} f_{lj}]_{\star_{3'}} + [f_{\nu l} f_{\mu i} f_{kj}]_{\star_{3'}}) - ([f_{\nu l} a_i \partial_j f_{\mu k}]_{\star_{3'}} + [f_{\mu k} a_i \partial_j f_{\nu l}]_{\star_{3'}} + [a_k \partial_l (f_{\mu i} f_{\nu j})]_{\star_{3'}}) \\ &+ [a_i\partial_j a_k \partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu} a_i \partial_j a_k]_{\star_{3'}} + [a_k a_i \partial_l \partial_j f_{\mu\nu}]_{\star_{3'}} - \frac{1}{2}([a_i\partial_k a_j \partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu} a_i \partial_k a_j]_{\star_{3'}})\Big].\end{aligned} \tag{65}$$

Here we notice a few facts. First, among all of the above terms in Eq. (65), the first two in the first line are manifestly invariant under the commutative gauge transformation and antisymmetric under the $\mu \leftrightarrow \nu$ permutation; therefore, they could be subject to the free variation, i.e., associated with a new deformation (weight) parameter κ_1.

Next, considering the next three terms in the first line of Eq. (65), with the help of Eq. (24) we find that the sum of these three terms together satisfy the following transformation property:

$$\delta_\lambda \frac{1}{2}\theta^{ij}\theta^{kl}([f_{\nu l} a_i \partial_j f_{\mu k}]_{\star_{3'}} + [f_{\mu k} a_i \partial_j f_{\nu l}]_{\star_{3'}} + [a_k \partial_l (f_{\mu i} f_{\nu j})]_{\star_{3'}}) = -i\theta^{kl}[\lambda \stackrel{\star}{,} f_{\mu k}\star_2 f_{\nu l}]. \tag{66}$$

Thus, the second fact is that they are a relevant subject for the κ deformation at the e^3 order.

These two facts lead us to an extended (κ, κ_1) deformation of the field strength (65):

$$F^{e^3}_{\mu\nu_{(\mathrm{I})}}(x)_{\kappa,\kappa_1} = \frac{e^3}{2}\theta^{ij}\theta^{kl}\Big[\kappa_1([f_{\mu k}f_{\nu i}f_{lj}]_{\star_{3'}} + [f_{\nu l}f_{\mu i}f_{kj}]_{\star_{3'}}) - \kappa([f_{\nu l}a_i\partial_j f_{\mu k}]_{\star_{3'}} + [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} + [a_k\partial_l(f_{\mu i}f_{\nu j})]_{\star_{3'}}) + [a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu}a_i\partial_j a_k]_{\star_{3'}} + [a_k a_i\partial_l\partial_j f_{\mu\nu}]_{\star_{3'}} - \frac{1}{2}([a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu}a_i\partial_k a_j]_{\star_{3'}})\Big]. \quad (67)$$

B. Gauge field strength from the Seiberg-Witten map (II)

Next we consider the e^3-order θ-exact gauge field strength from $A_{\mu_{(\mathrm{II})}}(x)$, which can be expressed as follows:

$$\begin{aligned} F^{e^3}_{\mu\nu_{(\mathrm{II})}}(x) = e^3\theta^{ij}\theta^{kl}\Big[& f_{\mu i}\star_2(f_{jk}\star_2 f_{l\nu}) + f_{l\nu}\star_2(f_{jk}\star_2 f_{\mu i}) - [f_{\mu i}f_{jk}f_{l\nu}]_{\star_3} \\ & - ((a_i\star_2\partial_j f_{\mu k})\star_2 f_{\nu l} + (a_i\star_2\partial_j f_{\nu l})\star_2 f_{\mu k} - [a_i\partial_j(f_{\mu k}f_{\nu l})]_{\star_3}) - a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l}) \\ & + (a_i\star_2\partial_j a_k)\star_2\partial_l f_{\mu\nu} + a_i\star_2(\partial_j a_k\star_2\partial_l f_{\mu\nu}) + a_i\star_2(a_k\star_2\partial_j\partial_l f_{\mu\nu}) - [a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_3} \\ & - \frac{1}{2}(a_i\star_2(\partial_k a_j\star_2\partial_l f_{\mu\nu}) + (a_i\star_2\partial_k a_j)\star_2\partial_l f_{\mu\nu} - [a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_3} + [a_i a_k\partial_j\partial_l f_{\mu\nu}]_{\star_3})\Big]. \end{aligned} \quad (68)$$

Using the basic relation (23), we can show that the infinitesimal commutative gauge transformation of the parentheses in the second line of Eq. (68),

$$\begin{aligned} &\delta_\lambda\theta^{ij}\theta^{kl}((a_i\star_2\partial_j f_{\mu k})\star_2 f_{\nu l} + (a_i\star_2\partial_j f_{\nu l})\star_2 f_{\mu k} - [a_i\partial_j(f_{\mu k}f_{\nu l})]_{\star_3}) \\ &\quad = \theta^{ij}\theta^{kl}(\partial_i\lambda\star_2\partial_j f_{\mu k})\star_2 f_{\nu l} + (\partial_i\lambda\star_2\partial_j f_{\nu l})\star_2 f_{\mu k} - [\partial_i\lambda\partial_j(f_{\mu k}f_{\nu l})]_{\star_3} \\ &\quad = i([\lambda \stackrel{\star}{,} f_{\mu k}]\star_2 f_{\nu l} + [\lambda \stackrel{\star}{,} f_{\nu l}]\star_2 f_{\mu k} + f_{\nu l}\star_2[f_{\mu k} \stackrel{\star}{,} \lambda] + f_{\mu k}\star_2[f_{\nu l} \stackrel{\star}{,} \lambda]) = 0, \end{aligned} \quad (69)$$

vanishes. One can further turn these parentheses into a manifestly gauge-invariant form with the help of the 3-products $\diamond_{2(n)}$:

$$\begin{aligned} &\theta^{ij}\theta^{kl}((a_i\star_2\partial_j f_{\mu k})\star_2 f_{\nu l} + (a_i\star_2\partial_j f_{\nu l})\star_2 f_{\mu k} - [a_i\partial_j(f_{\mu k}f_{\nu l})]_{\star_3}) \\ &\quad = \frac{1}{4}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}\Big([f_{pi}\partial_j\partial_r f_{\mu k}\partial_q\partial_s f_{\nu l}]_{\diamond_2(1)} + \frac{1}{4}\theta^{ab}\theta^{cd}[\partial_p\partial_a f_{ri}\partial_q\partial_c\partial_j f_{\mu k}\partial_s\partial_b\partial_d f_{\nu l} \\ &\qquad + \partial_p f_{ir}\partial_q\partial_a\partial_c\partial_j f_{\mu k}\partial_s\partial_b\partial_d f_{\nu l} + \partial_p f_{ir}\partial_q\partial_a\partial_c\partial_j f_{\nu l}\partial_s\partial_b\partial_d f_{\mu k}]_{\diamond_2(0)}\Big). \end{aligned} \quad (70)$$

Therefore, we conclude that the first two lines in Eq. (68) do not contribute to $\delta_\lambda F^{e^3}_{\mu\nu}$. Among the rest of the terms, we notice that the first one is compatible with the $\star_2$ commutator, since

$$\delta_\lambda(-\theta^{ij}\theta^{kl}a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l})) = i\theta^{kl}[\lambda \stackrel{\star}{,} f_{\mu k}\star_2 f_{\nu l}]. \quad (71)$$

Thus, this term alone gives the formal NC transformation of the fully commutative gauge field strength term $\theta^{ij}f_{\mu i}\star_2 f_{\nu j}$ at e^3 order. Therefore, multiplying Eq. (71) by the κ parameter ensures compatibility at the e^3 order.

It is also straightforward to notice that two more additional free variations could be performed on $F^{e^3}_{\mu\nu_{(\mathrm{II})}}(x)$ via a multiplication of the manifestly gauge-invariant first two lines of Eq. (68) by two new deformation parameters κ_1' and κ_2', respectively. This way, we obtain the $(\kappa, \kappa_1', \kappa_2')$-deformed extension for the gauge field strength at $F^{e^3}_{\mu\nu_{(\mathrm{II})}}(x)$,

$$
\begin{aligned}
F^{e^3}_{\mu\nu_{(\mathrm{II})}}(x)_{\kappa,\kappa'_1,\kappa'_2} = e^3\theta^{ij}\theta^{kl}\Bigg[&\kappa'_1\big(f_{\mu i}\star_2(f_{jk}\star_2 f_{l\nu}) + f_{l\nu}\star_2(f_{jk}\star_2 f_{\mu i}) - [f_{\mu i}f_{jk}f_{l\nu}]_{\star_3}\big)\\
&-\kappa'_2\frac{1}{4}\theta^{pq}\theta^{rs}\bigg([f_{pi}\partial_j\partial_r f_{\mu k}\partial_q\partial_s f_{\nu l}]_{\diamond_2(1)} + \frac{1}{4}\theta^{ab}\theta^{cd}[\partial_p\partial_a f_{ri}\partial_q\partial_c\partial_j f_{\mu k}\partial_s\partial_b\partial_d f_{\nu l}\\
&+\partial_p f_{ir}\partial_q\partial_a\partial_c\partial_j f_{\mu k}\partial_s\partial_b\partial_d f_{\nu l} + \partial_p f_{ir}\partial_q\partial_a\partial_c\partial_j f_{\nu l}\partial_s\partial_b\partial_d f_{\mu k}]_{\diamond_2(0)}\bigg) - \kappa a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l})\\
&+(a_i\star_2\partial_j a_k)\star_2\partial_l f_{\mu\nu} + a_i\star_2(\partial_j a_k\star_2\partial_l f_{\mu\nu}) + a_i\star_2(a_k\star_2\partial_j\partial_l f_{\mu\nu}) - [a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_3}\\
&-\frac{1}{2}\big(a_i\star_2(\partial_k a_j\star_2\partial_l f_{\mu\nu}) + (a_i\star_2\partial_k a_j)\star_2\partial_l f_{\mu\nu} - [a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_3} + [a_i a_k\partial_j\partial_l f_{\mu\nu}]_{\star_3}\big)\Bigg]. \qquad (72)
\end{aligned}
$$

Inspired by the form of Eq. (72), we add and subtract $a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l})$ in Eq. (65), then assign κ to the $a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l})$ term, and assign a new parameter κ_2 to the sum of the terms $([f_{\nu l}a_i\partial_j f_{\mu k}]_{\star_{3'}} + [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} + [a_k\partial_l(f_{\mu i}f_{\nu j})]_{\star_{3'}} - 2a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l}))$. This way, Eq. (67) is generalized into essentially the same form as Eq. (72). This leads to

$$
\begin{aligned}
F^{e^3}_{\mu\nu_{(\mathrm{I})}}(x)_{\kappa,\kappa_1,\kappa_2} = \frac{e^3}{2}\theta^{ij}\theta^{kl}\Bigg[&\kappa_1([f_{\mu k}f_{\nu i}f_{lj}]_{\star_{3'}} + [f_{\nu l}f_{\mu i}f_{kj}]_{\star_{3'}}) - 2\kappa a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l}) - \kappa_2([f_{\nu l}a_i\partial_j f_{\mu k}]_{\star_{3'}} + [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}}\\
&+[a_k\partial_l(f_{\mu i}f_{\nu j})]_{\star_{3'}} - 2a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l})) + [a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu}a_i\partial_j a_k]_{\star_{3'}} + [a_k a_i\partial_l\partial_j f_{\mu\nu}]_{\star_{3'}}\\
&-\frac{1}{2}([a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu}a_i\partial_k a_j]_{\star_{3'}})\Bigg]. \qquad (73)
\end{aligned}
$$

Note that the κ_2-proportional part can be expressed as follows using diamond $\diamond_1$ and $\diamond_{2(0,1)}$ products:

$$
\begin{aligned}
&([f_{\nu l}a_i\partial_j f_{\mu k}]_{\star_{3'}} + [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} + [a_k\partial_l(f_{\mu i}f_{\nu j})]_{\star_{3'}} - 2a_i\star_2\partial_j(f_{\mu k}\star_2 f_{\nu l}))\\
&=\frac{e^3}{4}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}\bigg([f_{pi}\partial_j\partial_r f_{\mu k}\partial_q\partial_s f_{\nu l}]_{\diamond_2(1)} - [\partial_r f_{pi}\partial_s\partial_j f_{\mu k}\partial_q f_{\nu l}]_{\diamond_1} + [\partial_r f_{pi}\partial_j f_{\mu k}\partial_s\partial_q f_{\nu l}]_{\diamond_1}\\
&\quad - [f_{pi}\partial_r\partial_j f_{\mu k}\partial_s\partial_q f_{\nu l}]_{\diamond_1} + \frac{1}{4}\theta^{ab}\theta^{cd}([\partial_p f_{ir}\partial_q\partial_a\partial_c\partial_j f_{\mu k}\partial_s\partial_b\partial_d f_{\nu l} + \partial_p f_{ir}\partial_q\partial_a\partial_c\partial_j f_{\nu l}\partial_s\partial_b\partial_d f_{\mu k}]_{\diamond_2(0)})\bigg). \qquad (74)
\end{aligned}
$$

Using Eq. (44), one can show that the difference between first two terms in Eq. (65) and the first line in Eq. (68) becomes

$$
\begin{aligned}
&\frac{1}{2}([f_{\mu k}f_{\nu i}f_{lj}]_{\star_{3'}} + [f_{\nu l}f_{\mu i}f_{kj}]_{\star_{3'}}) - (f_{\mu i}\star_2(f_{jk}\star_2 f_{l\nu}) + f_{l\nu}\star_2(f_{jk}\star_2 f_{\mu i}) - [f_{\mu i}f_{jk}f_{l\nu}]_{\star_3})\\
&=\frac{1}{4}\theta^{ij}\theta^{kl}\theta^{pq}\theta^{rs}\bigg([\partial_r\partial_p f_{\mu i}\partial_q f_{jk}\partial_s f_{\nu l}]_{\diamond_1} - [\partial_r\partial_p f_{\mu i}f_{jk}\partial_s\partial_q f_{\nu l}]_{\diamond_1} + [\partial_p f_{\mu i}\partial_r f_{jk}\partial_q\partial_s f_{\nu l}]_{\diamond_1}\\
&\quad + \frac{1}{4}\theta^{ab}\theta^{cd}[\partial_p\partial_r\partial_a f_{\mu i}\partial_q\partial_c f_{jk}\partial_s\partial_b\partial_d f_{\nu l}]_{\diamond_2(0)}\bigg). \qquad (75)
\end{aligned}
$$

Consequently, the difference between Eqs. (73) and (72) in the case without κ, κ_i, and κ'_i deformations (that is, for $\kappa = \kappa_{i=1,2} = \kappa'_{i=1,2} = 1$) gives exactly Eq. (52)[2], proving the consistency of our computations, as it should.

[2] The rest of the terms in Eqs. (67) and (72) receive no deformation; they all contain $f_{\mu\nu}$ and therefore they arise when the partial derivative $\partial_{\mu(\nu)}$ hits the gauge field carrying the external index $\nu(\mu)$, respectively. One can show that they are equal to the terms containing $f_{\mu\nu}$ in Eq. (52) following a procedure exactly identical to that in Sec. III.

V. DISCUSSION AND CONCLUSION

In this article we studied the e^3-order θ-exact SW map expansion of $\mathrm{U}_\star(1)$ gauge field theory, following important recent progress in solving the θ-exact SW map expansions for arbitrary gauge groups/representations [44,45]. We first focused on the ambiguities between two distinct θ-exact SW map expansions: the first expansion (I) is obtained by solving the SW differential equations θ exactly [1,44], and the other (II) is obtained by inverting a known SW solution [7]. Since the maps relate NC gauge orbits with ordinary ones, there are two types of freedoms: generic redefinitions or gauge transformations of the NC gauge fields, and only the former will contribute to the dynamics. The redefinition freedom should be (and it was) taken into account when dealing with pathologies in the photon and neutrino two-point functions [40,52]. We then used/applied these two SW map expansions up to the e^3 order to study the corresponding field-strength expansions and discuss the gauge-inspired freedom/deformation parameters in each of the field-strength expansions.

In the first part of the study, we managed to determine the ambiguity between these two maps explicitly and transformed it into the standard form given in Refs. [22,23], and showed that the difference between these two SW maps for the gauge field at the e^3 order (52) is generic rather than just gauge transformations. We found that the SW map ambiguity between these two $\mathrm{U}_\star(1)$ gauge field expansions is decomposed into totally symmetric momentum structures which have a lowest order at θ^4 or θ^6, which means that the ambiguity can be expressed in terms of several commutative (yet non-associative) 3-products. This crucial decomposition enabled us to perform permutation and relabeling within the commutative 3-products and obtain the θ-exact expression for the ambiguity in terms of the commutative field strength $f_{\mu\nu}$, as shown in the Sec. III. Our observation thus indicates that at the e^3 order (even just) the $\mathrm{U}_\star(1)$ SW map expansion can possess much more profound structures than the prior order.

In the next stage we extended the e^2-order gauge field strength deformation parameter κ [52] to the e^3 order. We identified that part of the e^3-order gauge field strength should be multiplied by κ to keep the consistency condition, while there are other parts which are invariant under the commutative gauge transformations by themselves, and thus each of them can be varied independently like the κ-proportional part in the e^2 order. This promotes the introduction (alongside κ) of the new parameters $\kappa_{1,2}$ and $\kappa'_{1,2}$ for maps (I) and (II), respectively. Each pair of $\kappa(\kappa')_i$-proportional parts bear an identical structure at the θ^2 order, yet they differ from the θ^4 order on. The difference between each κ_i and κ'_i pair can be put into a relatively compact form using the generalized star products $\diamond_1$, $\diamond_{2(0)}$, and $\diamond_{2(1)}$, defined in Sec. III. The total difference matches the result (52) derived from the gauge field ambiguity when all deformation parameters are switched off.

Besides its own manifestness, the results on the gauge field strength expansion in this paper can enable the construction of θ-exact, and (κ,κ_i) and/or (κ,κ'_i)-deformed $\mathrm{U}_\star(1)$ gauge theory (pure noncommutative Yang-Mills gauge theory action) up to the four-photon coupling term, which should then lead to the completion of the one-loop photon two-point function computation started in Ref. [40] by adding the four-photon tadpole diagram contributions. We hope that—like the κ parameter in the bubble diagram contribution to the photon polarization tensor [40]—exploring the extended deformation freedom parameter space (κ,κ_i) and/or (κ,κ'_i) would provide enough control over the pathological divergences in the four-photon-tadpole diagram. The same term should also contribute to the NC phenomenology at extreme energies, for example tree-level NCQED contributions to the $2\to 2$ scattering processes like $\gamma\gamma\to\gamma\gamma$, etc. [53–56].

Knowing the fact that prior studies based on the $\star_2$ product have given rise to profound pathologies in both theory and phenomenology (even more so with the presence of the gauge freedom parameter κ [38–40,43]), we positively expect that our work in this article will form a universal basis for future studies on the various potential physical effects of the generalized star products and the higher-order gauge freedom parameters κ_i's and κ'_i's.

Finally, it is worth noticing that despite its profound nature our study on the SW map ambiguity in this paper is limited to only two distinct SW map expansions. There should still be many other variants available. The current methods for solving SW map(s)—for example, open-Wilson line operators [7], string-/D-brane-inspired analyses [1,8–10], the Batalin-Vilkovisky formalism, and the (generalized) SW differential equations [14–16,22,23,44,45] etc.—are extremely powerful in finding specific (sometimes closed-form) solutions, yet normally they do not provide us with all possible maps simultaneously.[3] SW maps also relate Morita-equivalent star products on Poisson manifolds; their nonuniqueness can be understood as a local gauge freedom in this context [6,12], which may help to understand the background (in)dependence of the string theory. It would be delightful to see any progress alone this line in the near future.

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[3]The Kontsevich formality-based approach [12] does capture ambiguities in the SW map: they are related to (formal) changes of coordinates. (See the discussion in the context of the somewhat cryptic Fig. 1 in Ref. [12].) This procedure essentially produces all possible maps, yet it is difficult to derive the explicit expansion(s) needed for practical purposes. We are particularly grateful to Peter Schupp for comments on the Kontsevich formality approach and the connection between SW maps and the Morita equivalence among star products.

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