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Excitation of photons by inflationary gravitons

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We use a recent result for the graviton contribution to the one-loop vacuum polarization to solve the effective field equations for dynamical photons on a de Sitter background. Our results show that the electric field experiences a secular enhancement proportional to the number of inflationary e-foldings. We discuss the minimum effect this establishes for primordial inflation to seed cosmic magnetic fields.

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I. INTRODUCTION

Because photons have zero mass, it does not take much to affect the long wavelength modes. This has lead many to suspect that the explosive expansion of spacetime during primordial inflation might help to explain cosmic magnetic fields [1]. However, the Maxwell Lagrangian is conformally invariant, which means that free photons cannot locally sense the expansion of spacetime. The search for an inflationary connection has prompted investigations of explicit conformal breaking terms which might be present in the effective action [2]. Quantum effects from the conformal anomaly have been also studied [3].

Conformal breaking from other particles can be communicated to photons. No one knows the gravitational couplings of the charged partners of the Standard Model Higgs boson (which become the longitudinal polarizations of the $W^\pm$ at low energies), but it has been suggested that the inflationary production of minimally coupled Higgs scalars could endow the photon with a mass during inflation [4–6], and that this might seed the ubiquitous cosmic magnetic fields of the current epoch [7,8]. An explicit one-loop computation of the massless charged scalar contribution to the vacuum polarization on a de Sitter background [9,10] has confirmed the photon mass conjecture [11], although more work needs to be done to connect this to cosmic magnetic fields [12]. Similar one-loop results pertain as well when the scalar has a small mass [13,14].

Because inflation produces more and more charged scalars as time progresses (provided they are light and nearly minimally coupled), the effective photon mass grows. The scalar mass remains small during this process [15–17] until a static, nonperturbative limit is eventually reached [18]. The vacuum energy drops while this occurs [19], and there are dramatic changes in the electrodynamic forces exerted by point charges and current dipoles [20].

The effects of charged, minimally coupled scalars are fascinating but dependent upon assumptions about the unknown conformal coupling of the Higgs boson. Gravitons also break conformal invariance so they can also communicate the violence of a primordial inflation to the photon sector [21–23]. Graviton effects are weaker because they are mediated through derivative interactions, but they are universal. Hence, they serve to establish the minimum level at which primordial inflation *must* affect electromagnetism. The purpose of this paper is to complete the derivation of these minimum effects.

Our technique is based on a recent dimensionally regulated and fully renormalized computation of the one-loop graviton contribution to the vacuum polarization $i[{}^\mu\Pi^\nu](x;x')$ on a de Sitter background [24]. We use this to quantum correct Maxwell's equation,

$$\partial_\nu[\sqrt{-g}g^{\nu\rho}g^{\mu\sigma}F_{\rho\sigma}(x)] + \int d^4x'[{}^\mu\Pi^\nu](x;x')A_\nu(x') = J^\nu(x), \tag{1}$$

where $g_{\mu\nu}$ is the de Sitter metric, $F_{\rho\sigma} \equiv \partial_\rho A_\sigma - \partial_\sigma A_\rho$ is the usual field strength tensor, and $J^\mu(x)$ is the current density. With $J^\mu(x) \neq 0$, one can study how inflationary gravitons alter the electrodynamic response to standard sources, as has recently been done for a point charge and for a point magnetic dipole with the following results [25]:

(i) an observer comoving with respect to the sources (hence, at an exponentially increasing physical distance) perceives the magnitude of the point charge to increase linearly with comoving time and logarithmically with the comoving position;
(ii) the comoving observer reports only a negative logarithmic spatial variation in the one-loop field of the magnetic dipole;
(iii) an observer at a fixed invariant distance from the sources perceives no secular change of the point charge; and
(iv) the static observer reports a secular enhancement of the magnetic dipole moment.

For our study, we set $J^\mu(x) = 0$ and work out the one-loop corrections to dynamical photons.

This paper has four sections of which the first is this Introduction. In Sec. II, we use the vacuum polarization

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[24,25] to derive an equation for the one-loop correction to spatial plane wave photon mode functions. This equation is solved in Sec. III. In Sec. IV, we discuss the minimum effect our result sets for inflation to seed cosmic magnetic fields.

II. EFFECTIVE MODE EQUATION FOR PHOTONS

The purpose of this section is to convert the quantum corrected Maxwell equation (1) into a simple relation for the one-loop corrections to the mode function of a plane wave photon on a de Sitter background. We first specialize to plane wave photons at a one-loop order. Then we discuss the restrictions imposed by cosmology, by effective field theory, and by our lack of knowledge about the initial state.

A. Perturbative formulation

We work on spatially flat sections of the de Sitter manifold in conformal coordinates,

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}\cdot d\vec{x}), \tag{2}$$

where $a(\eta) = -\frac{1}{H\eta} = e^{Ht}$ is the scale factor and H is the Hubble parameter. Hence, the metric can be written as $g_{\mu\nu} = a^2\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric. Because the Maxwell Lagrangian is conformally invariant, all the scale factors cancel in the leftmost term of (1), and we can express it as $\partial_\nu F^{\nu\mu}(x)$, where we raise and lower indices with the Minkowski metric, $F^{\nu\mu} \equiv \eta^{\nu\rho}\eta^{\mu\sigma}F_{\rho\sigma}$.

We follow [24] in employing a noncovariant representation for the vacuum polarization [26],

$$\begin{aligned} i[{}^\mu\Pi^\nu](x;x') &= (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})\partial_\rho\partial'_\sigma F(x;x') \\ &\quad + (\bar{\eta}^{\mu\nu}\bar{\eta}^{\rho\sigma} - \bar{\eta}^{\mu\sigma}\bar{\eta}^{\nu\rho})\partial_\rho\partial'_\sigma G(x;x'), \end{aligned} \tag{3}$$

where $\bar{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^\mu_0\delta^\nu_0$ is the purely spatial part of the Minkowski metric. (For the transformation to a covariant representation, see [27].) Substituting (3) into (1) with $J^\mu = 0$ and partially integrating the primed derivatives on the right-hand side gives,

$$\begin{aligned} \partial_\nu F^{\nu\mu}(x) = -\partial_\nu \int d^4x' \{ &iF(x;x')F^{\nu\mu}(x') \\ &+ iG(x;x')F^{\bar{\nu}\bar{\mu}}(x')\}. \end{aligned} \tag{4}$$

Here, a barred index on any tensor means that its 0 components vanish, for example, $V^{\bar{\mu}} \equiv \bar{\eta}^{\mu\nu}V_\nu = V^\mu - \delta^\mu_0 V^0$.

Some nonperturbative statements can be made. For example, so long as the vacuum polarization is computed (as the one-loop correction in [24]) using electromagnetic and gravitational gauge conditions that respect homogeneity and isotropy then the structure functions $F(x;x')$ and $G(x;x')$ can depend upon the spatial coordinates $\vec{x}$ and $\vec{x}'$ only through the Euclidean norm of their difference $\Delta\vec{x} \equiv \vec{x} - \vec{x}'$. If spatial surface terms vanish (which will be shown in the next subsection), we can reflect external space derivatives onto the field strengths inside of the integral of expression (4),

$$\begin{aligned} \partial_\nu F^{\nu\mu}(x) = &-\partial_0 \int d^4x' iF(x;x')F^{0\mu} \\ &- \int d^4x' \{iF(x;x')\partial'_j F^{j\mu}(x') \\ &+ iG(x;x')\partial'_j F^{j\bar{\mu}}(x')\}. \end{aligned} \tag{5}$$

One simple consequence of (5) is the validity, to all orders, of the general form for the classical field strengths of plane wave photon solutions with a wave vector $\vec{k}$ and a transverse polarization vector $\varepsilon^i(\vec{k},\lambda)$,

$$\begin{aligned} F^{0i}_{\rm ph}(x) &= -\partial_0 u(\eta,k) \times \varepsilon^i e^{i\vec{k}\cdot\vec{x}}, \\ F^{ij}_{\rm ph}(x) &= u(\eta,k) \times i[k^i\varepsilon^j - k^j\varepsilon^i]e^{i\vec{k}\cdot\vec{x}}. \end{aligned} \tag{6}$$

To see this, first substitute (6) into the $\mu = 0$ component of Eq. (5) and act the space derivatives using $\partial_j F^{\mu\nu}_{\rm ph} = ik_j F^{\mu\nu}_{\rm ph}$,

$$ik_j F^{j0}_{\rm ph}(x) = 0 - \int d^4x' iF(x;x')ik_j F^{j0}_{\rm ph}(x') + 0. \tag{7}$$

Exploiting transversality ($k_j\varepsilon^j = 0$) reduces expression (7) to a tautology of the form $0 = 0$. Now substitute (6) into the $\mu = i$ component of (5) and again exploit transversality,

$$-(\partial_0^2 + k^2)u(\eta,k)\times\varepsilon^i e^{i\vec{k}\cdot\vec{x}} = \partial_0\int d^4x' iF(x;x')\partial'_0 u(\eta',k)\times\varepsilon^i e^{i\vec{k}\cdot\vec{x}'} + \int d^4x' i[F(x;x') + G(x;x')]k^2 u(\eta',k)\times\varepsilon^i e^{i\vec{k}\cdot\vec{x}'}, \tag{8}$$

$$= \varepsilon^i e^{i\vec{k}\cdot\vec{x}} \times \left\{\partial_0 \int d^4x' iF(x;x')\partial'_0 u(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}} + k^2\int d^4x'[iF(x;x') + iG(x;x')]u(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}}\right\}. \tag{9}$$

Dispensing with the now-redundant factors of $\varepsilon^i e^{i\vec{k}\cdot\vec{x}}$ results in an effective mode equation for $u(\eta, k)$,

$$(\partial_0^2 + k^2)u(\eta,k) = -\partial_0 \int d^4x' iF(x;x')\partial_0' u(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}} - k^2 \int d^4x' [iF(x;x') + iG(x;x')]u(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}}. \tag{10}$$

Because the structure functions can only depend upon the norm of $\Delta\vec{x}$, it is possible to quite generally reduce the right-hand side of the effective mode Eq. (10) to a double integral over η' and $r \equiv \|\vec{x} - \vec{x}\|$. However, we must at this point face the fact that the structure functions can only be computed to some finite order in the quantum gravitational loop counting parameter $\kappa^2 \equiv 16\pi G$,

$$F(x;x') = 0 + \kappa^2 F_{(1)}(x;x') + \kappa^4 F_{(2)}(x;x') + \cdots, \tag{11}$$

$$G(x;x') = 0 + \kappa^2 G_{(1)}(x;x') + \kappa^4 G_{(2)}(x;x') + \cdots \tag{12}$$

At this time, we have only the one-loop results [24,25],

$$\begin{aligned} iF_{(1)} = \frac{-1}{8\pi^2}\Big\{ H^2[\ln(a) + \alpha] - \Big[\frac{\ln(a)}{3a^2} - \frac{\beta}{a^2}\Big](\partial^2 + 2Ha\partial_0) + \frac{H}{3a}\partial_0 \Big\}\delta^4(x-x') \\ + \frac{a^{-1}\partial^6}{384\pi^3}\Big\{ \frac{\theta(\Delta\eta - \Delta x)}{a'}\Big(\ln\Big[\frac{1}{4}H^2(\Delta\eta^2 - \Delta x^2)\Big] - 1\Big)\Big\} - \frac{H^2}{32\pi^3}\Big\{\Big[\frac{\partial^4}{4} + \partial^2\partial_0^2\Big] \\ \times \theta(\Delta\eta - \Delta x)\ln\Big[\frac{1}{4}H^2(\Delta\eta^2 - \Delta x^2)\Big] - \Big[\frac{\partial^4}{4} - \partial^2\partial_0^2\Big]\theta(\Delta\eta - \Delta x)\Big\}, \end{aligned} \tag{13}$$

$$iG_{(1)} = \frac{H^2}{6\pi^2}\Big[\ln(a) + \frac{3}{4}\gamma\Big]\delta^4(x-x') + \frac{H^2\partial^4}{96\pi^3}\Big\{\theta(\Delta\eta - \Delta x)\Big(\ln\Big[\frac{1}{4}H^2(\Delta\eta^2 - \Delta x^2)\Big] - 1\Big)\Big\}. \tag{14}$$

In these and subsequent expressions, the coordinate separations are $\Delta\eta \equiv \eta - \eta'$ and $\Delta x \equiv \|\vec{x} - \vec{x}'\|$, and the flat space d'Alembertian is $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_0^2 + \nabla^2$.

Because the structure functions are only known to a finite order in κ^2, there is no alternative to making a similar expansion for the mode function,

$$u(\eta,k) = u_0(\eta,k) + \kappa^2 u_1(\eta,k) + \kappa^4 u_2(\eta,k) + \cdots \tag{15}$$

Substituting expansions (11)–(12) and (15) into (10) and segregating terms of the same order in κ^2 gives the tree order and one-loop relations,

$$(\partial_0^2 + k^2)u_0(\eta,k) = 0, \tag{16}$$

$$\begin{aligned} (\partial_0^2 + k^2)u_1(\eta,k) \\ = -\partial_0 \int d^4x' iF_{(1)}(x;x')\partial_0' u_0(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}} \\ - k^2 \int d^4x' [iF_{(1)}(x;x') + iG_{(1)}(x;x')]u_0(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}}. \end{aligned} \tag{17}$$

By conformal invariance, the tree order mode function is the same in de Sitter conformal coordinates as it is in flat space, $u_0(\eta,k) = e^{-ik\eta}/\sqrt{2k}$.

B. Schwinger-Keldysh formalism

The treatment [though not of course the explicit structure functions (13)–(14)] we have given so far applies as well for traditional quantum field theory on flat space (for example, see [28]). However, it is important to understand that the usual effective field equations describe matrix elements of the field operator between states which are free in the asymptotic past and future. These in-out matrix elements provide a correct description of the scattering processes in flat space, but they make little sense in cosmology because the universe began with a singularity at some finite time and no one knows how (or even if) it will end. Persisting with the in-out effective field equations for inflationary cosmology would result in two embarrassments from the nonlocal source term on the right-hand side of expression (17):

(i) because the in-out structure functions do not vanish for x'^μ outside the past light cone of x^μ, the right-hand side of Eq. (17) would be dominated by contributions from the far future when the inflated three volume is much larger;

(ii) because the in-out structure functions are complex, they couple the real and imaginary parts of the mode function $u(\eta, k)$, making real field strengths impossible.

The more meaningful effective field to study for cosmology is the true expectation value of the field operator in the presence of some state which is released at a finite time. The appropriate field equations for studying expectation values are those of the Schwinger-Keldysh formalism [29–37]. The associated one-loop structure functions were given in expressions (13)–(14). Note that they are manifestly real, and that the factors of $\theta(\Delta\eta - \Delta x)$ make each term vanish whenever the point x'^{μ} strays outside the past light cone of x^{μ}.[1] These are important features of the Schwinger-Keldysh formalism which the in-out formalism lacks.

The constants α, β, and γ which appear in expressions (13)–(14) represent the arbitrary finite parts of the three higher derivative counterterms which were needed to renormalize the vacuum polarization [24] because the combined theory of general relativity and Maxwell's theory, which from now on we will refer to "Einstein + Maxwell" equation is not perturbatively renormalizable [38,39]. (Appendix A discusses the noncovariant counterterm resulting from the use of a de Sitter breaking gauge to compute the vacuum polarization [24].) No physical principle can fix these constants because those counterterms cannot actually be present in a fundamental theory. They are the price we must pay for using the Einstein-Maxwell equation as a low energy effective field theory. In contrast, the logarithms of the scale factor with which the three constants are paired,

$$\ln(a) + \alpha, \qquad \frac{1}{3}\ln(a) - \beta, \qquad \ln(a) + \frac{3}{4}\gamma, \tag{18}$$

represent unique and reliable predictions of the theory which must persist in whatever is the correct ultraviolet completion of the Einstein + Maxwell. At late times, these logarithms dwarf the unknown constants, which means that we can make reliable predictions in the late time regime. We are of course making the usual assumption of a low energy effective field theory that α, β, and γ are of order one.

Another limit to the generality of our formalism is that the structure functions (13)–(14) were computed without correcting the free vacuum state. For in-out matrix elements, we typically do not worry about correcting the states because infinite time evolution is supposed to accomplish this in the weak operator sense. However, when the universe is released at a finite time, one must include at least the perturbative corrections to the initial state. In the Schwinger-Keldysh formalism, these corrections show up as new interaction vertices on the initial value surface [40]. Unlike the finite parts of the higher derivative counterterms, it is perfectly possible to work these corrections out [41–46]. However, there is no point in doing so because they give rise to surface terms which fall of the inflationary scale factor. We shall assume that these corrections simply serve to cancel the surface terms which would arise, at various points, from partial integrations.

III. SOLVING THE EQUATION

The purpose of this section is to solve Eq. (17) for $u_1(\eta, k)$ in the late time limit for which reliable predictions can be made. First, note that (17) can be expressed in terms of seven master integrals,

$$(\partial_0^2 + k^2)u_1(\eta, k) = ik\partial_0[I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7] + k^2\left[-I_1 + \frac{1}{3}I_2 - I_3 + \frac{1}{3}I_4 - I_5 - I_6 - I_7\right], \tag{19}$$

where the various integrals are,

$$I_1(\eta, k) \equiv -\frac{a^{-1}(\partial_0^2 + k^2)^3}{384\pi^3}\int d^4x' \frac{\Theta}{a'}\left[\ln\left[\frac{H^2}{4}(\Delta\eta^2 - \Delta x^2)\right] - 1\right]u_0(\eta', k)e^{-i\vec{k}\cdot\Delta\vec{x}}, \tag{20}$$

$$I_2(\eta, k) \equiv -\frac{H^2(\partial_0^2 + k^2)^2}{128\pi^3}\int d^4x'\Theta\left[\ln\left[\frac{H^2}{4}(\Delta\eta^2 - \Delta x^2)\right] - 1\right]u_0(\eta', k)e^{-i\vec{k}\cdot\Delta\vec{x}}, \tag{21}$$

$$I_3(\eta, k) \equiv \frac{H^2(\partial_0^2 + k^2)\partial_0^2}{32\pi^3}\int d^4x'\Theta\left[\ln\left[\frac{H^2}{4}(\Delta\eta^2 - \Delta x^2)\right] + 1\right]u_0(\eta', k)e^{-i\vec{k}\cdot\Delta\vec{x}}, \tag{22}$$

$$I_4(\eta, k) \equiv -\frac{H^2\ln(a)}{8\pi^2}\int d^4x'\delta^4(x - x')u_0(\eta', k)e^{-i\vec{k}\cdot\Delta\vec{x}}, \tag{23}$$

[1] One consequence is that spatial integration by parts produces no surface terms in the Schwinger-Keldysh formalism. Partial integration in time can and does produce surface terms at the initial time.

$$I_5(\eta,k) \equiv -\frac{a^{-2}\ln(a)(\partial_0^2+k^2)}{24\pi^2}\int d^4x'\delta^4(x-x')u_0(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}}, \tag{24}$$

$$I_6(\eta,k) \equiv \frac{Ha^{-1}\ln(a)\partial_0}{12\pi^2}\int d^4x'\delta^4(x-x')u_0(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}}, \tag{25}$$

$$I_7(\eta,k) \equiv -\frac{Ha^{-1}\partial_0}{24\pi^2}\int d^4x'\delta^4(x-x')u_0(\eta',k)e^{-i\vec{k}\cdot\Delta\vec{x}}. \tag{26}$$

To save space, we have defined the causality enforcing θ function as $\Theta \equiv \theta(\Delta\eta - \Delta x)$. Of course the delta function terms (23)–(26) are trivial, and most of the nonlocal contributions can be inferred from previous work [47]. Technical details can be found in Appendix B but the results are,

$$I_1(\eta,k) = \frac{H^2u_0(\eta,k)}{48\pi^2 a}\left\{\left[\frac{1+2ik\Delta\eta_i+e^{2ik\Delta\eta_i}}{H^2\Delta\eta_i^2}\right]+\left[\frac{1+e^{2ik\Delta\eta_i}}{H\Delta\eta_i}\right]\right.$$
$$\left.-\frac{4ik}{H}\ln(H\Delta\eta_i)-\frac{2ik}{H}\int_0^1\frac{dt}{t}[e^{2ik\Delta\eta_i t}-1]\right\}, \tag{27}$$

$$I_2(\eta,k) = \frac{H^2u_0(\eta,k)}{16\pi^2}\left\{-2\ln(H\Delta\eta_i)-\int_0^1\frac{dt}{t}[e^{2ik\Delta\eta_i t}-1]\right\}, \tag{28}$$

$$I_3(\eta,k) = \frac{H^2u_0(\eta,k)}{16\pi^2}\left\{[6-4ik\Delta\eta_i+2e^{2ik\Delta\eta_i}]\ln(H\Delta\eta_i)+e^{2ik\Delta\eta_i}+7-2ik\Delta\eta_i\right.$$
$$\left.+\int_0^1\frac{dt}{t}[(3-2ik\Delta\eta_i)(e^{2ik\Delta\eta_i t}-1)+e^{2ik\Delta\eta_i}(e^{-i2ik\Delta\eta_i t}-1)]\right\}, \tag{29}$$

$$I_4(\eta,k) = -\frac{H^2\ln(a)}{8\pi^2}\times u_0(\eta,k), \tag{30}$$

$$I_5(\eta,k) = 0, \tag{31}$$

$$I_6(\eta,k) = -\frac{ikH\ln(a)}{12\pi^2 a}\times u_0(\eta,k), \tag{32}$$

$$I_7(\eta,k) = \frac{ikH}{24\pi^2 a}\times u_0(\eta,k). \tag{33}$$

Here and henceforth, we define $\Delta\eta_i \equiv \eta-\eta_i = H^{-1}(1-\frac{1}{a})$, where η_i is the initial conformal time.

A few comments are in order regarding the inverse factors of $\Delta\eta_i$ which appear in expression (27) for $I_1(\eta,k)$. These factors diverge on the initial value surface and completely preclude any attempt to exactly solve the one-loop truncated effective field equations in their present form. That the problem has nothing to do with the de Sitter background is obvious from the fact that these very same divergences appear as well when the background is changed to flat space [28]. The problem arises because the vacuum polarization was computed in free (Bunch-Davies) vacuum $\Omega_0[A,h]$ [24]. Even in flat space, the true vacuum state wave functional $\Omega[A,h]$ requires perturbative corrections [41–46],

$$\Omega[A,h] = \Omega_0[A,h]\times\left\{1+\kappa\int d^3x_1\int d^3x_2\int d^3x_3\Omega^{\mu\nu\rho\sigma}(\vec{x}_1,\vec{x}_2,\vec{x}_1)h_{\mu\nu}(\eta_i,\vec{x}_1)A_\rho(\eta_i,\vec{x}_2)A_\sigma(\eta_i,\vec{x}_3)+O(\kappa^2)\right\}, \tag{34}$$

TABLE I. Leading late time limiting forms for the integrals defined in expressions (20)–(26) and their first time derivatives. Only the terms which show a secular growth are given explicitly.

i	$\sqrt{2k} I_i(\eta, k)$	$\sqrt{2k}\partial_0 I_i(\eta, k)$
1	$O(\frac{1}{a})$	$O(1)$
2	$O(1)$	$O(1)$
3	$O(1)$	$O(1)$
4	$-\frac{H^2\ln(a)}{8\pi^2}$	$-\frac{H^3 a}{8\pi^2}$
5	0	0
6	$O(\frac{\ln(a)}{a})$	$\frac{ikH^2\ln(a)}{12\pi^2}$
7	$O(\frac{1}{a})$	$O(1)$

where $\Omega^{\mu\nu\rho\sigma}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ is a $\mathbb{C}$-number function which could be worked out—but has not been—the same way one computes corrections to the simple harmonic oscillator wave functions when the Hamiltonian contains an anharmonic term. We stress that the problem derives from combining $(D = 4)$ interactions with an evolution from a finite time, and it cannot be solved by any clever choice Gaussian initial state $\Omega_0[A, h]$. It can be avoided in flat space by taking the initial time to $-\infty$ [28] but this is not an option for our de Sitter computation owing to the factors of $\ln(a)$ and $1/a$ which are evident in expressions (30) and (32)–(33).[2] On the other hand, the initial value divergences in expression (27) are all well behaved at late times,

$$\frac{1}{H^2\Delta\eta_i^2} \longrightarrow 1, \qquad \frac{1}{H\Delta\eta_i} \longrightarrow 1, \qquad \ln(H\Delta\eta_i) \longrightarrow 0. \tag{35}$$

The multiplicative factor of $1/a$ makes $I_1(\eta, k)$ go to zero at late times, so we can avoid computing initial state corrections by simply working consistently in the late time regime, which is in any case necessary owing to the unknown finite parts of the counterterms.

Table I gives the leading late time effect from each of the seven integrals and its first (conformal) time derivative. The dominant effect derives from the time derivative of $I_4(\eta, k)$,

$$(\partial_0^2 + k^2)u_1(\eta, k) = -\frac{ikH^3 a}{8\pi^2} \times u_0(\eta, k) + O(\ln(a)). \tag{36}$$

Hence, we find,

$$u_1(\eta, k) = \frac{ikH\ln(a)}{8\pi^2} \times u_0(\eta, k) + O\left(\frac{1}{a}\right). \tag{37}$$

[2]The physical origin of this mathematical obstacle is an inflationary particle production which results in very high occupation numbers $N(t, k) = [Ha(t)/2k]^2$ for graviton modes which have experienced a first horizon crossing. The earlier one releases the initial state, the more modes will have experienced first horizon crossing by any fixed late time.

From expression (6), it follows that the one-loop field strengths are

$$\kappa^2 F^{0i}_{(1)}(x) = \frac{\kappa^2 H^2}{8\pi^2}\{\ln(a) + O(1)\} \times F^{0i}_{(0)}(x), \tag{38}$$

$$\kappa^2 F^{ij}_{(1)}(x) = \frac{\kappa^2 H^2}{8\pi^2}\left\{\frac{ik\ln(a)}{Ha} + O\left(\frac{1}{a}\right)\right\} \times F^{ij}_{(0)}(x). \tag{39}$$

IV. DISCUSSION

We have employed a previous computation of the one-loop contribution to the vacuum polarization from inflationary gravitons [24] to derive what happens to photons during primordial inflation. Our results (38)–(39) for the field strengths show that the electric field experiences a secular enhancement, relative to its classical value. In contrast, the one-loop correction to the magnetic field falls off with respect to its classical counterpart. Both results are consistent with the one-loop photon wave function (37) relaxing to zero less slowly [by one factor of $\ln(a)$] than the classical mode function approaching a constant.

The enhancement we find seems to be derived from the buffeting of photons by inflationary gravitons. Even though the photon's kinetic energy redshifts to zero, its spin does not, and this permits it to continue interacting with inflationary gravitons even at late times. The same $\ln(a)$ enhancement was found for massless fermions [47–49] and was explicitly tied to the spin interaction [50]. In contrast, massless, minimally coupled scalars neither experience any significant effect from inflationary gravitons [51,52], nor do they induce a significant effect on inflationary gravitons [53–55]. Gravitons also have a spin so it would be very interesting to see what they do to themselves, as well as to the force of gravity.

An interesting technical detail concerns the comparison of our full one-loop computations with the result previously derived using the Hartree approximation [24]. Both calculations give the same time dependence, confirming the general reliability of the Hartree approximation for predicting the functional form. However, the sign of our exact computation differs from that of the Hartree result. This emphasizes the need for making exact computations and has clear implications for gravitons [56].

One important consequence of our result is that quantum gravitational perturbation theory must break down after an enormous number of e-foldings $\ln(a) \sim 1/\kappa^2 H^2$, which is larger than 10^{10} in standard inflation. Note first that this eventual breakdown in no way invalidates the use of a perturbation theory at earlier times. The reason for working in the late time regime is that we have not included the perturbative corrections to the initial state (although this could be done) and that we do not know, and cannot know, the finite parts α and β of the higher time derivative counterterms. However, as long as one makes the usual assumption that these finite parts are of an order one, it will be seen that there is an enormous range of times for which the late time limiting form (38) vastly dominates the unknown contributions, but it is still small compared with the classical result. For example, if it is agreed that the infrared logarithm $\ln(a)$ dominates when it has reached $\ln(a) \gtrsim 100$, and that it is still reliable for $\kappa^2 H^2 \ln(a) \lesssim 1/100$, then we see that the regime of validity for our result extends to a million e-foldings of inflation. For some perspective, it should be recalled that there is no currently recognized evidence for more than about 60 e-foldings of inflation.

Still, it is a fact that perturbation theory must eventually break down if inflation persists long enough, and we would like to know what takes place afterwards. The analogous problem for scalar potential models on a nondynamical de Sitter background [whose loop corrections also involve factors of $\ln(a)$] can be solved using Starobinsky's stochastic formalism [57,58]. A proof has been given that this method reproduces the leading secular growth factors at each order in the perturbation theory [59,60], and it has been extended to include scalars coupled to fermions [61] and to electromagnetism [18]. Detailed analysis of the nonperturbative resummation of this series of leading logarithms reveals that all three logical possibilities occur for the induced vacuum energy:

(i) it can approach a small positive constant [58];
(ii) it can approach a small negative constant [18]; or
(iii) it can decrease without bound, resulting in a big rip singularity [61].

The stochastic formalism is also very useful in debunking techniques which are sometimes proposed for evolving past the breakdown of perturbation theory, for example, using order reduction to solve the one-loop truncated effective field equations exactly [62][3] or employing variants of the renormalization group [63]. These techniques do give equations which can be evolved past the breakdown of perturbation theory, but the results are wrong [64].

Unfortunately, the presence of derivative interactions means that the proof of Starobinsky's formalism which was given for scalar potential models [59,60] does not apply to a quantum gravity. One might speculate that the stochastic formalism is nonetheless correct, and it has been employed on this basis to estimate the corrections to the inflationary power spectra [65]. Other plausible approximations were earlier invoked to arrive at somewhat different estimates [66–68]. And there has been recent work by Kitamoto and Kitazawa on secular growth corrections to gauge couplings [69–73]. The key question is, does any specific formalism reproduce the correct secular growth factors from inflationary gravitons? Absent a proof, one can never know except by comparison with complete and fully renormalized results such as the one (38) we have just derived. Indeed, expression (38) is only the second example of such a result so it effectively doubles what is reliably known about this fascinating phenomenon.

An appreciation of the importance of (38) comes by recalling what was learned from the only other complete and fully renormalized graviton loop which shows a factor of $\ln(a)$, the 2005–2006 computation of inflationary graviton corrections to massless fermions [47–49]. A diagram-by-diagram analysis of that result reached the following conclusions [50]:

(i) naive application of the stochastic formalism does not correctly predict the secular growth factor of $\ln(a)$;
(ii) the factor of $\ln(a)$ derives entirely from diagrams which involve the fermion spin connection; and
(iii) because the ultraviolet sector of the fermion field contributes to the factor of $\ln(a)$, one must leave the ultraviolet regularization on as well for the graviton.

One should also note the claim that false indications of secular corrections can come from neglecting diagrams [74].

It is not that approximate calculations are necessarily wrong but rather that we cannot currently judge their validity. When a technique is finally devised for isolating the leading secular effects at each order and resumming them, it will no doubt lead to simple approximations for quickly inferring just the leading effects, without enduring the tedium of a complete and fully renormalized computation. We have reached this point with scalar potential models through Starobinsky's stochastic formalism. However, we are not there yet for quantum gravity, and we are not likely to get there, or be sure that we have even made any progress, without the careful examination of exact results such as expression (38). These comparisons sometimes reveal subtle problems with plausible-sounding arguments such as the universal applicability of the stochastic formalism or the irrelevance of ultraviolet effects.

Finally, it is interesting to work out what our result says for the possibility of inflation seeding cosmic magnetic fields. We can use the 0-point energy to estimate the number of photons created by inflationary gravitons. Because the comoving time t is related to the conformal time η by $dt = a d\eta$, the physical Hamiltonian (which generates an evolution in t) is $1/a$ times the conformal one. The physical 0-point energy in a single photon polarization wave vector $\vec{k}$ is therefore,

[3]Order reduction is just a technique for preventing higher derivatives from introducing new solutions, which is a problem that does not even arise in our one-loop solution. The problem comes in solving the one-loop truncated equations exactly. This is only valid if no equally strong contributions occur in the primitive contributions from higher loops. The stochastic formalism shows that they do.

$$\frac{1}{2a}[F^{0i}_{\rm ph}\times F^{0i*}_{\rm ph}+\frac{1}{2}F^{ij}_{\rm ph}\times F^{ij*}_{\rm ph}]=\frac{k^2uu^*}{2a}, \tag{40}$$

$$\longrightarrow\frac{k}{2a}\times\left\{1+\frac{\kappa^2H^2\ln(a)}{8\pi^2}+O(\kappa^4)\right\}. \tag{41}$$

The occupation number $N(\eta,k)$ is defined by equating the 0-point energy (41) to $(N+\frac{1}{2})\times\frac{k}{a}$,

$$N=\frac{\kappa^2H^2\ln(a)}{16\pi^2}+O(\kappa^4). \tag{42}$$

The remainder of the computation is the same as the analysis [12] for the much larger effect from inflationary scalars (if the Higgs boson is minimally coupled and still light at inflationary energy scales). Substituting our value (42) for the occupation number into Eq. (118) of [12] results in the following estimate for the magnetic strength on the scale ℓ_0,

$$B^2(\ell_0)\sim\frac{\hbar^2GH_I^2N_I}{8\pi^3c^4\ell_0^4}, \tag{43}$$

where H_I is the inflationary Hubble parameter, and N_I is the value reached by $\ln(a)$. Plugging in the numbers with $N_I\sim 50$ and $\ell_0\sim 10$ kpc gives,

$$B(\ell_0)\sim 10^{-61}\ \text{G}. \tag{44}$$

This is far too small to have seeded today's cosmic magnetic fields, but it does serve to establish the absolute minimum effect which *must* be present from inflation.

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APPENDIX A: DE SITTER BREAKING GAUGES

Our analysis is based on an earlier computation [24] of the one-loop vacuum polarization from inflationary gravitons that was made using electromagnetic and gravitational gauge fixing terms which respect the homogeneity and the isotropy, and also the dilatation invariance, but not the three remaining symmetries of the de Sitter group. The special feature of these gauge fixing terms is that the photon and graviton propagators consist of linear combinations of constant tensor factors times scalar propagators [75,76]. This makes the computation tractable but it also means that renormalization can involve noninvariant counterterms, and one such does occur [24].[4] Similar noninvariant counterterms arose from the graviton contribution to the one-loop fermion self-energy [48] [see Eqs. (74), (216), and (221)] and the one-loop self-mass squared of a massless, minimally coupled scalar [51] [see Eq. (145) and Table 8].

Those accustomed to modern techniques of renormalization in covariant gauges sometimes find the appearance of the noninvariant counterterms to be disconcerting. However, it is important to realize that they pose no problem of the principle. The divergent part of the counterterm is of course fixed by the primitive divergences it is to remove, and the finite part can be determined to enforce physical symmetries. (In our case, the focus on late times obviates the need for this as long as the finite part of the counterterm is assumed to be of order one.) The procedure is explained in older standard texts on quantum field theory, for example, [78]. And it is important to recognize that many of the classic computations of quantum electrodynamics were in fact performed using noncovariant gauges [79].

Finally, it should be noted that an impressive amount of evidence has been accumulated to support the consistency of this particular gauge. This evidence includes:

(i) an explicit check of the tree order gravitational Ward identity [80];
(ii) an explicit check of the one graviton loop gravitational Ward identity [81];
(iii) a detailed physical comparison between the noncovariant gauge result for the one (photon) loop contribution to a charged scalar self-mass squared [15,16] and the analogous covariant guage result [17]; and
(iv) a computation of the linearized Weyl-Weyl correlator in both the noncovariant [82] and covariant gauges [83].

APPENDIX B: INTEGRALS FROM SEC. III

The purpose of this appendix is to summarize the necessary details for evaluating the nonlocal terms

[4]The vacuum polarization was recently computed in a manifestly de Sitter invariant gauge. Strangely enough, the noninvariant counterterm is still required owing the time ordering of interactions, the fact that gravitational interactions contain two derivatives, and the fact that the coincident graviton propagator diverges in de Sitter background [77].

(20)–(22) and show that they indeed make no significant contribution in the late time regime. The first step is to perform the angular integrations using the formula,

$$\int d^3x'\Theta f(\Delta x)e^{-i\vec{k}\cdot\Delta\vec{x}} = 4\pi\int_0^{\Delta\eta} drr^2 f(r)\frac{\sin(kr)}{kr}. \tag{B1}$$

Employing relation (B1) in (20)–(22) allows us to write,

$$I_1 = -\frac{a^{-1}(\partial_0^2+k^2)^3}{96\pi^2 k}\int_{\eta_i}^{\eta} d\eta'\frac{u(\eta',k)}{a'}\int_0^{\Delta\eta} drr\,\sin(kr)\left\{\ln\left[\frac{H^2}{4}(\Delta\eta^2-r^2)\right]-1\right\}, \tag{B2}$$

$$I_2 = -\frac{H^2(\partial_0^2+k^2)^2}{32\pi^2 k}\int_{\eta_i}^{\eta} d\eta' u(\eta',k)\int_0^{\Delta\eta} drr\sin(kr)\left\{\ln\left[\frac{H^2}{4}(\Delta\eta^2-r^2)\right]-1\right\}, \tag{B3}$$

$$I_3 = \frac{H^2(\partial_0^2+k^2)\partial_0^2}{8\pi^2 k}\int_{\eta_i}^{\eta} d\eta' u(\eta',k)\int_0^{\Delta\eta} drr\sin(kr)\left\{\ln\left[\frac{H^2}{4}(\Delta\eta^2-r^2)\right]+1\right\}, \tag{B4}$$

where $\eta_i \equiv -H^{-1}$ denotes the initial time. The next step is to perform the two independent radial integrations,

$$J_1(\Delta\eta,k) \equiv \int_0^{\Delta\eta} drr\sin(kr) = \frac{T(k\Delta\eta)}{k^2}, \tag{B5}$$

$$J_2(\Delta\eta,k) \equiv \int_0^{\Delta\eta} drr\sin(kr)\ln\left[\frac{H^2}{4}(\Delta\eta^2-r^2)\right], \tag{B6}$$

$$= \frac{2\ln(H\Delta\eta)}{k^2}T(k\Delta\eta) + \frac{1}{k^2}\int_0^1\frac{dt}{t}\{T[k\Delta\eta(1-2t)]-T(k\Delta\eta)\}, \tag{B7}$$

where $T(x) \equiv \sin(x) - x\cos(x) = \frac{1}{3}x^3 + O(x^5)$. This allows us to express the integrals $I_{1-3}(\eta,k)$ as,

$$I_1(\eta,k) = -\frac{a^{-1}(\partial_0^2+k^2)^3}{96\pi^2 k}\int_{\eta_i}^{\eta} d\eta'\frac{u(\eta',k)}{a'}\{J_2(\Delta\eta,k)-J_1(\Delta\eta,k)\}, \tag{B8}$$

$$I_2(\eta,k) = -\frac{H^2(\partial_0^2+k^2)^2}{32\pi^2 k}\int_{\eta_i}^{\eta} d\eta' u(\eta',k)\{J_2(\Delta\eta,k)-J_1(\Delta\eta,k)\}, \tag{B9}$$

$$I_3(\eta,k) = \frac{H^2\partial_0^2(\partial_0^2+k^2)}{8\pi^2 k}\int_{\eta_i}^{\eta} d\eta' u(\eta',k)\{J_2(\Delta\eta,k)+J_1(\Delta\eta,k)\}. \tag{B10}$$

Because the various integrands of (B8)–(B10) vanish like $\Delta\eta^3\ln(\Delta\eta)$ at $\eta'=\eta$, we can pass one factor of the differential operator $(\partial_0^2+k^2)$ through the integration to act on $J_1(\Delta\eta,k)$ and $J_2(\Delta\eta,k)$,

$$I_1(\eta,k) = -\frac{a^{-1}(\partial_0^2+k^2)^2}{48\pi^2 k}\int_{\eta_i}^{\eta} d\eta'\frac{u(\eta',k)}{a'}\left\{2\sin(k\Delta\eta)\ln(H\Delta\eta)+\int_0^1\frac{dt}{t}[\sin[k\Delta\eta(1-2t)]-\sin(k\Delta\eta)]\right\}, \tag{B11}$$

$$I_2(\eta,k) = -\frac{H^2(\partial_0^2+k^2)}{16\pi^2 k}\int_{\eta_i}^{\eta} d\eta' u(\eta',k)\left\{2\sin(k\Delta\eta)\ln(H\Delta\eta)+\int_0^1\frac{dt}{t}[\sin[k\Delta\eta(1-2t)]-\sin(k\Delta\eta)]\right\}, \tag{B12}$$

$$I_3(\eta,k) = \frac{H^2\partial_0^2}{4\pi^2 k}\int_{\eta_i}^{\eta} d\eta' u(\eta',k)\left\{2\sin(k\Delta\eta)\ln(H\Delta\eta)+2\sin(k\Delta\eta)+\int_0^1\frac{dt}{t}[\sin[k\Delta\eta(1-2t)]-\sin(k\Delta\eta)]\right\}. \tag{B13}$$

We can pass one more derivative through the integration using the identities,

$$(\partial_0^2 + k^2)[u(\eta', k)f(\Delta\eta)] = -(\partial_0 - ik) \times \partial_0'[u(\eta', k)f(\Delta\eta)], \tag{B14}$$

$$(\partial_0^2 + k^2)^2\left[\frac{u(\eta', k)}{a'}f(\Delta\eta)\right] = -(\partial_0^2 + k^2)(\partial_0 - ik) \times \partial_0'\left[\frac{u(\eta', k)}{a'}f(\Delta\eta)\right] + H(\partial_0 - ik)^2 \times \partial_0'[u(\eta', k)f(\Delta\eta)]. \tag{B15}$$

The final results are (31)–(33).

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