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Gravitational radiation-reaction in arbitrary dimension

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We use effective field theory tools to study nonconservative effects in the gravitational two-body problem in general spacetime dimension. Using the classical version of the Closed Time Path formalism, we treat both the radiative gravitational field and its dynamical sources within a single action principle. New results include the radiation-reaction effective action in arbitrary dimensions to leading and +1PN orders, as well as the generalized quadrupole formula to order +1PN.

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I. INTRODUCTION

In Ref. [1], an effective field theory (EFT) formalism for the simultaneous, action-level treatment of radiation and radiation-reaction (RR) effects was developed (see also Ref. [2] for a pedagogical introduction). The EFT was explicitly constructed for systems of localized objects coupled to scalar, electromagnetic (EM) and gravitational fields. The method was generalized to arbitrary spacetime dimensions in the scalar and EM cases [3]. In particular, for a single point particle, it yields the higher-dimensional analogue of the Abraham-Lorentz-Dirac self-force (SF). In this article, we complete the generalization for *general relativistic* (GR) post-Newtonian (PN) systems in arbitrary dimension by applying our method and constructing the associated EFT. The GR case is more involved than the (free) scalar and EM cases in two main respects: first, the higher spin of the gravitational field, which complicates the construction of gauge-invariant fields and corresponding sources, both central features of our formalism; and second, the intrinsic nonlinearity of GR.

The PN approximation of the gravitational two-body problem in four spacetime dimensions has been studied, over the past few decades, to very high accuracy (for a wide review of the field, see Ref. [4]). The state of the art is the +4PN (corrected to order v^8) effective action, recently completed in Ref. [5]. Since Goldberger and Rothstein's groundbreaking paper [6], much progress has been made on the EFT approach to the PN binary problem (including Refs. [7–14]); for a more detailed review section, see Ref. [1]. Less attention has been given to gravitational radiation and radiation reaction in spacetime dimensions other than four[1]; these will be the focus of this paper. Although not directly relevant for astrophysical gravitating binaries, we feel that studying this rather fundamental problem is well motivated, both because an understanding of a system's behavior in general dimension complements and enhances its understanding in 4D (see for example Refs. [19–21]), and because higher-dimensional gravitational scenarios frequently emerge in theoretical physics, for example in string theory. Gravitational radiation in higher dimensions was treated in Refs. [22–24], including in EFT methods. Specifically, the leading-order expression for gravitational wave emission in higher dimensions, i.e., the equivalent of Einstein's quadrupole formula [25], was computed. Equation (4.25) of Ref. [24] [also Eqs. (38) and (39) of Ref. [22]] gives the energy output in gravitational quadrupole radiation in d dimensions as

$$\frac{dE}{d\omega} = \mathbf{G}_d \frac{2^{2-d}\pi^{-(d-5)/2} d(d-3)}{(d-2)(d+1)\Gamma[\frac{d-1}{2}]} \omega^{d+2} |Q_{ij}(\omega)|^2. \quad (1.1)$$

In this paper we develop an effective action for the radiation reaction of gravitation in general dimension (3.2). From this action we derive both the SF and the dissipated power, up to subleading (+1PN) order. At the leading order, the energy output we find [(3.6) and (3.7)] matches exactly the result (1.1). As an additional test of our results at +1PN, we substitute $d = 4$ and compare with Refs. [1,26], finding a match.

It is important to stress a relevant qualitative difference between $d = 4$ and $d > 4$ spacetime dimensions. For PN astrophysical binaries RR is weak, hence it must influence the system for a long time for its effect to be significant; this is the case for bound orbits. In higher dimensions, it is well known that there are no stable bound gravitational orbits for a binary system of gravitating compact objects.[2] Nevertheless, high-dimensional systems can be stabilized by nongravitational forces (in particular, short-range repulsive forces); RR may also have a substantial effect for trajectories close to the unstable circular orbit, as well as for systems with more than two bodies. As we are neither specifying the sources' trajectories nor trying to solve for them, the question of binding (and its mechanisms) do not affect the results of this paper.

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[1]For earlier work on a fully relativistic treatment of SF for fields of various spin, see Refs. [15–18].

[2]For point particles, for example, there exist unstable circular orbits.

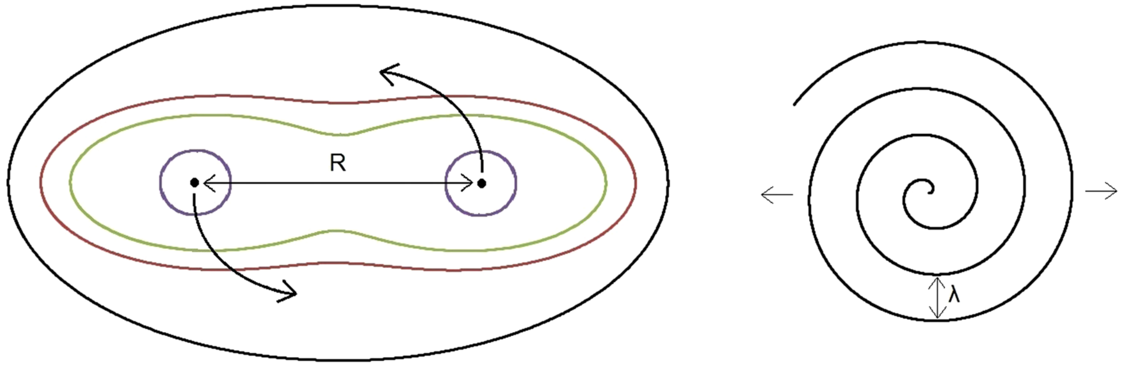


FIG. 1 (color online). The two relevant zones: The system zone is on the left, with a typical stationary-like field configuration. The radiation zone with its typical outspiraling waves is on the right.

A. Method

In this work we treat the gravitational two-body problem in arbitrary spacetime dimension and calculate explicitly its radiation source multipoles, the RR effective action and important physical quantities derivable from it: the outgoing radiation, dissipated energy and RR force acting on the system. Our method [1] divides the problem into two zones, the system zone and the radiation zone (Fig. 1), each with different enhanced symmetries. The system zone enjoys approximate stationarity (time independence), since by assumption all velocities are nonrelativistic; we thus use the "nonrelativistic gravitational" (NRG) fields [11] to describe it. The radiation zone enjoys an approximate spherical symmetry, as from its point of view the system has shrunk to a point. Hence, we use *gauge-invariant* spherical field variables [27–31] as in Ref. [1] and unlike the plane-wave decomposition used in most previous EFT works [7–10,14]. An important ingredient of our method is the matching of these two (system and radiation) zones at the level of the action, using "two-way multipoles" [1]. These are degrees of freedom we introduce—"integrate in"—to couple the two zones. From the radiation zone point of view, we think of them as sources situated at the origin; while in the system zone view, they reside at infinity. At the end of the day we integrate out (eliminate) all the other degrees of freedom in the problem and remain with an effective action which is a function of these multipoles precisely. Schwinger-Keldysh field doubling [32], which was beautifully adapted to the classical context in general in Ref. [33] and was used in studies of the binary problem in Refs. [1,14], is an essential ingredient, since it allows the derivation of dissipative effects from action principles. As we have shown in Ref. [3] and as is further exemplified here, a clear advantage of our formalism is its ability to naturally extend to general spacetime dimensions.

B. Conventions and nomenclature

We follow the conventions of Ref. [3]. Thus, the flat d-dimensional spacetime metric $\eta_{\mu\nu}$'s signature is mostly plus, and also $D \coloneqq d-1$, $\hat{d} \coloneqq d-3$. Regarding the $\hat{d}+1$-dimensional unit sphere, we define $\Omega_{\hat{d}+1}$ to be its area; on it $g_{\Omega\Omega'}$ is the metric, D_Ω is the covariant derivative,[3] and $\Delta_{\hat{d}+1} = D_\Omega D^\Omega$ is the Laplace-Beltrami operator. The eigenfunctions of this operator on the $\hat{d}+1$ unit sphere will be given by various multipoles enumerated by an order ℓ; in treating their eigenvalues we shall make use of $c_s = \ell(\ell+\hat{d})$ and $\hat{c}_s = c_s - (\hat{d}+1)$. We shall designate by ϵ the sector (scalar, vector, tensor) of different multipoles, with either $\epsilon \in \{S, V, T\}$ or $\epsilon \in \{0, 1, 2\}$, respectively.

Lowercase greek letters stand for spacetime indices $\{0..D\}$, lowercase latin letters for spatial indices $\{1..D\}$, uppercase greek letters for indices on the sphere $\{1..(\hat{d}+1)\}$, hebrew letters (א, ב) for different vectorial and tensorial multipoles on the sphere, and uppercase latin letters for spatial multi-indices, i.e., $I \equiv I_\ell \coloneqq (i_\ell \ldots i_\ell)$, where each $i_k \in \{1..D\}$ is an ordinary spatial index, and ℓ is the number of indices. We use the multi-index summation convention with implied $\frac{1}{\ell!}$,

$$P_I Q_I \coloneqq \sum_\ell P_{I_\ell} Q_{I_\ell} \coloneqq \sum_\ell \frac{1}{\ell!} P_{i_1\ldots i_\ell} Q_{i_1\ldots i_\ell}, \tag{1.2}$$

as well as the multi-index delta function defined through $\delta_{I_\ell J_\ell} \coloneqq \ell! \delta_{i_1 j_1} \ldots \delta_{i_\ell j_\ell}$, so that factors of $\ell!$ are accounted for automatically.

While as usual $c = 1$, we specifically wish to keep the gravitational constant in d dimensions, marked G_d. We choose it and the normalization for the gravitational action so that the Newtonian potential is always $-G_d M / r^{\hat{d}}$, and

[3]At some points we will be interested in the covariant derivative with respect to an angular variable Ω in the full d-dimensional space, and we will mark it D^d_Ω. D_Ω will be reserved for the more common case of variation restricted to the $\Omega_{\hat{d}+1}$ sphere. Likewise, the full d-dimensional metric, when it relates spherical coordinates, will be marked $g^d_{\Omega\Omega'}$. We note that $g^d_{\Omega\Omega'} = r^2 g_{\Omega\Omega'}$, where r is the radial coordinate.

the action is given by (2.1). It is related to the commonly used constant[4] $\mathbf{G}_d$ by $G_d = \frac{8\pi}{(\hat{d}+1)\Omega_{\hat{d}+1}}\mathbf{G}_d$. Both definitions identify for $d = 4$ [24,34]. The constant G_d also has units of $L^{\hat{d}}/M$.

We use the convention that the Feynman rules are real [35], as befits a nonquantum field theory. We also denote

$$\sum\!\!\!\!\!\!\int \triangleq \int \frac{d\omega}{2\pi}\sum_L. \tag{1.3}$$

II. FROM EINSTEIN'S ACTION TO FEYNMAN RULES

A. Action for metric perturbations

In GR, the action for the metric $\tilde{g}_{\mu\nu}$ coupled to matter is given by the sum of the Einstein-Hilbert (EH) action and a matter term:

$$S = \int d^d x\sqrt{-\tilde{g}}\left[\frac{1}{2(\hat{d}+1)\Omega_{\hat{d}+1}G_d}R + \mathcal{L}_M\right], \tag{2.1}$$

where $\mathcal{L}_M$ is the matter Lagrangian. We shall calculate the generation and reaction of gravitational waves (GWs) propagating out to infinity in asymptotically flat space in spherical coordinates. The metric splits to $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ describes the background flat spacetime in spherical coordinates[5] and $h_{\mu\nu}$ is a perturbation on it. The EH term is the kinetic part of the action and describes the propagation of GWs. The matter term describes the generation of GWs from (matter) sources. We expand the kinetic (source) action to quadratic (linear) order in $h_{\mu\nu}$ to find

$$S_{\rm EH} = \frac{1}{4(\hat{d}+1)\Omega_{\hat{d}+1}G_d}\int d^d x\sqrt{-g}\left[\frac{1}{2}h^{\alpha}_{[\alpha;}{}^{\gamma}h^{\beta}_{\beta];\gamma} + h^{\gamma}_{[\beta;}{}^{\beta}h^{\alpha}_{\gamma];\alpha}\right], \tag{2.2}$$

$$S_{\rm mat} = \int d^d x\sqrt{-g}h_{\mu\nu}T^{\mu\nu}. \tag{2.3}$$

We use the action composed of (2.2) and (2.3) to construct our *far zone* action in terms of spherical variables. As we compute corrections only up to +1PN, and as far zone nonlinearities enter only at orders higher than +1PN (see discussion in Sec. III C 2), we need not take them into account here; this justifies expanding the action to quadratic order in h. The *near zone* (aka "system zone") nonlinearities *do* enter at +1PN, and are treated in Sec. II C.

We use spherical harmonics to decompose the fields and sources to 11 families of fields, comprised of the seven scalar families h_{tt}, h_{tr}, h_{rr}, h_{tS}, h_{rS}, h_S, $\tilde{h}_S$, the three vector families $h_{t\aleph}$, $h_{r\aleph}$, $h_{\aleph}$, and the single tensor family $h_{\aleph\beth}$, defined through (see Appendix B)

$$\begin{aligned}
h_{tt} &= \sum\!\!\!\!\!\!\int h^{L\omega}_{tt}n_L e^{-i\omega t} \triangleq \int\frac{d\omega}{2\pi}\sum_L h^{L\omega}_{tt}n_L e^{-i\omega t},\\
h_{tr} &= \sum\!\!\!\!\!\!\int h^{L\omega}_{tr}n_L e^{-i\omega t}, \qquad h_{rr} = \sum\!\!\!\!\!\!\int h^{L\omega}_{rr}n_L e^{-i\omega t},\\
h_{t\Omega} &= \sum\!\!\!\!\!\!\int (h^{L\omega}_{t}\partial_\Omega n_L + h^{L\omega}_{t\aleph}n^L_{\aleph\Omega})e^{-i\omega t}, \qquad h_{r\Omega} = \sum\!\!\!\!\!\!\int (h^{L\omega}_{r}\partial_\Omega n_L + h^{L\omega}_{r\aleph}n^L_{\aleph\Omega})e^{-i\omega t},\\
h_{\Omega\Omega'} &= \sum\!\!\!\!\!\!\int [h^{L\omega}_S n^L_{\Omega\Omega'} + \tilde{h}^{L\omega}_S\tilde{n}^L_{\Omega\Omega'} + h^{L\omega}_{\aleph}n^L_{\aleph\Omega\Omega'} + h^{L\omega}_{\aleph\beth}n^L_{\aleph\beth\Omega\Omega'}]e^{-i\omega t},
\end{aligned} \tag{2.4}$$

where we use the scalar multipoles n^L, the vector multipoles $n^L_{\aleph\Omega}$ and the tensor multipoles $n^L_{\aleph\aleph'\Omega\Omega'}$ described in Appendix B. They are dimensionless and depend only on the angular coordinates. We shall at times omit L, ω indices for brevity.

Substituting the new fields (2.4) into the homogenous action (2.2) and using the definitions (B4), derivative relations (B6)–(B11) and normalization relations (B12), we find the homogenous action decomposes into three independent sectors, namely spherical scalar, vector and tensor.

At this point we follow the prescription generally outlined in Ref. [27] and applied in Ref. [28] for higher-dimensional black holes to write the action in terms of *gauge-invariant* fields. We solve for the algebraic fields [three scalars (C10) and $\hat{d}$ vectors (C17)] and reduce further over the gauge degrees of freedom (again, $\hat{d}$ free functions in the vector sector and three in the scalar). For the full derivation, see Appendix C. Using our d gauge (diffeo) functions [each one eliminating two degrees of freedom

[4] In terms of which the overall prefactor of the Einstein-Hilbert action is $\frac{1}{16\pi\mathbf{G}_d}$ in arbitrary dimension and black hole entropy is given by $\frac{A_H}{4\mathbf{G}_d}$.

[5] See Appendix A; henceforth, covariant derivatives are associated with this metric.

([27])], after the dust settles we are left with three types of gauge-invariant master fields, one type in each sector [36–42]; we call them $\mathfrak{h}^{L\omega}$, $\mathfrak{h}^{L\omega}_{\aleph}$, $\mathfrak{h}^{L\omega}_{\aleph\beth}$. These appear in the action in similar forms [see Appendix C, in particular (C5), (C16), and (C26)]:

$$S_{\rm EH} = \sum\!\!\!\!\!\!\int \frac{N_{\ell,\hat d}}{8(\hat d+1)G_d}\int dr r^{2\ell+\hat d+1}\left[\frac{4(\hat d+1)(\ell-1)\ell}{\hat d(\ell+\hat d+1)(\ell+\hat d)}\mathfrak{h}^*\mathfrak{L}\mathfrak{h} + \frac{4(\ell-1)(\ell+\hat d)}{(\ell+\hat d+1)\ell}\mathfrak{h}^*_{\aleph}\mathfrak{L}\mathfrak{h}_{\aleph} + \hat d^2 c_s(c_s-\hat d)\mathfrak{h}^*_{\aleph\beth}\mathfrak{L}\mathfrak{h}_{\aleph\beth}\right], \tag{2.5}$$

$$\mathfrak{L} = \left(\omega^2 + \partial_r^2 + \frac{2\ell+\hat d+1}{r}\partial_r\right), \tag{2.6}$$

where $N_{\ell,\hat d} = \frac{\Gamma(1+\hat d/2)}{2^\ell\Gamma(\ell+1+\hat d/2)} = \frac{\hat d!!}{(2\ell+\hat d)!!}$. We note that for $d=4$ the scalar and vector expressions match correspondingly Eq. (3.80) of [1]; there is no tensor sector in 4D.

For the inhomogenous (source) part of the action (2.3), we similarly decompose the sources $T^{\mu\nu}$ to T^{tt}, T^{tr}, T^{rr}, T^t, T^r, T^S, $\tilde T^S$, $T^{t\aleph}$, $T^{r\aleph}$, $T^{\aleph}$, $T^{\aleph\beth}$ as in (D1). Collecting the sources into the combinations $\mathcal{T}$, $\mathcal{T}^{\aleph}$, $\mathcal{T}^{\aleph\beth}$ matching the gauge-invariant master fields [see Appendix D, in particular (D15)–(D18)], we find that the action for the interaction of the fields with their sources is given by (D14), and the complete action in our gauge-invariant variables is comprised of three parts:

$$S = S^S + S^V + S^T, \tag{2.7}$$

where

$$S^\epsilon = \frac{1}{2}\sum\!\!\!\!\!\!\int \int dr\left[\frac{N_{\ell,\hat d}}{G_d R^\epsilon_{\ell,\hat d}} r^{2\ell+\hat d+1}\mathfrak{h}^*_\epsilon\mathfrak{L}\mathfrak{h}^\epsilon - (\mathfrak{h}^*_\epsilon\mathcal{T}^\epsilon + \text{c.c.})\right] \tag{2.8}$$

for all sectors $\epsilon\in\{S,V,T\}$, and

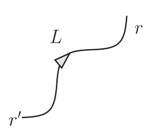
$$R^S_{\ell,\hat d} = \frac{\hat d(\ell+\hat d+1)(\ell+\hat d)}{(\ell-1)\ell},$$
$$R^V_{\ell,\hat d} = \frac{(\hat d+1)(\ell+\hat d+1)\ell}{2(\ell-1)(\ell+\hat d)},$$
$$R^T_{\ell,\hat d} = \frac{8(\hat d+1)}{\hat d^2 c_s(c_s-\hat d)}. \tag{2.9}$$

Variation of (2.8) yields the master wave equation [compare Eqs. (2.14), (3.18), (3.26) of Ref. [3], and Eqs. (2.9), (3.8), (3.49), (3.57) of Ref. [1][6]]

$$0 = \frac{\delta S}{\delta\mathfrak{h}^{L\omega*}_\epsilon} = \frac{N_{\ell,\hat d}}{G_d R^\epsilon_{\ell,\hat d}} r^{2\ell+\hat d+1}\left(\omega^2+\partial_r^2+\frac{2\ell+\hat d+1}{r}\partial_r\right)\mathfrak{h}^\epsilon_{L\omega} - \mathcal{T}^\epsilon_{L\omega}. \tag{2.10}$$

B. Feynman rules: Propagators and vertices

The solutions to the homogenous part of such wave equations are composed using the origin-normalized Bessel functions $\tilde j_{\ell+\hat d/2}(\omega r)$ and $\tilde h^+_{\ell+\hat d/2}(\omega r)$ (see Appendix E and Ref. [3]). Hence, in the language of Feynman diagrams, we can represent the retarded propagator for the various waves using the diagram



$$\equiv G^\epsilon_{ret}(r',r) = -i\omega^{2\ell+\hat d}\; G_d\; C^\epsilon_{\ell,\hat d}\;\tilde j_{\ell+\hat d/2}(\omega r_1)\;\tilde h^{\pm}_{\ell+\hat d/2}(\omega r_2)\;; \tag{2.11}$$

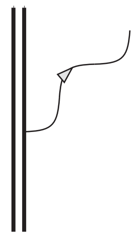
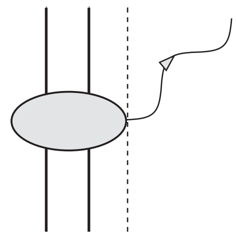
with $r_1 \coloneqq \min\{r',r\}$, $r_2 \coloneqq \max\{r',r\}$, $C^\epsilon_{\ell,\hat d} = M_{\ell,\hat d}R^\epsilon_{\ell,\hat d}$ and

$$M_{\ell,\hat d} = \frac{\pi}{2^{\ell+1+\hat d}\Gamma(1+\hat d/2)\Gamma(\ell+1+\hat d/2)}. \tag{2.12}$$

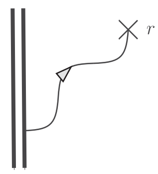
In particular, $M_{\ell,\hat d} = [\hat d!!(2\ell+\hat d)!!]^{-1}$ in odd $\hat d$; $M_{\ell,\hat d} = \frac{\pi}{2}[\hat d!!(2\ell+\hat d)!!]^{-1}$ in even $\hat d$.

[6]Notice that our subscripts $\ell,\hat d$ in $R^\epsilon_{\ell,\hat d}$ carry a different meaning than the subscript s in Ref. [1]'s R^ϵ_s. The s in Ref. [1] denotes the field type, which for gravity is always 2; we choose to emphasize the $\ell,\hat d$ dependence.

The vertices (source multipole terms) in the radiation zone are found by matching with the system zone according to the diagrammatic definition

$$-Q^{\epsilon}_{L\omega} =$$  $$:=$$  , (2.13)

where the blob on the right-hand side means one should sum over all possible near-zone diagrams with an outgoing radiation leg. From the radiation zone's point of view the sources $Q^{L\omega}$ are located at the origin $r = 0$; hence the radiation zone field can be written as

$$\mathfrak{h}^{\epsilon,EFT}_{L\omega}(r) =$$ 

$$= -Q^{L\omega}_{\epsilon}\left(-i\omega^{2\ell+\hat{d}}\, G_d\, C^{\epsilon}_{\ell,\hat{d}}\right)\tilde{h}^{+}_{\ell+\hat{d}/2}(\omega r)\ . \tag{2.14}$$

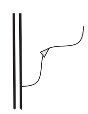
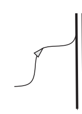
On the other hand, in the full theory we may also use spherical waves to obtain the field outside the source as

$$\begin{aligned}\mathfrak{h}^{\epsilon}_{L\omega}(r) &= -\int dr' \mathcal{T}^{\epsilon}_{L\omega}(r') G^{\epsilon}_{\text{ret}}(r', r) \\ &= -\left[\int dr' \mathcal{T}^{\epsilon}_{L\omega}(r')\tilde{j}_{\ell+\hat{d}/2}(\omega r')\right](-i\omega^{2\ell+\hat{d}} G_d C^{\epsilon}_{\ell,\hat{d}})\tilde{h}^{+}_{\ell+\hat{d}/2}(\omega r).\end{aligned} \tag{2.15}$$

From the comparison between (2.14) and (2.15) we read off the source multipoles as

$$Q^{L\omega}_{\epsilon} = \int dr \tilde{j}_{\ell+\hat{d}/2}(\omega r)\mathcal{T}^{\epsilon}_{L\omega}(r). \tag{2.16}$$

Following Refs. [1,26], we call this matching process a "zoom-out balayage"[7] of the original source distribution $\mathcal{T}^{\epsilon}_{\omega}(\vec{r})$ out to the multipole $Q^{L\omega}_{\epsilon}$, carried out through propagation with $\tilde{j}_{\ell+\hat{d}/2}(\omega r)$. Thus, we find both the source vertex and vertex for the doubled (hatted) source,

 $$= -Q^{L\omega}_{\epsilon}$$,  $$= -\hat{Q}^{L\omega}_{\epsilon *}$$. (2.17)

At this point we first introduce field doubling and employ the classically adapted Closed Time Path formalism; we work in the Keldysh basis. For a detailed description, see Refs. [1,33].

It is worthy of note that the series expansions of both $\tilde{j}_{\ell+\hat{d}/2}$ (E3) and of the sources $\mathcal{T}^{\epsilon}_{L\omega}$ (D15), (D17), (D18) include only integer powers of ω. This implies analyticity of the multipoles $Q(\omega)$ as functions of (complex) frequency, for any ℓ and $\hat{d}$, which in turn results in local-in-time expressions for the time-domain multipoles $Q(t)$—that is, the multipoles at a certain time t themselves depend on the energy-momentum tensor at the same t alone and do not contain tails. The effective action $\hat{S}$ *can* still contain tails—this will happen in the case of noneven dimension [45–47]. Using the inverse transformations (D2)–(D5),

[7]French for "sweeping away"; Poincaré coined the term [43,44], describing the process where a charge distribution in some spatial region is "swept away" to the boundary of that region, leaving the potential outside unchanged.

source terms (D15)–(D18), integration by parts and the modified Bessel equation (E2), we find the *radiation source multipoles* in the frequency domain:

$$Q_S^{L\omega} = \frac{(\hat{d}+1)\ell}{\hat{d}c_s(\ell+\hat{d}+1)}\int d^Dx x^L_{\mathrm{STF}}\Big[\hat{d}T^{tt}_\omega\Big(\frac{c_s}{\hat{d}+1}+\ell+\hat{d}-\frac{\omega^2r^2}{\hat{d}}+r\partial_r\Big) + 2i\omega r T^{tr}_\omega(\ell+\hat{d}+1+r\partial_r) - \omega^2r^2T^{rr}_\omega + T^{aa}_\omega\Big(\frac{c_s}{\hat{d}+1}+\ell+\hat{d}+r\partial_r\Big)\Big]\tilde{j}_{\ell+\hat{d}/2}(\omega r), \tag{2.18}$$

$$Q_\aleph^{L\omega} = \frac{2\epsilon^{(D)}_{\aleph abk_\ell}}{\ell+\hat{d}+1}\int d^Dx[r^{-(\ell+\hat{d})}x^{bL-1}T^{ta}_\omega\partial_r(r^{\ell+\hat{d}+1}\tilde{j}_{\ell+\hat{d}/2}(\omega r)) + i\omega T^{ac}_\omega x^{bcL-1}\tilde{j}_{\ell+\hat{d}/2}(\omega r)], \tag{2.19}$$

$$Q_{\aleph\beth}^{L\omega} = \frac{\ell(\ell-1)}{2}\epsilon^{(D)}_{\aleph abk_\ell}\epsilon^{(D)}_{\beth a'b'k_{\ell'}}\int d^Dx x^{bb'L-2}T^{aa'}_\omega\tilde{j}_{\ell+\hat{d}/2}(\omega r). \tag{2.20}$$

Fourier-transforming gives the multipoles in the time domain, recorded explicitly in (4.1), (4.2), (4.3). These may also be written using the $\hat{d}$-dimensional generating time-weighted function $\delta_{\ell,\hat{d}}(z)$ defined in Ref. [3] (following the 4D definition [48,49]) as

$$Q^L_\epsilon = \int d^Dx x^L_{\mathrm{STF}}\int_{-1}^1 dz\delta_{\ell,\hat{d}}(z)\mathcal{T}^\epsilon(\vec{r},u+zr). \tag{2.21}$$

The multipoles represent the gravitational source scalar, vector and tensor multipole moments. A simple test of substituting $d=4$ shows these multipoles coincide (for any ℓ) with the scalar and vector multipoles given by Refs. [1,26] [see Eqs. (3.88) and (3.89) of Ref. [1]]; there is no tensor sector in 4D. In general d, for a system of N masses at leading PN order (low velocities, weak potential), the leading multipoles (of order $\ell=2$, or quadrupole) of each sector are the mass (scalar, or electric) quadrupole, the gravitational source current (vector, or magnetic, +1PN) quadrupole,[8] and the mass-tensor quadrupole (+2PN), respectively:

$$Q_S^{ij} \to \sum_{A=1}^n m_A\Big(x^ix^j-\frac{1}{D}\delta^{ij}x^2\Big)_A, \tag{2.22}$$

$$Q_\aleph^{ij} \to 2\sum_{A=1}^n m_A(x^i\epsilon^j_{\aleph ab}x^av^b)^{TF}_A, \tag{2.23}$$

$$Q_{\aleph\beth}^{ij} \to \sum_{A=1}^n m_A(\epsilon^i_{\aleph ab}x^av^b\epsilon^j_{\aleph cd}x^cv^d)^{TF}_A. \tag{2.24}$$

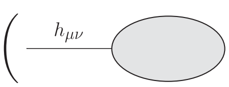
We note again at this point that in order to go *beyond* the +1PN approximation, more Feynman rules that correspond to radiation zone interactions need to be included; see the discussion in Secs. II A and III C.

[8]The factor of 2 between the gravitational source current and the mass current follows Appendix A.3 of Ref. [1].

C. Source nonlinearities

In order to describe the radiative multipoles of a system of masses beyond the leading orders, we prescribe how the system's $T^{\mu\nu}$ is given by the masses and their coordinates. From the inhomogenous coupling (2.3), $T^{\mu\nu}$ is the source for $h_{\mu\nu}$, represented as a one-point diagram (compare Refs. [1,50]),

$$T^{\mu\nu}(x) := -2\frac{\delta S_{eff}}{\delta h_{\mu\nu}(x)} \equiv -2\Big(\quad \Big). \tag{2.25}$$



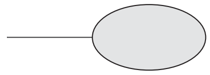
In the system zone we describe the metric using the three NRG fields $h_{\mu\nu}\leftrightarrow(\phi,\vec{A},\overleftrightarrow{\sigma})$, defined by

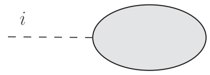
$$ds^2 = e^{2\phi}(dt-2\vec{A}\cdot d\vec{x})^2 - e^{-2\phi/\hat{d}}(\delta_{ij}+\sigma_{ij})dx^idx^j. \tag{2.26}$$

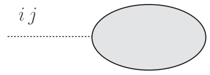
This definition almost matches that of Refs. [11,35], but it includes a factor of 2 in the definition of the coupling of the gravitomagentic vector potential $\vec{A}$, as introduced in the 4D case in Eq. (A10) of Ref. [1]. The corresponding sources are given by $T^{\mu\nu}\leftrightarrow(\rho_\phi,\vec{J},\overleftrightarrow{\Sigma})$, defined by

$$S_{\mathrm{mat}} = -\rho_\phi\phi + \vec{J}\cdot\vec{A} - \frac{1}{2}\Sigma^{ij}\sigma_{ij}. \tag{2.27}$$

This implies

$$T^{tt} + \frac{1}{\hat{d}} T^{ii} = \rho_\phi = -\Big($$  $$\Big),$$

$$T^{ti} = J^i =$$  $$,$$

$$T^{ij} = -\Sigma^{ij} = 2$$  $$. \quad (2.28)$$

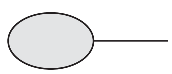
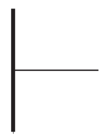
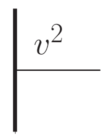
We refer to ρ_ϕ as the gravitational mass density, to $\vec{J}$ as the gravitational source current, and to Σ^{ij} the gravitational stress; together they comprise the NRG source fields. We further mark $\Sigma = \Sigma^{aa}$ and note that $T^{tt} = \rho_\phi + \frac{1}{\hat{d}}\Sigma$. In terms of these source fields, we rewrite the expression for the radiation source multipoles (2.18), (2.19), and (2.20)—after Fourier-transforming to the time domain—as follows:

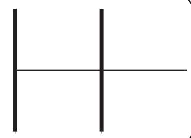
$$Q_S^L = \frac{(\hat{d}+1)}{\hat{d}(\ell+\hat{d})(\ell+\hat{d}+1)} \int d^D x x_{\mathrm{STF}}^L \Big[\hat{d}\Big(\frac{(\ell+\hat{d})(\ell+\hat{d}+1)}{\hat{d}+1} + r\partial_r\Big)\tilde{j}_{\ell+\hat{d}/2}(ir\partial_t)\rho_\phi - 2r^{-(\ell+\hat{d})}(r^{\ell+\hat{d}+1}\tilde{j}_{\ell+\hat{d}/2}(ir\partial_t))'\vec{x}\cdot\dot{\vec{J}} + \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t)\Big(r^2\Big(\ddot{\rho}_\phi + \frac{1}{\hat{d}}\ddot{\Sigma}\Big) - x_a x_b \ddot{\Sigma}^{ab}\Big)\Big], \quad (2.29)$$

$$Q_{\aleph}^L = \frac{\epsilon^{(D)}_{\aleph a b k_\ell}}{\ell+\hat{d}+1} \int d^D x \Big[(r^{\ell+\hat{d}+1}\tilde{j}_{\ell+\hat{d}/2}(ir\partial_t))' \frac{x^{bL-1}}{r^{\ell+\hat{d}}} J^a + \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t)\dot{\Sigma}^{ac} x^{bcL-1}\Big], \quad (2.30)$$

$$Q_{\aleph\beth}^L = -\frac{\ell(\ell-1)}{2} \epsilon^{(D)}_{\aleph a b k_\ell} \epsilon^{(D)}_{\beth a' b' k_{\ell'}} \int d^D x x^{bb'L-2} \Sigma^{aa'}(\vec{r},t) \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t). \quad (2.31)$$

We remark that upon substitution of $d=4$, equations (2.29), (2.30), and (2.31) reproduce Eq. (3.105) of Ref. [1] [using (E2)]. To use the NRG source fields defined in (2.28), we expand the diagrams perturbatively in PN orders. We now focus on the N-body problem of masses $\{m_A\}_{A=1}^N$ whose trajectories are described by $\{\vec{x}_A(t)\}_{A=1}^N$ and influenced by their gravitational interactions. For such systems, the gravitational mass density at leading and next-to-leading (+1PN) orders is found by Taylor-expanding (2.26) and varying with respect to ϕ [using (2.27)] to be

$$\rho_\phi(\vec{x},t) = -\Big($$  $$\Big) = -\Big($$  $$+$$ 

$$+$$  $$\Big)$$

$$= \sum_{A=1}^{n} m_A\,\delta(\vec{x} - \vec{x}_A(t))\left(1 + \frac{\hat{d}+2}{2\hat{d}} v_A^2(t) - \sum_{B\neq A} \frac{G_d m_B}{\|\vec{x}_A(t) - \vec{x}_B(t)\|^{\hat{d}}}\right). \quad (2.32)$$

When other (not only gravitational) interactions are involved in maintaining the bound system (see Sec. I), they also contribute at +1PN. Their potential energy is to be included in the nonlinear correction (2.32), similarly to its last term (or in its stead, if the gravitational potential is much weaker than the PN parameter, $\phi_{\text{gravity}} \ll v^2 \sim \phi_{\text{other}}$).

Nonlinear contributions to the additional NRG source fields $\vec{J}$ and Σ^{ij} are to be derived in similar fashion. As we here calculate only up to +1PN, the leading orders of $\vec{J}$ and Σ^{ij} suffice:

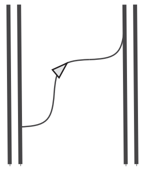
$$\vec{J} = 2\sum_{A=1}^{n} m_A \vec{v}_A \delta(x - x_A), \quad (2.33)$$

$$\Sigma^{ij} = \sum_{A=1}^{n} m_A v_A^i v_A^j \delta(x - x_A). \quad (2.34)$$

These, of course, match Eqs. (3.107) and (3.108) of Ref. [1] for $\hat{d} = 1$.

III. APPLICATION AND RESULTS

Having set up the apparatus of Feynman rules (by sector ϵ and by multipole order ℓ, in Sec. II B), we now diagrammatically construct the RR effective action [1–3]. We expand order by order in the PN expansion, i.e., in the small parameter $v^2 \sim \frac{G_d M}{r^{\hat{d}}}$, and recalling that $\omega r \sim v^{1/2}$ counts as half an order. A general contribution to this effective action is given by a vertex-propagator-vertex diagram,



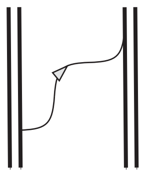
$$\equiv Q_\epsilon^{L\omega} G_{ret}^\epsilon (r=0, r'=0) \hat{Q}_\epsilon^{L\omega\,*} = -i\omega^{2\ell+\hat{d}}\, G_d\, C_{\ell,\hat{d}}^\epsilon\, Q_\epsilon^{L\omega} \hat{Q}_\epsilon^{L\omega\,*} + c.c. \tag{3.1}$$

Below, we calculate the RR effective action to the leading and next-to-leading orders; we then discuss higher-order contributions.

A. Leading order

The only contribution at leading order (LO) is that of the scalar quadrupole ("mass quadrupole," $s=0, \ell=2$, also called E2):

$$S_{eff,\omega}^{LO} =$$



$$= -i\omega^{\hat{d}+4}\, G_d\, C_{2,\hat{d}}^S\, Q_S^{L_2\omega} \hat{Q}_{S*}^{L_2\omega} + c.c.$$

$$= -i\omega^{\hat{d}+4}\, G_d \frac{\pi\, \hat{d}(\hat{d}+3)(\hat{d}+2)}{2^{\hat{d}+4}\, \Gamma(3+\hat{d}/2)\Gamma(1+\hat{d}/2)}\, Q_S^{L_2\omega} \hat{Q}_{S*}^{L_2\omega} + c.c. \tag{3.2}$$

Following the procedure of Ref. [1] in Eqs. (2.57)–(2.61) and Sec. III. C. 1, we can immediately find from this expression the RR force on the system's constituent masses, the effective potential, and the energy dissipated as radiation. In the remainder of this section we will focus, for clarity, on time-domain results for even spacetime dimensions d (odd $\hat{d}$), where (3.2) reduces to

$$S_{\rm eff}^{\rm LO,even} = (-)^{\frac{\hat{d}+1}{2}} \frac{\hat{d}(\hat{d}+3)(2+\hat{d})}{2\hat{d}!!(\hat{d}+4)!!} G_d \hat{Q}_S^{L_2} \partial_t^{\hat{d}+4} Q_S^{L_2}. \tag{3.3}$$

The LO RR force is given by[9]

$$F_{\rm SF}^{i,\rm LO,even} = \frac{\delta S_{\rm eff}^{\rm LO,even}}{\delta \hat{x}^i(t)}$$
$$= (-)^{\frac{\hat{d}+1}{2}} G_d \frac{\hat{d}(\hat{d}+3)(2+\hat{d})}{4\hat{d}!!(\hat{d}+4)!!} \frac{\delta \hat{Q}_S^{ij}}{\delta \hat{x}^i(t)} \partial_t^{\hat{d}+4} Q_S^{ij}, \tag{3.4}$$

and it can be represented by a potential which generalizes the Burke-Thorne potential [51] to any even dimension,

$$V_{\rm BT}^{\rm even} = (-)^{\frac{\hat{d}+1}{2}} G_d m x^i x^j \frac{\hat{d}(\hat{d}+3)(2+\hat{d})}{4\hat{d}!!(\hat{d}+4)!!} \partial_t^{\hat{d}+4} Q_S^{ij}. \tag{3.5}$$

The LO dissipated energy is given by

$$\frac{dE^{\rm LO,even}}{d\omega} = \omega^{\hat{d}+5} \frac{\hat{d}(\hat{d}+3)(2+\hat{d})}{4\hat{d}!!(\hat{d}+4)!!} G_d |Q_S^{ij}(\omega)|^2, \tag{3.6}$$

which in the time domain gives the dissipated power as

$$\langle P_{\rm rad}^{\rm LO,even} \rangle = \frac{\hat{d}(\hat{d}+3)(2+\hat{d})}{4\hat{d}!!(\hat{d}+4)!!} G_d \langle \partial_t^{\frac{\hat{d}+5}{2}} Q_S^{ij}(t) \rangle^2, \tag{3.7}$$

using $Q_{ij}^S(t)$ to LO as given in (2.22). We note that for $d=4$, (3.4) and (3.5) reproduce precisely the well-known results for the Burke-Thorne potential and RR force, and that for any d, (3.6) reproduces precisely the generalized quadrupole formula of Cardoso *et al.*, (1.1) [24][10]; (3.7) matches Einstein's quadrupole formula in $d=4$ [25] and generalizes it to any even dimension.

Equation (3.2) gives the leading-order RR effective action (and allows the other quantities to be equally derived from it) in any dimension. Below, we will discuss the higher-order contributions; we compute the +1PN correction and give an outline of higher-order contributions. Although in this section we focus on even d, our frequency domain results hold for odd d[11] as well. The main difference in odd dimensions appears when Fourier-transforming back to the time domain, where due to nonanalytic features

[9] Notice that a factor of $\frac{1}{\ell!} = \frac{1}{2}$ from switching from multi-index notation L_2 to usual indices ij; see Sec. I B.

[10] In order to revert from G_d to G, see Sec. I B.

[11] One may also formally discuss the RR effective action in noninteger d; see Ref. [3].

(branch cuts) of the effective action as a function of (complex) frequency, the transformation introduces non-local (in time) tail terms in the effective action (see Secs. II. C and IV of Ref. [3] for a thorough discussion, and see Sec. IV here).

B. Next-to-leading order (+1PN)

At the next-to-leading post-Newtonian order (NLO, which is +1PN), four effects must be considered; we shall label them E3, E2δ^1, E2$\hat{\delta}^1$ and M2, as we now detail. The scalar quadrupole is supplemented by the scalar octupole ($s = 0, \ell = 3$, aka E3), found exactly as in (3.2), but with $\ell = 3$. In addition, the scalar quadrupole itself becomes more complicated, as it must include +1PN corrections (specialized to the N-body problem),

$$Q_S^{L_2} \to Q_S^{L_2} + \delta^1 Q_S^{L_2} = \sum_{A=1}^{n} m_A x_A^{L_2} + \delta^1 Q_S^{L_2}, \tag{3.8}$$

where $\delta^1 Q_S^{L_2}$ includes five possible corrections: the +1PN nonlinear corrections to the gravitational mass ($\delta^1 Q_S^{L_2} NL1$), the contribution of the gravitational source current ($\delta^1 Q_S^{L_2} J$), the contribution of a retardation effect from expanding the Bessel function $\tilde{j}_{\ell+\hat{d}/2}(\omega r)$ to sub-leading order ($\delta^1 Q_S^{L_2} b$), a term with double time derivatives of ρ_ϕ ($\delta^1 Q^S_{L_2} \partial^2$), and a term with double time derivatives from the derivative of the Bessel function ($\delta^1 Q_S^{L_2} \partial b$). Altogether, for any $\ell, \hat{d}$ we find the first correction to be

$$\begin{aligned}\delta^1 Q_S^L &= \delta^1 Q_S^L NL1 + \delta^1 Q_S^L J + \delta^1 Q_S^L b + \delta^1 Q_S^L \partial^2 + \delta^1 Q_S^L \partial b \\ &= \sum_{A=1}^{n} \left[m x^L \left(\left(\frac{\hat{d}+2}{2\hat{d}} v^2 - \sum_{B\neq A} \frac{G_d m_B}{\|\vec{x}-\vec{x}_B\|^{\hat{d}}} \right) + \frac{2(\hat{d}+1)}{\hat{d}(\ell+\hat{d})} i\omega(\vec{x}\cdot\vec{v}) - \frac{(\hat{d}^2+(\ell+4)\hat{d}+4)\omega^2 r^2}{2\hat{d}(\ell+\hat{d})(2\ell+\hat{d}+2)} \right) \right]_A,\end{aligned} \tag{3.9}$$

and specifically for $\ell = 2$ (the correction relevant to +1PN order),

$$\delta^1 Q_S^{L_2\omega} = \sum_{A=1}^{n} \left[m x^{L_2} \left(\left(\frac{\hat{d}+2}{2\hat{d}} v^2 - \sum_{B\neq A} \frac{G_d m_B}{\|\vec{x}-\vec{x}_B\|^{\hat{d}}} \right) + \frac{2(\hat{d}+1)}{\hat{d}(\hat{d}+2)} i\omega(\vec{x}\cdot\vec{v}) - \frac{(\hat{d}^2+6\hat{d}+4)\omega^2 r^2}{2\hat{d}(\hat{d}+2)(\hat{d}+6)} \right) \right]_A. \tag{3.10}$$

This exactly coincides with Eqs. (3.109) and (3.110) of Ref. [1] for $d = 4$, as obtained in Ref. [52] and reproduced in Ref. [8] within the EFT approach. At +1PN, The correction $\delta^1 Q^S_{L_2}$ itself can appear in either the source vertex Q^S (E2δ^1) or the hatted vertex $\hat{Q}^S$ (E2$\hat{\delta}^1$).

Finally, the +1PN action includes also the leading-order vector quadrupole ($s = 1, \ell = 2$, aka M2), which includes different $Q_{\aleph}^{L_2\omega} \hat{Q}_{\aleph}^{L_2\omega}$ terms. We sum over the $\aleph$ indices to define the bivector multipoles

$$Q_M^{L\omega} = \frac{1}{\ell+\hat{d}+1} \int d^D x \left[\vec{r} \wedge \left(\frac{1}{r^{\ell+\hat{d}}} (r^{\ell+\hat{d}+1} \tilde{j}_{\ell+\hat{d}/2}(\omega r))' \vec{J} + \tilde{j}_{\ell+\hat{d}/2}(\omega r) \vec{r} \cdot \dot{\overset{\leftrightarrow}{\Sigma}} \right) \right]^{(k_\ell} x^{L-1)}. \tag{3.11}$$

We accordingly define $\hat{Q}_M^L$ and replace $C^V_{\ell,\hat{d}}$ with

$$C^M_{\ell,\hat{d}} = C^V_{\ell,\hat{d}} * \frac{D_\ell(\hat{d}+1,1)}{D_\ell(\hat{d}+1,0)}, \tag{3.12}$$

which accounts for the summation over the different $\aleph$ combinations using (B5), following Ref. [3]. At +1PN order, for the vector contributions we can set $\tilde{j}_{\ell+\hat{d}/2}(\omega r) = 1$ and $\overset{\leftrightarrow}{\Sigma} = 0$; the other terms only enter at +2PN and onwards.

With these definitions, the four contributions combine to produce the radiation-reaction effective action to NLO, given by

$$\begin{aligned}S^{\mathrm{NLO}}_{\mathrm{eff},\omega} = &-i\omega^{\hat{d}+4} G_d C^S_{2,\hat{d}} (Q_S^{L_2\omega} \hat{Q}_S^{L_2\omega *} + \delta^1 Q_S^{L_2\omega} \hat{Q}_S^{L_2\omega *} + Q_S^{L_2\omega} \delta^1 \hat{Q}_S^{L_2\omega *}) \\ &- i\omega^{\hat{d}+6} G_d C^S_{3,\hat{d}} Q^S_{L_3\omega} \hat{Q}^{S*}_{L_3\omega} - i\omega^{\hat{d}+4} G_d C^M_{2,\hat{d}} Q_M^{L_2\omega} \hat{Q}_M^{L_2\omega *} + \mathrm{c.c.}\end{aligned} \tag{3.13}$$

C. Beyond +1PN

Our method allows identification and enumeration of higher-PN-order corrections as well. While we do not calculate them explicitly here, it is convenient to see how they too fall under the general prescription. Effects such as higher-order multipoles (ℓ), higher-spin (s) sectors, retardation effects or nonlinearities in the source terms all introduce integer PN order corrections, and will thus only enter at +2PN. Radiation-zone nonlinear corrections and the effects of spin (intrinsic angular momentum) enter at different PN orders depending on the dimension, as we describe below.

1. +2PN

At +2PN, our formalism captures easily all the relevant near-zone contributions. The tensor field first enters at this order via the tensor quadrupole ($s = 2, \ell = 2$, aka T2). The vector sector now also includes the vector octupole ($s = 1, \ell = 3$, aka M3), as well as +1PN corrections to the vector quadrupole (M2δ^1) and (M2$\hat{\delta}^1$), which arise from the contributions of the gravitational stress ($\delta^1 Q^V_{L_2}\Sigma$), the contribution of two time derivatives of the gravitational current ($\delta^1 Q^V_{L_2}\partial^2$), a retardation effect ($\delta^1 Q^V_{L_2} b$), and a derivative of the Bessel function ($\delta^1 Q^V_{L_2}\partial b$). The scalar sector now goes up to the scalar hexadecapole ($s = 0, \ell = 4$, aka E4), while corrections to the lower scalar multipoles include five analog corrections to the octupole ($\delta^1 Q^S_{L_3}\partial^2$NL1, $\delta^1 Q^S_{L_3}$J, $\delta^1 Q^S_{L_3}\partial^2$, $\delta^1 Q^S_{L_3} b$, $\delta^1 Q^S_{L_3}\partial b$) and several new corrections to the quadrupole. For example, the contribution of $T2$ is given by

$$S^{T2}_{\mathrm{eff},\omega} = -i\omega^{\hat{d}+4} G_d C^T_{2,\hat{d}} Q^{L_2\omega}_{\aleph\beth} \hat{Q}^{L_2\omega*}_{\aleph\beth}. \tag{3.14}$$

2. Radiation zone corrections

We have thus far taken the propagator to be linear in the radiation zone (2.11). However, there are nonlinear corrections to the propagator, which may be included through systematic inclusion of nonlinear terms in the action (2.2); see also the discussion in Sec. II A. The first of these contributions is represented by Fig. 2 and is interpreted as scattering of the outgoing waves off the background curvature generated by the entire system's total mass M (labeling the vertex in Fig. 2). Its value includes a factor of the gravitational potential $G_d M/\lambda^{\hat{d}}$, where λ is the typical wavelength for radiation. As $\lambda \sim \omega^{-1}$, the value of such contributions is suppressed by at least $(\omega r)^{\hat{d}} G_d M/r^{\hat{d}} \sim v^{\hat{d}} G_d M/r^{\hat{d}}$ relative to the leading order. In PN terms, this is equivalent to $+(1 + \frac{\hat{d}}{2})$PN order. In $d = 4$, it implies a +1.5PN contribution; it is suppressed even further in higher dimensions. For further discussion and calculations for the 4D case, see Refs. [1,8,10,53,54]. In the high-d limit, linearized gravity suffices.

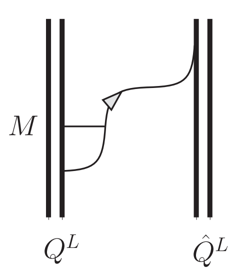


FIG. 2. Correction to the radiation reaction due to nonlinear interaction in the radiation zone.

3. Spin effects

Depending on the system's parameters and nature of its constituents, spin effects may also need to be considered. This adds complexity and introduces new interesting effects which are worth exploring also in higher dimensions. The important scales determining the PN order—in addition to the typical orbital separation R and typical orbital velocity v—are the typical size of the spinning body r_0, its moment of inertia $I \sim m r_0^2$, and its typical angular frequency ω_s; together these give the spin

$$S \sim I\omega_s \sim m r_0^2 \omega_s. \tag{3.15}$$

Spins can couple to the orbital angular momentum

$$L \sim mvR \tag{3.16}$$

or to each other. Couplings of the first type (spin-orbit, S-O) appear in the action (at leading order and up to dimensionless factors composed of the masses) as

$$S_{SO} = \#_{SO} \cdot \int dt \frac{G}{r^{\hat{d}+2}} S \cdot L \sim \int dt \frac{Gm^2}{R^{\hat{d}}} v \frac{r_0^2 \omega_s}{R}, \tag{3.17}$$

while couplings of the second type (spin-spin, S-S) appear (again at LO and up to the masses) as

$$S_{SS} = \#_{SS} \cdot \int dt \frac{G}{r^{\hat{d}+2}} S \cdot S \sim \int dt \frac{Gm^2}{R^{\hat{d}}} \frac{r_0^4 \omega_s^2}{R^2}, \tag{3.18}$$

where in the above equations we omit the indices of spin and angular momentum tensors since we are interested only in orders of magnitude. In terms of the post-Newtonian parameter v^2, we recall that $\frac{Gm^2}{R^{\hat{d}}}$ is the leading order, and that

$$\frac{r_0}{R} \gtrsim v^{2/\hat{d}}, \tag{3.19}$$

where similarity occurs in the case of highly compact objects such as black holes (see Ref. [55]). We focus on two interesting cases: corotation ($\omega_s \sim \omega \sim \frac{v}{R}$) and maximal spin

($\omega_s \sim \frac{v_s}{r_0} \sim \frac{1}{r_0}$). In the case of corotation, we find that S-O effects enter with a suppression of $v^2(\frac{r_0}{R})^2$ corresponding to PN order $1 + 2/\hat{d}$, while S-S effects enter with a suppression of $v^2(\frac{r_0}{R})^4$, which means PN order $1 + 4/\hat{d}$. For maximal spin, S-O coupling incurs a suppression of $v\frac{r_0}{R}$ ($0.5 + 1/\hat{d}$ PN) and S-S a suppression of $(\frac{r_0}{R})^2$ ($2/\hat{d}$ PN). Thus, for corotation, spin effects only enter beyond +1PN in any dimension. For maximal rotation, while in 4D S-O enters at +1.5PN and S-S at +2PN (compare Refs. [4,12,13,56–64]), as the dimension grows, both effects become more and more important, with both effects at +1PN for $d = 5$, and both entering before +1PN at $d > 5$. Also, for $d > 5$, spin-spin interactions become more dominant than spin-orbit.

IV. SUMMARY OF RESULTS

In general spacetime dimension d, the radiative field decouples, at the linear level, into three sectors—scalar, vector, and tensor—with respect to the $\Omega_{\hat{d}+1}$ sphere. The radiation is generated by the corresponding multipole moments of the source. The linearized-GR multipoles are given in the time domain by

$$Q_S^L = \frac{(\hat{d}+1)}{\hat{d}(\ell+\hat{d})(\ell+\hat{d}+1)} \int d^D x x_L^{\mathrm{STF}} \left[\hat{d} \left(\frac{(\ell+\hat{d})(\ell+\hat{d}+1)}{\hat{d}+1} + r\partial_r + \frac{r^2}{\hat{d}} \partial_t^2 \right) \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t) T^{tt} - 2x_a(\ell+\hat{d}+1+r\partial_r) \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t) \partial_t T^{ta} + \left(\frac{(\ell+\hat{d})(\ell+\hat{d}+1)}{\hat{d}+1} + r\partial_r \right) \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t) T^{aa} + x_a x_b \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t) \partial_t^2 T^{ab} \right], \tag{4.1}$$

$$Q_{\aleph}^L = \frac{2\epsilon^{(D)}_{\aleph a b k_\ell}}{\ell+\hat{d}+1} \int d^D x \left[(r^{\ell+\hat{d}+1} \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t))' \frac{x^{bL-1}}{r^{\ell+\hat{d}}} T^{ta} - \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t) \partial_t T^{ac} x^{bcL-1} \right], \tag{4.2}$$

$$Q_{\aleph\beth}^L = \frac{\ell(\ell-1)}{2} \epsilon^{(D)}_{\aleph a b k_\ell} \epsilon^{(D)}_{\beth a' b' k_{\ell'}} \int d^D x x^{bb'L-2} T^{aa'} \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t), \tag{4.3}$$

where $T^{\mu\nu}(\vec{r}, t)$ is the energy-momentum tensor. It is useful to sum over the $\aleph$ coordinates to replace the vector multipoles with

$$Q_M^L = \frac{1}{\ell+\hat{d}+1} \int d^D x \left[\vec{r} \wedge \left(\frac{1}{r^{\ell+\hat{d}}} (r^{\ell+\hat{d}+1} \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t))' \vec{J} + \tilde{j}_{\ell+\hat{d}/2}(ir\partial_t) \vec{r} \cdot \dot{\overleftrightarrow{\Sigma}} \right) \right]^{(k_\ell} x^{L-1)}. \tag{4.4}$$

These multipoles share the same form in any spacetime dimension. The Feynman propagator (2.11) introduces $2\ell + \hat{d}$ factors of ω into the RR effective action (3.1). In even dimensions, these are transformed to the time domain to become $2\ell + \hat{d}$ time derivatives.

For the N-body problem at leading post-Newtonian order, the radiation-reaction effective action includes only the leading order of the scalar quadrupole[12] $Q_S^{ij} = \sum_{A=1}^n m_A (x_A^{ij})^{TF}$ (2.22), and the action reproduces the generalized Burke-Thorne potential, as well as Eq. (1.1), as described in Sec. III A.

At +1PN order, contributions arise from the scalar (mass) octupole,

$$Q_S^{L_3} = Q_S^{ijk} = \sum_{A=1}^n m_A \left[x^i x^j x^k - \frac{1}{D+2} (\delta^{ij} x^k + \delta^{ik} x^j + \delta^{jk} x^i) x^2 \right]_A, \tag{4.5}$$

from the vector (current) quadrupole,

$$Q_M^{L_2} = Q_M^{ij} = 2 \sum_{A=1}^n [m(\vec{r} \wedge \vec{J})^{(i} x^{j)}]_A, \tag{4.6}$$

and from corrections to the scalar quadrupole,

$$\delta^1 Q_S^{L_2} = \sum_{A=1}^n m_A \left[\left(\frac{\hat{d}+2}{2\hat{d}} v_A^2 - \sum_{B\neq A} \frac{G_d m_B}{\|\vec{x}_A - \vec{x}_B\|^{\hat{d}}} \right) x_A^{L_2} - \frac{2(\hat{d}+1)}{\hat{d}(\hat{d}+2)} \partial_t (\vec{x}_A \cdot \vec{v}_A x_A^{L_2}) + \frac{(\hat{d}^2 + 6\hat{d} + 4)}{2\hat{d}(\hat{d}+2)(\hat{d}+6)} \partial_t^2 (r_A^2 x_A^{L_2}) \right]. \tag{4.7}$$

[12] The LO quadrupoles, as well as the LO radiation reaction in each sector, are gauge invariant; for further discussion, see Sec. 3.3.1 of Ref. [1].

Taken together, we find the radiation-reaction effective action to +1PN, which in the case of even dimension d is

$$S_{\mathrm{eff}}^{\mathrm{NLO}} = (-)^{\frac{\hat{d}+1}{2}} G_d \int dt \left[\frac{\hat{d}(\hat{d}+2)(\hat{d}+3)}{2\hat{d}!!(\hat{d}+4)!!} (\hat{Q}_S^{L_2} \partial_t^{d+1} Q_S^{L_2} + \hat{Q}_S^{L_2} \partial_t^{d+1} \delta^1 Q_S^{L_2} + \delta^1 \hat{Q}_S^{L_2} \partial_t^{d+1} Q_S^{L_2}) - \frac{\hat{d}(\hat{d}+4)(\hat{d}+3)}{6\hat{d}!!(\hat{d}+6)!!} \hat{Q}_S^{L_3} \partial_t^{d+3} Q_S^{L_3} + \frac{2\hat{d}(\hat{d}+3)}{3\hat{d}!!(\hat{d}+4)!!} \hat{Q}_M^{L_2} \partial_t^{d+1} Q_M^{L_2} \right]. \tag{4.8}$$

This is the main result of this paper. It matches Eq. (3.111) of Ref. [1] in $d = 4$ (upon reintroduction of $1/\ell!$ from the summation convention) and extends it to different dimensions. The energy dissipated from the system, as well as the gravitational SF acting on it, can be read off from this equation.

While all the results of this paper, presented in the frequency domain, are valid for any spacetime dimension, for odd dimensions the Fourier transformation to the time domain introduces nonlocal "tail" expressions in the action. This happens due to the appearance of branch cuts in the frequency domain effective action. For a thorough discussion of this, see Eqs. (4.7), (4.8), (4.14), (4.15) of Ref. [3]; as an example, the time-domain effective action in odd d is given by

$$\hat{S}_{\mathrm{eff}} = G_d \int dt \sum_L (-)^{\ell + \frac{\hat{d}+1}{2}} [C^S_{\ell,\hat{d}} S^{(S)}(t) + C^M_{\ell,\hat{d}} S^{(M)}(t) + C^T_{\ell,\hat{d}} S^{(T)}(t)], \tag{4.9}$$

$$S^\epsilon(t) = \hat{Q}_L^\epsilon(t) \left[\left(\frac{1}{2} H(2\ell + \hat{d}) - H\left(\ell + \frac{\hat{d}}{2}\right) \right) \partial_t^{2\ell+\hat{d}} Q_\epsilon^L(t) - \int_{-\infty}^t dt' \left(\frac{1}{t - t'} \partial_{t'}^{2\ell+\hat{d}} Q_\epsilon^L(t') \right) \Big|_{\mathrm{regularized}} \right], \tag{4.10}$$

where $\epsilon \in \{S, V, T\}$, $H(n)$ is the nth harmonic number, and regularization (hence the effective coefficient of the local part) depends on short-distance details of the system, and hence should be determined by matching with the system zone. For even d (and specifically 4D), this nonuniversality of the effective coupling is well known to first appear in the +1.5 PN tail correction (cf. Refs. [3,4,8,19,65]); in the odd d case, however, it comes up already at *leading order*. Moreover, no nonlinearities are required for this nonuniversality to arise (it will occur, for example, for a free scalar field as in Ref. [3]). The odd-dimensional setting may therefore be considered as a simpler setting to tackle such matching.

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APPENDIX A: PROPERTIES OF SPHERICAL DERIVATIVES

The generalized spherical metric for coordinates $\{t, r, \Omega_1, \Omega_2, \ldots, \Omega_{\hat{d}+1}\}$ is

$$g_{\mu\nu} = \mathrm{diag}\{-1, 1, r^2\Pi_1, r^2\Pi_2, \ldots, r^2\Pi_{\hat{d}+1}\}, \quad \Pi_i = \prod_{j=1}^{i-1} \sin^2\Omega_j, \tag{A1}$$

from which we find all the Christoffel symbols

$$\Gamma^r_{\Omega_a\Omega_b} = -r g_{\Omega_a\Omega_b}, \qquad \Gamma^{\Omega_a}_{r\Omega_b} = \Gamma^{\Omega_b}_{r\Omega_a} = \frac{1}{r} g^{\Omega_a}_{\Omega_b} = \frac{1}{r}\delta^{\Omega_a}_{\Omega_b}, \qquad \Gamma^t_{\alpha\beta} = \Gamma^\alpha_{t\beta} = 0,$$
$$\Gamma^{\Omega_a}_{\Omega_b\Omega_c} = \delta^{\Omega_a}_{\Omega_b}\theta(a-c)\cot\Omega_c + \delta^{\Omega_a}_{\Omega_c}\theta(a-b)\cot\Omega_b - \delta^{\Omega_b}_{\Omega_c}\theta(b-a)\frac{\Pi_b}{\Pi_a}\cot\Omega_a, \tag{A2}$$

where $\theta(a-b) = 1$ for $a > b$, otherwise 0. The Riemann tensor on the sphere is

$$R_{\Omega_a\Omega_b\Omega_c\Omega_d} = g_{\Omega_a\Omega_c} g_{\Omega_b\Omega_d} - g_{\Omega_a\Omega_d} g_{\Omega_b\Omega_c}. \tag{A3}$$

We shall also make use of the relation

$$g^{\Omega\Omega'}g_{\Omega\Omega'} = \hat{d}+1 \tag{A4}$$

and note that

$$g_{\Omega\Omega',\Omega} = 0. \tag{A5}$$

The volume element is

$$\sqrt{-g}d^dx = r^{\hat{d}+1}\prod_{i=1}^{\hat{d}}(\sin\Omega_i)^{\hat{d}+1-i}dtdrd\Omega_1\cdots\Omega_{\hat{d}+1} \triangleq dtr^{\hat{d}+1}drd\Omega_{\hat{d}+1}. \tag{A6}$$

Using the Riemann tensor we find the commutation relations for a tensor on the sphere:

$$[D_{\Omega_a}, D_{\Omega_b}]V_{\Omega_c} = R^{\Omega_d}_{\Omega_c\Omega_b\Omega_a}V_{\Omega_d} = g_{\Omega_c[\Omega_a}V_{\Omega_b]}. \tag{A7}$$

The commutators for tensors give

$$[D_{\Omega_a}, D_{\Omega_b}]T_{\Omega_c\Omega_d} = g_{\Omega_a\Omega_c}T_{\Omega_b\Omega_d} - g_{\Omega_b\Omega_c}T_{\Omega_a\Omega_d} + g_{\Omega_a\Omega_d}T_{\Omega_b\Omega_c} - g_{\Omega_b\Omega_d}T_{\Omega_a\Omega_c}, \tag{A8}$$

$$[D_{\Omega''}, D_Q](D_{\Omega''}V_P) = (\hat{d}+1)D_QV_P - g_{QP}D_{\Omega''}V_{\Omega''}. \tag{A9}$$

APPENDIX B: SPHERICAL FIELDS

For a rank-2 tensor $A_{\mu\nu}$ in d dimensions, we use the following spherical decomposition (A's L, ω indices are suppressed):

$$A_{\alpha\beta} = \begin{pmatrix} A_{tt} & A_{tr} & A_{t\Omega} \\ A_{tr} & A_{rr} & A_{r\Omega} \\ A_{t\Omega} & A_{r\Omega} & A_{\Omega\Omega'} \end{pmatrix}$$
$$= \sum\!\!\!\!\!\!\int \begin{pmatrix} A_{tt}n_L & A_{tr}n_L & A_t\partial_\Omega n_L + A_{t\aleph}n^L_{\aleph\Omega} \\ \cdots & A_{rr}n_L & A_r\partial_\Omega n_L + A_{r\aleph}n^L_{\aleph\Omega} \\ \cdots & \cdots & A_Sn^L_{\Omega\Omega'} + \tilde{A}_S\tilde{n}^L_{\Omega\Omega'} + A_\aleph n^L_{\aleph\Omega\Omega'} + A_{\aleph\beth}n^L_{\aleph\beth\Omega\Omega'} \end{pmatrix} e^{-i\omega t}, \tag{B1}$$

where we use the scalar multipoles n^L, the divergenceless vector multipoles $n^L_{\aleph\Omega}$ [13] and the tensor multipoles $n^L_{\aleph\aleph'\Omega\Omega'}$ (which are symmetric, traceless and divergenceless).[14] These multipoles are all dimensionless and depend only on the angular coordinates. They are related to the scalar, vector and tensor spherical harmonics:

[13]The vector multipoles are enumerated by an antisymmetric multi-index $\aleph$ taken from the hebrew alphabet, representing $D-3$ *spherical* indices:

$$n^L_{\aleph\Omega} = \epsilon^{(\hat{d}+1)}_{\aleph\Omega\Omega'}D^{\Omega'}n^L = (\star(\vec{r}\wedge\vec{\nabla}))_{\aleph\Omega}n^L, \tag{B2}$$

where $\epsilon^{(\hat{d}+1)}_{\Omega_1\cdots\Omega_{\hat{d}+1}}$ is the completely antisymmetric symbol on the $\Omega_{\hat{d}+1}$-sphere, $\wedge$ is the exterior product and $\star$ is the Hodge duality operator [66–69]. The spatial Levi-Cività tensor will be marked $\epsilon^{(D)}_{a_1\cdots a_D}$.

[14]The tensor multipoles (of rank 2, generalizing Ref. [70]) are enumerated by two antisymmetric multi-indices $\aleph, \aleph'$:

$$n^L_{\aleph\aleph'\Omega\Omega'} = \epsilon^{(\hat{d}+1)}_{\aleph\Omega\Psi}\epsilon^{(\hat{d}+1)}_{\aleph'\Omega'\Psi'}D^\Psi D^{\Psi'}n^L. \tag{B3}$$

$$n^L=\frac{x^L}{r^\ell}=Y^S,\qquad D_\Omega n^L=Y^S_\Omega,\qquad n^L_{\Omega\Omega'}=g_{\Omega\Omega'}n^L,\qquad \tilde{n}^L_{\Omega\Omega'}=\left(D_\Omega D_{\Omega'}+\frac{c_s}{\hat{d}+1}g_{\Omega\Omega'}\right)n^L,$$
$$n^L_{\aleph\Omega}=\frac{x^L_{\aleph\Omega}}{r^\ell}=Y^{\aleph}_\Omega,\qquad n^L_{\aleph\Omega\Omega'}=\frac{1}{2}(D_\Omega n^L_{\aleph\Omega'}+D_{\Omega'}n^L_{\aleph\Omega}),\qquad n^L_{\aleph\beth\Omega\Omega'}=\frac{x^L_{\aleph\beth\Omega\Omega'}}{r^\ell}=Y^{\aleph\beth}_{\Omega\Omega'}. \tag{B4}$$

Using Refs. [3,71,72], the numbers $D_\ell(\hat{d}+1,\epsilon)$ of such independent multipoles of order ℓ and sector ϵ on the $\Omega_{\hat{d}+1}$-sphere are

$$D_\ell(\hat{d}+1,0)=\frac{(2\ell+\hat{d})(\ell+\hat{d}-1)!}{\hat{d}!\ell!},\qquad D_\ell(\hat{d}+1,1)=\frac{\ell(\ell+\hat{d})(2\ell+\hat{d})(\ell+\hat{d}-2)!}{(\hat{d}-1)!(\ell+1)!},$$
$$D_\ell(\hat{d}+1,2)=\frac{(\hat{d}+2)(\hat{d}-1)(\ell+\hat{d}+1)(\ell-1)(2\ell+\hat{d})(\ell+\hat{d}-2)!}{2\hat{d}!(\ell+1)!}. \tag{B5}$$

The spherical multipoles satisfy the derivative properties (using Appendix A)

$$\partial_t h_{L\omega}=-i\omega h,\qquad \partial_r h_{L\omega}=h'_{L\omega},\qquad \partial_r n^L_X=0,\qquad \partial_r x^L_X=\frac{\ell}{r}x^L_X, \tag{B6}$$

$$D_\Omega h_{L\omega}=0,\qquad D_\Omega n^L_{\aleph\Omega}=0,\qquad D_\Omega n^L_{\aleph\beth\Omega\Omega'}=D_{\Omega'}n^L_{\aleph\beth\Omega\Omega'}=0, \tag{B7}$$

$$g^{\Omega\Omega'}n^L_{\aleph\Omega\Omega'}=0,\qquad D_\Omega\tilde{n}^L_{\Omega\Omega'}=-\frac{\hat{d}\hat{c}_s}{\hat{d}+1}D_{\Omega'}n^L,\qquad D_\Omega n^L_{\aleph\Omega\Omega'}=-\frac{\hat{c}_s}{2}n^L_{\aleph\Omega'}, \tag{B8}$$

where a prime denotes r derivatives ($'\coloneqq\partial_r$). They are eigenfunctions of the Laplace-Beltrami operator on the $(\hat{d}+1)$-sphere, $\Delta_{\hat{d}+1}=D_\Omega D^\Omega$, with eigenvalues

$$\Delta_{\hat{d}+1}n^L=-c_s n^L,\qquad \Delta_{\hat{d}+1}n^L_{\Omega\Omega'}=-c_s n^L_{\Omega\Omega'},\qquad \Delta_{\hat{d}+1}D_\Omega x^L=-(c_s-\hat{d})D_\Omega n^L, \tag{B9}$$

$$\Delta_{\hat{d}+1}n^L_{\aleph\Omega}=-(c_s-1)n^L_{\aleph\Omega},\qquad \Delta_{\hat{d}+1}n^L_{\aleph\Omega\Omega'}=-(c_s-d)n^L_{\aleph\Omega\Omega'}, \tag{B10}$$

$$\Delta_{\hat{d}+1}\tilde{n}^L_{\Omega\Omega'}=-(c_s-2(\hat{d}+1))\tilde{n}^L_{\Omega\Omega'},\qquad \Delta_{\hat{d}+1}n^L_{\aleph\beth\Omega\Omega'}=-(c_s-2)n^L_{\aleph\beth\Omega\Omega'}. \tag{B11}$$

The scalar, vector and tensor basis elements are orthogonal to each other, and we use the following normalization conditions in d dimensions [71,72]:

$$\begin{aligned}
\int d\Omega_{\hat{d}+1}n_{L_\ell}(\Omega_{\hat{d}+1})n^{L'_{\ell'}}(\Omega_{\hat{d}+1})&=N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}},\\
\int d\Omega_{\hat{d}+1}g^{\Omega\Omega'}D_\Omega n_{L_\ell}D_{\Omega'}n^{L'_{\ell'}}&=c_s\cdot N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}},\\
\int d\Omega_{\hat{d}+1}g^{\Omega\Omega'}n^{L_\ell}_{\aleph\Omega}n^{L'_{\ell'}}_{\aleph'\Omega'}&=c_s\cdot N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}}\delta_{\aleph\aleph'},\\
\int d\Omega_{\hat{d}+1}g^{\Omega\Omega'}g^{\Psi\Psi'}n^{L_\ell}_{\Omega\Psi}n^{L'_{\ell'}}_{\Omega'\Psi'}&=(\hat{d}+1)N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}},\\
\int d\Omega_{\hat{d}+1}g^{\Omega\Omega'}g^{\Psi\Psi'}\tilde{n}^{L_\ell}_{\Omega\Psi}\tilde{n}^{L'_{\ell'}}_{\Omega'\Psi'}&=\frac{\hat{d}c_s\hat{c}_s}{\hat{d}+1}\cdot N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}},\\
\int d\Omega_{\hat{d}+1}g^{\Omega\Omega'}g^{\Psi\Psi'}n^{L_\ell}_{\aleph\Omega\Psi}n^{L'_{\ell'}}_{\aleph'\Omega'\Psi'}&=\frac{c_s\hat{c}_s}{2}\cdot N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}},\\
\int d\Omega_{\hat{d}+1}g^{\Omega\Omega'}g^{\Psi\Psi'}n^{L_\ell}_{\aleph\beth\Omega\Psi}n^{L'_{\ell'}}_{\aleph'\beth'\Omega'\Psi'}&=\hat{d}^2c_s(c_s-\hat{d})\cdot N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\delta_{\ell\ell'}\delta_{L_\ell L'_{\ell'}}\delta_{\aleph\aleph'}\delta_{\beth\beth'}.
\end{aligned} \tag{B12}$$

1. More spherical expressions

The d-dimensional trace of $A_{\mu\nu}$ is given by

$$A \coloneqq g^{\alpha\tilde{\alpha}} A_{\alpha\tilde{\alpha}} = \sum\!\!\!\!\!\!\int \left(-A_{tt} + A_{rr} + \frac{\hat{d}+1}{r^2} A_S \right) n^L e^{-i\omega t} \triangleq \sum\!\!\!\!\!\!\int A n^L e^{-i\omega t}, \tag{B13}$$

and its divergence is given by

$$(\mathrm{div} A)_\alpha = A_{\alpha\beta;}{}^{\beta} = -\partial_t A_{t\alpha} + (A_{r\alpha,r} - \Gamma^\mu_{r\alpha} A_{\mu r}) + \frac{g^{\Omega\Omega'}}{r^2} (A_{\alpha\Omega,\Omega'} - \Gamma^\mu_{\Omega'\alpha} A_{\mu\Omega} - \Gamma^\mu_{\Omega\Omega'} A_{\mu\alpha}). \tag{B14}$$

Explicitly, its components are given by

$$(\mathrm{div} A)_t = \sum\!\!\!\!\!\!\int e^{-i\omega t} \left[i\omega A_{tt} + \left(\partial_r + \frac{\hat{d}+1}{r} \right) A_{tr} - \frac{c_s}{r^2} A_t \right] n_L, \tag{B15}$$

$$(\mathrm{div} A)_r = \sum\!\!\!\!\!\!\int e^{-i\omega t} \left[i\omega A_{tr} + \left(\partial_r + \frac{\hat{d}+1}{r} \right) A_{rr} - \frac{c_s}{r^2} A_r - \frac{\hat{d}+1}{r^3} A_S \right] n_L, \tag{B16}$$

$$\begin{aligned}(\mathrm{div} A)_\Omega = \sum\!\!\!\!\!\!\int e^{-i\omega t} \Bigg\{ &\left[i\omega A_t + \left(\partial_r + \frac{\hat{d}+1}{r} \right) A_r + \frac{A_S}{r^2} - \frac{\hat{d}\hat{c}_s}{(\hat{d}+1) r^2} \tilde{A}_S \right] D_\Omega n_L \\ &+ \left[i\omega A_{t\aleph} + \left(\partial_r + \frac{\hat{d}+1}{r} \right) A_{r\aleph} - \frac{\hat{c}_s}{2r^2} A_\aleph \right] n^L_{\aleph\Omega} \Bigg\},\end{aligned} \tag{B17}$$

where we have used

$$D^{\Omega'} A_{\Omega\Omega'} = \sum\!\!\!\!\!\!\int e^{-i\omega t} \left[\left(A_S - \frac{\hat{d}\hat{c}_s}{\hat{d}+1} \tilde{A}_S \right) D_\Omega n^L - \frac{\hat{c}_s}{2} n^L_{\aleph\Omega} A_\aleph \right]. \tag{B18}$$

APPENDIX C: ACTION IN SPHERICAL FIELDS: KINETIC TERMS

In this appendix we express the linearized GR kinetic term with gauge-invariant spherical fields. We start with the action (2.2). Note that the terms containing traces of the perturbation $\sim D h^\alpha_\alpha$ can be expressed only with d scalars, and thus do not contribute to any vectorial or tensorial terms; and that in fact tensorial terms arise only from the term $\sim D^\gamma h^{\alpha\beta} D_\gamma h_{\alpha\beta}$. After expansions, partial integrations and use of the normalization conditions, we find the homogenous part of the action (L, ω indices will be suppressed throughout much of this appendix),

$$S_{\mathrm{EH}} = \sum\!\!\!\!\!\!\int_{L,\omega} \int dr L(r) = \sum\!\!\!\!\!\!\int_{L,\omega} \frac{N_{\ell,\hat{d}}}{8(\hat{d}+1) G_d} \int r^{\hat{d}+1} dr \tilde{L}(r), \tag{C1}$$

where the action decomposes into scalar, vector and tensor sectors:

$$\tilde{L}(r) = \tilde{L}_S(r) + \tilde{L}_V(r) + \tilde{L}_T(r). \tag{C2}$$

The tensor sector includes a single contribution,

$$\begin{aligned}\tilde{L}_T &= h^*_{\aleph\beth} E_{\aleph\beth}, \\ E_{\aleph\beth} &= \frac{\hat{d}^2 c_s (c_s - \hat{d})}{2r^4} \left(\omega^2 + \partial_r^2 + \frac{\hat{d}-3}{r} \partial_r - \frac{c_s + 2(\hat{d}-2)}{r^2} \right) h_{\aleph\beth}.\end{aligned} \tag{C3}$$

This matches Eq. (5.6) of Ref. [28], if we set $\Phi_{\aleph\beth} \sim r^{-2} h_{\aleph\beth}$. Defining

$$\mathfrak{h}_{\aleph\beth} = r^{-(\ell+2)} h_{\aleph\beth}, \tag{C4}$$

it can also be presented in the master form

$$L_T = \frac{N_{\ell,\hat{d}}}{8(\hat{d}+1)G_d} \frac{\hat{d}^2 c_s(c_s - \hat{d})}{2} r^{2\ell+\hat{d}+1} \mathfrak{h}^*_{\aleph\beth} \mathfrak{L} \mathfrak{h}_{\aleph\beth}, \tag{C5}$$

where $\mathfrak{L}$ is the master operator defined in (2.6).

The vector sector is comprised of three terms:

$$\tilde{L}_V = h^*_{t\aleph} E_{t\aleph} + h^*_{r\aleph} E_{r\aleph} + h^*_{\aleph} E_{\aleph}, \tag{C6}$$

$$E_{t\aleph} = -\frac{c_s}{r^2}\left[\left(\partial_r^2 + \frac{\hat{d}-1}{r}\partial_r - \frac{c_s+\hat{d}-1}{r^2}\right)h_{t\aleph} + i\omega\left(\left(\partial_r + \frac{\hat{d}+1}{r}\right)h_{r\aleph} - \frac{\hat{c}_s}{2r^2}h_{\aleph}\right)\right], \tag{C7}$$

$$E_{r\aleph} = \frac{c_s}{r^2}\left[\left(\omega^2 - \frac{\hat{c}_s}{r^2}\right)h_{r\aleph} - i\omega\left(\partial_r - \frac{2}{r}\right)h_{t\aleph} + \frac{\hat{c}_s}{2r^2}\left(\partial_r - \frac{2}{r}\right)h_{\aleph}\right], \tag{C8}$$

$$E_{\aleph} = \frac{c_s\hat{c}_s}{4r^4}\left[\left[\frac{4}{r} - 2\left(\partial_r + \frac{\hat{d}+1}{r}\right)\right]h_{r\aleph} - 2i\omega h_{t\aleph} + \left[\omega^2 + \partial_r^2 + \frac{\hat{d}-3}{r}\partial_r + 2\frac{2-\hat{d}}{r^2}\right]h_{\aleph}\right]. \tag{C9}$$

The field $h_{r\aleph}$ appears in the equation $E_{r\aleph} = 0$ (C8) without derivatives. Thus, we can solve for it algebraically:

$$h_{r\aleph} = \frac{1}{M_V(r)}\left[i\omega r^2\left(\partial_r - \frac{2}{r}\right)h_{t\aleph} - \frac{\hat{c}_s}{2}\left(\partial_r - \frac{2}{r}\right)h_{\aleph}\right], \tag{C10}$$

where $M_V(r) = \omega^2 r^2 - \hat{c}_s$ [compare Eqs. (5.9) and (5.10) of Ref. [28]]. We thus see $h_{r\aleph}$ is an auxiliary field. We plug this expression into the equations for $E_{t\aleph}$, $E_{\aleph}$, L_V:

$$\begin{aligned}\tilde{L}_V &= h^*_{t\aleph} E_{t\aleph} + h^*_{\aleph} E_{\aleph} \\ &= \frac{c_s\hat{c}_s}{4r^2 M_V}(r^2\Phi_{\aleph})^* \mathfrak{L}_V (r^2\Phi_{\aleph}) = \frac{c_s\hat{c}_s}{4}\Phi^*_{\aleph}\left[\Phi_{\aleph} + \frac{1}{r^{\hat{d}+1}}\partial_r\left(\frac{r^{\hat{d}+3}}{M_V}\partial_r\Phi_{\aleph}\right)\right],\end{aligned} \tag{C11}$$

where $\mathfrak{L}_V = \partial_r^2 + (\hat{d} - 3 - \frac{2\hat{c}_s}{M_V})\frac{1}{r}\partial_r + (2 - \hat{d} + \frac{2\hat{c}_s}{M_V} + \frac{M_V}{2})\frac{2}{r^2}$, and the vector part of the action is seen to depend only on the *single* gauge-invariant field [compare Eqs. (5.12) and (5.13) of Ref. [28], and Ref. [41]]

$$\Phi_{\aleph} = r^{-2}(2h_{t\aleph} + i\omega h_{\aleph}). \tag{C12}$$

Since $\tilde{L}_V$ only appears in the integral (C1), we can integrate by parts to obtain

$$\tilde{L}_V = \frac{c_s\hat{c}_s}{4}\left[(\Phi_{\aleph})^2 - \frac{r^2}{M_V(r)}(\partial_r\Phi_{\aleph})^2\right]. \tag{C13}$$

Defining the field $\mathfrak{h}_{\aleph}$ using the canonical transformations

$$\begin{aligned}(2(\ell-1)(\ell+\hat{d})r^{\ell+\hat{d}+1}\mathfrak{h}_{\aleph}) &\coloneqq \frac{\partial L}{\partial(\Phi^*_{\aleph})'} = -\frac{c_s\hat{c}_s}{2M_V}r^{\hat{d}+3}(\Phi_{\aleph})' \\ &\Rightarrow (\Phi_{\aleph})' = -\frac{4r^{\ell}}{\ell(\ell+\hat{d}+1)}\left(\omega^2 - \frac{\hat{c}_s}{r^2}\right)\mathfrak{h}_{\aleph},\end{aligned} \tag{C14}$$

$$\begin{aligned}(2(\ell-1)(\ell+\hat{d})r^{\ell+\hat{d}+1}\mathfrak{h}_{\aleph})' &\coloneqq \frac{\partial L}{\partial(\Phi^*_{\aleph})} = \frac{1}{2}c_s\hat{c}_s r^{\hat{d}+1}\Phi_{\aleph} \\ &\Rightarrow \Phi_{\aleph} = \frac{4}{\ell(\ell+\hat{d}+1)r^{\hat{d}+1}}(r^{\ell+\hat{d}+1}\mathfrak{h}_{\aleph})',\end{aligned} \tag{C15}$$

and again using integration by parts, we can present the vector contribution as well in the master form [compare with (C5), differing only in the prefactor constants]:

$$L_V = \frac{N_{\ell,\hat{d}}}{8(\hat{d}+1)G_d}\frac{8(\ell-1)(\ell+\hat{d})}{(\ell+\hat{d}+1)\ell} r^{2\ell+\hat{d}+1}\mathfrak{h}_{\aleph}^{*}\mathfrak{L}\mathfrak{h}_{\aleph}. \quad \text{(C16)}$$

In the remaining scalar sector, we also expect a reduction to a single gauge-invariant field. We first notice that three of the fields (h_{tr}, h_{rr}, h_r) appear without r derivatives in the action; thus, variations with respect to their complex conjugates $(h_{tr}^*, h_{rr}^*, h_r^*)$ give us algebraic equations.[15] Hence, these fields can be found by solving the system given by the (Hermitian) matrix A,[16]

$$A\begin{pmatrix} h_{tr} \\ h_{rr} \\ \frac{c_s}{r}h_r \end{pmatrix} = B; \qquad A := \begin{pmatrix} c_s & -(\hat{d}+1)i\omega r & i\omega r \\ (\hat{d}+1)i\omega r & \frac{\hat{d}(\hat{d}+1)}{2} & -\hat{d} \\ -i\omega r & -\hat{d} & \frac{\omega^2 r^2+2\hat{d}}{c_s} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ B_3, \end{pmatrix}, \quad \text{(C17)}$$

with

$$\begin{aligned} B_1 &:= c_s\partial_r h_t - (\hat{d}+1)i\omega\left(\partial_r - \frac{1}{r}\right)h_S, \\ B_2 &:= \frac{c_s-(\hat{d}+1)r\partial_r}{2}h_{tt} + i\omega c_s h_t - \left(\frac{\hat{d}(c_s+\hat{d}+1)}{2r^2} - \frac{\hat{d}(\hat{d}+1)}{2r}\partial_r - \frac{\hat{d}+1}{2}\omega^2\right)h_S - \frac{\hat{d}c_s\hat{c}_s\tilde{h}_S}{2(\hat{d}+1)r^2}, \\ B_3 &:= (r\partial_r - 1)h_{tt} + i\omega(r\partial_r - 2)h_t + \frac{\hat{d}}{r}\left(\frac{2}{r} - \partial_r\right)h_S - r\partial_r\left(\frac{\hat{d}\hat{c}_s}{(\hat{d}+1)r^2}\tilde{h}_S\right). \end{aligned} \quad \text{(C18)}$$

We solve these simultaneously to find

$$\begin{aligned} \begin{pmatrix} h_{tr} \\ h_{rr} \\ \frac{c_s}{r}h_r \end{pmatrix} &= A^{-1}B = \frac{1}{2M_S(r)}\tilde{A}^{-1}B, \\ \tilde{A}^{-1} &= \begin{pmatrix} \hat{d}(2\hat{d}\hat{c}_s - (\hat{d}+1)\omega^2 r^2) & 2i\omega r(\hat{d}(c_s - 2\hat{d} - 2) - (\hat{d}+1)\omega^2 r^2) & -i\omega r c_s\hat{d}(\hat{d}+1) \\ * & -4c_s\hat{d} & 2c_s((\hat{d}+1)\omega^2 r^2 - \hat{d}c_s) \\ * & * & c_s(\hat{d}+1)(2(\hat{d}+1)\omega^2 r^2 - c_s\hat{d}) \end{pmatrix}, \\ M_S(r) &:= c_s\hat{c}_s\hat{d}^2 - 2\hat{c}_s\hat{d}(\hat{d}+1)\omega^2 r^2 + (\hat{d}+1)^2\omega^4 r^4, \end{aligned} \quad \text{(C19)}$$

where the matrix terms under the main diagonal (in asterisks) are completed by Hermiticity. Plugging (C19) back into the action (C1) and (C2), we find that h_S drops out entirely, and the three remaining fields h_{tt}, h_t and $\tilde{h}_S$ appear only combined in the single gauge invariant field [compare Eq. (4.30) of Ref. [28] in the flat space limit]

$$\Phi := h_{tt} + 2i\omega h_t - \omega^2\tilde{h}_S. \quad \text{(C20)}$$

The scalar action is thus given by

$$\tilde{L}_S = \frac{1}{M_S(r)}[\alpha\partial_r\Phi^*\partial_r\Phi + \beta(\partial_r\Phi^*\Phi + \Phi^*\partial_r\Phi) + \gamma\Phi^*\Phi], \quad \text{(C21)}$$

[15] The complete derivation of the scalar sector of the action, along with the collection of terms to the gauge invariant combination, can be found in Mathematica code in our additional notes online [73].

[16] We mark that this matrix is identical to Eq. (4.35) of Ref. [28], adapted to a flat background—up to ordering of the fields, scalings in the field definitions, and sign mismatches stemming from the choice of Lorentzian over Euclidean signature. Using our definitions, the matrices A, A^{-1}, $\tilde{A}^{-1}$ are dimensionless.

$$\alpha = -\hat{d}(\hat{d}+1)c_s\hat{c}_s, \qquad \beta = \frac{c_s\hat{c}_s}{r}(c_s\hat{d} - (\hat{d}+1)\omega^2 r^2), \qquad \gamma = \frac{c_s\hat{c}_s}{r^2}(2(\hat{d}+1)\omega^2 r^2 - c_s\hat{d}). \tag{C22}$$

Following Ref. [28], we canonically transform to $\tilde{\Phi}$ with the generating function

$$F[\Phi, \Phi^*, \tilde{\Phi}, \tilde{\Phi}^*] = -\hat{c}_s r^{\frac{\hat{d}-1}{2}}(\tilde{\Phi}^*\Phi + \tilde{\Phi}\Phi^*), \tag{C23}$$

and after partial integrations find

$$L_S = \frac{N_{\ell,\hat{d}}}{8(\hat{d}+1)G_d}\frac{\hat{d}(\hat{d}+1)\hat{c}_s}{c_s}\tilde{\Phi}^*\left[\partial_r^2 + \omega^2 + \left(c_s + \frac{\hat{d}^2-1}{4}\right)\frac{1}{r^2}\right]\tilde{\Phi}, \tag{C24}$$

which we recognize as the Zerilli action [38]. Defining the field

$$\mathfrak{h} = -\frac{\hat{d}(\ell+\hat{d}+1)}{2\ell}r^{-(\ell+\frac{\hat{d}+1}{2})}\tilde{\Phi}, \tag{C25}$$

we find the homogenous scalar action in the familiar master form [compare (C5) and (C16)]

$$L_S = \frac{N_{\ell,\hat{d}}}{8(\hat{d}+1)G_d}\frac{4(\hat{d}+1)(\ell-1)\ell}{\hat{d}(\ell+\hat{d}+1)(\ell+\hat{d})}r^{2\ell+\hat{d}+1}\mathfrak{h}^*\mathfrak{L}\mathfrak{h}. \tag{C26}$$

APPENDIX D: ACTION IN SPHERICAL FIELDS: SOURCE TERMS

We wish to construct the inhomogeneous part of the action S_{mat} (2.3), which describes the interaction of the gravitational field with matter sources, in terms of the gauge invariant spherical fields and corresponding source functions. We decompose the stress-energy tensor as

$$\begin{aligned}
T^{tt} &= \sum\!\!\!\!\!\!\int T^{tt}_{L\omega}n_L e^{-i\omega t}, \qquad T^{tr} = \sum\!\!\!\!\!\!\int T^{tr}_{L\omega}n_L e^{-i\omega t}, \qquad T^{rr} = \sum\!\!\!\!\!\!\int T^{rr}_{L\omega}n_L e^{-i\omega t}, \\
T^{t\Omega} &= \sum\!\!\!\!\!\!\int (T^t_{L\omega}\partial^\Omega n_L + T^{t\aleph}_{L\omega}n_L^{\aleph\Omega})e^{-i\omega t}, \qquad T^{r\Omega} = \sum\!\!\!\!\!\!\int (T^r_{L\omega}\partial^\Omega n_L + T^{r\aleph}_{L\omega}n_L^{\aleph\Omega})e^{-i\omega t}, \\
T^{\Omega\Omega'} &= \sum\!\!\!\!\!\!\int [T^S_{L\omega}n_L^{\Omega\Omega'} + \tilde{T}^S_{L\omega}\tilde{n}_L^{\Omega\Omega'} + T^{\aleph}_{L\omega}n_L^{\aleph\Omega\Omega'} + T^{\aleph\beth}_{L\omega}n_L^{\aleph\beth\Omega\Omega'}]e^{-i\omega t}.
\end{aligned} \tag{D1}$$

We shall also use the inverse transformations,

$$T^{(tt/tr/rr)}_{L\omega}(r) = (N_{\ell,\hat{d}}\Omega_{\hat{d}+1})^{-1}\int\int d\Omega_{\hat{d}+1}dte^{i\omega t}n_L T^{(tt/tr/rr)}(\vec{r}, t), \tag{D2}$$

$$T_{L\omega}(r) = (N_{\ell,\hat{d}}\Omega_{\hat{d}+1}(\hat{d}+1)r^2)^{-1}\int\int d\Omega_{\hat{d}+1}dte^{i\omega t}n_L[T^{aa}(\vec{r}, t) - T^{rr}(\vec{r}, t)], \tag{D3}$$

$$T^{(t/r)\aleph}_{L\omega}(r) = (c_s N_{\ell,\hat{d}}\Omega_{\hat{d}+1})^{-1}\int\int d\Omega_{\hat{d}+1}dte^{i\omega t}\epsilon^{(D)}_{\aleph abk_\ell}\frac{\ell}{r}n^{bL-1}T^{(t/r)a}, \tag{D4}$$

$$T^{\aleph\beth}_{L\omega}(r) = \frac{1}{N_{\ell,\hat{d}}\Omega_{\hat{d}+1}\hat{d}^2 c_s(c_s-\hat{d})}\int\int d\Omega_{\hat{d}+1}dte^{i\omega t}\epsilon^{(D)}_{\aleph abk_\ell}\epsilon^{(D)}_{\beth a'b'k_{\ell'}}\frac{\ell(\ell-1)}{r^2}n^{bb'L-2}T^{aa'}, \tag{D5}$$

where $T^{\mu r} = T^{r\mu} = \frac{x^c}{r}T^{\mu c}$ and $T^{rr} = \frac{x^c x^d}{r^2}T^{cd}$.

Using the divergence expressions (B14)–(B17), current conservation $D_\mu T^{\mu\nu} = 0$ is recast as the d equations (we henceforth suppress the L, ω indices):

$$0 = -i\omega T^{tt} + \left(\partial_r + \frac{\hat{d}+1}{r}\right)T^{tr} - c_s T^t, \tag{D6}$$

$$0 = -i\omega T^{tr} + \left(\partial_r + \frac{\hat{d}+1}{r}\right)T^{rr} - c_s T^r - (\hat{d}+1)rT^S, \tag{D7}$$

$$0 = -i\omega T^t + \left(\partial_r + \frac{\hat{d}+3}{r}\right)T^r + T^S - \frac{\hat{d}\hat{c}_s}{(\hat{d}+1)}\tilde{T}^S, \tag{D8}$$

$$0 = -i\omega T^{t\aleph} + \left(\partial_r + \frac{\hat{d}+3}{r}\right)T^{r\aleph} - \frac{\hat{c}_s}{2}T^{\aleph}. \tag{D9}$$

From these we can eliminate the fields T^t, T^r, $\tilde{T}^S$, and $T^{\aleph}$, retaining only the scalars T^{tt}, T^{tr}, T^{rr}, T^S; the vectors $T^{t\aleph}$, $T^{r\aleph}$; and the tensor $T^{\aleph\beth}$:

$$T^t = \frac{1}{c_s}\left[-i\omega T^{tt} + \left(\partial_r + \frac{\hat{d}+1}{r}\right)T^{tr}\right], \tag{D10}$$

$$T^r = \frac{1}{c_s}\left[-i\omega T^{tr} + \left(\partial_r + \frac{\hat{d}+1}{r}\right)T^{rr} - (\hat{d}+1)rT^S\right], \tag{D11}$$

$$\tilde{T}^S = \frac{(\hat{d}+1)}{\hat{d}c_s\hat{c}_s}\left[(c_s - (\hat{d}+1)(r\partial_r + \hat{d}+4))T^S - \omega^2 T^{tt} - 2i\omega\left(\partial_r + \frac{\hat{d}+2}{r}\right)T^{tr} + \left(\partial_r^2 + 2\frac{\hat{d}+2}{r}\partial_r + \frac{(\hat{d}+1)(\hat{d}+2)}{r^2}\right)T^{rr}\right], \tag{D12}$$

$$T^{\aleph} = \frac{2}{\hat{c}_s}\left[-i\omega T^{t\aleph} + \left(\partial_r + \frac{\hat{d}+3}{r}\right)T^{r\aleph}\right]. \tag{D13}$$

Plugging the expansions (2.4), (D1) into (2.3) and using (D6)–(D9) and (C10), (C19), we find

$$S_{\mathrm{mat}} = -\frac{1}{2}\sum\!\!\!\!\!\!\int \int [\mathfrak{h}\mathcal{T}^* + \mathfrak{h}_{\aleph}\mathcal{T}^{\aleph *} + \mathfrak{h}_{\aleph\beth}\mathcal{T}^{\aleph\beth *} + \mathrm{c.c.}]dr, \tag{D14}$$

where the source functions (scalar, vector, tensor) are given by

$$\mathcal{T} \coloneqq -\frac{N_{\ell,\hat{d}}\Omega_{\hat{d}+1}(\hat{d}+1)r^{\ell+\hat{d}+3}}{\hat{d}(\ell+\hat{d})(\ell+\hat{d}+1)}\left[\frac{\hat{d}}{r}\partial_r + \omega^2 - \frac{\hat{d}(c_s - 2(\hat{d}+1))}{(\hat{d}+1)r^2}\right]J, \tag{D15}$$

$$\begin{aligned} J \coloneqq{}& T^{tt} + \frac{2i\omega}{r^{\hat{d}+1}}\left(\frac{(\hat{d}+1)(-\hat{d}\hat{c}_s + (\hat{d}+1)r^2\omega^2)r^{\hat{d}+3}}{M_S}T^{tr}\right)' + \frac{2i(\hat{d}+1)^2 r^3\omega^3}{M_S}T^{tr} - (\hat{d}+1)^2\left[\frac{1}{r^{\hat{d}+1}}\left(\frac{\omega^2 r^{\hat{d}+5}}{M_S}T^{rr}\right)'\right]' \\ &- (\hat{d}+1)^2\left(\frac{r^3\omega^2(c_s\hat{c}_s(\hat{d}-1)\hat{d}^2 - 2\hat{d}(\hat{d}+1)^2\hat{c}_s r^2\omega^2 + (\hat{d}+3)(\hat{d}+1)^2 r^4\omega^4)}{M_S^2}T^{rr}\right)' \\ &+ \frac{1}{M_S^2}[c_s^2\hat{c}_s^2\hat{d}^3 - c_s\hat{c}_s\hat{d}^2(\hat{d}+1)(2c_s + (\hat{d}+1)(\hat{d}^2+\hat{d}-4))r^2\omega^2 \\ &+ \hat{c}_s\hat{d}(\hat{d}+1)^2(c_s + 2\hat{d}(\hat{d}+1)(\hat{d}+3))r^4\omega^4 - (\hat{d}+1)^4(\hat{d}+2)(\hat{d}+3)r^6\omega^6]T^{rr} \\ &+ \frac{(\hat{d}+1)^3}{r^{\hat{d}+2}}\left(\frac{\omega^2 r^{\hat{d}+7}}{M_S}T^S\right)' + \frac{(\hat{d}+1)\hat{c}_s r^2(c_s\hat{d} - (\hat{d}+1)\omega^2 r^2)}{M_S}T^S, \end{aligned} \tag{D16}$$

$$\mathcal{T}^{\aleph} \coloneqq -\frac{2N_{\ell,\hat{d}}\Omega_{\hat{d}+1}(\ell+\hat{d})r^{\ell+\hat{d}+1}}{\ell+\hat{d}+1}\left[r^2 T^{t\aleph} + \frac{1}{r^{\hat{d}+1}}\left(\frac{i\omega r^{\hat{d}+5}}{M_V}T^{r\aleph}\right)'\right]', \tag{D17}$$

$$\mathcal{T}^{\aleph\beth} \coloneqq \frac{N_{\ell,\hat{d}}\Omega_{\hat{d}+1}}{2}\hat{d}^2 c_s(c_s - \hat{d}) r^{\ell+\hat{d}+3} T^{\aleph\beth}. \tag{D18}$$

APPENDIX E: ORIGIN-NORMALIZED BESSEL FUNCTIONS

For B a Bessel function of the first or second kind or a Hankel function, i.e., $B \in \{J, Y, H^{\pm}\}$, and with α representing its order, we define the origin-normalized Bessel functions $\tilde{b}_\alpha$ as

$$\tilde{b}_\alpha \coloneqq \Gamma(\alpha+1) 2^\alpha \frac{B_\alpha(x)}{x^\alpha}. \tag{E1}$$

These functions satisfy the equation [compare (2.10)]

$$\left[\partial_x^2 + \frac{2\alpha+1}{x}\partial_x + 1\right]\tilde{b}_\alpha(x) = 0. \tag{E2}$$

The purpose of the definition is to have $\tilde{j}_\alpha$ normalized to 1 at the origin $x = 0$. Around the origin, it is given by the Taylor expansion (which contains only even powers)

$$\begin{aligned}\tilde{j}_\alpha(x) &= \sum_{p=0}^{\infty} \frac{(-)^p (2\alpha)!!}{(2p)!!(2p+2\alpha)!!} x^{2p} \\ &= 1 - \frac{x^2}{2(2\alpha+2)} + \cdots. \end{aligned} \tag{E3}$$

The asymptotic form for $x \to \infty$ is best stated in terms of the Hankel functions $\tilde{h}^{\pm} \coloneqq \tilde{j} \pm i\tilde{y}$:

$$\tilde{h}_\alpha^{\pm}(x) \sim (\mp i)^{\alpha+1/2} \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\sqrt{\pi}} \frac{e^{\pm ix}}{x^{\alpha+1/2}}. \tag{E4}$$

For more details, see Appendix B.2 of Ref. [3].

[1] O. Birnholtz, S. Hadar, and B. Kol, Theory of post-Newtonian radiation and reaction, Phys. Rev. D **88**, 104037 (2013).

[2] O. Birnholtz, S. Hadar, and B. Kol, Radiation reaction at the level of the action, Int. J. Mod. Phys. A **29**, 1450132 (2014).

[3] O. Birnholtz and S. Hadar, Action for reaction in general dimension, Phys. Rev. D **89**, 045003 (2014).

[4] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativity **5**, 3 (2002); Living Rev. Relativity **9**, 4 (2006); L. Blanchet, Post-Newtonian theory and the two-body problem, Fundam. Theor. Phys. **162**, 125 (2011).

[5] T. Damour, P. Jaranowski, and G. Schäfer, Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems, Phys. Rev. D **89**, 064058 (2014).

[6] W. D. Goldberger and I. Z. Rothstein, An effective field theory of gravity for extended objects, Phys. Rev. D **73**, 104029 (2006).

[7] W. D. Goldberger and I. Z. Rothstein, Dissipative effects in the worldline approach to black hole dynamics, Phys. Rev. D **73**, 104030 (2006).

[8] W. D. Goldberger and A. Ross, Gravitational radiative corrections from effective field theory, Phys. Rev. D **81**, 124015 (2010).

[9] C. R. Galley and M. Tiglio, Radiation reaction and gravitational waves in the effective field theory approach, Phys. Rev. D **79**, 124027 (2009).

[10] S. Foffa and R. Sturani, Tail terms in gravitational radiation reaction via effective field theory, Phys. Rev. D **87**, 044056 (2013).

[11] B. Kol and M. Smolkin, Non-relativistic gravitation: From Newton to Einstein and back, Classical Quantum Gravity **25**, 145011 (2008).

[12] R. A. Porto, Post-Newtonian corrections to the motion of spinning bodies in NRGR, Phys. Rev. D **73**, 104031 (2006).

[13] R. A. Porto and I. Z. Rothstein, The Hyperfine Einstein-Infeld-Hoffmann Potential, Phys. Rev. Lett. **97**, 021101 (2006).

[14] C. R. Galley and A. K. Leibovich, Radiation reaction at 3.5 post-Newtonian order in effective field theory, Phys. Rev. D **86**, 044029 (2012).

[15] A. Mironov and A. Morozov, Radiation beyond four space-time dimensions, Theor. Math. Phys. **156**, 1209 (2008).

[16] A. Mironov and A. Morozov, On the problem of radiation friction beyond 4 and 6 dimensions, Int. J. Mod. Phys. A **23**, 4677 (2008).

[17] D. V. Gal'tsov and P. A. Spirin, Radiation reaction in curved even-dimensional spacetime, Gravitation Cosmol. **13**, 241 (2007).

[18] D. Gal'tsov, Radiation reaction and energy-momentum conservation, Fundam. Theor. Phys. **162**, 367 (2011).

[19] L. Blanchet, T. Damour, G. Esposito-Farese, and B. R. Iyer, Dimensional regularization of the third post-Newtonian gravitational wave generation from two point masses, Phys. Rev. D **71**, 124004 (2005).

[20] V. Asnin, D. Gorbonos, S. Hadar, B. Kol, M. Levi, and U. Miyamoto, High and low dimensions in the black hole negative mode, Classical Quantum Gravity **24**, 5527 (2007).

[21] R. Emparan, R. Suzuki, and K. Tanabe, The large-*D* limit of general relativity, J. High Energy Phys. 06 (2013) 009.

[22] V. Cardoso, O. J. C. Dias, and J. P. S. Lemos, Gravitational radiation in *D*-dimensional space-times, Phys. Rev. D **67**, 064026 (2003).

[23] V. Cardoso, S. Yoshida, O. J. C. Dias, and J. P. S. Lemos, Late time tails of wave propagation in higher dimensional space-times, Phys. Rev. D **68**, 061503 (2003).

[24] V. Cardoso, O. J. C. Dias, and P. Figueras, Gravitational radiation in $d > 4$ from effective field theory, Phys. Rev. D **78**, 105010 (2008).

[25] A. Einstein, Über Gravitationswellen, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin **1918**, 154 (1918); [*On Gravitational Waves* (Prussian Academy of Science, Berlin, 1918)]; Translation is available in Albert Einstein, *The Collected Papers of Albert Einstein, The Berlin Years: Writings, 1918–1921*, Vol. 7, edited by (Princeton University Press, Princeton, NJ, 2002). We remind the reader that this work was missing a factor of 2.

[26] A. Ross, Multipole expansion at the level of the action, Phys. Rev. D **85**, 125033 (2012).

[27] B. Kol, Perturbations around backgrounds with one non-homogeneous dimension, arXiv:hep-th/0609001; Higher order perturbations around backgrounds with one non-homogeneous dimension, arXiv:0709.3254.

[28] V. Asnin and B. Kol, Dynamical versus auxiliary fields in gravitational waves around a black hole, Classical Quantum Gravity **24**, 4915 (2007).

[29] K. S. Thorne, Multipole expansions of gravitational radiation, Rev. Mod. Phys. **52**, 299 (1980).

[30] H. Kodama and A. Ishibashi, A Master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions, Prog. Theor. Phys. **110**, 701 (2003).

[31] A. Ishibashi and H. Kodama, Stability of higher-dimensional Schwarzschild black holes, Prog. Theor. Phys. **110**, 901 (2003).

[32] J. S. Schwinger, Brownian motion of a quantum oscillator, J. Math. Phys. (N.Y.) **2**, 407 (1961); K. T. Mahanthappa, Multiple production of photons in quantum electrodynamics, Phys. Rev. **126**, 329 (1962); L. V. Keldysh, Diagram technique for nonequilibrium processes, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].

[33] C. R. Galley, The Classical Mechanics of Non-conservative Systems, Phys. Rev. Lett. **110**, 174301 (2013).

[34] R. Emparan and H. S. Reall, Black holes in higher dimensions, Living Rev. Relativity **11**, 6 (2008).

[35] B. Kol and M. Smolkin, Classical effective field theory and caged black holes, Phys. Rev. D **77**, 064033 (2008).

[36] U. H. Gerlach and U. K. Sengupta, Gauge invariant perturbations on most general spherically symmetric space-times, Phys. Rev. D **19**, 2268 (1979); U. H. Gerlach and U. K. Sengupta, Gauge invariant coupled gravitational, acoustical, and electromagnetic modes on most general spherical space-times, Phys. Rev. D **22**, 1300 (1980).

[37] T. Regge and J. A. Wheeler, Stability of a Schwarzschild singularity, Phys. Rev. **108**, 1063 (1957).

[38] F. J. Zerilli, Effective Potential for Even-Parity Regge-Wheeler Gravitational Perturbation Equations, Phys. Rev. Lett. **24**, 737 (1970).

[39] V. Moncrief, Gravitational perturbations of spherically symmetric systems: I. The exterior problem, Ann. Phys. (N.Y.) **88**, 323 (1974).

[40] C. T. Cunningham, R. H. Price, and V. Moncrief, Radiation from collapsing relativistic stars: I. Linearized odd-parity radiation, Astrophys. J. **224**, 643 (1978).

[41] H. Kodama and A. Ishibashi, Master equations for perturbations of generalized static black holes with charge in higher dimensions, Prog. Theor. Phys. **111**, 29 (2004).

[42] K. Martel and E. Poisson, Gravitational perturbations of the Schwarzschild spacetime: A practical covariant and gauge-invariant formalism, Phys. Rev. D **71**, 104003 (2005).

[43] E. D. Solomentsev (originator), Balayage method, Encyclopedia of Mathematics (online), URL: http://www.encyclopediaofmath.org/index.php?title=Balayage_method&oldid=11819.

[44] H. Poincaré, Sur les Équations aux Dérivées Partielles de la Physique Mathématique, Am. J. Math. **12**, 211 (1890); *Theorie du potentiel Newtonien* (Carré et Naud, Paris, 1899), pp. 211–294.

[45] N. L. Balazs, Wave propagation in even and odd dimensional spaces, Proc. Phys. Soc. London Sect. A **68**, 521 (1955).

[46] E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Wave propagation in gravitational systems: Late time behavior, Phys. Rev. D **52**, 2118 (1995).

[47] Y. Z. Chu, Scalar wave tails in even-dimensional weakly curved static Newtonian spacetimes, arXiv:1407.2252.

[48] L. Blanchet and T. Damour, Post-Newtonian generation of gravitational waves, Ann. Inst. Henri Poincaré, A **50**, 377 (1989).

[49] T. Damour and B. R. Iyer, Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors, Phys. Rev. D **43**, 3259 (1991).

[50] L. Blanchet, On the multipole expansion of the gravitational field, Classical Quantum Gravity **15**, 1971 (1998).

[51] W. L. Burke and K. S. Thorne, Gravitational radiation damping, in *Relativity* edited by M. Carmeli, S. I. Fickler, and L. Witten (Plenum, New York, 1970) pp. 209–228; W. L. Burke, Ph.D. dissertation, California Institute of Technology, 1969; Gravitational radiation damping of slowly moving systems calculated using matched asymptotic expansions, J. Math. Phys. (N.Y.) **12**, 401 (1971).

[52] L. Blanchet and G. Schäfer, Higher-order gravitational radiation losses in binary systems, Mon. Not. R. Astron. Soc. **239**, 845 (1989).

[53] L. Blanchet and T. Damour, Tail transported temporal correlations in the dynamics of a gravitating system, Phys. Rev. D **37**, 1410 (1988).

[54] L. Blanchet, Time asymmetric structure of gravitational radiation, Phys. Rev. D **47**, 4392 (1993).

[55] R. C. Myers and M. J. Perry, Black holes in higher-dimensional space-times, Ann. Phys. (N.Y.) **172**, 304 (1986).

[56] B. M. Barker and R. F. O'Connell, Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments, Phys. Rev. D **12**, 329 (1975).

[57] B. M. Barker and R. F. O'Connell, The gravitational interaction: Spin, rotation, and quantum effects: A review, Gen. Relativ. Gravit. **11**, 149 (1979).

[58] L. E. Kidder, C. M. Will, and A. G. Wiseman, Spin effects in the inspiral of coalescing compact binaries, Phys. Rev. D **47**, R4183 (1993).

[59] L. E. Kidder, Coalescing binary systems of compact objects to post-Newtonian 5/2 order: 5. Spin effects, Phys. Rev. D **52**, 821 (1995).
[60] B. J. Owen, H. Tagoshi, and A. Ohashi, Nonprecessional spin orbit effects on gravitational waves from inspiraling compact binaries to second post-Newtonian order, Phys. Rev. D **57**, 6168 (1998).
[61] H. Tagoshi, A. Ohashi, and B. J. Owen, Gravitational field and equations of motion of spinning compact binaries to 2.5 post-Newtonian order, Phys. Rev. D **63**, 044006 (2001).
[62] J. Steinhoff, S. Hergt, and G. Schäfer, On the next-to-leading order gravitational spin(1)-spin(2) dynamics, Phys. Rev. D **77**, 081501 (2008).
[63] M. Levi, Next-to-leading-order gravitational spin(1)-spin(2) coupling with Kaluza-Klein reduction, Phys. Rev. D **82**, 064029 (2010).
[64] E. Barausse, E. Racine, and A. Buonanno, Hamiltonian of a spinning test-particle in curved spacetime, Phys. Rev. D **80**, 104025 (2009); **85**, 069904(E) (2012).
[65] W. D. Goldberger, Les Houches lectures on effective field theories and gravitational radiation, arXiv:hep-ph/0701129.
[66] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics* (D. Reidel Publishing Company, Dordrecht, 1984).
[67] D. Hestenes, *New Foundations for Classical Mechanics* (Kluwer Academic Publishers, Dordrecht, 1990).
[68] W. E. Baylis, *Electrodynamics, A Modern Geometric Approach* (Birkhäuser, Boston, 1999).
[69] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003).
[70] J. Applequist, Traceless Cartesian tensor forms for spherical harmonic functions: New theorems and applications to electrostatics of dielectric media, J. Phys. A **22**, 4303 (1989).
[71] M. A. Rubin and C. R. Ordonez, Eigenvalues and degeneracies for n-dimensional tensor spherical harmonics, J. Math. Phys. (N.Y.) **25**, 2888 (1984); M. A. Rubin and C. R. Ordonez, Symmetric tensor eigenspectrum of the Laplacian on $\boldsymbol{n}$ spheres, J. Math. Phys. (N.Y.) **26**, 65 (1985).
[72] A. Higuchi, Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group $SO(N,1)$, J. Math. Phys. (N.Y.) **28**, 1553 (1987); **43**, 6385(E) (2002).
[73] http://phys.huji.ac.il/~ofek/PNRR/GDGR/.