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3D gravity with dust: Classical and quantum theory

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We study the Einstein gravity and dust system in three spacetime dimensions as an example of a nonperturbative quantum gravity model with local degrees of freedom. We derive the Hamiltonian theory in the dust time gauge and show that it has a rich class of exact solutions. These include the Bañados–Teitelboim–Zanelli black hole, static solutions with naked singularities, and traveling wave solutions with dynamical horizons. We give a complete quantization of the wave sector of the theory, including a definition of a self-adjoint spacetime metric operator. This operator is used to demonstrate the quantization of deficit angle and the fluctuation of dynamical horizons.

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I. INTRODUCTION

The difficulty in formulating a four-dimensional theory of quantum gravity has led to the study of many simpler models. These include symmetry reductions of four-dimensional general relativity [1] and dimensional reduction to 3D gravity [2–7]. There is a large volume of literature on the latter, which includes pure gravity with point defects and/or topological degrees of freedom [8–11], topologically massive gravity [12], and higher spin gravity. While some of these (lower-dimensional) simplifications have allowed for covariant quantization, there is relatively little work on the canonical quantization of any gravity-matter model.

Our purpose in this paper is to develop a 3D model of gravity with matter which has the potential for complete quantization. This would reveal insights into quantum gravity in a setting with local degrees of freedom. The pressureless dust matter we use is perhaps the simplest such model, but it is sufficiently nontrivial in that there is a rich class of classical solutions, including ones with dynamical horizons. Such solutions are of much interest at the quantum level; questions such as what is the quantum analog of a classical dynamical horizon remain unanswered and are key to understanding what is a "quantum black hole" or a "quantum trapped surface" [13].

With these issues in mind, we begin by formulating a canonical theory of 3D gravity coupled to pressureless dust. This is a special case of the Brown–Kuchar [14] model which is designed to provide a dynamical matter reference system for gravity in $3+1$ dimensions. It was used to give a physical Hilbert space setting for loop quantum gravity in the dust time gauge [15–17] and added as an additional world sheet field in the bosonic string to yield a curious extension of that theory [18].

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We will see that in the $2+1$ model the dust time gauge gives a physical Hamiltonian that describes the dynamics of one local geometry degree of freedom; this remains in the circularly symmetric setting we consider in detail. The model also provides an example of the transfer of a matter degree of freedom to a geometric one; this may provide a useful viewpoint for quantum gravity in a more general setting, distinct from the strict conventional separation of matter and geometry degrees of freedom. In Sec. II we develop the circular-symmetry-reduced theory, and in Sec. III we give the gauge fixed theory. In Sec. IV we give several types of classical solutions, followed by the construction of a quantum theory of the system in Sec. V, with focus on the traveling wave solutions. The concluding section is a summary of our results and a discussion of further questions.

II. HAMILTONIAN THEORY

In units where $8G=c=1$, the action for gravity and dust is a sum of the two components

$$S=S_G+S_D. \tag{1}$$

Let us consider this action defined on a three-dimensional manifold with topology $\Sigma\times\mathbb{R}$. The gravitational part of the action is

$$S_G=\frac{1}{2\pi}\int dx^3\sqrt{g}({}^{(3)}R-2\Lambda), \tag{2}$$

where ${}^{(3)}R$ is the scalar curvature of spacetime and Λ is the cosmological constant. The dust action is

$$S_D=-\frac{1}{4\pi}\int dx^3\sqrt{g}m(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+1), \tag{3}$$

where $m(x)$ is a function of the spacetime coordinates.

To derive the Hamiltonian formulation, we use the Arnowitt–Deser–Misner (ADM) parametrization of the line element

$$ds^2 = -N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \quad (4)$$

where q_{ab} is the space metric, N is the lapse function, and N^a is the shift vector. With this the gravitational part of the action takes the well-known canonical form (see e.g. Ref. [6])

$$S_G = \frac{1}{2\pi}\int dx^3(\tilde{\pi}^{ab}\dot{q}_{ab} - N\mathcal{H}^G - N^a\mathcal{C}^G_a), \quad (5)$$

where $\tilde{\pi}^{ab}$ is the (density weight 1) momentum conjugate to q_{ab}. N and N^a appear as the Lagrange multipliers corresponding respectively to the Hamiltonian and diffeomorphism constraints

$$\mathcal{H}^G = \sqrt{q}\left(-{}^{(2)}R + \frac{1}{q}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \tilde{\pi}^2) + 2\Lambda\right), \quad (6)$$

$$\mathcal{C}^G_a = -2\nabla_a \tilde{\pi}^a_b, \quad (7)$$

where $q \equiv \det q_{ab}$, $\tilde{\pi} \equiv \tilde{\pi}^a_a$, and ${}^{(2)}R$ is the Ricci scalar of the spatial hypersurface.

The canonical dust action is obtained starting with the momentum

$$P_\phi := \frac{\delta S_D}{\delta\dot{\phi}} = \frac{\sqrt{q}m}{N}(\dot{\phi} - N^a\partial_a\phi), \quad (8)$$

which leads to

$$S_D = \frac{1}{2\pi}\int dx^3(P_\phi\dot{\phi} - N\mathcal{H}^D - N^a\mathcal{C}^D_a), \quad (9)$$

$$\mathcal{H}^D = \frac{1}{2}\left(\frac{P^2_\phi}{m\sqrt{q}} + m\sqrt{q}(q^{ab}\partial_a\phi\partial_b\phi + 1)\right), \quad (10)$$

$$\mathcal{C}^D_a = P_\phi\partial_a\phi. \quad (11)$$

The variation of m in the dust action gives the equation of motion

$$m = \pm\frac{P_\phi}{\sqrt{q(q^{ab}\partial_a\phi\partial_b\phi + 1)}}. \quad (12)$$

Using this we eliminate m from the Hamiltonian by writing the dust part of the scalar constraint as

$$\mathcal{H}^D = \pm P_\phi\sqrt{q^{ab}\partial_a\phi\partial_b\phi + 1}. \quad (13)$$

The sign ambiguity will be determined below when we fix a time gauge.

A. Imposing circular symmetry

Let us now impose circular symmetry. A parametrization of the ADM phase space variables $(q_{ab}, \tilde{\pi}^{ab})$ for circular symmetry may be prescribed by using the flat 2D metric $e_{ab}dx^a dx^b = dr^2 + r^2 d\theta^2$ and the radial vector $s^a = [1, 0]$ and $s_a = s^b e_{ab} = [1, 0]$. In these coordinates

$$q_{ab}(t,r) = \Omega^2(t,r))s_a s_b + \frac{\rho^2(t,r)}{r^2}(e_{ab}(r) - s_a s_b) \quad (14)$$

$$\tilde{\pi}^{ab}(t,r) = \frac{P_\Omega(t,r)}{2\Omega(t,r)}s^a s^b + \frac{r^2 P_\rho(t,r)}{2\rho(t,r)}(e^{ab}(r) - s^a s^b). \quad (15)$$

With these definitions the symplectic term in the gravitational action is

$$\frac{1}{2\pi}\int drd\theta dt\tilde{\pi}^{ab}\dot{q}_{ab} = \int drdt(P_\rho\dot{\rho} + P_\Omega\dot{\Omega}). \quad (16)$$

The ADM metric becomes

$$ds^2 = -(N^2 - (\Omega N^r)^2)dt^2 + 2\Omega^2 N^r drdt + \Omega^2 dr^2 + \rho^2 d\theta^2, \quad (17)$$

and the Ricci scalar on the slice is

$${}^{(2)}R = -\frac{2}{\Omega\rho}\left(\frac{\rho'}{\Omega}\right)'. \quad (18)$$

Adding the gravitational and dust parts, with $\sqrt{q} = |\Omega\rho|$, gives the symmetry-reduced action

$$S = \int drdt(P_\rho\dot{\rho} + P_\Omega\dot{\Omega} + P_\phi\dot{\phi} - N\mathcal{H} - N^r\mathcal{C}_r), \quad (19)$$

$$\mathcal{H} = \mathrm{sgn}(\Omega\rho)\left(2\left(\frac{\rho'}{\Omega}\right)' - \frac{1}{2}P_\Omega P_\rho\right) + 2\Lambda|\Omega\rho| \pm P_\phi\sqrt{\left(\frac{\phi'}{\Omega}\right)^2 + 1}, \quad (20)$$

$$\mathcal{C}_r = \rho' P_\rho - \Omega P'_\Omega + P_\phi\phi', \quad (21)$$

where we have used "primes" to denote derivatives with respect to the radial coordinate. As one would expect, the angular component of the diffeomorphism constraint is identically zero ($\mathcal{C}_\phi \equiv 0$) so that only radial diffeomorphisms play a role in the symmetry-reduced theory.

The Poisson algebra of the constraints is first class

$$\{H(N), H(M)\} = C_r(NM' - MN'), \quad (22)$$

$$\{C_r(N^r), C_r(M^r)\} = C_r(N^r(M^r)' - M^r(N^r)'), \qquad (23)$$

$$\{H(N), C_r(N^r)\} = -H(N'N^r), \qquad (24)$$

being the reduced version of the Dirac/ADM algebra. At this point, with gauge freedom remaining, there are three pairs of conjugate variables parametrizing the six-dimensional phase space. The physical theory obtained by a Dirac gauge reduction, which fixes the constraints and removes the gauge ambiguity, will leave only one pair of conjugate variables in the two-dimensional physical phase space. In the following we consider the case of noncompact spatial slices with full gauge fixing and appropriate boundary terms to obtain a well-defined variational principle for the canonical $2+1$ action.

III. GAUGE FIXING AND PHYSICAL HAMILTONIAN

In this section our goal is to obtain the Hamiltonian theory of the local physical degrees of freedom by fixing gauges and solving the constraints.

We first fix the radial coordinate gauge by imposing $\chi_\rho \coloneqq \rho - r \approx 0$. This is a standard choice in spherical symmetry and is second class with the diffeomorphsim constraint

$$\{\chi_\rho, C_r(N^r)\} = N^r. \qquad (25)$$

Keeping this constraint preserved dynamically gives a relation between the lapse and shift functions,

$$N^r - \mathrm{sgn}(\Omega)\frac{NP_\Omega}{2} = 0. \qquad (26)$$

Solving the diffeomorphism constraint and imposing this gauge condition removes ρ and P_ρ from the system. We have

$$P_\rho = \Omega P'_\Omega - P_\phi \phi', \qquad (27)$$

which leads to the partially gauge-fixed action

$$S = \int drdt(P_\Omega \dot{\Omega} + P_\phi \dot{\phi} - N\mathcal{H}), \qquad (28)$$

$$\mathcal{H} = \frac{1}{2}\mathrm{sgn}(\Omega)P_\Omega(P_\phi\phi' - \Omega P'_\Omega) + 2\Lambda|\Omega|r + \left(\frac{2}{|\Omega|}\right)' \pm P_\phi\sqrt{\left(\frac{\phi'}{\Omega}\right)^2 + 1}. \qquad (29)$$

In writing the Hamiltonian, we assume that $\mathrm{sgn}(\Omega)$ is fixed since we must have $|\Omega(t,r)| > 0$ for the metric be nondegenerate.

We now choose the dust time gauge by adding the constraint $\chi_\phi \coloneqq \phi - t \approx 0$, a condition which is second class with the Hamiltonian constraint:

$$\{\chi_\phi, H(N)\} = \pm N. \qquad (30)$$

Requiring that this gauge is dynamically preserved leads to

$$N = \pm 1. \qquad (31)$$

Recalling now the definition of the momentum P_ϕ (8), we see that the signs of m and N are linked in this gauge by $m\sqrt{q} = NP_\phi$. Therefore, choosing $N = 1$, which generates dynamics forward in time, fixes [19] the sign ambiguity arising from (12): $P_\phi = +m\sqrt{q}$ and the solution of the Hamiltonian constraint give the physical Hamiltonian density

$$-\mathcal{H}_{\mathrm{phys}} = P_\phi = |\Omega|\left(\frac{P_\Omega^2}{4}\right)' - 2\Lambda|\Omega|r - \left(\frac{2}{|\Omega|}\right)'. \qquad (32)$$

The shift function is also fixed via (26)

$$N = +1 \Leftrightarrow N^r = \mathrm{sgn}(\Omega)\frac{P_\Omega}{2}. \qquad (33)$$

A. Reduced action

The reduced action is obtained by substituting the gauge fixing conditions and the solutions of the constraint (32) into the starting action (19). This gives

$$S = \int dt \int_\Sigma dr(P_\Omega\dot{\Omega} - \mathcal{H}_{\mathrm{phys}}) - \int_{\partial\Sigma} dt \frac{2}{|\Omega|}, \qquad (34)$$

where

$$\mathcal{H}_{\mathrm{phys}} = 2\Lambda|\Omega|r - |\Omega|\left(\frac{P_\Omega^2}{4}\right)'. \qquad (35)$$

The boundary term arises from the total derivative present in (32) and comes ultimately from the Ricci scalar density $\sqrt{q}^{(2)}R = (2/|\Omega|)'$.

If Σ_t is asymptotically flat, as will be the case for some but not all solutions to the equations of motion, this term evaluated at a fixed radius

$$\left.\frac{2}{|\Omega|}\right|_{r=r_0} \qquad (36)$$

determines the energy within a disc of radius r_0 [20], and as we shall see below, this term also gives a measure of the deficit angle at the origin in the limit $r_0 \to 0$. This is because in 3D gravity a conical defect represents a point source of energy at the origin; the relationship between deficit angle α and the energy M of the point source (in units $8G = 1$) is $M = \alpha/\pi$ [2].

Many interesting solutions in 3D gravity are singular at the origin [5], and in order to allow for these solutions, we excise the origin $r = 0$. This ensures that the metric and curvature are well defined everywhere on each Σ_t. We handle this excision by restricting the radial coordinate to the range $r \in (0, r_{\max}]$. Thus, each spatial slice Σ_t is taken to have an outer and an inner boundary.

With these considerations in hand, we turn to a discussion of the functional differentiability of the action. This requires specifying what variations are to be fixed on the boundaries and may require the addition of more boundary terms [21]. Variation of the action (34) with respect to Ω gives the boundary terms

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{r_{\max}} dr[P_\Omega \delta\Omega]_{t=t_i}^{t=t_f} \tag{37}$$

for some initial and final times t_i and t_f, and

$$\lim_{\epsilon\to 0}\int_{t_i}^{t_f} dt\left[\frac{2}{\Omega^2}\delta|\Omega|\right]_{r=\epsilon}^{r=r_{\max}}. \tag{38}$$

And variation with respect to P_Ω gives the boundary term

$$\lim_{\epsilon\to 0}\int dt\left[\frac{|\Omega|}{2}P_\Omega \delta P_\Omega\right]_{r=\epsilon}^{r=r_{\max}}. \tag{39}$$

We define the variational principle by fixing Ω at the end points by

$$\Omega(t, r_{\max}) = a(t), \qquad \Omega(t,\epsilon) = b(t). \tag{40}$$

With this choice the $\delta\Omega$ variation is well defined. The surface term arising from the symplectic piece is zero because initial data and its subsequent evolution fix Ω at t_i and t_f. Lastly, to keep P_Ω free at the boundaries, we add a surface term to cancel the δP_Ω variation. The final gauge fixed action is

$$\begin{aligned} S = &\int dt\int_\Sigma dr(P_\Omega\dot{\Omega} - \mathcal{H}_{\rm phys}) \\ &- \lim_{\epsilon\to 0}\int dt\left[\frac{2}{|\Omega|} + \frac{1}{4}|\Omega|P_\Omega^2\right]_{r=\epsilon}^{r=r_{\max}}. \end{aligned} \tag{41}$$

The summary so far is that, in the process of deriving this action, the dust field and its conjugate momentum have been eliminated from the system, and the remaining metric field and its conjugate momentum (Ω, P_Ω) describe the geometry. Furthermore, as noted in Ref. [15], the dust time gauge results in the conversion of the former Hamiltonian constraint of pure gravity into a nonvanishing true Hamiltonian.

There are other possibilities for fixing the variational principle. For example, we could have gone without adding the second boundary term and instead fixed the momentum P_Ω on the boundaries. The choice above is the simplest since it requires boundary conditions for Ω only and still permits a large class of interesting solutions.

With the variational principle well defined, the equations of motions are

$$\dot{\Omega} = \frac{P_\Omega}{2}|\Omega|', \tag{42}$$

$$\dot{P}_\Omega = \mathrm{sgn}(\Omega)\left(\frac{P_\Omega}{2}P_\Omega' - 2\Lambda r\right). \tag{43}$$

B. Physical conditions

Let us consider the spacetime metric and physical properties resulting from these gauge choices. In fixing a gauge for the field variables, we obtained conditions which fix the lapse and shift functions (33). The resulting metric is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{\Omega^2 P_\Omega^2}{4}\right)dt^2 + \mathrm{sgn}(\Omega)\Omega^2 P_\Omega dr dt \\ &+ \Omega^2 dr^2 + r^2 d\theta^2. \end{aligned} \tag{44}$$

The metric is nondegenerate so long as $\Omega^2 > 0$, which implies that $\mathrm{sgn}(\Omega)$ is constant.

1. Deficit angle

Consider the ratio $\mathcal{F}$ of the circumference of a circle with radius $r_0 > 0$ divided by the proper radius

$$\mathcal{F} \coloneqq \lim_{\epsilon\to 0}\frac{\int_{r=r_0+\epsilon}\sqrt{g_{\theta\theta}}d\theta}{\int_\epsilon^{r_0+\epsilon}\sqrt{g_{rr}}dr}. \tag{45}$$

In flat space this is 2π, but in general we may have $\mathcal{F} = 2\pi - \alpha(r_0)$ where $\alpha(r_0)$ is a deficit angle given by

$$\alpha(r_0) = 2\pi\left(1 - \frac{r_0}{\lim_{\epsilon\to 0}\int_\epsilon^{r_0+\epsilon}|\Omega|dr}\right). \tag{46}$$

The limit of vanishing radius r_0, after first taking the limit $\epsilon \to 0$, gives $\alpha = 2\pi(1 - \frac{1}{|\Omega(t,0)|})$. This implies that constant-time slices of the above metric generically describe a conical geometry near the origin with a time-dependent deficit angle. Note that in 3D gravity a negative deficit angle corresponds to a point source with negative energy [2,6]. To have a positive semidefinite energy at the origin, one would require that

$$|\Omega(t,0)| \geq 1. \tag{47}$$

2. Energy density

The stress-energy tensor may also be written in terms of the phase space variables. From the action we find that

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_D}{\delta g^{\mu\nu}} = \frac{m}{2\pi}\delta^t_\mu \delta^t_\nu. \tag{48}$$

We see that m is the time-time component of the above. From the metric, or using (32) and (12) with the positive choice of sign to solve for m, we obtain the energy density

$$2\pi\mathfrak{T}_{tt} = \sqrt{|g|}m = |\Omega|\dot{P}_\Omega - \left(\frac{2}{|\Omega|}\right)', \tag{49}$$

where the equations of motion (53), (54) have been used to simplify the expression. A positive definite energy density requires that the right-hand side be greater than or equal to zero. The spacetime Ricci scalar is

$${}^{(3)}R = m = \frac{\dot{P}_\Omega}{r} - \frac{1}{r}\left(\frac{1}{\Omega^2}\right)'. \tag{50}$$

Since we have excised the point $r = 0$ from the spatial manifold, the curvature scalar is missing a delta function contribution at the origin when there is a conical singularity.

3. Horizons

Congruences of future directed outgoing and ingoing radial null geodesics are

$$u^\mu = \left(1, \mathrm{sgn}(\Omega)\left(\frac{1}{\Omega} - \frac{P_\Omega}{2}\right), 0\right),$$
$$v^\mu = \left(1, -\mathrm{sgn}(\Omega)\left(\frac{1}{\Omega} - \frac{P_\Omega}{2}\right), 0\right). \tag{51}$$

These satisfy $u^\mu v_\mu = -2$ and provide the null expansions. In our context these are *physical* phase space observables which are potentially useful in a quantum theory [13]. The outward null expansion of circles embedded in a spatial slice with unit outward normal $s^\mu = (0, 1, 0)/\Omega$ is

$$\Theta := (q^{\mu\nu} - s^\mu s^\nu)\nabla_\mu u_\nu = \frac{1}{r}\left(\frac{P_\Omega}{2} - \frac{1}{|\Omega|}\right). \tag{52}$$

Dynamical apparent horizons are obtained by solving $\Theta(t, r) = 0$ to give the horizon radius $r_h(t)$. This may have multiple solutions on a given time slice (see e.g. Refs. [22,23] for explicit examples).

IV. CLASSICAL SOLUTIONS

In this section we discuss classical solutions to our model. We find a large class of exact solutions and provide several examples. In particular we obtain a static solution for $\Lambda < 0$ which describes the Bañados–Teitelboim–Zanelli (BTZ) black hole, and for $\Lambda = 0$ we find traveling wave solutions.

To this point we have left the sign of Ω arbitrary. As noted above we must require that $\mathrm{sgn}(\Omega)$ is constant throughout the spacetime in order for the action to be well defined. This implies that the solution space is split into sectors with $\mathrm{sgn}(\Omega) = \pm 1$. To keep the presentation simple, we assume $\Omega > 0$ for the remainder of the article. The solution space for $\Omega < 0$ is nearly identical with only trivial differences.

A. $\Lambda \neq 0$

For the case of nonzero cosmological constant, the equations of motion are

$$\dot{\Omega} = \frac{P_\Omega}{2}\Omega', \tag{53}$$

$$\dot{P_\Omega} = \frac{P_\Omega}{2}P'_\Omega - 2\Lambda r. \tag{54}$$

The second equation is similar to the inviscid Burger's equation, but with a source term coming from the cosmological constant; it contains only the momentum and can be solved independently. This is coupled to the first equation which resembles the advection equation but with a variable speed of propagation given by $P_\Omega/2$. As we will see, any solution for the momentum then determines how initial data $\Omega(0, r)$ evolve.

1. General solution

There is an auxilliary, flat spacetime with Lorentzian signature defined by the (t, r) plane. On the auxilliary spacetime the momentum equation (54) is a conservation equation $\partial_a j_1^a = 0$ for the current,

$$j_1^a = \left[-P_\Omega, \frac{P_\Omega^2}{4} - \Lambda r^2\right], \tag{55}$$

which has an associated conserved charge given by

$$q_1 = \lim_{\epsilon\to 0}\int_\epsilon^{r_{\max}} P_\Omega dr. \tag{56}$$

Considering the system as a whole, there is another conserved current,

$$j_2^a = \left[\Omega\left(2\Lambda r - \frac{P_\Omega}{2}P'_\Omega\right), \Omega\frac{P_\Omega}{2}\left(\frac{P_\Omega}{2}P'_\Omega - 2\Lambda r\right)\right], \tag{57}$$

where the conserved charge is

$$q_2 = \lim_{\epsilon\to 0}\int_{\epsilon}^{r_{\max}} \Omega\left(\frac{P_\Omega}{2}P'_\Omega - 2\Lambda r\right)dr. \tag{58}$$

It is well known that conservative equations can be solved by the method of characteristics. To employ this method, we consider characteristic lines parametrized by s, described in terms of parametric equations for the coordinates $r(s)$ and $t(s)$. Differentiating with respect to s, we obtain

$$\frac{d}{ds}P_\Omega = \dot{P}_\Omega\frac{\partial t}{\partial s} + P'_\Omega\frac{\partial r}{\partial s} = -2\Lambda r. \tag{59}$$

This is equivalent to the equation of motion if we have the following equations along each characteristic:

$$\frac{\partial t}{\partial s} = 1, \qquad \frac{\partial r}{\partial s} = -\frac{P_\Omega}{2}, \qquad \frac{d}{ds}P_\Omega = -2\Lambda r. \tag{60}$$

These equations are solved by

$$t = s, \qquad r = r_0\cosh\sqrt{\Lambda}s - \frac{P_0}{2\sqrt{\Lambda}}\sinh\sqrt{\Lambda}s,$$
$$P_\Omega = P_0\cosh\sqrt{\Lambda}s - 2\sqrt{\Lambda}r_0\sinh\sqrt{\Lambda}s, \tag{61}$$

where the initial values are $r_0 = r(s=0)$ and $P_0 = P_\Omega(s=0)$. Each characteristic is labelled by the "starting point" r_0, and initial data for the momentum are a function of the radial points on the initial slice $P_\Omega(t=0,r) = P_0(r_0)$. (This solution was used in Ref. [24] to construct an Oppenheimer–Snyder model in 3D gravity).

Given a solution for the momentum, we can solve (53) for Ω using the characteristic method again. The characteristics for this equation are the same as those for the momentum equation of motion, but here we have

$$\frac{d}{ds}\Omega = 0 \tag{62}$$

so that Ω is constant along each characteristic. This implies that, given some initial data $\Omega(0,r)$, the configuration field simply flows along the characteristic lines defined by the momentum.

2. Examples

Static solutions are obtained by setting $\dot{\Omega} = \dot{P}_\Omega = 0$ in (53), (54). This gives

$$\Omega = C_1, \tag{63}$$

$$P_\Omega = \pm 2\sqrt{C_2 + \Lambda r^2}. \tag{64}$$

The metric may be put in a more succinct form as follows. Rescale C_2 and the r and θ coordinates by absorbing the constant C_1 as $\tilde{r} = C_1 r$, $\tilde{\theta} = \theta/C_1$, and $\tilde{C}_2 = C_1^2 C_2$. With this rescaling the angular coordinate has a range $0 \le \tilde{\theta} \le 2\pi/C_1$ so that a choice of $C_1 > 0$ implies a deficit angle defined by Ω as described in the preceding section. The line element becomes

$$ds^2 = -f dt^2 \pm 2\sqrt{1-f}\,d\tilde{r}dt + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2, \tag{65}$$

where $f(\tilde{r}) \equiv 1 - \tilde{C}_2 - \Lambda\tilde{r}^2$.

This solution remains well defined for any choice of $C_2 > 0$, which is required for P_Ω to be real at each point. For de Sitter spacetime ($\Lambda > 0$), there are no additional restrictions, but for the anti-de Sitter (AdS) case ($\Lambda < 0$), the radial coordinate has a limited extent in order to keep P_Ω nonimaginary:

$$0 < \tilde{r} \le \sqrt{\frac{\tilde{C}_2}{|\Lambda|}}. \tag{66}$$

Let us consider the AdS case further. The above line element is in fact a generalization of the BTZ black hole which allows for a deficit angle due to the choice of C_1. This can be seen by transforming to a new time coordinate,

$$\tilde{t} = t \pm \int_0^{\tilde{r}} \frac{\sqrt{1-f(x)}}{f(x)}dx, \tag{67}$$

which puts the line element in the form

$$ds^2 = -f d\tilde{t}^2 + f^{-1}d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2, \tag{68}$$

where we note again that the angular range is $0 \le \tilde{\theta} < 2\pi/C_1$. This spacetime has an event horizon at $\tilde{r} = \sqrt{(\tilde{C}_2 - 1)/|\Lambda|}$, and when $C_1 = 1$ it is the BTZ spacetime in flat slice coordinates.

B. $\Lambda = 0$

With zero cosmological constant, the equations of motion have a remarkable symmetric form,

$$\dot{\Omega} = \frac{1}{2}P_\Omega\Omega', \tag{69}$$

$$\dot{P}_\Omega = \frac{1}{2}P_\Omega P'_\Omega. \tag{70}$$

The momentum equation of motion is now Burger's equation with vanishing viscosity. There is a substantial volume of literature on the subject. Most notably this equation gives shock waves when characteristics cross.

1. General solution

The equations of motion are again conservation equations $\partial_a j^a = 0$ in the auxiliary flat Lorentzian spacetime defined by the (t, r) plane. With $\Lambda = 0$ the currents are

$$j_1^a = \left[-P_\Omega, \frac{P_\Omega^2}{4}\right], \quad j_2^a = \left[-\Omega\left(\frac{P_\Omega^2}{4}\right)', \Omega\frac{P_\Omega^2}{4}P'_\Omega\right], \tag{71}$$

and the corresponding conserved charges are

$$q_1 = \lim_{\epsilon\to 0}\int_\epsilon^{r_{\max}} P_\Omega dr, \qquad q_2 = \lim_{\epsilon\to 0}\frac{1}{2}\int_\epsilon^{r_{\max}} \Omega P_\Omega P'_\Omega dr. \tag{72}$$

The equation of motions can again be solved by the methods of characteristics; one needs only to put $\Lambda = 0$ in the equations from the last section. There are two classes of solutions: 1) P_Ω is constant, and $\Omega = h(r + \frac{P_\Omega}{2}t)$, and 2) P_Ω is not constant, and $\Omega = h(P_\Omega)$, for an arbitrary function h. This last fact is a remarkable consequence of the structure of the $\Lambda = 0$ equations. When P_Ω is constant, the characteristics are guaranteed not to cross.

We also note that for a vanishing cosmological constant the parametric equations (61) can be inverted to yield the following:

$$2r + P_\Omega t - 2f(P_\Omega) = 0. \tag{73}$$

Given any choice of function $f(P_\Omega)$ of the momentum, solutions to (70) are given by the roots to this equation.

2. Examples

Let us note three types of solutions. The first is a class of static solutions obtained by setting $P_\Omega = 0$ in (69)–(70). This implies $\Omega = f(r)$, a nowhere-vanishing but otherwise arbitrary function. The resulting metric is

$$ds^2 = -dt^2 + f(r)^2 dr^2 + r^2 d\theta^2. \tag{74}$$

The energy density is given by $2\pi\mathfrak{T}_{tt} = \frac{2f'}{f^2}$, and its sign is determined by the sign of f'. The spacetime Ricci scalar is ${}^{(3)}R = \frac{f'}{rf^3}$, and the r^{-1} factor indicates that solutions are generally singular at $r = 0$, except for the particular choice $f(r) = \pm(C_1 - C_2 r^2)^{-1/2}$. Constant time slices are cones with deficit angle $\alpha = 2\pi(1 - 1/f(0))$, and there are no horizons in this spacetime.

The second is a self-similar solution obtained by setting $f(P_\Omega) = 0$ in (73) to obtain

$$P_\Omega = -\frac{2r}{t}. \tag{75}$$

It is immediate that this solves (70) and leads to the Ω equation of motion

$$\dot\Omega = -\frac{r}{t}\Omega'. \tag{76}$$

One solution of this is $\Omega = 1$, which gives the metric

$$ds^2 = -\left(1 - \frac{r^2}{t^2}\right)dt^2 - \frac{2r}{t}drdt + dr^2 + r^2 d\theta^2. \tag{77}$$

The Ricci scalar from (50) is ${}^{(3)}R = 2/t^2$, so there is a spacelike curvature singularity at $t = 0$. Looking at the condition (52), we find horizons at $r = -t$. Constant t slices are flat without any deficit angle due to the choice of $\Omega = 1$.

A third class of solutions is obtained by setting $P_\Omega = 2v$ for some constant $v \in \mathbb{R}$. The Ω equation reduces to the advection equation

$$\dot\Omega = v\Omega', \tag{78}$$

which has the general solution

$$\Omega = h(r + vt) \equiv h(u) \tag{79}$$

for arbitrary h and no restriction on v, where $u \equiv r + vt$ labels each (straight) characteristic. If we choose $\Omega = C$, a constant, we have a flat metric with deficit angle $\alpha = 2\pi(1 - 1/C)$ as described in (46).

For a nonconstant Ω, the $v > 0$ and $v < 0$ solutions describe respectively radially ingoing and outgoing profiles. The $v > 0$ wave metric is

$$ds^2 = -[1 - (v\Omega)^2]dt^2 + 2v\Omega^2 drdt + \Omega^2 dr^2 + r^2 d\theta^2. \tag{80}$$

We note the following features of these "wave" solutions. The Ricci scalar from (50) is

$${}^{(3)}R = \frac{2}{r}(\ln\Omega)' = \frac{2}{rh}\frac{dh}{du}, \tag{81}$$

and there are dynamical horizons if $\Omega(t, r) = 1/v$. Thus, horizons will be present if the maximum value of the wave profile exceeds $1/v$ and the minimum is less than $1/v$.

From the expression for energy density (49), we see that the energy flux is positive where $\Omega' > 0$ and negative where $\Omega' < 0$. There is a conical singularity at the origin with deficit angle $\alpha = 2\pi(1 - 1/\Omega(t, 0))$. As the energy flux reaches the origin, positive flux adds to the deficit angle (which represents the mass of the singularity), and negative flux reduces the deficit angle.

V. QUANTUM THEORY

In this section we describe a nonperturbative quantization of the $\Lambda = 0$ theory. The circularly symmetric sector of the model we are considering has one local physical degree of freedom (Ω), and as we have shown in the last section, the classical theory can be solved by the method of characteristics with the P_Ω solution providing a local (t, r)-dependent speed for the Ω equation. The full quantum theory of this sector is more challenging. Although there is a physical Hamiltonian and no constraints, the Hamiltonian is unconventional in the sense that there is no separation of pure kinetic and potential terms. We can, however, achieve a full quantization of the $P_\Omega = \text{constant}$ sector of the solution space discussed in the last section.

As we noted, this sector of the solution space describes either purely ingoing or outgoing waves. A first challenge is that, since P_Ω is a constant, the symplectic structure we have been using up to now is not available. This is overcome by noting that we can obtain a new symplectic structure starting from the solution space of a differential equation [25]. The basic idea involves defining geometric structures on the solutions space that leads to a conserved symplectic current. The integral of this current on an initial value surface defines the desired symplectic form.

For completeness, we now derive the symplectic 2-form for the circularly symmetric, $\Lambda = 0$, $P_\Omega = 2v$ (where v is a constant), sector of the solution space and refer the reader to Ref. [25] for details. Once the symplectic structure is obtained, we move on to the canoncial quantum theory.

In our case the differential equation is

$$\dot{\Omega} = v\Omega', \tag{82}$$

on the half plane $t \in (-\infty, \infty)$, $r \in (0, \infty)$. For our purposes we take this half-plane to define an auxilliary spacetime M with a flat Lorenzian metric given by $\eta = \mathrm{diag}(-v^2, 1)$.

Consider the space Z of solutions to (82). A point $\Omega \in Z$ represents a solution to this equation, and a tangent vector $\delta\Omega$ at this point is a small displacement which must also be in the solution space Z. Writing the displacement of the solution as $\Omega + \delta\Omega$, we find that the tangent vector $\delta\Omega$ must also satisfy (82).

The space of 1-forms on Z is dual to the tangent space. If we label the spacetime points $x \in M$, then the 1-form dual to $\delta\Omega$ is given by $\delta\Omega(x)$. It is important to note that these 1-forms are anticommutative,

$$\delta\Omega(x)\delta\Omega(y) + \delta\Omega(y)\delta\Omega(x) = 0. \tag{83}$$

The symplectic current is defined by

$$J_a(x) = \delta\Omega(x)\partial_a\delta\Omega(x), \tag{84}$$

which, due to the equation of motion (82) and the anticommutivity of one-forms, obeys the conservation equation $\eta^{ab}\partial_a J_b = 0$. The associated conserved charge is given by integrating over a spatial hypersurface:

$$\omega = \int drJ_t = \int dr\delta\Omega\delta\dot{\Omega}. \tag{85}$$

This conserved charge is the symplectic 2-form we seek. It implies that the momentum conjugate to Ω is $\Pi \coloneqq \dot{\Omega}$, with the equal-time Poisson algebra

$$\{\Omega(r,t), \Pi(r',t)\} = \delta(r - r'),$$
$$\{\Omega(r,t), \Omega(r',t)\} = \{\Pi(r,t), \Pi(r',t)\} = 0. \tag{86}$$

These are equivalent to the Poisson brackets for free scalar field theory, but the Hilbert space we define next will differ in that it includes only the ingoing *or* the outgoing modes, depending on the sign of v chosen.

Having obtained the symplectic structure, let us consider the Hilbert space we will use for quantization. Consider the positive energy (dust time) mode functions

$$\psi_k^\pm(r,t) = e^{-ivkt}(e^{-ikr} \pm e^{ikr}), \qquad k > 0. \tag{87}$$

These sets have different boundary conditions at $r = 0$: $\psi^+(t, r = 0) = 2e^{-ivkt}$ and $\psi^-(t, r = 0) = 0$. Both sets are orthogonal and complete on the half-line,

$$\int_0^\infty dr\bar{\psi}_k^\pm(r,t)\psi_{k'}^\pm(r,t) = 2\pi\delta(k - k'),$$
$$\int_0^\infty dk\bar{\psi}_k^\pm(r,t)\psi_k^\pm(r',t) = 2\pi\delta(r - r'), \tag{88}$$

and also satisfy

$$\int_0^\infty dr\bar{\psi}_k^\pm(r,t)\psi_{k'}^\mp(r,t) = 0. \tag{89}$$

Let $\mathcal{H}_v^\pm$ denote the Hilbert spaces with the bases $\psi_k^\pm$, and let $\mathcal{H}_v = \mathcal{H}_v^+ \oplus \mathcal{H}_v^-$. The purely ingoing (outgoing) wave solutions are obtained by the normalized linear combination

$$g_k(r,t) \coloneqq \frac{1}{\sqrt{2\pi}}(\psi_k^+(r,t) + \psi_k^-(r,t)) = \frac{1}{\sqrt{\pi}}e^{-ik(vt+r)}. \tag{90}$$

Clearly $g_k \in \mathcal{H}_v$ are solutions of our model. They may be viewed as "quasiparticles" if the bases given above for $\mathcal{H}^+$ and $\mathcal{H}^-$ are viewed as "particles." The Hilbert space of the entire wave sector (all ingoing and outgoing modes labelled by $v \in \mathbb{R}$) is the tensor product

$$\mathcal{H} = \otimes_v \mathcal{H}_v. \tag{91}$$

Let us demonstrate the quantization in the $v = 1$ component. $\Omega(r,t)$ and its conjugate $\Pi(r,t)$ may be represented as operators in $\mathcal{H}_1$, in the manner that is standard in field theory:

$$\hat{\Omega}(r,t) = \int_0^\infty dk \frac{1}{\sqrt{k}} (\hat{a}_k g_k(r,t) + \hat{a}_k^\dagger \bar{g}_k(r,t)), \quad (92)$$

$$\hat{\Pi}(r,t) = -i \int_0^\infty dk \sqrt{k} (\hat{a}_k g_k(r,t) - \hat{a}_k^\dagger \bar{g}_k(r,t)). \quad (93)$$

Their commutator algebra implied by the Poisson algebra (86) leads to the usual commutators for ladder operators

$$[\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = [\hat{a}_k, \hat{a}_{k'}] = 0 \qquad [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta(k - k'). \quad (94)$$

Fock states are constructed by starting with the vacuum state $|0_k\rangle$ defined by $\hat{a}_k|0_k\rangle = 0$,and the n-particle states are defined by

$$|n_k\rangle = \frac{1}{\sqrt{n_k}} [\hat{a}_k^\dagger]^{n_k} |0_k\rangle. \quad (95)$$

The Fock basis is given by products of the n_k-particle states with different k values.

This completes the specification of the quantum theory for the $P_\Omega =$ constant sector of the solution space. What it shows is that this nonperturbative sector of $2+1$ gravity coupled to pressureless dust in spherical symmetry is *exactly dual* to the quantum theory of a massless scalar field on the half-line.

A. Metric operator

With the above quantization, we can now proceed to describe the "quantum geometries" for this spherically symmetric sector of the model. The metric contains only the function Ω, so it is possible to define the metric operator in the Hilbert space $\mathcal{H}$ by $\hat{g}_{ab} \coloneqq g_{ab}(\hat{\Omega})$. A notion of geometry is given by the expectation value of this operator in a quantum state. There is thus an infinite set of possible geometries depending on the choice of state.

The metric contains the factor $\hat{\Omega}^2$, so we need to select an operator ordering of $\hat{a}_k, \hat{a}_k^\dagger$ to define it. This is provided by imposing the physical requirement that expectation values in semiclassical states give recognizable classical solutions. One choice for such states is the coherent states defined by

$$\hat{a}_k|\alpha_k\rangle = \alpha_k|\alpha_k\rangle. \quad (96)$$

These states are explicitly given by [26]

$$|\alpha_k\rangle = e^{-|\alpha_k|^2/2} \sum_{n_k=0}^\infty \frac{\alpha_k^{n_k}}{\sqrt{n_k!}} |n_k\rangle. \quad (97)$$

It is known that the expectation values of normal ordered operator in these states give the corresponding classical results. We therefore define

$$\hat{g}_{ab} \coloneqq g_{ab}(:\widehat{\Omega^2}:). \quad (98)$$

As an explicit example, let us consider the state which is the vacuum for all modes except k, and the coherent state for mode k,

$$|\psi\rangle = |\alpha_k\rangle \prod_{j\neq k} |0_j\rangle. \quad (99)$$

This gives

$$\langle :\widehat{\Omega^2}: \rangle = \frac{1}{k}((g_k(r,t))^2 \alpha_k^2 + (g_k^*(r,t))^2 (\alpha_k^*)^2 + 2|\alpha_k|^2), \quad (100)$$

where α_k are any complex numbers specifying a classical solution. The complex number α_k must be such that the right-hand side is positive definite in order to avoid a degenerate metric, and depending upon the value of α_k, there may be apparent horizons. The quantum fluctuations in these states

$$\Delta(\Omega^2) = \langle :\widehat{\Omega^2}::\widehat{\Omega^2}: \rangle - \langle :\widehat{\Omega^2}: \rangle^2 \quad (101)$$

are of course not zero, since $|\alpha_k\rangle$ are the minimum uncertainty states.

The expectation value of the metric in the n_k-particle state $|n_k\rangle$ on the other hand gives the metric

$$ds^2 = -\left(1 - \frac{2n_k}{k}\right) dt^2 + \frac{4n_k}{k} drdt + \frac{2n_k}{k} dr^2 + r^2 d\theta^2. \quad (102)$$

The constant time slices are cones with deficit angle

$$\alpha = 2\pi\left(1 - \sqrt{\frac{k}{2n_k}}\right). \quad (103)$$

Recall that in 3D gravity a conical singularity corresponds to a point source with a mass proportional to the deficit angle [2,6]. This implies a discrete mass spectrum of the n_k-particle states determined by the wave number k. It asymptotes to 2π as n_k gets large and has a positive or zero energy for $2n_k \geq k$.

For special values of the parameters satisfying $2n_k = k$, the apparent horizon function vanishes everywhere $\Theta(r,t) = 0$. With these values, the conical defect of the spacetime slicing is such that the outward going null geodesics remain at constant radius.

The self-adjoint metric operator defined above using the creation and annihilation operators provides, via the expectation value, a correspondence between quantum states and spacetime geometries. The coherent states lead to classical geometries with fluctuations. There is also the interesting possibility of obtaining "macroscopically entangled geometries" by using states that describe entangled superpositions.

Construction of entangled states requires either two or more systems described by a tensor product Hilbert space (such as the Hilbert space of 2 or more spin 1/2 particles), or the division of Hilbert space into subsystems.

In the quantum theory of the wave sector we have discussed, it is possible to produce entangled states by considering for example states in the subspace $\mathcal{H}_1 \otimes \mathcal{H}_{-1}$ of the full Hilbert space (91). Let $|\alpha_k\rangle_1$ denote a semiclassical state in $\mathcal{H}_1$ and $|\beta_k\rangle_{-1}$ a semiclassical state in $\mathcal{H}_{-1}$. Then an example of a macroscopic entangled state of spacetime geometries is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha_k\rangle_1|\beta_k\rangle_{-1} + |\beta_k\rangle_1|\alpha_k\rangle_{-1}). \tag{104}$$

There are numerous examples of this type involving two or more subsystems, even within a fixed v sector of the Hilbert space, but with states labelled by different k values. In a full quantum theory of gravity, it would presumably be possible to divide the physical Hilbert space into sectors corresponding to, for example, black hole and cosmological geometries. It would then be possible to construct interpretationally challenging macroscopically entangled states.

B. Quantum horizons

From the forgoing we can consider the idea of a "quantum horizon" defined by a horizon operator [13]

$$\hat{h} =: \widehat{\Omega^2} : -\frac{1}{v^2} I, \tag{105}$$

which is the operator analog of the classical apparent horizon condition. In our gauge fixed setting, this is a physical observable. It is clear that the fluctuation of this operator is nonzero in a coherent state, and so the corresponding dynamical horizons are not sharply defined as they are in the classical theory. For a coherent state with an α_k, such that the horizon condition $\langle\hat{h}\rangle = 0$ is marginally satisfied, fluctuations of the metric operator can lead to uncertainty in whether or not horizons are present at all.

VI. SUMMARY AND DISCUSSION

We studied a new model for gravity in $2+1$ dimensions. Unlike most of the existing literature on 3D gravity, the model has a local degree of freedom which manifests itself as a metric function in the dust time gauge. The resulting theory has novel aspects in circular symmetry. The equations of motion are simple, yielding several interesting classes of solutions, including waves; the latter provide a "midisuperspace" sector of the solution space which is amenable to Fock quantization.

The quantization provides some interesting and precise results. Among these is the observation that horizons fluctuate, which we showed using semiclassical states. It is natural to expect that this result goes over to four dimensions where it could inform issues such as the so-called information loss problem in black hole evaporation. In particular metric fluctuations imply that the separation of the Hilbert space into states which are strictly inside/outside the horizon, as is common in computing entanglement entropy, is an ambiguous procedure. Metric fluctuations further imply that the time of apparent horizon formation may be ambiguous for any choice of time coordinate.

Metric fluctuations also inform the "firewall" issue, which at its core requires exactly null nonfluctuating horizons as a fundamental assumption. If a horizon is leaky because it has fluctuations, then it is clear that the central argument based on the impossibility of simultaneous perfect correlation between modes across a horizon on the one hand, and perfect correlation between early and late time Hawking radiation on the other, ceases to be an issue: no perfect null horizon implies no monogamy problem. It is possible that horizon fluctuations are small for large black holes if the appropriate semiclassical state is sharply peaked. But if a firewall were to form, its accompanying backreaction on the metric would obviously lead to high curvature fluctuations, and in turn to large horizon fluctuations in the early stages of its formation.

As a last comment, our quantization also demonstrates an exact duality between (a sector) of the model and $1+1$ free scalar field theory on the half-line. This in turn is dual to fermionic theory via the well-known Bose–Fermi correspondence in two spacetime dimensions. This means that the quantization we have presented likely has a fermionic description.

This work is a first exploration of the use of dust time to study quantization of gravity. Natural extensions of our approach are to the $3+1$ theory Bianchi models, spherically symmetry, and other reduced sectors, such as the Gowdy models.

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