Creation of spin 1/2 particles and renormalization in FLRW spacetime

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Within the framework of adiabatic regularization, we present a simple formalism to calculate number density and renormalized energy-momentum density of spin 1/2 particles in spatially flat FLRW spacetimes using an appropriate WKB ansatz for the adiabatic expansion for the field modes. The conformal and axial anomalies thus found are in exact agreement with those obtained from other renormalization methods. This formalism can be considered as an appropriate extension of the techniques originally introduced for scalar fields, applicable to fermions in curved space.

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I. INTRODUCTION

Quantum field theory in curved spacetime [1–5] has been developed as an approximate quantum theory of gravity in order to study particle creation by evolving universes [6–9] and black holes [10], as well as inhomogeneities in the cosmic microwave background radiation and the large-scale structure of the Universe [11]. Parker [6] conceptualized the so-called adiabatic vacuum in order to obtain a notion of particles in curved space that comes closest to the usual one in flat space. For scalar fields in spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes: (i) in a comoving volume, the particle number density ($|\beta_k|^2$) is an adiabatic invariant, (ii) particles of conformally invariant field with zero mass will not be created, (iii) the total number density of created particles of specific mass, summed over all modes, is ultraviolet (UV) divergent and (iv) the stress tensor ($T_{\mu\nu}$) of created particles has quadratic and logarithmic UV divergences in addition to the expected quartic divergence.

Various renormalization methods were developed to tame these infinities. The concept of adiabatic regularization was introduced by Parker [12] to make the total particle number density for scalar particles finite, and was later extended to tame the UV divergences in $T_{\mu\nu}$ by Parker and Fulling [13]. In adiabatic regularization, the physically relevant finite expression is obtained from the formal one containing UV divergences by subtracting mode by mode (under the integral sign) each term in the adiabatic expansion of the integrand that contains at least one UV divergent part for arbitrary values of the parameters of the theory. The number of time derivatives of the cosmological scale factor $a(t)$ that appear in a term of the expansion is called the adiabatic order of the term. The adiabatic regularization scheme is particularly useful for numerical calculations. In [13], the authors also showed that the adiabatic regularization is equivalent to the n-wave regularization (which is essentially a variant of Pauli-Villars regularization [5]) used by Zeldovich and Starobinsky [14] to renormalize the divergent $T_{\mu\nu}$ for scalar fields in an anisotropic universe. Like adiabatic regularization scheme, this method is also particularly suited for numerical computations [15,16].

Among other standard techniques, proper-time regularization, point-splitting regularization (particularly by the Hadamard method), zeta-function regularization and dimensional regularization have been applied to curved space [1–5]. The DeWitt-Schwinger point-splitting regularization [17–21] has been recently used in [22,23] to construct an approximate $T_{\mu\nu}$ of the quantized massive scalar, spinor and vector fields in the spatially flat FLRW universe using asymptotic expansion of the Green function constructed within the framework of the n-wave regularization [14,24] and reproduced the leading-order contribution to the stress tensor derived in [25]. All these methods are equivalent and lead to the same output [5]. Production of spin 1/2 particles in various cosmological scenarios have been studied by many [26–31]. Recently, fundamental issues like the problem of defining a preferred vacuum state at a given time have been addressed in [32,33]. Analysis of the approximate definition of the particle number via an adiabatic WKB ansatz can be found in [34]. A systematic adiabatic expansion for spin 1/2 modes has been recently constructed in [35–37] to analyze $|\beta_k|^2$ for fermions and the corresponding renormalization of $T_{\mu\nu}$ in a FLRW universe, and it is used to prove the equivalence between adiabatic regularization and point-splitting DeWitt-Schwinger renormalization [38]. It was argued in [35] that a WKB ansatz is specifically designed to preserve the Klein-Gordon product and the associated Wronskian condition, but not the Dirac product or the normalization condition.

In the following, we present a simple formalism to determine $|\beta_k|^2$ and also to regularize the resulting $T_{\mu\nu}$ for spin 1/2 particles in spatially flat FLRW universe. The essential difference between our formalism and that introduced in [35] is explicit in the corresponding expressions of so-called "out" states. In our case, the entire nonadiabaticity

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is incorporated in the Bogoliubov coefficients, whereas in [35], the Bogoliubov coefficients are defined to be of particular adiabatic order. Further, we have expressed quantum fields and other quantities as functions of conformal time, which is useful particularly in conformally flat spacetimes and leads us to simple structure of field equations. We define our "in" state as the adiabatic vacuum given by the WKB solution to the field equations and the "out" state as a mixture of positive and negative frequency in states via time-dependent Bogoliubov coefficients $\alpha_k(t)$ and $\beta_k(t)$. Next we use the field equations to derive the governing equations for $|\beta_k|^2$ in terms of a set of three *real* and independent variables s_k, u_k and τ_k[1] (to be defined later), which were introduced in [14] in the context of scalar particle creation during anisotropic collapse. It is then straightforward to regularize $T_{\mu\nu}$ by subtracting leading-order terms from the adiabatic mode expansions of these variables. The renormalized quantities thus derived match exactly with the known results found by other methods [1,5,35].

II. DIRAC FIELD IN FLRW SPACETIME

The homogeneous and isotropic FLRW spacetime geometry is given by

$$ds^2 = a^2(t)(-dt^2 + d\vec{x}^2), \tag{1}$$

where t is the conformal time and $a(t)$ is the conformal scale factor. The Dirac equation in generic curved spacetime for a field $\psi(\vec{x}, t)$ with mass m is given by [1–5],

$$(e^\mu_a \gamma^a \nabla_\mu - m)\psi = 0, \tag{2}$$

where e^a_μ are the vierbeins, γ^a's are standard Dirac matrices (defined in terms of usual Pauli matrices σ^i) in Minkowski space satisfying $\{\gamma^a, \gamma^b\} = \eta^{ab}$ and $\nabla_\mu = \partial_\mu - \Gamma_\mu$ is the covariant derivative, with Γ_μ being the spin connection [1,5]. The Dirac matrices (compatible with signature $-, +, +, +$) in the Dirac-Pauli representation is given by

$$\gamma^0 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3}$$

where σ^i are the usual Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

For metric (1), Eq. (2) leads to

$$\left[\gamma^0\left(\partial_0 + \frac{3\dot{a}}{2a}\right) + \gamma^i\partial_i + ma\right]\psi = 0. \tag{5}$$

The Dirac field ψ can be written in terms of a time-dependent annihilation operator for particles $[B_{\vec{k}\lambda}(t)]$ and a creation operator for antiparticles $[D^\dagger_{\vec{k}\lambda}(t)]$ as

$$\psi = \sum_\lambda \int d^3k (B_{\vec{k}\lambda} u_{\vec{k}\lambda} + D^\dagger_{\vec{k}\lambda} v_{\vec{k}\lambda}), \tag{6}$$

where momentum expansion of the eigenfunctions $u_{\vec{k}\lambda}(\vec{x}, t)$ and $v_{\vec{k}\lambda}(\vec{x}, t)$, which is obtained by a charge conjugation ($v = \gamma^2 u^*$) operation on $u_{\vec{k}\lambda}(\vec{x}, t)$, are given by, in terms of two-component spinors [35],

$$u_{\vec{k}\lambda}(\vec{x}, t) = \frac{e^{i\vec{k}.\vec{x}}}{(2\pi a)^{3/2}} \begin{pmatrix} h^I_k(t)\xi_\lambda(k) \\ h^{II}_k(t)\frac{\vec{\sigma}.\vec{k}}{k}\xi_\lambda(k) \end{pmatrix} \tag{7}$$

$$v_{\vec{k}\lambda}(\vec{x}, t) = \frac{e^{-i\vec{k}.\vec{x}}}{(2\pi a)^{3/2}} \begin{pmatrix} -h^{II*}_k(t)\frac{\vec{\sigma}.\vec{k}}{k}\xi_{-\lambda}(k) \\ -h^{I*}_k(t)\xi_{-\lambda}(k) \end{pmatrix} \tag{8}$$

where $\xi_\lambda(k)$ is the normalized two-component spinor satisfying $\xi^\dagger_\lambda \xi_\lambda = 1$ and $\frac{\vec{\sigma}.\vec{k}}{2k}\xi_\lambda = \lambda\xi_\lambda$ where $\lambda = \pm 1/2$ represents the helicity.[2] The normalization condition in terms of the Dirac product for ψ, $(u_{\vec{k}\lambda}, u_{\vec{k}'\lambda'}) = (v_{\vec{k}\lambda}, v_{\vec{k}'\lambda'}) = \delta_{\lambda\lambda'}\delta(\vec{k} - \vec{k}')$, implies

$$|h^I_k(t)|^2 + |h^{II}_k(t)|^2 = 1. \tag{9}$$

This condition guarantees the standard anticommutation relations for creation and annihilation operators. Putting Eq. (7) in Eq. (5) we get the following first-order coupled differential equations:

$$\dot{h}^I_k + imah^I_k + ikh^{II}_k = 0, \tag{10}$$

$$\dot{h}^{II}_k - imah^{II}_k + ikh^I_k = 0, \tag{11}$$

and the Wronskian is given by

$$\dot{h}^I_k h^{II*}_k - h^I_k \dot{h}^{II*}_k = -ik. \tag{12}$$

Here ($\dot{}$) means derivative with respect to the conformal time t. Equations (10) and (11) lead to the following decoupled second-order equations:

$$\ddot{h}^I_k + [\Omega^2_k(t) + iQ(t)]h^I_k = 0, \tag{13}$$

$$\ddot{h}^{II}_k + [\Omega^2_k(t) - iQ(t)]h^{II}_k = 0, \tag{14}$$

where $\Omega_k(t) = \sqrt{m^2a^2 + k^2}$ and $Q(t) = m\dot{a}$. Note that the following methodology is applicable to generic backgrounds where equations of structure similar to Eqs. (13) and (14)

[1]Similarly, in [35], one has to solve for three complex quantities, namely, ω, F and G.

[2]Note that using either value of helicity, or either $u_{\vec{k}\lambda}$ or $v_{\vec{k}\lambda}$, leads to exactly same end results in the following calculations.

appear. The "in" adiabatic vacuum (i.e. the state of adiabatic order zero) is given by the WKB solution (which naturally generalizes the standard Minkowski space solution) of the field equations (see Appendix A),

$$h_k^{I(0)}(t) = f_1 e_-, \qquad h_k^{II(0)}(t) = f_2 e_- \tag{15}$$

with $f_1 = \sqrt{\frac{\Omega_k + ma}{2\Omega_k}}$, $f_2 = \sqrt{\frac{\Omega_k - ma}{2\Omega_k}}$ and $e_\pm = \exp(\pm i \int \Omega_k dt)$. This implies that we shall seek the general solution corresponding to the out state in the form

$$h_k^I(t) = \alpha_k(t) h^{I(0)} - \beta_k(t) h^{II(0)*}, \tag{16}$$

$$h_k^{II}(t) = \alpha_k(t) h^{II(0)} + \beta_k(t) h^{I(0)*}, \tag{17}$$

where $\alpha_k(t)$ and $\beta_k(t)$ are the Bogoliubov coefficients. Equations (16) and (17) further imply

$$\alpha_k(t) = (f_1 h_k^I + f_2 h_k^{II}) e_+, \tag{18}$$

$$\beta_k(t) = (f_1 h_k^{II} - f_2 h_k^I) e_-. \tag{19}$$

The normalization condition (9) leads to

$$|\alpha_k(t)|^2 + |\beta_k(t)|^2 = 1. \tag{20}$$

Then the average number of spin $1/2$ particles of specific helicity and charge with momentum $\vec{k}$ created per unit volume is given by [1]

$$\langle N_{\vec{k}} \rangle = \langle B^\dagger_{\vec{k}\lambda} B_{\vec{k}\lambda} \rangle = \langle D^\dagger_{\vec{k}\lambda} D_{\vec{k}\lambda} \rangle = |\beta_k(t)|^2. \tag{21}$$

Note that, the WKB solutions (15) obey the following Wronskian condition:

$$\dot{h}_k^{I(0)} h_k^{II(0)*} - h_k^{I(0)} \dot{h}_k^{II(0)*} = F - ik, \qquad F = \frac{kQ}{2\Omega_k^2}. \tag{22}$$

The function $F(t)$ is of adiabatic order 1 and contains the factor $Q(t)$, which breaks the conformal invariance in the field equations. Thus, the Wronskian is satisfied in the adiabatic limit and $F(t)$ is a measure of nonadiabaticity of the cosmological evolution. Equation (22) also implies that the WKB ansatz is not an exact solution of the field equation during the nonadiabatic expansion, which is the desired condition for any particle creation [12]. In Eq. (6), the creation and annihilation operators carry this nonadiabaticity, and so do the Bogoliubov coefficients in Eqs. (16) and (17). Thus $|\beta_k|^2$ is expected to depend on $F(t)$, as particle creation can be considered as a result of this nonadiabaticity. Putting Eqs. (16) and (17) in Eqs. (10) and (11) and simplifying, a system of two linear first-order differential equations is obtained for $\alpha_k(t)$ and $\beta_k(t)$,

$$\dot{\alpha}_k = -F \beta_k e_+^2, \qquad \dot{\beta}_k = F \alpha_k e_-^2, \tag{23}$$

which were first derived by Parker [9] in exactly this particular form. It is obvious from Eq. (23) that creation of massless particles in conformally flat spacetimes is prohibited, and it also indicates that fermions at rest shall not be created. Similar results were found in [39], where the authors used Newman-Penrose formalism.

To determine $|\beta_k|^2$ and the resulting $T_{\mu\nu}$, let us define the following three *real* and independent variables [14] (for later convenience), in terms of the two complex variables α_k and β_k that are related by condition (20):

$$\begin{aligned} s_k &= |\beta_k|^2, \\ u_k &= \alpha_k \beta_k^* e_-^2 + \alpha_k^* \beta_k e_+^2, \\ \tau_k &= i(\alpha_k \beta_k^* e_-^2 - \alpha_k^* \beta_k e_+^2). \end{aligned} \tag{24}$$

For these variables one gets a system of three linear first-order differential equations,

$$\dot{s}_k = F u_k, \tag{25}$$

$$\dot{u}_k = 2F(1 - 2s_k) - 2\Omega_k \tau_k, \tag{26}$$

$$\dot{\tau}_k = 2\Omega_k u_k, \tag{27}$$

with initial conditions $s_k = u_k = \tau_k = 0$ at some suitably chosen $t = t_0$.

From Eqs. (25)–(27) one can further decouple the equation for $|\beta_k|^2$ or s_k given as

$$\dddot{s}_k + F_1 \ddot{s}_k + F_2 \dot{s}_k + F_3(1 - 2s_k) = 0, \tag{28}$$

where

$$F_1 = -\left(\frac{2\dot{F}}{F} + \frac{\dot{\Omega}_k}{\Omega_k}\right), \tag{29}$$

$$F_2 = \left(4F^2 + 4\Omega_k^2 + \frac{2\dot{F}^2}{F} + \frac{\dot{F}\dot{\Omega}_k}{F\Omega_k} - \frac{\ddot{F}}{F}\right), \tag{30}$$

$$F_3 = -2F^2\left(\frac{\dot{F}}{F} - \frac{\dot{\Omega}_k}{\Omega_k}\right). \tag{31}$$

Note that once h_k^I and h_k^{II} are derived (analytically or numerically) from the field equations, one can find $|\beta_k|^2$ directly from Eq. (19). Alternatively, when such a closed form solution to the field equations is not available, one can solve the set of equations (25)–(27) numerically (or analytically whenever possible). Below we discuss how one can find the renormalized stress tensor by solving Eqs. (25)–(27).

A. Energy-momentum tensor

The energy-momentum tensor for the Dirac field in curved spacetime is given by

$$T_{\mu\nu} = \frac{i}{2}[\bar{\psi}\gamma_{(\mu}\nabla_{\nu)}\psi - (\nabla_{(\mu}\bar{\psi})\gamma_{\nu)}\psi]. \quad (32)$$

The independent components of $T_{\mu\nu}$ are given by

$$T^0_0 = -\frac{i}{2a}(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi), \quad (33)$$

$$T^i_i = \frac{i}{2a}(\bar{\psi}\gamma^i\psi' - \bar{\psi}'\gamma^i\psi), \quad (34)$$

where ($'$) denotes derivative with respect to x^i. The vacuum expectation value of the above quantities leads to

$$\langle T^0_0\rangle = \frac{1}{(2\pi a)^3}\int d^3k\rho_k, \quad (35)$$

$$\langle T^i_i\rangle = \frac{1}{(2\pi a)^3}\int d^3k p_k, \quad (36)$$

with energy density ρ_k and pressure density p_k are given, respectively, as

$$\rho_k = \frac{i}{a}(h^I_k\dot{h}^{I*}_k + h^{II}_k\dot{h}^{II*}_k - h^{I*}_k\dot{h}^I_k - h^{II*}_k\dot{h}^{II}_k), \quad (37)$$

$$p_k = \frac{2k}{3a}(h^I_k h^{II*}_k + h^{I*}_k h^{II}_k). \quad (38)$$

Using Eqs. (16), (17), (20) and (24), we get from Eqs. (37) and (38)

$$\rho_k = -\frac{2\Omega_k}{a}(1-2s_k), \quad (39)$$

$$p_k = \frac{2k}{3a}\left[\frac{k}{\Omega_k}(1-2s_k) + \frac{ma}{\Omega_k}u_k\right]. \quad (40)$$

The vacuum energy (when $s_k = u_k = \tau_k = 0$) matches with the standard result. We discuss below how to remove the divergences in $T_{\mu\nu}$ by subtracting the leading-order terms from the adiabatic expansion of s_k, u_k and τ_k.

B. Renormalization

Let us consider the case of large momenta ($\Omega_k \to \infty$) and expand the solutions of the system of Eqs. (25)–(27) in an asymptotic series in powers of Ω_k^{-1}. This is essentially the same as the adiabatic expansion that is valid in the quasiclassical region where $|\dot{\Omega}_k| \ll \Omega_k^2$. It is straightforward to see that $\tau_k = \tau_k^{(1)} + \tau_k^{(3)} + \ldots$, $u_k = u_k^{(2)} + u_k^{(4)} + \ldots$ and $s_k = s_k^{(2)} + s_k^{(4)} + \ldots$, where the superscripts inside the brackets indicate the adiabatic order (Appendix B). Equations (25)–(27) lead to the following recursion relations:

$$u_k^{(r)} = \frac{\dot{\tau}_k^{(r-1)}}{2\Omega_k}, \quad (41)$$

$$s_k^{(r)} = \int F u_k^{(r)} dt, \quad (42)$$

$$\tau_k^{(r+1)} = -\frac{4Fs_k^{(r)} + \dot{u}_k^{(r)}}{2\Omega_k}, \quad (43)$$

with $r = 2, 4, \ldots$ and $\tau_k^{(1)} = \frac{F}{\Omega_k}$. It is straightforward to solve these equations analytically to arbitrary order. Further, as $k \to \infty$, we have

$$s_k^{(r)} \sim k^{-(r+2)}, \quad u_k^{(r)} \sim k^{-(r+1)}, \quad \tau_k^{(r)} \sim k^{-(r+1)}. \quad (44)$$

This implies the well-known logarithmic UV divergences of the total energy and pressure density. Note that no quadratic divergence appears for fermions as it does for scalar fields [1]. To remove these infinities, we need to subtract leading terms up to second order from the expansion of s_k and u_k. This prescription is thus equivalent to adiabatic regularization. The total particle number density of a specific mass with summed-over momenta is simply given as

$$N_m = \frac{1}{(2\pi a)^3}\int d^3k s_k. \quad (45)$$

The renormalized total energy and momentum density (after subtracting the vacuum contribution i.e. the quartic divergence) are given by

$$\langle T^0_0\rangle_{\text{ren}} = \frac{2}{\pi^2 a^4}\int dk k^2\Omega_k(s_k - s_k^{(2)}), \quad (46)$$

$$\langle T^i_i\rangle_{\text{ren}} = \frac{1}{3\pi^2 a^4}\int dk\frac{k^3}{\Omega_k}\Big[-2k(s_k - s_k^{(2)}) + ma(u_k - u_k^{(2)})\Big]. \quad (47)$$

Note that in more generic spacetimes the fourth-order adiabatic terms may give rise to proper UV divergences [21] and the renormalized quantities become

$$\langle T^0_0\rangle_{\text{ren}} = \frac{2}{\pi^2 a^4}\int dk k^2\Omega_k(s_k - s_k^{(2)} - s_k^{(4)}), \quad (48)$$

$$\langle T^i_i\rangle_{\text{ren}} = \frac{1}{3\pi^2 a^4}\int dk\frac{k^3}{\Omega_k}[-2k(s_k - s_k^{(2)} - s_k^{(4)}) + ma(u_k - u_k^{(2)} - u_k^{(4)})]. \quad (49)$$

According to the standard approach of regularization one considers the fourth-order adiabatic terms as *potentially* divergent [1,5]; to compute the trace anomaly, Eqs. (48) and (49) are used instead of Eqs. (46) and (47).

C. Conformal and axial anomalies

The trace of the energy-momentum tensor (32) is $T^\mu_\mu = m\bar{\psi}\psi$. Thus, the trace vanishes for massless fields. However, the renormalization procedure renders the quantum counterpart of T^μ_μ finite. This phenomenon is known as the conformal anomaly. The vacuum expectation value of the trace of stress tensor is given by

$$\langle T^\mu_\mu \rangle = \frac{1}{(2\pi a)^3} \int d^3k \langle T^\mu_\mu \rangle_k \tag{50}$$

with

$$\langle T^\mu_\mu \rangle_k = -2m(|h^I_k|^2 - |h^{II}_k|^2) \tag{51}$$

$$= -2m\left[\frac{ma}{\Omega_k}(1-2s_k) - \frac{k}{\Omega_k}u_k\right], \tag{52}$$

where we have again used Eqs. (16), (17), (20) and (24). One can also derive Eq. (52) using the identity $\langle T^\mu_\mu \rangle_k = \rho_k + 3p_k$. In the limit $m \to 0$, it is enough to subtract terms up to the second order (i.e. $s_k^{(2)}$ and $u_k^{(2)}$) to remove the UV divergence from $\langle T^\mu_\mu \rangle$. After subtracting the vacuum contribution, the resulting renormalized trace anomaly is given by

$$\langle T^\mu_\mu \rangle_{\mathrm{ren}} = \lim_{m\to 0} \frac{2m}{(2\pi a)^3} \int d^3k \left[\frac{2ma}{\Omega_k} s_k^{(4)} + \frac{k}{\Omega_k} u_k^{(4)}\right], \tag{53}$$

as only the fourth-order term in the expansions of $s_k(t)$ and $u_k(t)$ survives in the $m \to 0$ limit and in fact is independent of m. Using explicit expressions of $s_k^{(4)}$ and $u_k^{(4)}$ in Eq. (53), we get

$$\langle T^\mu_\mu \rangle_{\mathrm{ren}} = \frac{11\dot{a}^4 - 29a\dot{a}^2\ddot{a} + 12a^2\dot{a}\dddot{a} + 9a^2\ddot{a}^2 - 3a^3\ddddot{a}}{240\pi^2 a^8}. \tag{54}$$

To cross-check the above expression, note that the conformal anomaly can be expressed in terms of the curvature invariants by the following generic expression [1,5]:

$$\langle T^\mu_\mu \rangle_{\mathrm{ren}} = \frac{1}{(4\pi)^2}(AC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + BG + C\Box R), \tag{55}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, R is the Ricci scalar and G is the Gauss-Bonnet invariant, given by $G = -2(R_{\alpha\beta}R^{\alpha\beta} - R^2/3)$ with $R_{\alpha\beta}$ as the Ricci tensor. For conformally flat spacetimes (Appendix C), the Weyl tensor vanishes identically. Equating Eq. (55) with Eq. (54), we get $B = -11/360$ and $C = 1/30$, which agrees with the known results [1,5]. This proves the viability of the methodology presented here.

The classical axial current ($J^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ where $\gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^4$) is conserved for a massless Dirac field. The quantum counterpart of the divergence of the axial current is given by

$$\langle \nabla_\mu j^\mu \rangle = 2im\langle \bar{\psi}\gamma^5\psi \rangle \tag{56}$$

$$= -\frac{4im}{(2\pi a)^3} \int d^3k (h^{I*}_k h^{II}_k - h^I_k h^{II*}_k) \tag{57}$$

$$= \frac{4m}{(2\pi a)^3} \int d^3k \tau_k. \tag{58}$$

This implies that the renormalized axial anomaly is given by

$$\langle \nabla_\mu j^\mu \rangle_{\mathrm{ren}} = \lim_{m\to 0} \frac{4m}{(2\pi a)^3} \int d^3k (\tau_k - \tau_k^{(1)}). \tag{59}$$

In $m \to 0$, none of the terms in the right-hand side of Eq. (59) survive and the resulting axial anomaly vanishes as expected [1].

III. SUMMARY

We have constructed a simple formalism, within the framework developed in [6,13,14], to compute number density and renormalized energy-momentum density of spin 1/2 particles created during the evolution of spatially flat FLRW universes. We introduced an appropriate WKB ansatz that satisfies the normalization condition and the Wronskian condition up to the desired adiabatic order. The role of nonadiabaticity is crucial in defining the out vacuum. Here the Bogoliubov coefficients carry all the adiabatic orders (so to speak), unlike [35], where the Bogoliubov coefficients are defined to be of some particular adiabatic order. We have expressed the physical quantities as simple linear combinations of three real and independent variables s_k, u_k and τ_k, which are defined in terms of the usual Bogoliubov coefficients. The role of these variables is a distinguishing feature of the algorithm presented here and makes the process of renormalization simple. The evolution of these three variables is governed by three linear first-order coupled differential equations. It is easy to solve these equations with appropriate boundary conditions. Further, using adiabatic approximation, one can find the adiabatic expansion of these variables in powers of momenta. Subtracting up to necessary leading-order terms from the expansion of $\langle T_{\mu\nu} \rangle$, renormalization is achieved in the usual manner. The conformal and axial anomalies thus found are in exact agreement with those obtained from other renormalization methods that involve tedious calculations. To carry out all the steps, one need not solve the field equations analytically for the out vacuum, and the whole process is also suitable for numerical calculations. This work gives us a simple alternative to [35] as well as an appropriate extension and unification of standard techniques (within the framework of adiabatic regularization), originally introduced for scalar fields, that are applicable to fermions in curved space. Application of this formalism to interesting cosmological scenarios and the corresponding results will be reported elsewhere.

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APPENDIX A: WKB SOLUTION

To find the WKB solution to

$$\ddot{h}^I_k + [\Omega^2_k(t) + iQ(t)]h^I_k = 0, \tag{A1}$$

$$\ddot{h}^{II}_k + [\Omega^2_k(t) - iQ(t)]h^{II}_k = 0, \tag{A2}$$

let us assume

$$h^I_k(t) \sim \exp\left[\int (X(t) + iY(t))dt\right] \tag{A3}$$

where

$$X(t) = \frac{1}{\hbar}\sum_{n=0}^{\infty} \hbar^n X_n(t), \qquad Y(t) = \frac{1}{\hbar}\sum_{n=0}^{\infty} \hbar^n Y_n(t). \tag{A4}$$

Putting Eq. (A3) in Eq. (A1) and equating the terms of zeroth order in n, we get

$$X_0^2 - Y_0^2 + \Omega_k^2 = 0, \tag{A5}$$

$$2X_0Y_0 + Q = 0. \tag{A6}$$

Similarly, solving for the first order in n leads to

$$\dot{X}_0 + 2X_0X_1 - 2Y_0Y_1 = 0, \tag{A7}$$

$$\dot{Y}_0 + 2X_0Y_1 - 2Y_0X_1 = 0. \tag{A8}$$

Higher-order terms can be neglected in the adiabatic approximation. Solving Eqs. (A5) and (A6) we get

$$X_0 \approx \frac{Q}{2\Omega_k}, \qquad Y_0 \approx \Omega_k. \tag{A9}$$

Similarly, Eqs. (A7) and (A8) give

$$X_1 \approx -\frac{\dot{\Omega}_k}{2\Omega_k}, \qquad Y_1 \approx 0. \tag{A10}$$

This leads to

$$h^I_k(t) \sim \sqrt{\frac{\Omega_k + ma}{2\Omega_k}}\exp\left[i\int \Omega_k dt\right]. \tag{A11}$$

One can solve Eq. (A2) similarly. Note that the approximations made above are valid in the adiabatic limit; therefore, Eq. (15) represents the adiabatic vacuum.

APPENDIX B: s_k, u_k AND τ_k OF DIFFERENT ADIABATIC ORDERS

Terms in the adiabatic expansion of s_k, u_k and τ_k can be derived solving Eqs. (25)–(27) in the following way. For $k \to \infty$, s_k, u_k, τ_k and their temporal variations must tend to zero. Therefore, Eq. (26) for large k implies

$$0 \sim 2F - 2\Omega_k\tau_k, \tag{B1}$$

which further implies that the leading term in the adiabatic expansion of τ_k is of adiabatic order 1, i.e.

$$\tau_k^{(1)} \sim \frac{F}{\Omega_k} = \frac{mk\dot{a}}{2\Omega_k^3}. \tag{B2}$$

Putting Eq. (B2) in Eq. (27) we get the leading term in the adiabatic expansion of u_k, which is of order 2,

$$u_k^{(2)} \sim \frac{\dot{\tau}_k^{(1)}}{2\Omega_k} = -\frac{3m^3ka\dot{a}^2}{4\Omega_k^6} + \frac{mk\ddot{a}}{4\Omega_k^4}. \tag{B3}$$

Similarly, by putting Eq. (B3) in Eq. (25) we get the leading term in the adiabatic expansion of s_k, which is again of order 2,

$$s_k^{(2)} \sim \int Fu_k^{(2)}dt = \frac{m^2k^2\dot{a}^2}{16\Omega_k^6}. \tag{B4}$$

Now putting Eq. (B4) again back in Eq. (26) we get the next-to-leading term in the adiabatic expansion of τ_k, which is of adiabatic order 3. This iteration leads to Eqs. (41)–(43) that give e.g.

$$\tau_k^{(3)} = \frac{5m^3k^3\dot{a}^3}{16\Omega_k^9} - \frac{15m^5ka^2\dot{a}^3}{8\Omega_k^9} + \frac{5m^3k\dot{a}\,\ddot{a}}{4\Omega_k^7} - \frac{mk\dddot{a}}{8\Omega_k^5}, \tag{B5}$$

$$
\begin{aligned}
u_k^{(4)} = {} & \frac{35m^3k^5\dot{a}^2\ddot{a}}{32\Omega_k^{12}} + \frac{5m^7ka^5\ddot{a}^2}{8\Omega_k^{12}} - \frac{15m^7ka^5\dot{a}\dddot{a}}{16\Omega_k^{12}} + \frac{5m^3k^5a\ddot{a}^2}{8\Omega_k^{12}} \\
& - \frac{105m^5k^3a\dot{a}^4}{32\Omega_k^{12}} + \frac{15m^3k^5a\dot{a}\dddot{a}}{16\Omega_k^{12}} + \frac{105m^7ka^3\dot{a}^4}{16\Omega_k^{12}} \\
& + \frac{5m^5k^3a^3\ddot{a}^2}{4\Omega_k^{12}} + \frac{15m^5k^3a^3\dot{a}\dddot{a}}{8\Omega_k^{12}} - \frac{mk^7\ddddot{a}}{16\Omega_k^{12}} \\
& - \frac{175m^5k^3a^2\dot{a}^2\ddot{a}}{32\Omega_k^{12}} - \frac{3m^3k^3a^2\ddddot{a}}{16\Omega_k^{10}} - \frac{105m^7k\dot{a}^2\ddot{a}}{16\Omega_k^{12}},
\end{aligned}
\tag{B6}
$$

$$
\begin{aligned}
s_k^{(4)} = {} & \frac{m^4k^4\dot{a}^4}{16\Omega_k^{12}} - \frac{m^6k^2a^2\dot{a}^4}{4\Omega_k^{12}} + \frac{7m^4k^2a\dot{a}^2\ddot{a}}{32\Omega_k^{10}} \\
& + \frac{m^2k^2\ddot{a}^2}{64\Omega_k^{8}} - \frac{m^2k^2\dot{a}\dddot{a}}{32\Omega_k^{8}}.
\end{aligned}
\tag{B7}
$$

Higher-order terms can be derived in similar way. The *Mathematica* file containing these results is available on correspondence. This particular methodology, introduced in [14] for scalars, has not been extended to deal with fermions as such.

APPENDIX C: USEFUL CURVATURE QUANTITIES

The following are a few useful formulas for FLRW geometry:

$$R = 6\frac{\ddot{a}}{a}, \tag{C1}$$

$$\Box R = 6\left(\frac{3\ddot{a}^2}{a^6} - \frac{6\dot{a}^2\ddot{a}}{a^7} + \frac{4\dot{a}\dddot{a}}{a^6} - \frac{\dddot{a}}{a^5}\right), \tag{C2}$$

$$R_{\mu\nu}R^{\mu\nu} = 12\left(\frac{\dot{a}^4}{a^8} - \frac{\dot{a}^2\ddot{a}}{a^7} + \frac{\ddot{a}^2}{a^6}\right). \tag{C3}$$

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