Classifying BPS states in supersymmetric gauge theories coupled to higher derivative chiral models

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We study $\mathcal{N}=1$ supersymmetric gauge theories coupled with higher derivative chiral models in four dimensions in the off-shell superfield formalism. We solve the equation of motion for the auxiliary fields and find two distinct on-shell structures of the Lagrangian that we call the canonical and noncanonical branches characterized by zero and nonzero auxiliary fields, respectively. We classify Bogomol'nyi-Prasado-Sommerfield (BPS) states of the models in Minkowski and Euclidean spaces. In Minkowski space, we find Abelian and non-Abelian vortices, vortex lumps (or gauged lumps with fractional lump charges) as $1/2$ BPS states in the canonical branch, and higher derivative generalization of vortices and vortex-(BPS)baby Skyrmions (or gauged BPS baby Skyrmions with fractional baby Skyrme charges) as $1/4$ BPS states in the noncanonical branch. In four-dimensional Euclidean space, we find Yang-Mills instantons trapped inside a non-Abelian vortex, intersecting vortices, and intersecting vortex-(BPS)baby Skyrmions as $1/4$ BPS states in the canonical branch but no BPS states in the noncanonical branch other than those in the Minkowski space.

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I. INTRODUCTION

Low-energy effective theories play an important role in the study of nonperturbative effects of quantum field theory, such as the chiral Lagrangian of QCD [1]. In certain supersymmetric gauge theories, low-energy effective theories are determined exactly, offering full quantum spectra of Bogomol'nyi-Prasado-Sommerfield (BPS) states [2]. BPS states preserve a part of supersymmetry, belonging to so-called short multiplets of supersymmetry algebra, and consequently they are stable against quantum corrections perturbatively and nonperturbatively [3]. The low-energy effective field theories are constructed by a derivative expansion and are usually complemented by higher derivative corrections, as in the chiral perturbation theory [1].

Recently, in our previous paper, BPS states in the supersymmetric chiral models with higher derivative terms have been classified in $\mathcal{N}=1$ supersymmetric theories in four dimensions [4]. The purpose of this paper is to classify BPS states in $\mathcal{N}=1$ supersymmetric gauge theories coupled with higher derivative chiral models in four-dimensional Minkowski and Euclidean spaces.

Higher derivative corrections to supersymmetric field theories have a long history because of the auxiliary field problem. The auxiliary fields F in the off-shell superfield formalism of higher derivative models are generically acted on by space-time derivatives and consequently cannot be eliminated algebraically to obtain on-shell actions. Supersymmetric higher derivative terms free from the auxiliary field problem have been studied individually in various contexts: the Wess-Zumino-Witten term [5–8], low-energy effective action [9–17], $\mathbb{C}P^1$ (Faddeev-Skyrme) model [18,19], Dirac-Born-Infeld (DBI) action [20,21], k-field theory [22,23], low-energy effective action on BPS solitons [24], BPS baby Skyrme model [4,25–27], and nonlinear realizations of Nambu-Goldstone fields [28]. In the framework of supergravity, higher derivative terms [29–33] have been applied to ghost condensations [29,30] and the Galileon inflation models [31]. Among those, the four derivative term first found in Ref. [9], that can be constructed from a (2, 2) Kähler tensor, was rediscovered in Refs. [29,30] and has recently been used in various contexts. By using a Kähler tensor containing space-time derivatives, one can construct higher derivative terms with an arbitrary number of space-time derivatives [28].

In our previous paper [4], the auxiliary field equations were found to admit at least two distinct solutions that we called canonical and noncanonical branches with $F=0$ and $F\neq 0$, respectively. In particular, BPS baby Skyrmions (compactons) [25,26] have been found to be $1/4$ BPS states in the noncanonical branch, while BPS lumps are $1/2$ BPS states in the canonical branch [24], although both of them saturate the same Bogomol'nyi bound. In the former, the on-shell Lagrangian contains no usual kinetic term and consists of only a four derivative term, while in the latter, higher derivative corrections disappear in solutions and energy. BPS baby Skyrmions as compactons are currently paid much attention [34,35].

In this paper, we classify BPS states in $\mathcal{N}=1$ supersymmetric gauge theories coupled with higher derivative chiral models in four-dimensional Minkowski and

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Euclidean spaces. Here, we concentrate on the cases where superpotentials are absent for simplicity. As in the previous cases without gauge fields, we find canonical and noncanonical branches corresponding to solutions $F = 0$ and $F \neq 0$ of auxiliary field equations, respectively. We find that $1/2$ BPS states that exist in theories without higher derivative terms remain $1/2$ BPS in the canonical branch and that corresponding BPS states in the noncanonical branch are $1/4$ BPS states. On the other hand, we also find that $1/4$ BPS states that exist in theories without higher derivative terms remain $1/4$ BPS in the canonical branch but there are no corresponding BPS states in the noncanonical branch. More precisely, we find that $1/2$ BPS equations in the canonical branch do not receive higher derivative corrections for an Abrikosov-Nielsen-Olesen (ANO) vortex [36] at the critical (BPS) coupling, a non-Abelian vortex [37], lumps [38], and vortex lumps (gauged lumps with fractional lump charges) [39,40]. We then show that higher derivative generalization of vortices (that we may call compact vortices) and vortex-baby Skyrmion (or gauged baby Skyrmions with fractional baby Skyrme charges) are $1/4$ BPS states in the noncanonical branch. In four-dimensional Euclidean space, we find $1/2$ BPS Yang-Mills instantons, $1/4$ BPS Yang-Mills instantons trapped inside a non-Abelian vortex, and $1/4$ BPS intersecting vortices with instanton charges in the canonical branch. These configurations were known in supersymmetric theories with eight supercharges without higher derivative terms in $4+1$ or $5+1$ dimensions [41–44], and so what we confirm here is that they are still $1/4$ BPS states in theories with four supercharges in Euclidean four dimensions and that higher derivative terms are canceled out in the BPS equations and energy bound. Further, as new configurations, we find $1/4$ BPS vortex-lump string intersections with Yang-Mills instanton charges. We find no BPS states in the noncanonical branch other than those in Minkowski space.

This paper is organized as follows. In Sec. II, we give a supersymmetric Lagrangian in the superfield formalism. The first subsection is devoted to a review for higher derivative chiral models of chiral multiplets without coupling to gauge fields. In the second subsection, we introduce vector multiplets and coupling of vector and chiral multiplets. In Sec. III, we classify BPS states in four-dimensional Minkowski space. In Sec. IV, BPS states in four-dimensional Euclidean space are discussed. Section V is devoted to a summary and discussion. Notations and conventions are summarized in Appendix A. Explicit supersymmetry variations of fermions in Euclidean space are found in Appendix B.

II. HIGHER DERIVATIVE CHIRAL MODEL

In this section, we introduce the four-dimensional $\mathcal{N}=1$ supersymmetric higher derivative chiral model [4,29] and its coupling to the vector multiplet. The supersymmetric higher derivative chiral model consists of chiral superfields $\Phi^i (i = 1, \ldots, n)$ with arbitrary Kähler potential K, superpotential W, and a symmetric (2, 2) Kähler tensor $\Lambda_{ik\bar{j}\bar{l}}$. The tensor $\Lambda_{ik\bar{j}\bar{l}}$ is an arbitrary function of $\Phi^i, \Phi^{\dagger\bar{j}}$ and its space-time derivatives. Among other things, the purely bosonic part of the model never contains the space-time derivatives of the auxiliary fields F^i. Then all the auxiliary fields are integrated out by the algebraic equation of motion, and one finds explicit on-shell Lagrangians. When global symmetries in the model are gauged, the higher derivative term couples to the vector multiplet. In the following, we provide the explicit Lagrangian of the nongauged higher derivative chiral model and its coupling to the vector multiplet (gauged model).

A. Higher derivative chiral models without gauge coupling

We first start from the nongauged $\mathcal{N}=1$ supersymmetric higher derivative model with chiral superfields Φ^i. We employ the Wess-Bagger convention [45] in this paper, and detailed conventions and notations are summarized in Appendix A. The component expansion of the chiral superfield in the chiral base $y^m = x^m + i\theta\sigma^m\bar{\theta}$ is

$$\Phi^i = \varphi^i(y) + \theta\psi^i(y) + \theta^2 F^i(y). \tag{2.1}$$

Here φ^i is the complex scalar field, ψ^i is the Weyl fermion, and F^i is the auxiliary complex scalar field. The Lagrangian of the nongauged higher derivative chiral model is given by

$$\begin{aligned}\mathcal{L} = & \int d^4\theta K(\Phi^i, \Phi^{\dagger\bar{j}}) \\ & + \frac{1}{16}\int d^4\theta \Lambda_{i\bar{j}k\bar{l}}(\Phi, \Phi^\dagger) D^\alpha\Phi^i D_\alpha\Phi^k \bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{j}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} \\ & + \left(\int d^2\theta W(\Phi^i) + (\text{H.c.})\right),\end{aligned} \tag{2.2}$$

where K is the Kähler potential, $\Lambda_{ik\bar{j}\bar{l}}$ is a symmetric (2, 2) Kähler tensor, and W is the superpotential. The fourth derivative part in the Lagrangian is evaluated as

$$\begin{aligned}& D^\alpha\Phi^i D_\alpha\Phi^k \bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{j}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} \\ & \quad = 16\theta^2\bar{\theta}^2\Big[(\partial_m\varphi^i\partial^m\varphi^k)(\partial_n\bar{\varphi}^{\bar{j}}\partial^n\bar{\varphi}^{\bar{l}}) \\ & \qquad - \frac{1}{2}(\partial_m\varphi^i F^k + F^i\partial_m\varphi^k)(\partial^m\bar{\varphi}^{\bar{j}}\bar{F}^{\bar{l}} + \bar{F}^{\bar{j}}\partial^m\bar{\varphi}^{\bar{l}}) \\ & \qquad + F^i\bar{F}^{\bar{j}}F^k\bar{F}^{\bar{l}}\Big] + I_f.\end{aligned} \tag{2.3}$$

Here I_f stands for terms that contain fermions. Since the purely bosonic part in Eq. (2.3) saturates the Grassmann coordinate, only the lowest components in $\Lambda_{ik\bar{j}\bar{l}}$ contribute to the bosonic part of the Lagrangian. Then, the bosonic part of the Lagrangian is

$$\mathcal{L}_b = g_{i\bar{j}}(-\partial_m\varphi^i\partial^m\bar{\varphi}^{\bar{j}} + F^i\bar{F}^{\bar{j}}) + \frac{\partial W}{\partial\varphi^i}F^i + \frac{\partial\bar{W}}{\partial\bar{\varphi}^{\bar{j}}}\bar{F}^{\bar{j}} + \Lambda_{ik\bar{j}\bar{l}}(\varphi,\bar{\varphi})\{(\partial_m\varphi^i\partial^m\varphi^k)(\partial_n\bar{\varphi}^{\bar{j}}\partial^n\bar{\varphi}^{\bar{l}}) - 2\partial_m\varphi^iF^k\partial^m\bar{\varphi}^{\bar{j}}\bar{F}^{\bar{l}} + F^i\bar{F}^{\bar{j}}F^k\bar{F}^{\bar{l}}\}. \quad (2.4)$$

Here $g_{i\bar{j}} = \frac{\partial^2 K}{\partial\varphi^i\partial\bar{\varphi}^{\bar{j}}} > 0$ is the Kähler metric. In order to find the on-shell Lagrangian, we integrate out the auxiliary fields F^i. Since the Lagrangian does not contain space-time derivatives of the auxiliary fields F^i, one can solve the equation of motion for F^i and find the explicit form of the purely bosonic part of the on-shell Lagrangian.[1] The equation of motion for the auxiliary fields is

$$g_{i\bar{j}}F^i - 2\partial_m\varphi^iF^k\Lambda_{ik\bar{j}\bar{l}}\partial^m\bar{\varphi}^{\bar{l}} + 2\Lambda_{ik\bar{j}\bar{l}}F^iF^k\bar{F}^{\bar{l}} + \frac{\partial\bar{W}}{\partial\bar{\varphi}^{\bar{j}}} = 0. \quad (2.5)$$

As we have advertised, Eq. (2.5) is an algebraic equation, and it can be solved in principle. There are distinct on-shell branches associated with different solutions to Eq. (2.5). In general, there are two classes of solutions. The first class has a smooth limit $\Lambda_{ik\bar{j}\bar{l}} \to 0$ to the ordinary (i.e. without higher derivative terms) theory. For this class of solutions, higher derivative terms are introduced as perturbations to the ordinary (with second space-time derivatives) theory in the on-shell Lagrangian. We call this case the canonical (perturbative) branch. On the other hand, the second class of solutions does not have a smooth limit $\Lambda_{ik\bar{j}\bar{l}} \to 0$ to the ordinary theory. For this class of solutions, the higher derivative terms enter into the on-shell Lagrangian nonperturbatively. We call this case the noncanonical (nonperturbative) branch. In Ref. [4], we studied on-shell structures of the Lagrangian (2.4) for the single chiral superfield model. When $W \neq 0$, the equation of motion for the auxiliary field becomes that of the cubic power of F, and the solutions can be obtained by Cardano's method [21]. The explicit solutions are quite nonlinear in K, Λ, W, and $\partial_m\varphi$. Therefore, the on-shell Lagrangian becomes a highly complicated function of the scalar field φ. In the following, we consider models with $W = 0$ and show the explicit on-shell Lagrangians in the canonical and noncanonical branches.

Canonical branch.—It is apparent that $F^i = 0$ is always a solution to Eq. (2.5). In this case, the bosonic part of the on-shell Lagrangian is

$$\mathcal{L}_b = -g_{i\bar{j}}\partial_m\varphi^i\partial^m\bar{\varphi}^{\bar{j}} + \Lambda_{ik\bar{j}\bar{l}}(\varphi,\bar{\varphi})(\partial_m\varphi^i\partial^m\varphi^k)(\partial_n\bar{\varphi}^{\bar{j}}\partial^n\bar{\varphi}^{\bar{l}}). \quad (2.6)$$

The tensor $\Lambda_{ik\bar{j}\bar{l}}$ determines higher derivative terms in the Lagrangian. Since $\Lambda_{ik\bar{j}\bar{l}}$ is an arbitrary function of $\varphi, \bar{\varphi}$, one can construct arbitrary higher derivative terms for $n = 1$ models. For example, the scalar part of the $\mathcal{N} = 1$ supersymmetric Dirac-Born-Infeld action [20] is obtained by the single chiral superfield model with a flat Kähler potential and

$$\Lambda = \frac{1}{1 + A + \sqrt{(1 + A^2) - B}}, \quad A = \partial_m\Phi\partial^m\Phi^\dagger, \qquad B = \partial_m\Phi\partial^m\Phi\partial_n\Phi^\dagger\partial^n\Phi^\dagger. \quad (2.7)$$

The supersymmetric Faddeev-Skyrme model is obtained by the $\mathbb{C}P^1$ Fubuni-Study metric $K_{\varphi\bar{\varphi}} = \frac{1}{(1+|\varphi|^2)^2}$ and [4]

$$\Lambda = (\partial_m\Phi\partial^m\Phi\partial_n\Phi^\dagger\partial^n\Phi^\dagger)^{-1} \times \frac{1}{(1+\Phi\Phi^\dagger)^4}[(\partial_m\Phi^\dagger\partial^m\Phi)^2 - \partial_m\Phi\partial^m\Phi\partial_n\Phi^\dagger\partial^n\Phi^\dagger]. \quad (2.8)$$

This does not contain an additional term other than Faddeev-Skyrme term, in contrast to Refs. [18,19] that contain an additional term. The other examples include a supersymmetric completion of the Galileon inflation model [29], the ghost condensation [30], and the effective action of the supersymmetric Wess-Zumino model and QCD [6,10].

Noncanonical branch.—Although it is not easy to find explicit solutions $F^i \neq 0$ for the $n > 1$ case, one finds the solution for a single chiral superfield model [4]:

$$F = e^{i\eta}\sqrt{-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial^m\bar{\varphi}}, \quad (2.9)$$

where η is a phase factor and $K_{\varphi\bar{\varphi}} = \frac{\partial^2 K}{\partial\varphi\partial\bar{\varphi}}$. Then the bosonic part of the on-shell Lagrangian in the noncanonical branch is

$$\mathcal{L}_b = \Lambda|\partial_m\varphi\partial^m\varphi|^2 - \Lambda(\partial_m\varphi\partial^m\bar{\varphi})^2 - \frac{K_{\varphi\bar{\varphi}}^2}{4\Lambda}. \quad (2.10)$$

In this case, the ordinary canonical (second space-time derivative) kinetic term cancels out, and the on-shell Lagrangian contains higher derivative terms only. An example is the BPS baby Skyrme model [26], where Λ is given by

[1] There are space-time derivatives of the auxiliary fields F^i in the fermion term I_f. Solutions to F^i that include fermions are obtained order by order of the fermions. Since we are interested in the classical configurations of fields, these fermionic contributions are irrelevant in this paper.

$$\Lambda = \frac{1}{(1+\Phi\Phi^\dagger)^4}. \tag{2.11}$$

The Kähler metric is the Fubini-Study metric of $\mathbb{C}P^1$.

A few comments are in order for the noncanonical branch. First, since $F\bar{F} \geq 0$, the fields satisfy the constraint

$$\partial_m \varphi \partial^m \bar{\varphi} - \frac{K_{\varphi\bar{\varphi}}}{2\Lambda} \geq 0. \tag{2.12}$$

Second, the last term in Eq. (2.10) is considered as the scalar potential when Λ does not contain a space-time derivative term. One can introduce an arbitrary scalar potential without the superpotential W or the D-term potential in the noncanonical branch. This is an alternative way to introduce the scalar potential in supersymmetric models [32].

B. Gauged higher derivative chiral models

In this subsection, we study couplings of the gauge field to the higher derivative chiral models. We consider the higher derivative model of the type (2.2) where some global symmetries are assumed. Let us consider the chiral superfields $\Phi^{ia}(a=1,\ldots,\dim G)$ belonging to the fundamental representation of global symmetry group G with an additional flavor index i.[2] Then the fourth derivative term which preserves the global symmetry G is

$$\frac{1}{16}\int d^4\theta \Lambda_{ik\bar{j}\bar{l},ab}{}^{cd} D^\alpha \Phi^{ia} D_\alpha \Phi^{kb} \bar{D}_{\dot{\alpha}} \Phi^{\dagger\bar{j}}_c \bar{D}^{\dot{\alpha}} \Phi^{\dagger\bar{l}}_d, \tag{2.13}$$

where the Kähler tensor $\Lambda_{ik\bar{j}\bar{l},ab}{}^{cd}$ has indices of the (anti) fundamental representation of G.

The gauge field is introduced by the $\mathcal{N}=1$ vector superfield V with gauge group G. The generators $T^{\hat{a}}(\hat{a}=0,1,\ldots,\dim\mathcal{G}-1)$ of the gauge algebra $\mathcal{G}$ are normalized as $\mathrm{Tr}[T^{\hat{a}}T^{\hat{b}}] = k\delta^{\hat{a}\hat{b}}(k>0)$. The component expansion of $V = V^{\hat{a}}T^{\hat{a}}$ in the Wess-Zumino gauge is

$$V = -(\theta\sigma^m\bar{\theta})A_m(x) + i\theta^2\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x). \tag{2.14}$$

Here, A_m is the gauge field, $\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}$ are the gauginos, and D is the auxiliary real scalar field. All the fields belong to the adjoint representation of G. The coupling of the gauge field to the higher derivative terms is introduced by gauge covariantizing the supercovariant derivatives in Eq. (2.13). The gauge covariantized supercovariant derivative is defined by

$$\mathcal{D}_\alpha \Phi^{ia} = D_\alpha \Phi^{ia} + (\Gamma_\alpha)^a{}_b \Phi^{ib}. \tag{2.15}$$

Here Γ_α is the gauge connection defined by

$$\Gamma_\alpha = e^{-2gV} D_\alpha e^{2gV}, \tag{2.16}$$

where g is the gauge coupling constant. The gauge transformations of the superfields are

$$\Phi^i \to e^{-i\Theta}\Phi^i, \qquad e^{2gV} \to e^{-i\Theta^\dagger} e^{2gV} e^{i\Theta}, \tag{2.17}$$

where $\Theta = \Theta^{\hat{a}}(x,\theta,\bar{\theta})T^{\hat{a}}$ is a gauge parameter chiral superfield. Then the quantities $\mathcal{D}_\alpha\Phi^i, \bar{\mathcal{D}}_{\dot{\alpha}}\Phi^{\dagger\bar{i}}$ are transformed covariantly under the gauge transformation:

$$\mathcal{D}_\alpha \Phi^i \to e^{-i\Theta}\mathcal{D}_\alpha\Phi^i, \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\Phi^{\dagger\bar{i}} \to \bar{\mathcal{D}}_{\dot{\alpha}}\Phi^{\dagger\bar{i}} e^{i\Theta^\dagger}. \tag{2.18}$$

We note that the Kähler tensor $\Lambda_{ik\bar{j}\bar{l},ab}{}^{cd}$ becomes a function of $\Phi, \Phi^\dagger$ and V, in general.

Now we look for the concrete realizations of the gauge invariant generalization of the higher derivative term (2.13). We find a manifestly gauge invariant generalization of (2.13) is given by

$$-\frac{1}{16}\int d^4\theta \Lambda_{ik\bar{j}\bar{l}}(\Phi,\Phi^\dagger,V)(\bar{\mathcal{D}}_{\dot{\alpha}}\Phi^{\dagger\bar{j}} e^{2gV}\mathcal{D}^\alpha\Phi^i) \times (\bar{\mathcal{D}}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} e^{2gV}\mathcal{D}_\alpha\Phi^k), \tag{2.19}$$

where the Kähler tensor is

$$\Lambda_{ik\bar{j}\bar{l}\,ab}{}^{cd} = \Lambda_{ik\bar{j}\bar{l}}(\Phi,\Phi^\dagger,V)(e^{2gV})^c{}_a(e^{2gV})^d{}_b \tag{2.20}$$

and $\Lambda_{ik\bar{j}\bar{l}}$ is a gauge invariant (2, 2) Kähler tensor which is a function of $\Phi, \Phi^\dagger, V$.

The component expansion of the fourth derivative term (2.19) is

$$\begin{aligned}
&-\frac{1}{16}(\bar{\mathcal{D}}_{\dot{\alpha}}\Phi^{\dagger\bar{j}} e^{2gV}\mathcal{D}^\alpha\Phi^i)(\bar{\mathcal{D}}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} e^{2gV}\mathcal{D}_\alpha\Phi^k) \\
&\quad = \theta^2\bar{\theta}^2\Big[(D^m\bar{\varphi}^{\bar{j}}_a D^n\varphi^{ia})(D_m\bar{\varphi}^{\bar{l}}_b D_n\varphi^{kb}) \\
&\qquad -\frac{1}{2}(D_m\varphi^{ia}F^{kb} + F^{ia}D_m\varphi^{kb})(D^m\bar{\varphi}^{\bar{j}}_a\bar{F}^{\bar{l}}_b + \bar{F}^{\bar{j}}_a D^m\bar{\varphi}^{\bar{l}}_b) \\
&\qquad + F^{ia}\bar{F}^{\bar{j}}_a F^{kb}\bar{F}^{\bar{l}}_b\Big] + I'_f,
\end{aligned} \tag{2.21}$$

where $I_f{}'$ is terms that contain fermions. Again, there are no auxiliary fields with space-time derivatives in the purely bosonic terms. Since the bosonic terms in $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi^\dagger\mathcal{D}^\alpha\Phi\bar{\mathcal{D}}^{\dot{\alpha}}\Phi^\dagger\mathcal{D}_\alpha\Phi$ already saturate the Grassmann coordinate, the factor e^{2gV} does not contribute to the purely bosonic sector of the Lagrangian. However, the factor e^{2gV} is necessary for the gauge invariance of the higher

[2]It is straightforward to generalize the result in this subsection to other representations. Therefore, we consider the fundamental representation of G for the chiral superfield Φ^a throughout this paper.

derivative terms, and this indeed contributes to the fermionic part I'_f in Eq. (2.21). We also note that the lowest components in $\Lambda_{ik\bar{j}\bar{l}}$ come from the chiral superfields only. This is because the lowest component in the vector superfield V contains the Grassmann coordinate θ in the Wess-Zumino gauge (2.14). In Ref. [26], a three-dimensional analogue of the gauge invariant higher derivative model for a $U(1)$ gauge group was discussed.

Introducing the ordinary kinetic terms for Φ^{ia} and the gauge field, the total Lagrangian we consider is

$$\begin{aligned}\mathcal{L} = &\int d^4\theta K(\Phi^\dagger, \Phi, V) \\ &-\frac{1}{16}\int d^4\theta \Lambda_{ik\bar{j}\bar{l}}(\Phi, \Phi^\dagger, V)(\bar{\mathcal{D}}_{\dot{\alpha}}\Phi^{\dagger\bar{j}}e^{2gV}\mathcal{D}^\alpha\Phi^i) \\ &\times(\bar{\mathcal{D}}^{\dot{\alpha}}\Phi^{\dagger\bar{l}}e^{2gV}\mathcal{D}_\alpha\Phi^k) + \frac{1}{16kg^2}\mathrm{Tr}\left[\int d^2\theta W^\alpha W_\alpha + (\mathrm{H.c.})\right] \\ &-2\kappa g\int d^4\theta \mathrm{Tr}V. \end{aligned} \tag{2.22}$$

Here we have introduced the Fayet-Iliopoulos parameter κ for the purpose of later discussions. The field strength of the vector superfield V is defined by

$$W_\alpha = -\frac{1}{4}\bar{D}^2(e^{-2gV}D_\alpha e^{2gV}). \tag{2.23}$$

Throughout this paper, we consider the gauge invariant Kähler potential of the form $K(\Phi^\dagger, \Phi, V) = \frac{1}{2}(K(\Phi^\dagger e^{2gV}, \Phi) + K(\Phi^\dagger, e^{2gV}\Phi))$ and general gauge group G if not mentioned. Then, the bosonic component of the Lagrangian (2.22) is

$$\begin{aligned}\mathcal{L}_b = &-\frac{\partial^2 K}{\partial\bar{\varphi}^{\bar{j}}_a\partial\varphi^{ib}}D_m\bar{\varphi}^{\bar{j}}_a D^m\varphi^{ib} - \frac{\partial^2 K}{\partial\bar{\varphi}^{\bar{j}}_a\partial\varphi^{ib}}\bar{F}^{\bar{j}}_a F^{ib} \\ &+\frac{g}{2}D^{\hat{a}}\left(\bar{\varphi}^{\bar{j}}_c(T^{\hat{a}})^c{}_d\frac{\partial K}{\partial\bar{\varphi}^{\bar{j}}_d} + \frac{\partial K}{\partial\varphi^{ic}}(T^{\hat{a}})^c{}_d\varphi^{id} - 2\kappa\delta^{\hat{a}}{}_0\right) \\ &+\frac{1}{k}\mathrm{Tr}\left[-\frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}D^2\right] \\ &+\Lambda_{ik\bar{j}\bar{l}}(\varphi,\bar{\varphi})\Big[(D^m\bar{\varphi}^{\bar{j}}_a D^n\varphi^{ia})(D_m\bar{\varphi}^{\bar{l}}_b D_n\varphi^{kb}) \\ &-\frac{1}{2}(D_m\varphi^{ia}F^{kb} + F^{ia}D_m\varphi^{kb})(D^m\bar{\varphi}^{\bar{j}}_a\bar{F}^{\bar{l}}_b + \bar{F}^{\bar{j}}_a D^m\bar{\varphi}^{\bar{l}}_b) \\ &+F^{ia}\bar{F}^{\bar{j}}_a F^{kb}\bar{F}^{\bar{l}}_b\Big], \end{aligned} \tag{2.24}$$

where we have assigned the $U(1)$ generator to T^0. The gauge field strength is

$$F_{mn} = \partial_m A_n - \partial_n A_m + ig[A_m, A_n]. \tag{2.25}$$

The equation of motion for the auxiliary field D is[3]

$$D^{\hat{a}} + \frac{g}{2}\left(\bar{\varphi}^{\bar{j}}_c(T^{\hat{a}})^c{}_d\frac{\partial K}{\partial\bar{\varphi}^{\bar{j}}_d} + \frac{\partial K}{\partial\varphi^{ic}}(T^{\hat{a}})^c{}_d\varphi^{id}\right) - g\kappa\delta^{\hat{a}}{}_0 = 0. \tag{2.26}$$

The equation of motion for $\bar{F}^{\bar{j}}_a$ is

$$\begin{aligned}&\frac{\partial^2 K}{\partial\bar{\varphi}^{\bar{j}}_a\partial\varphi^{ib}}F^{ib} - \Lambda_{ik\bar{j}\bar{l}}(\varphi,\bar{\varphi})[D_m\varphi^{ib}D^m\bar{\varphi}^{\bar{j}}_b F^{ka} \\ &\quad + D_m\varphi^{ia}D^m\bar{\varphi}^{\bar{l}}_b F^{kb} - 2F^{ia}F^{kb}\bar{F}^{\bar{l}}_b] = 0. \end{aligned} \tag{2.27}$$

As in the case of the nongauged chiral superfield models, there are two on-shell branches associated with solutions to Eq. (2.27).

Canonical branch.—We first consider the canonical branch. One finds that $F^{ia} = 0$ is always a solution. Then, the on-shell Lagrangian in the canonical branch is

$$\begin{aligned}\mathcal{L}_b = &-\frac{\partial^2 K}{\partial\bar{\varphi}^{\bar{j}}_a\partial\varphi^{ib}}D_m\bar{\varphi}^{\bar{j}}_a D^m\varphi^{ib} \\ &+\Lambda_{ik\bar{j}\bar{l}}(\varphi,\bar{\varphi})(D^m\bar{\varphi}^{\bar{j}}_a D^n\varphi^{ia})(D_m\bar{\varphi}^{\bar{l}}_b D_n\varphi^{kb}) \\ &-\frac{g^2}{2}\left(\frac{1}{2}\bar{\varphi}^{\bar{j}}_c(T^{\hat{a}})^c{}_d\frac{\partial K}{\partial\bar{\varphi}^{\bar{j}}_d} + \frac{1}{2}\frac{\partial K}{\partial\varphi^{ic}}(T^{\hat{a}})^c{}_d\varphi^{id} - \kappa\delta^{\hat{a}}{}_0\right)^2 \\ &-\frac{1}{4k}\mathrm{Tr}F_{mn}F^{mn}. \end{aligned} \tag{2.28}$$

The vacuum of the model is determined by the D-term condition

$$\bar{\varphi}^{\bar{i}}_c(T^{\hat{a}})^c{}_d\varphi^{id} - \kappa\delta^a{}_0 = 0. \tag{2.29}$$

We stress that $\Lambda_{ik\bar{j}\bar{l}}$ does not contain the space-time derivatives on Φ ($\Phi^\dagger$), unlike the nongauged cases for which the space-time derivative can act on Φ ($\Phi^\dagger$) in $\Lambda_{ik\bar{j}\bar{l}}$. This is because the gauge covariant derivative of a chiral superfield $D_m\Phi^{ia}$ does not provide supersymmetric couplings of the gauge field. From now on, we therefore consider the tensor $\Lambda_{ik\bar{j}\bar{l}}$ which never contains the space-time derivatives of the superfields.

Noncanonical branch.—It is not so easy to find a $F^{ia} \neq 0$ solution even for the single chiral superfield model. However, we find that a $F^a \neq 0$ solution can be explicitly written down for single chiral superfield models with a $U(1)$ gauge group as

[3] We never introduce higher derivative terms of the vector superfield V. Therefore, the equation of motion for D is always linear and can be solved trivially.

$$F^0 = e^{i\eta}\sqrt{-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + D_m\varphi D^m\bar{\varphi}}, \tag{2.30}$$

where η is a phase factor. The solution in Eq. (2.30) is just the gauge covariantized counterpart of that in Eq. (2.9). The fields satisfy the gauge covariantized constraint (2.12):

$$|F^0|^2 = -\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + D_m\varphi D^m\bar{\varphi} \geq 0. \tag{2.31}$$

Then the bosonic part of the on-shell Lagrangian in the noncanonical branch is

$$\mathcal{L}_b = -\frac{1}{4}F_{mn}F^{mn} - \frac{g^2}{2}\left(\frac{1}{2}\bar{\varphi}\frac{\partial K}{\partial\bar{\varphi}} + \frac{1}{2}\frac{\partial K}{\partial\varphi}\varphi - \kappa\right)^2 + \Lambda(|D_m\varphi D^m\varphi|^2 - (D_m\varphi D^m\bar{\varphi})^2) - \frac{(K_{\varphi\bar{\varphi}})^2}{4\Lambda}, \tag{2.32}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$ is the field strength of the $U(1)$ gauge field. An example of the Lagrangian (2.32) is a supersymmetric generalization of the gauged BPS baby Skyrme model [35] whose potential term is determined by the Kähler potential K through the D term and the term $K^2_{\varphi\bar{\varphi}}/\Lambda$. In this case, the explicit function Λ is given in Eq. (2.11).

III. BPS STATES IN MINKOWSKI SPACE

In this section, we investigate BPS configurations of the model (2.22) in four-dimensional Minkowski space. BPS configurations in supersymmetric theories preserve parts of supersymmetry. BPS equations are obtained from the condition that the on-shell supersymmetry transformation of the fermions in the model vanishes: $\delta^{\text{on}}_\xi\psi_\alpha = \delta^{\text{on}}_\xi\lambda_\alpha = 0$. Here δ^{on}_ξ (δ^{off}_ξ) is the on-shell (off-shell) supersymmetry transformation by the parameters ξ_α, $\bar{\xi}^{\dot{\alpha}}$. The off-shell supersymmetry variation of the fermions ψ, λ is

$$\delta^{\text{off}}_\xi\psi^{ia}_\alpha = \sqrt{2}i(\sigma^m)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}D_m\varphi^{ia} + \sqrt{2}\xi_\alpha F^{ia}, \tag{3.1}$$

$$\delta^{\text{off}}_\xi\lambda_\alpha = i\xi_\alpha D + (\sigma^{mn})_\alpha{}^\beta\xi_\beta F_{mn}. \tag{3.2}$$

The on-shell supersymmetry transformations are obtained by substituting the solutions of the auxiliary fields equations into F and D. Therefore, they have distinct structures in the canonical and noncanonical branches.

In Ref. [4], we studied BPS equations in the nongauged higher derivative models given in Eq. (2.4) where no gauge fields are present. We derived the $1/2$ BPS domain wall and lump equations in the canonical branch. These equations are the same for the ordinary (without higher derivative term) theory. We calculated the BPS bound of the on-shell action associated with these configurations. Then we found that the BPS bound is given by the ordinary tension of the domain wall and the lump (topological) charge, respectively. Namely, higher derivative effects are totally canceled in the $1/2$ BPS domain wall and lump. In the noncanonical branch, we found $1/4$ BPS configurations for the domain wall junctions and lump-type solitons. The equation for the domain wall junction receives higher derivative contributions, while the associated BPS bound of the Lagrangian is expressed by the ordinary domain wall tension and the junction charge. For the lump-type soliton, it is considered as a compacton, which is a soliton with a compact support. Indeed, when the Kähler potential K and Λ are chosen appropriately, the $1/4$ BPS equations in Ref. [4] have compacton-type solutions [26].

In the following subsections, we proceed with the analysis of the BPS configurations for the gauged higher derivative chiral models given in Eq. (2.22). For the ordinary $\mathcal{N}=1$ supersymmetric gauge theory with fundamental matter in Minkowski space, there are BPS vortices which are codimension-two solitons. We study codimension-two vortex configurations in the canonical and noncanonical branches of the model (2.22).

A. Canonical branch

We start from the flat Kähler potential $K = \Phi^{\dagger\bar{i}}e^{2gV}\Phi^i$ and look for the vortex configurations. The static ansatz for the vortex is given by

$$\varphi^{ia} = \varphi^{ia}(x^1, x^2), \qquad F_{12} \neq 0, \tag{3.3}$$

where the other components of F_{mn} all vanish. In the canonical branch, we have the solution $F^{ia} = 0$. Then, the on-shell supersymmetry variations of the fermions are

$$\delta\psi^i = \sqrt{2}i\begin{pmatrix}(D_1 - iD_2)\varphi^i\bar{\xi}^{\dot{2}} \\ (D_1 + iD_2)\varphi^i\bar{\xi}^{\dot{1}}\end{pmatrix} = 0, \tag{3.4}$$

$$\delta\lambda = -i\begin{pmatrix}\xi_1 F_{12} - \xi_1 D \\ -\xi_2 F_{12} - \xi_2 D\end{pmatrix} = 0, \tag{3.5}$$

where $D^{\hat{a}} = -g(\bar{\varphi}^{\bar{i}}_c(T^{\hat{a}})^c{}_d\varphi^{id} - \kappa\delta^{\hat{a}}{}_0)$. The vortex configuration is obtained by imposing the following projection condition on the supersymmetry parameter:

$$\frac{1}{2}(\sigma^1 + i\sigma^2)\bar{\xi} = 0. \tag{3.6}$$

This is equivalent to the condition $\bar{\xi}^{\dot{2}} = \xi_1 = 0$ so that the projection (3.6) leaves a half of $\mathcal{N}=1$ supersymmetry. Therefore, we obtain the following BPS equations:

$$\bar{D}_z\varphi^{ia} = 0, \qquad F^{\hat{a}}_{12} - g(\bar{\varphi}^{\bar{i}}_c(T^{\hat{a}})^c{}_d\varphi^{id} - \kappa\delta^{\hat{a}}{}_0) = 0. \tag{3.7}$$

Here we have defined $z \equiv \frac{1}{2}(x^1 + ix^2)$ and $D_z \equiv D_1 - iD_2$, $\bar{D}_z \equiv D_1 + iD_2$. This is just the ordinary $1/2$ BPS Abelian (ANO) or non-Abelian vortex equation [37]. Now we calculate the Lagrangian bound[4] associated with the BPS equations (3.7). Using the first condition in Eq. (3.7), we find the higher derivative terms vanish:

$$\begin{aligned}&\Lambda_{ik\bar{j}\bar{l}}(D^m\bar{\varphi}^{\bar{j}}_a D^n\varphi^{ia})(D_m\bar{\varphi}^{\bar{l}}_b D_n\varphi^{kb})\\&\quad=\frac{1}{4}\Lambda_{ik\bar{j}\bar{l}}(D_z\varphi^{ia}\bar{D}_z\varphi^{kb}+\bar{D}_z\varphi^{ia}D_z\varphi^{kb})\\&\qquad\times(D_z\bar{\varphi}^{\bar{j}}_a\bar{D}_z\bar{\varphi}^{\bar{l}}_b+\bar{D}_z\bar{\varphi}^{\bar{j}}_a D_z\bar{\varphi}^{\bar{l}}_b)\\&\quad=0.\end{aligned} \tag{3.8}$$

Then, by using the first and the second equations in (3.7), we obtain the Lagrangian bound

$$\mathcal{L}=\kappa g F^0_{12}. \tag{3.9}$$

Here F^0_{12} is the $U(1)$ flux density in the (x^1,x^2) plane. Integrating it in the (x^1,x^2) plane, we obtain the ordinary vortex topological charge. Therefore, in the canonical branch, all the higher derivative corrections to the 1/2 BPS vortex are canceled in both the equations (3.7) and the Lagrangian bound (3.9). This is a conceivable result, since the BPS nature is determined by the supersymmetry algebra. The model (2.28) includes higher derivative terms, but supersymmetry is manifestly realized. Then we expect that the BPS structure is protected against higher derivative corrections. A typical example is the world-volume theory of D-branes where BPS states in super Yang-Mills theory linearize the non-Abelian DBI action canceling the higher derivative corrections [47]. While the higher derivative corrections exist in the non-Abelian vortex effective theory, the higher derivative effects are canceled in the BPS equation and energy of $\mathbb{C}P^{N-1}$ lumps inside a non-Abelian vortex [24]. We also comment that this is the same conclusion discussed in the domain wall and lump in the nongauged chiral models [4].

We next consider the general gauge invariant Kähler potential of the form $K(\Phi^\dagger,\Phi,V)=\frac{1}{2}(K(\Phi^\dagger e^{2gV},\Phi)+K(\Phi^\dagger,e^{2gV}\Phi))$. The BPS equations for the 1/2 BPS projection condition (3.6) are

$$\begin{aligned}&\bar{D}_z\varphi^{ia}=0,\\&F^{\hat{a}}_{12}-\frac{g}{2}\left(\bar{\varphi}^{\bar{j}}_c(T^{\hat{a}})^c{}_d\frac{\partial K}{\partial\bar{\varphi}^{\bar{j}}_d}+\frac{\partial K}{\partial\varphi^{ic}}(T^{\hat{a}})^c{}_d\varphi^{id}-\kappa\delta^{\hat{a}}{}_0\right)=0.\end{aligned} \tag{3.10}$$

By using the first condition in (3.10), we find that the higher derivative terms vanish. Then, the Lagrangian bound associated with the BPS condition (3.10) is

$$\begin{aligned}\mathcal{L}&=-\frac{1}{2}\frac{\partial^2 K}{\partial\bar{\varphi}^{\bar{j}}_a\partial\varphi^{ib}}\bar{D}_z\bar{\varphi}^{\bar{j}}_a D_z\varphi^{ib}\\&\quad-\frac{g^2}{2}\left(\frac{1}{2}\bar{\varphi}^{\bar{j}}_c(T^{\hat{a}})^c{}_d\frac{\partial K}{\partial\bar{\varphi}^{\bar{j}}_d}+\frac{1}{2}\frac{\partial K}{\partial\varphi^{ic}}(T^{\hat{a}})^c{}_d\varphi^{id}-\kappa\delta^{\hat{a}}{}_0\right)^2\\&\quad-\frac{1}{2}(F^{\hat{a}}_{12})^2\\&=-\varepsilon^{st}\partial_s\mathcal{N}_t+\kappa g F^0_{12},\end{aligned} \tag{3.11}$$

where we have defined the following quantity:

$$\mathcal{N}_s=\frac{i}{2}\left(\frac{\partial K}{\partial\bar{\varphi}^{\bar{j}}_a}D_s\bar{\varphi}^{\bar{j}}_a-\frac{\partial K}{\partial\varphi^{ia}}D_s\varphi^{ia}\right)\quad(s,t=1,2). \tag{3.12}$$

The first term in Eq. (3.11) is the gauge covariant generalization of the lump charge density. Then the Lagrangian bound is given by the sum of the lump and the vortex charge densities. The BPS configurations whose energy bound is given by Eq. (3.11) have been studied in the gauged nonlinear sigma models where higher derivative corrections are absent [39,40]. In there, the configurations admit fractional lump charges. Once again, we find that all the higher derivative effects are canceled on the 1/2 BPS states (3.10).

B. Noncanonical branch

We next consider BPS equations in the noncanonical branch. The Lagrangian is given by (2.32) where the gauge group is $U(1)$ and $K=\Phi^\dagger e^{2gV}\Phi$. The nonzero solution of the auxiliary field F^0 is given in Eq. (2.30). The supersymmetry variation of the fermions is

$$\delta\psi=\sqrt{2}\begin{pmatrix}i(D_1-iD_2)\varphi\bar{\xi}^{\dot{2}}+\xi_1F^0\\ i(D_1+iD_2)\varphi\bar{\xi}^{\dot{1}}+\xi_2F^0\end{pmatrix}=0, \tag{3.13}$$

$$\delta\lambda=-i\begin{pmatrix}\xi_1F_{12}-\xi_1D\\ -\xi_2F_{12}-\xi_2D\end{pmatrix}=0. \tag{3.14}$$

Since the auxiliary field F^0 is nonzero in the noncanonical branch, the 1/2 BPS projection (3.6) gives the equations (3.7) and the following additional condition:

$$F^0=e^{i\eta}\sqrt{-\frac{1}{2\Lambda}+D_m\varphi D^m\bar{\varphi}}=0. \tag{3.15}$$

Solutions that satisfy the ordinary vortex equations (3.7) do not satisfy the condition in Eq. (3.15) for general Λ.[5] We therefore look for another BPS condition. A natural candidate is the gauge covariantized generalization of the BPS lumps in the noncanonical branch. Following

[4]When the Lagrangian (2.28) contains higher order time derivatives of fields, the positive energy Hamiltonian is not defined in general [46]. Therefore, we calculate the Lagrangian bound, rather than the energy bound, for the BPS configurations.

[5]However, when Λ is chosen appropriately, it is possible that the ordinary vortex solution satisfies the condition (3.15).

the BPS lumps studied in Ref. [4], we consider the $1/4$ BPS projection conditions

$$\frac{1}{2}(\sigma^1 + i\sigma^2)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} = 0, \qquad \frac{1}{2}(\sigma^1 - i\sigma^2)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} = i\xi_\alpha. \tag{3.16}$$

Then, from the variation of the fermions, we find a set of $1/4$ BPS equations:

$$\bar{D}_z\varphi = -ie^{i\eta}\sqrt{-\frac{1}{2\Lambda} + \frac{1}{2}(D_z\bar{\varphi}\bar{D}_z\varphi + \bar{D}_z\bar{\varphi}D_z\varphi)},$$
$$F^0_{12} - g(\bar{\varphi}\varphi - \kappa) = 0. \tag{3.17}$$

The first equation is the gauge covariantized generalization of the compacton-type equation, while the second equation is that for the ANO vortex. We call solutions to these equations as higher derivative vortices. These equations may admit a vortex with a compact support for the scalar fields (that we may call a compact vortex). See Ref. [48] for a vortex with a compact support which are non-BPS in nonsupersymmetric theories.

We then calculate the Lagrangian bound associated with the BPS condition (3.20). Using the first condition in Eq. (3.20), we obtain the following relation:

$$\Lambda\{(D_m\varphi D^m\varphi)(D_n\bar{\varphi}D^n\bar{\varphi}) - (D_m\varphi D^m\bar{\varphi})^2\}$$
$$= -\frac{1}{4}\Lambda(\bar{D}_z\varphi D_z\bar{\varphi} - D_z\varphi\bar{D}_z\bar{\varphi})^2 = -\frac{1}{4\Lambda}. \tag{3.18}$$

By using this relation and the second equation in Eq. (3.17), we calculate the BPS bound of the Lagrangian as

$$\mathcal{L} = \kappa g F^0_{12}. \tag{3.19}$$

This is the topological vortex charge density. Therefore, Eqs. (3.17) correspond to the higher derivative generalization of the ANO vortex rather than the compacton. We comment that the higher derivative terms cancel out in the Lagrangian bound even in the noncanonical branch. However, the BPS equation (3.20) receives higher derivative corrections. The situation is quite similar to the $1/4$ BPS domain wall junction and the compacton in the nongauged model [4]. In there, there are higher derivative corrections to the BPS equations. However, the bounds for the BPS states do not receive higher derivative corrections.

Now we consider the general gauge invariant Kähler potential. A set of $1/4$ BPS equations is obtained as

$$\bar{D}_z\varphi = -ie^{i\eta}\sqrt{-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \frac{1}{2}(D_z\bar{\varphi}\bar{D}_z\varphi + \bar{D}_z\bar{\varphi}D_z\varphi)},$$
$$F^0_{12} - \frac{g}{2}\left(\bar{\varphi}\frac{\partial K}{\partial\bar{\varphi}} + \frac{\partial K}{\partial\varphi}\varphi - \kappa\right) = 0. \tag{3.20}$$

Using the first condition in Eq. (3.20), we find that the higher derivative terms cancel out in the Lagrangian bound. The result is

$$\mathcal{L} = -\varepsilon^{st}\partial_s\mathcal{N}_t + \kappa g F^0_{12} \quad (s,t = 1,2), \tag{3.21}$$

where

$$\mathcal{N}_s = \frac{i}{2}(K_{\bar{\varphi}}D_s\bar{\varphi} - K_\varphi D_s\varphi). \tag{3.22}$$

This is precisely the sum of the lump and the vortex charges. We therefore expect that Eqs. (3.20) describe composite states of the higher derivative ANO vortex and the BPS baby Skyrmions, or simply gauged BPS baby Skyrmions. Solutions should carry fractional baby Skyrmion charges as for the vortex lumps in the canonical branch. BPS states in Minkowski space are summarized in Table I.

IV. BPS STATES IN EUCLIDEAN SPACE

In four-dimensional Euclidean space, one can consider codimension-four objects. Typical examples are the Yang-Mills instantons and the instantons trapped inside (intersecting) vortices. In this section, we study codimension-four BPS configurations of the higher derivative model (2.22) in Euclidean space. The off-shell supersymmetry variation of the fermions in Euclidean space is

$$\delta_\xi\psi^i_\alpha = \sqrt{2}i(\sigma^m_{\mathrm{E}})_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}D_m\varphi^i + \sqrt{2}\xi_\alpha F^i, \tag{4.1}$$

$$\delta_\xi\lambda_\alpha = i\xi_\alpha D + (\sigma^{mn}_{\mathrm{E}})_\alpha{}^\beta\xi_\beta F_{mn}, \tag{4.2}$$

where $m = 1, 2, 3, 4$ and the sigma matrices in the Euclidean space are defined by

TABLE I. BPS states in the gauged higher derivative (HD) chiral model and super Yang-Mills (SYM) with gauged nonlinear sigma model (SUSY NLSM). Theories are defined in Minkowski space. The BPS states are classified into the lump (L) type, the vortex (V) type, and the vortex-lump (VL) type.

	SYM + SUSY NLSM	Canonical	Noncanonical
L type	1/2 BPS lump	1/2 BPS lump	1/4 BPS baby Skyrmion
V type	1/2 BPS vortex	1/2 BPS vortex	1/4 BPS HD vortex
VL type	1/2 BPS vortex lump	1/2 BPS vortex lump	1/4 BPS vortex-baby Skyrmion

$$(\sigma^m_{\rm E})_{\alpha\dot\alpha} = (i\vec\tau, \mathbf{1}), \qquad (\bar\sigma^m_{\rm E})^{\dot\alpha\alpha} = (-i\vec\tau, \mathbf{1}). \tag{4.3}$$

Here, $\vec\tau$ are the Pauli matrices. The explicit supersymmetry variations of the fermions are found in Appendix B. We note that, in Euclidean space, ξ^α and $\bar\xi^{\dot\alpha}$ are independent from each other and they are not complex conjugate anymore. Then it is possible to consider BPS projections that drop a chiral half of $\mathcal{N}=1$ supersymmetry $\xi^\alpha = 0$, $\bar\xi^{\dot\alpha} \neq 0$. Indeed, the standard Yang-Mills instantons exist in our model (2.22), that preserve the (anti)chiral half of supersymmetry and are $1/2$ BPS configurations. Since BPS states with codimensions less than four in Euclidean space are the same as those in Minkowski space, discussed in the previous section, we focus on codimension-four BPS states in the higher derivative model in the following subsections.

A. Canonical branch

We start from the Lagrangian (2.22) where the Kähler potential is flat. We consider the $1/4$ BPS projection condition[6]

$$\bar\xi^{\dot 1} \neq 0, \qquad \bar\xi^{\dot 2} = \xi_1 = \xi_2 = 0. \tag{4.4}$$

Then from the supersymmetry variation of the fermions, we obtain the following set of $1/4$ BPS equations in the canonical branch:

$$\begin{aligned} &\bar D_z \varphi^i = \bar D_w \varphi^i = 0, \\ &F^{\hat a}_{12} - F^{\hat a}_{34} = g(\bar\varphi^{\bar i}_c (T^{\hat a})^c{}_d \varphi^{id} - \delta^{\hat a}{}_0 \kappa), \\ &F^{\hat a}_{13} + F^{\hat a}_{24} = F^{\hat a}_{14} - F^{\hat a}_{23} = 0, \end{aligned} \tag{4.5}$$

where we have defined complex coordinates and derivatives with respect to them by

$$\begin{aligned} &z \equiv \frac{1}{2}(x^1 + ix^2), \qquad w \equiv \frac{1}{2}(x^4 + ix^3), \\ &D_z \equiv D_1 - iD_2, \qquad D_w \equiv D_4 - iD_3. \end{aligned} \tag{4.6}$$

Using the condition $\bar D_z \varphi^i = \bar D_w \varphi^i = 0$, we find that the higher derivative terms vanish for the BPS configuration (4.5):

$$\begin{aligned} &\Lambda_{ik\bar j\bar l}(D_m \bar\varphi^{\bar j}_a D^m \bar\varphi^{\bar l}_b)(D_n \varphi^{ib} D^n \varphi^{kb}) \\ &= \frac{1}{4}\Lambda_{ik\bar j\bar l}(D_z\varphi^{ia}\bar D_z\varphi^{kb} + \bar D_z\varphi^{ia} D_z\varphi^{kb} + D_w\varphi^{ia}\bar D_w\varphi^{kb} \\ &\quad + \bar D_w\varphi^{ia} D_w\varphi^{kb})(D_z\bar\varphi^{\bar j}_a \bar D_z\bar\varphi^{\bar l}_b + \bar D_z\bar\varphi^{\bar j}_a D_z\bar\varphi^{\bar l}_b \\ &\quad + D_w\bar\varphi^{\bar j}_a \bar D_w\bar\varphi^{\bar l}_b + \bar D_w\bar\varphi^{\bar j}_a D_w\bar\varphi^{\bar l}_b) \\ &= 0. \end{aligned} \tag{4.7}$$

[6]The other combinations, for example, $\xi_2 \neq 0, \xi_1 = \bar\xi^{\dot 1} = \bar\xi^{\dot 2} = 0$ and so on, give essentially the same form of the BPS equations.

Then the BPS bound of the Lagrangian associated with the configuration (4.5) is

$$\mathcal{L}_{\rm E} = -\kappa g(F^0_{12} - F^0_{34}) + \frac{1}{4k}{\rm Tr}[F_{mn}\tilde F^{mn}], \tag{4.8}$$

where $\tilde F_{mn} = \frac{1}{2}\varepsilon_{mnpq}F^{pq}$ is the Hodge dual of the gauge field strength F_{mn}. We note that the sign of the Lagrangian in Euclidean space is flipped from that in Minkowski space. The first and the second terms in (4.8) correspond to the vortex charge densities in the (x^1, x^2) and (x^3, x^4) planes, respectively. The last term is the instanton charge density. Therefore, solutions to Eq. (4.5) are the Yang-Mills instantons trapped inside intersecting vortices. A set of these equations were first found in Refs. [41–44] for supersymmetric theories with eight supercharges without higher derivative terms, and configurations were shown to be $1/4$ BPS states [42]. Solutions can be constructed in terms of the moduli matrix [43] and are mathematically characterized in terms of amoeba and tropical geometry [44].

We next consider the general gauge invariant Kähler potential. In this case, a set of $1/4$ BPS equations that we obtain is

$$\begin{aligned} &\bar D_z \varphi^i = \bar D_w \varphi^i = 0, \\ &F^{\hat a}_{12} - F^{\hat a}_{34} = \frac{g}{2}\left(\bar\varphi^{\bar j}_c (T^{\hat a})^c{}_d \frac{\partial K}{\partial \bar\varphi^{\bar j}_d} + \frac{\partial K}{\partial \varphi^{ic}}(T^{\hat a})^c{}_d \varphi^{id} - \kappa\delta^{\hat a}{}_0\right), \\ &F^{\hat a}_{13} + F^{\hat a}_{24} = F^{\hat a}_{14} - F^{\hat a}_{23} = 0. \end{aligned} \tag{4.9}$$

Using Eqs. (4.9), the BPS bound of the Lagrangian can be evaluated as

$$\begin{aligned} \mathcal{L}_{\rm E} = &\varepsilon^{st}\partial_s \mathcal{N}_t - \varepsilon^{s't'}\partial_{s'}\mathcal{N}_{t'} - \kappa g(F^0_{12} - F^0_{34}) \\ &+ \frac{1}{4k}{\rm Tr}[F_{mn}\tilde F^{mn}], \end{aligned} \tag{4.10}$$

where $s, t = 1, 2$ and $s', t' = 3, 4$. The first and the second terms correspond to the gauge covariantized extension of the lump charge densities in the (x^1, x^2) and (x^3, x^4) planes, respectively. The third and the fourth terms are vortex charge densities in the (x^1, x^2) and (x^3, x^4) planes, respectively, and the last term is the Yang-Mills instanton charge density. Note that, when the gauge field vanishes, the configuration corresponds to the intersecting topological vortex lumps in the (x^1, x^2) and (x^3, x^4) planes.

B. Noncanonical branch

Finally, we consider the noncanonical branch where the gauge group is $U(1)$. The $1/4$ BPS configurations in the two-dimensional subspaces are constructed by the same ways discussed in the Minkowski case. We now look for codimension-four BPS states. Since the solution of the

TABLE II. BPS states in the gauged higher derivative (HD) chiral model and super Yang-Mills with gauged nonlinear sigma model. Theories are defined in Euclidean space. Here L, V, I, VL, HDV, bS, and HDVbS stand for lumps, vortices, instantons, vortex lumps, higher derivative vortices, BPS baby Skyrmions, and higher derivative vortex-BPS baby Skyrmions, respectively. The subscript stands for subspaces that the soliton is defined.

	SYM + SUSY NLSM	Canonical	Noncanonical
L type	1/2 BPS L_{12}	1/2 BPS L_{12}	1/4 BPS bS_{12}
V type	1/2 BPS V_{12}	1/2 BPS V_{12}	1/4 BPS HDV_{12}
VL type	1/2 BPS VL_{12}	1/2 BPS VL_{12}	1/4 BPS $HDVbS_{12}$
V-V-I type	1/4 BPS V_{12}-V_{34}-I	1/4 BPS V_{12}-V_{34}-I	No
VL-VL-I type	1/4 BPS VL_{12}-VL_{34}-I	1/4 BPS VL_{12}-VL_{34}-I	No
L-L type	1/4 BPS L_{12}-L_{34}	1/4 BPS L_{12}-L_{34}	No

auxiliary field is not zero in the noncanonical branch, the 1/4 BPS projection (4.4) gives the BPS equations (4.5) and the additional condition $F^0 = 0$ (3.15). As in the case of the Minkowski space, the solutions to Eqs. (4.5) do not satisfy the condition (3.15) for general Λ. Therefore, the 1/4 BPS configurations associated with the projection (4.4) do not exist in the noncanonical branch. BPS states in Euclidean space are summarized in Table II.

V. SUMMARY AND DISCUSSION

In this paper, we have classified BPS states in $\mathcal{N} = 1$ supersymmetric gauge theories coupled with higher derivative chiral models in four Minkowski and Euclidean dimensions. We have found canonical and noncanonical branches corresponding to solutions $F = 0$ and $F \neq 0$ of auxiliary field equations, respectively. 1/2 BPS states in theories without higher derivative terms remain 1/2 BPS in the canonical branch, and the corresponding BPS states in the noncanonical branch are 1/4 BPS states. 1/4 BPS states in theories without higher derivative terms remain 1/4 BPS in the canonical branch, but there are no corresponding BPS states in the noncanonical branch. We have obtained 1/2 BPS equations for an ANO vortex, a non-Abelian vortex, a lump, and a vortex lump in the canonical branch and 1/4 BPS higher derivative generalization of the ANO vortices in the noncanonical branch. In four Euclidean dimensions, we have obtained the 1/4 BPS Yang-Mills instantons trapped inside a non-Abelian vortex, and 1/4 BPS intersecting vortices or vortex-lump intersections with instanton charges in the canonical branch and no codimension-four BPS states in the noncanonical branch.

While we have given the superfield Lagrangian of gauged multicomponent chiral models, we have been able to obtain on-shell Lagrangian only for the cases of a single component because of difficulty solving the equations of motion for the auxiliary fields for the multicomponent cases. Obtaining on-shell Lagrangians for gauged or nongauged multicomponent chiral models, in particular in the presence of an isometry large enough, remains a future problem. Our method will give a simple way to construct higher derivative nonlinear sigma models on Kähler manifolds by gauging chiral fields with flat target spaces for which auxiliary field equations of motions are easy to solve. In the strong gauge coupling limit, vector superfields do not have gauge kinetic terms becoming auxiliary superfields and can be eliminated by their equations of motion. This procedure is known as the Kähler quotients; see Ref. [49] for constructions of Hermitian symmetric spaces. Thus, it will be possible to construct higher derivative nonlinear sigma models on Hermitian symmetric spaces, as a generalization of the Faddeev-Skyrme $\mathbb{C}P^1$ model.

In this paper, we have not introduced superpotentials while we introduced them for nongauged chiral models in our previous paper [4]. In the presence of a superpotential, there are more varieties of BPS topological solitons such as domain walls [50] in $U(N)$ gauge theories [51], domain wall junctions [52,53] or networks [54], and vortices ending on or stretched between domain walls [55,56]. In these cases, the auxiliary field equation can be solved at most perturbatively even for a single component, as was so for nongauged chiral models [4].

We also comment that, in our gauged model, $\Lambda_{ik\bar{j}\bar{l}}$ does not contain space-time derivatives of the chiral superfields, unlike the nongauged cases for which it is possible as for the supersymmetric Dirac-Born-Infeld action in Eq. (2.7) and the supersymmetric Faddeev-Skyrme model in Eq. (2.8). A simple gauge covariant generalization of the form (2.7) or (2.8) does not provide supersymmetric interactions of the vector superfield. It is interesting to introduce the gauge covariant derivatives of Φ in a supersymmetric way in the Kähler tensor $\Lambda_{ik\bar{j}\bar{l}}$, in order to construct a gauged Dirac-Born-Infeld action [57] or a gauged Faddeev-Skyrme model.

In Ref. [58], 1/2, 1/4, and 1/8 BPS states were classified in $\mathcal{N} = 2$ supersymmetric field theories without higher derivative terms. Extension to $\mathcal{N} = 2$ supersymmetric field theories with higher derivative terms should be an interesting future problem. In particular, 1/4 BPS states in the canonical branch may have 1/8 BPS state counterparts in the noncanonical branch. While off-shell supersymmetry for eight supercharges is a hard task, because one needs harmonic superfield or projective superfield formalisms, partially off-shell supersymmetry that BPS solitons preserve can be used to construct an effective theory of BPS solitons [59].

Extension to supergravity is also interesting for application to cosmology such as the ghost condensations and the Galileon inflation models in supersymmetric theories along the line in Refs. [29–33].

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APPENDIX A: NOTATION AND CONVENTIONS

We use the convention in the textbook of Wess and Bagger [45]. The component expansion of the $\mathcal{N}=1$ chiral superfield in the x basis is

$$\Phi(x,\theta,\bar{\theta}) = \varphi + i\theta\sigma^m\bar{\theta}\partial_m\varphi + \frac{1}{4}\theta^2\bar{\theta}^2\Box\varphi + \theta^2 F, \tag{A1}$$

where only the bosonic components are presented. The supercovariant derivatives are defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m. \tag{A2}$$

The sigma matrices are $\sigma^m = (\mathbf{1}, \vec{\tau})$. Here $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$ are Pauli matrices. The bosonic components of the supercovariant derivatives of Φ^i are

$$\begin{aligned} D^\alpha\Phi^i D_\alpha\Phi^j = & -4\bar{\theta}^2\partial_m\varphi^i\partial^m\varphi^j + 4i(\theta\sigma^m\bar{\theta})(\partial_m\varphi^i F^j + F^i\partial_m\varphi^j) - 4\theta^2 F^i F^j \\ & + 2\theta^2\bar{\theta}^2(\Box\varphi^i F^j + F^i\Box\varphi^j - \partial_m\varphi^i\partial^m F^j - \partial_m F^i\partial^m\varphi^j), \end{aligned} \tag{A3}$$

$$\begin{aligned} \bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{i}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{j}} = & -4\theta^2\partial_m\bar{\varphi}^{\bar{i}}\partial^m\bar{\varphi}^{\bar{j}} - 4i(\theta\sigma^m\bar{\theta})(\partial_m\bar{\varphi}^{\bar{i}}\bar{F}^{\bar{j}} + \bar{F}^{\bar{i}}\partial_m\bar{\varphi}^{\bar{j}}) + 4\bar{\theta}^2\bar{F}^{\bar{i}}\bar{F}^{\bar{j}} \\ & + 2\theta^2\bar{\theta}^2(\bar{F}^{\bar{i}}\Box\bar{\varphi}^{\bar{j}} + \Box\bar{\varphi}^{\bar{i}}\bar{F}^{\bar{j}} - \partial_m\bar{\varphi}^{\bar{i}}\partial^m\bar{F}^{\bar{j}} - \partial_m\bar{F}^{\bar{i}}\partial^m\bar{\varphi}^{\bar{j}}), \end{aligned} \tag{A4}$$

$$D^\alpha\Phi^i D_\alpha\Phi^k\bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{j}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} = 16\theta^2\bar{\theta}^2\left[(\partial_m\varphi^i\partial^m\varphi^k)(\partial_m\bar{\varphi}^{\bar{j}}\partial^m\bar{\varphi}^{\bar{l}}) - \frac{1}{2}(\partial_m\varphi^i F^k + F^i\partial_m\varphi^k)(\partial^n\bar{\varphi}^{\bar{j}}\bar{F}^{\bar{l}} + \bar{F}^{\bar{j}}\partial^n\bar{\varphi}^{\bar{l}}) + F^i\bar{F}^{\bar{j}}F^k\bar{F}^{\bar{l}}\right]. \tag{A5}$$

When the supercovariant derivative is gauged, we obtain

$$\mathcal{D}_\alpha\Phi = 2i(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}D_m\varphi + 2\theta_\alpha F + 2\theta_\alpha\bar{\theta}^2(\Box\varphi + gD\varphi) - \frac{1}{2}(\sigma^m)_{\alpha\dot{\alpha}}(\bar{\sigma}^n)^{\dot{\alpha}}{}_{\dot{\beta}}\theta^\beta\bar{\theta}^2(\partial_m\partial_n\varphi - 2ig\partial_m A_n\varphi) + i\theta^2(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m F. \tag{A6}$$

Using this expression, we obtain Eq. (2.21).

APPENDIX B: SUPERSYMMETRY VARIATION OF FERMIONS

The explicit supersymmetry variation of the fermions in the Euclidean space is given by

$$\delta_\xi\psi^i_\alpha = \sqrt{2}i\begin{pmatrix} (\partial_4 + i\partial_3)\varphi^i\bar{\xi}^{\dot{1}} + i(\partial_1 - i\partial_2)\varphi^i\bar{\xi}^{\dot{2}} - i\xi_1 F^i \\ (\partial_4 - i\partial_3)\varphi^i\bar{\xi}^{\dot{2}} + i(\partial_1 + i\partial_2)\varphi^i\bar{\xi}^{\dot{1}} - i\xi_2 F^i \end{pmatrix}, \tag{B1}$$

$$\delta_\xi\bar{\psi}^{\dot{\alpha}i} = \sqrt{2}i\begin{pmatrix} (\partial_4 - i\partial_3)\bar{\varphi}^i\xi_1 - i(\partial_1 - i\partial_2)\bar{\varphi}^i\xi_2 - i\bar{\xi}^{\dot{1}}\bar{F}^i \\ (\partial_4 + i\partial_3)\bar{\varphi}^i\xi_2 - i(\partial_1 + i\partial_2)\bar{\varphi}^i\xi_1 - i\bar{\xi}^{\dot{2}}\bar{F}^i \end{pmatrix}, \tag{B2}$$

$$\delta_\xi\lambda_\alpha = \begin{pmatrix} i\xi_1 D + i\xi_1(F_{12} + F_{34}) - \xi_2(F_{13} - iF_{14} - iF_{23} - F_{24}) \\ i\xi_2 D - i\xi_2(F_{12} + F_{34}) + \xi_1(F_{13} + iF_{14} + iF_{23} - F_{24}) \end{pmatrix}, \tag{B3}$$

$$\delta_\xi\bar{\lambda}^{\dot{\alpha}} = \begin{pmatrix} -i\bar{\xi}^{\dot{1}}D - i\bar{\xi}^{\dot{1}}(F_{12} - F_{34}) + \bar{\xi}^{\dot{2}}(F_{13} + iF_{14} - iF_{23} + F_{24}) \\ -i\bar{\xi}^{\dot{2}}D + i\bar{\xi}^{\dot{2}}(F_{12} - F_{34}) - \bar{\xi}^{\dot{1}}(F_{13} - iF_{14} + iF_{23} + F_{24}) \end{pmatrix}. \tag{B4}$$

[1] H. Leutwyler, On the foundations of chiral perturbation theory, Ann. Phys. (N.Y.) **235**, 165 (1994).

[2] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory, Nucl. Phys. **B426**, 19 (1994); **430**, 485(E) (1994); Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD**B431**, 484 (1994).

[3] E. Witten and D. I. Olive, Supersymmetry algebras that include topological charges, Phys. Lett. B **78**, 97 (1978).

[4] M. Nitta and S. Sasaki, BPS states in supersymmetric chiral models with higher derivative terms, Phys. Rev. D **90**, 105001 (2014).

[5] D. Nemeschansky and R. Rohm, Anomaly constraints on supersymmetric effective lagrangians, Nucl. Phys. **B249**, 157 (1985).

[6] S. J. Gates, Jr., Why auxiliary fields matter: The strange case of the 4-D, $N = 1$ supersymmetric QCD effective action, Phys. Lett. B **365**, 132 (1996); Why auxiliary fields matter: The strange case of the 4-D, $N = 1$ supersymmetric QCD effective action. 2., Nucl. Phys. **B485**, 145 (1997).

[7] S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt, and S. Penati, The Superspace WZNW action for 4-D, $N = 1$ supersymmetric QCD, Phys. Lett. B **503**, 349 (2001); S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt, S. Penati, and H. Suzuki, Supersymmetric gauge anomaly with general homotopic paths, Nucl. Phys. **B596**, 315 (2001); S. J. Gates, Jr., M. T. Grisaru, and S. Penati, Holomorphy, minimal homotopy and the 4-D, $N = 1$ supersymmetric Bardeen-Gross-Jackiw anomaly, Phys. Lett. B **481**, 397 (2000).

[8] M. Nitta, A note on supersymmetric WZW term in four dimensions, Mod. Phys. Lett. A **15**, 2327 (2000).

[9] I. L. Buchbinder, S. Kuzenko, and Z. Yarevskaya, Supersymmetric effective potential: Superfield approach, Nucl. Phys. **B411**, 665 (1994).

[10] S. M. Kuzenko and S. J. Tyler, The one-loop effective potential of the Wess-Zumino model revisited, J. High Energy Phys. 09 (2014) 135.

[11] A. T. Banin, I. L. Buchbinder, and N. G. Pletnev, On quantum properties of the four-dimensional generic chiral superfield model, Phys. Rev. D **74**, 045010 (2006).

[12] I. Antoniadis, E. Dudas, and D. M. Ghilencea, Supersymmetric models with higher dimensional operators, J. High Energy Phys. 03 (2008) 045.

[13] I. L. Buchbinder, S. M. Kuzenko, and A. Yu. Petrov, Superfield chiral effective potential, Phys. Lett. **B321**, 372 (1994).

[14] M. Matone, Modular Invariance and Structure of the Exact Wilsonian Action of $N = 2$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **78**, 1412 (1997).

[15] D. Bellisai, F. Fucito, M. Matone, and G. Travaglini, Nonholomorphic terms in $N = 2$ supersymmetric Wilsonian actions and the renormalization group equation, Phys. Rev. **D6**, 5218 (1997).

[16] M. Gomes, J. R. Nascimento, A. Yu. Petrov, and A. J. da Silva, On the effective potential in higher-derivatives superfield theories, Phys. Lett. **B82**, 229 (2009).

[17] F. S. Gama, M. Gomes, J. R. Nascimento, A. Yu. Petrov, and A. J. da Silva, Higher-derivative supersymmetric gauge theory, Phys. Rev. **D4**, 045001 (2011).

[18] E. A. Bergshoeff, R. I. Nepomechie, and H. J. Schnitzer, Supersymmetric Skyrmions in four-dimensions, Nucl. Phys. **B249**, 93 (1985).

[19] L. Freyhult, The supersymmetric extension of the Faddeev model, Nucl. Phys. **B681**, 65 (2004).

[20] M. Rocek and A. A. Tseytlin, Partial breaking of global $D = 4$ supersymmetry, constrained superfields, and three-brane actions, Phys. Rev. D **59**, 106001 (1999).

[21] S. Sasaki, M. Yamaguchi, and D. Yokoyama, Supersymmetric DBI inflation, Phys. Lett. B **718**, 1 (2012).

[22] C. Adam, J. M. Queiruga, J. Sanchez-Guillen, and A. Wereszczynski, Supersymmetric K field theories and defect structures, Phys. Rev. D **84**, 065032 (2011).

[23] C. Adam, J. M. Queiruga, J. Sanchez-Guillen, and A. Wereszczynski, BPS bounds in supersymmetric extensions of K field theories, Phys. Rev. D **86**, 105009 (2012).

[24] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, and N. Sakai, Higher derivative corrections to non-Abelian vortex effective theory, Prog. Theor. Phys. **128**, 67 (2012).

[25] C. Adam, J. M. Queiruga, J. Sanchez-Guillen, and A. Wereszczynski, $N = 1$ supersymmetric extension of the baby Skyrme model, Phys. Rev. D **84**, 025008 (2011).

[26] C. Adam, J. M. Queiruga, J. Sanchez-Guillen, and A. Wereszczynski, Extended supersymmetry and BPS solutions in baby Skyrme models, J. High Energy Phys. 05 (2013) 108.

[27] S. Bolognesi and W. Zakrzewski, Baby Skyrme model, near-BPS approximations and supersymmetric extensions, Phys. Rev. D **91**, 045034 (2015).

[28] M. Nitta and S. Sasaki, Higher derivative corrections to manifestly supersymmetric nonlinear realizations, Phys. Rev. D **90**, 105002 (2014).

[29] J. Khoury, J.-L. Lehners, and B. Ovrut, Supersymmetric $P(X, \phi)$ and the ghost condensate, Phys. Rev. D **83**, 125031 (2011).

[30] M. Koehn, J.-L. Lehners, and B. Ovrut, Ghost condensate in $N = 1$ supergravity, Phys. Rev. D **87**, 065022 (2013).

[31] J. Khoury, J.-L. Lehners, and B. A. Ovrut, Supersymmetric Galileons, Phys. Rev. D **84**, 043521 (2011).

[32] M. Koehn, J.-L. Lehners, and B. A. Ovrut, Higher-derivative chiral superfield actions coupled to $N = 1$ supergravity, Phys. Rev. D **86**, 085019 (2012).

[33] F. Farakos and A. Kehagias, Emerging potentials in higher-derivative gauged chiral models coupled to $N = 1$ supergravity, J. High Energy Phys. 11 (2012) 077.

[34] C. Adam, P. Klimas, J. Sanchez-Guillen, and A. Wereszczynski, Compact baby Skyrmions, Phys. Rev. D **80**, 105013 (2009); C. Adam, T. Romanczukiewicz, J. Sanchez-Guillen, and A. Wereszczynski, Investigation of restricted baby Skyrme models, Phys. Rev. D **81**, 085007 (2010); C. Adam, J. Sanchez-Guillen, A. Wereszczynski, and W. J. Zakrzewski, Topological duality between vortices and planar Skyrmions in BPS theories with area-preserving diffeomorphism symmetries, Phys. Rev. D **87**, 027703 (2013); C. Adam, T. Romanczukiewicz, J. Sanchez-Guillen, and A. Wereszczynski, Magnetothermodynamics of BPS baby Skyrmions, J. High Energy Phys. 11 (2014) 095.

[35] C. Adam, C. Naya, J. Sanchez-Guillen, and A. Wereszczynski, The gauged BPS baby Skyrme model, Phys. Rev. D **86**, 045010 (2012); C. Adam, C. Naya,

T. Romanczukiewicz, J. Sanchez-Guillen, and A. Wereszczynski, Topological phase transitions in the gauged BPS baby Skyrme model, arXiv:1501.03817.

[36] A. A. Abrikosov, On the magnetic properties of superconductors of the second group, Zh. Eksp. Teor. Fiz. **32**, 1442 (1957) [Sov. Phys. JETP **5**, 1174 (1957)]; H. B. Nielsen and P. Olesen, Vortex line models for dual strings, Nucl. Phys. **B61**, 45 (1973).

[37] A. Hanany and D. Tong, Vortices, instantons and branes, J. High Energy Phys. 07 (2003) 037; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Non-Abelian superconductors: Vortices and confinement in $N = 2$ SQCD, Nucl. Phys. **B673**, 187 (2003); M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Moduli Space of Non-Abelian Vortices, Phys. Rev. Lett. **96**, 161601 (2006); M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi, Non-Abelian vortices of higher winding numbers, Phys. Rev. D **74**, 065021 (2006).

[38] A. M. Polyakov and A. A. Belavin, Metastable states of two-dimensional isotropic ferromagnets, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].

[39] B. J. Schroers, Bogomolny solitons in a gauged O(3) sigma model, Phys. Lett. B **356**, 291 (1995); The spectrum of Bogomol'nyi solitons in gauged linear sigma models, Nucl. Phys. **B475**, 440 (1996); J. M. Baptista, Vortex equations in Abelian gauged sigma-models, Commun. Math. Phys. **261**, 161 (2006); A. Alonso-Izquierdo, W. G. Fuertes, and J. M. Guilarte, Two species of vortices in massive gauged non-linear sigma models, J. High Energy Phys. 02 (2015) 139.

[40] M. Nitta and W. Vinci, Decomposing instantons in two dimensions, J. Phys. A **45**, 175401 (2012).

[41] A. Hanany and D. Tong, Vortex strings and four-dimensional gauge dynamics, J. High Energy Phys. 04 (2004) 066.

[42] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Instantons in the Higgs phase, Phys. Rev. D **72**, 025011 (2005).

[43] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Solitons in the Higgs phase: The moduli matrix approach, J. Phys. A **39**, R315 (2006).

[44] T. Fujimori, M. Nitta, K. Ohta, N. Sakai, and M. Yamazaki, Intersecting solitons, amoeba and tropical geometry, Phys. Rev. D **78**, 105004 (2008).

[45] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton University, Princeton, NJ, 1992).

[46] M. Ostrogradski, Memoires sur les equations differentielles relatives au probleme des isoperimetres, Mem. Ac. St. Petersbourg, 385 (1850).

[47] D. Brecher, BPS states of the non-Abelian Born-Infeld action, Phys. Lett. B **442**, 117 (1998).

[48] C. Adam, P. Klimas, J. Sanchez-Guillen, and A. Wereszczynski, Compact gauge K vortices, J. Phys. A **42**, 135401 (2009).

[49] K. Higashijima and M. Nitta, Supersymmetric nonlinear sigma models as gauge theories, Prog. Theor. Phys. **103**, 635 (2000).

[50] G. R. Dvali and M. A. Shifman, Domain walls in strongly coupled theories, Phys. Lett. B **396**, 64 (1997); **407**, 452(E) (1997).

[51] Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Construction of Non-Abelian Walls and Their Complete Moduli Space, Phys. Rev. Lett. **93**, 161601 (2004); Non-Abelian walls in supersymmetric gauge theories, Phys. Rev. D **70**, 125014 (2004); M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, D-brane construction for non-Abelian walls, Phys. Rev. D **71**, 125006 (2005); M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, and Y. Tachikawa, Global structure of moduli space for BPS walls, Phys. Rev. D **71**, 105009 (2005).

[52] G. W. Gibbons and P. K. Townsend, A Bogomolny Equation for Intersecting Domain Walls, Phys. Rev. Lett. **83**, 1727 (1999); S. M. Carroll, S. Hellerman, and M. Trodden, Domain wall junctions are 1/4—BPS states, Phys. Rev. D **61**, 065001 (2000); A. Gorsky and M. A. Shifman, More on the tensorial central charges in $N = 1$ supersymmetric gauge theories (BPS wall junctions and strings), Phys. Rev. D **61**, 085001 (2000).

[53] H. Oda, K. Ito, M. Naganuma, and N. Sakai, An exact solution of BPS domain wall junction, Phys. Lett. B **471**, 140 (1999); K. Ito, M. Naganuma, H. Oda, and N. Sakai, Nonnormalizable zero modes on BPS junctions, Nucl. Phys. **B586**, 231 (2000); M. Naganuma, M. Nitta, and N. Sakai, BPS walls and junctions in SUSY nonlinear sigma models, Phys. Rev. D **65**, 045016 (2002).

[54] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Webs of walls, Phys. Rev. D **72**, 085004 (2005); Non-Abelian webs of walls, Phys. Lett. B **632**, 384 (2006); M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, and N. Sakai, Dynamics of domain wall networks, Phys. Rev. D **76**, 125025 (2007).

[55] J. P. Gauntlett, R. Portugues, D. Tong, and P. K. Townsend, D-brane solitons in supersymmetric sigma models, Phys. Rev. D **63**, 085002 (2001); M. Shifman and A. Yung, Domain walls and flux tubes in $N = 2$ SQCD: D-brane prototypes, Phys. Rev. D **67**, 125007 (2003).

[56] Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, All exact solutions of a 1/4 Bogomol'nyi-Prasad-Sommerfield equation, Phys. Rev. D **71**, 065018 (2005); M. Eto, T. Fujimori, T. Nagashima, M. Nitta, K. Ohashi, and N. Sakai, Dynamics of strings between walls, Phys. Rev. D **79**, 045015 (2009).

[57] S. Sasaki, On non-linear action for gauged M2-brane, J. High Energy Phys. 02 (2010) 039.

[58] M. Eto, Y. Isozumi, M. Nitta, and K. Ohashi, 1/2, 1/4 and 1/8 BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations, Nucl. Phys. **B752**, 140 (2006).

[59] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, Manifestly supersymmetric effective Lagrangians on BPS solitons, Phys. Rev. D **73**, 125008 (2006).