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General-relativistic rotation laws in rotating fluid bodies

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We formulate new general-relativistic extensions of Newtonian rotation laws for self-gravitating stationary fluids. They have been used to rederive, in the first post-Newtonian approximation, the well-known geometric dragging of frames. We derive two other general-relativistic weak-field effects within rotating tori: the recently discovered dynamic antidragging and a new effect that measures the deviation from the Keplerian motion and/or the contribution of the fluids self-gravity. One can use the rotation laws to study the uniqueness and the convergence of the post-Newtonian approximations as well as the existence of the post-Newtonian limits.

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I. INTRODUCTION

Stationary Newtonian hydrodynamic configurations are characterized by a variety of rotation curves. The angular momentum per unit mass j can be any function of r, where r is the distance from the rotation axis. Other restrictions arise from stability considerations [1]. In contrast to that, for a long time, the only known rotation law in general-relativistic hydrodynamics had been that with j as a linear function of the angular velocity. Recently Galeazzi, Yoshida, and Eriguchi [2] have found a nonlinear angular velocity profile, that may approximate the Newtonian monomial rotation curves $\Omega_0 = w/r^\lambda$ in the nonrelativistic limit. In this paper we define general-relativistic rotation curves $j = j(\Omega)$ that in the nonrelativistic limit exactly coincide with $\Omega_0 = w/r^\lambda$ ($0 \leq \lambda \leq 2, \lambda \neq 1$). We are able to obtain the general-relativistic Keplerian rotation law that possesses the first post-Newtonian limit (1PN) and exactly encompasses the solution corresponding to the massless disk of dust in the Schwarzschild spacetime.

II. HYDRODYNAMICAL EQUATIONS

We recapitulate, following Ref. [3], the equations of general-relativistic hydrodynamics. Einstein equations, with the signature of the metric $(-,+,+,+)$, read

$$R_{\mu\nu} - g_{\mu\nu}\frac{R}{2} = 8\pi \frac{G}{c^4} T_{\mu\nu}, \tag{1}$$

where $T_{\mu\nu}$ is the stress-momentum tensor. The *stationary* metric reads

$$ds^2 = -e^{\frac{2\nu}{c^2}}(dx^0)^2 + r^2 e^{\frac{2\beta}{c^2}}\left(d\phi - \frac{\omega}{c^3}(r,z)dx^0\right)^2 + e^{\frac{2\alpha}{c^2}}(dr^2 + dz^2). \tag{2}$$

Here $x^0 = ct$ is the rescaled time coordinate, and r, z, ϕ are cylindrical coordinates. We assume axial symmetry and employ the stress-momentum tensor

$$T^{\alpha\beta} = \rho(c^2 + h)u^\alpha u^\beta + p g^{\alpha\beta}, \tag{3}$$

where ρ is the baryonic rest-mass density, h is the specific enthalpy, and p is the pressure. The 4-velocity u^α is normalized, $g_{\alpha\beta}u^\alpha u^\beta = -1$. The coordinate (angular) velocity reads $\vec{v} = \Omega\partial_\phi$, where $\Omega = u^\phi/u^t$.

We assume a barotropic equation of state $p = p(\rho)$. To be more concrete, one can take the polytropic equation of state $p(\rho, S) = K(S)\rho^\gamma$, where S is the specific entropy of fluid. Then one has $h(\rho, S) = K(S)\frac{\gamma}{\gamma-1}\rho^{\gamma-1}$. The entropy is assumed to be constant.

Define the square of the linear velocity

$$V^2 = r^2\left(\Omega - \frac{\omega}{c^2}\right)^2 e^{2(\beta-\nu)/c^2}. \tag{4}$$

The potentials α, β, ν, and ω satisfy equations that have been found by Komatsu, Eriguchi, and Hachisu [3]. They constitute an overdetermined, but consistent, set of equations. The general-relativistic Euler equations are solvable, assuming an integrability condition—that the angular momentum per unit mass,

$$j = u_\phi u^t = \frac{V^2}{(\Omega - \frac{\omega}{c^2})(1 - \frac{V^2}{c^2})}, \tag{5}$$

depends only on the angular velocity Ω; $j \equiv j(\Omega)$. In such a case, the Euler equations reduce to a general-relativistic integroalgebraic Bernoulli equation, that embodies the hydrodynamic information carried by the continuity equations $\nabla_\mu T^{\mu\nu} = 0$ and the baryonic mass conservation $\nabla_\mu(\rho u^\mu) = 0$. It is given by the expression

$$\ln\left(1+\frac{h}{c^2}\right)+\frac{\nu}{c^2}+\frac{1}{2}\ln\left(1-\frac{V^2}{c^2}\right)+\frac{1}{c^2}\int d\Omega j(\Omega)=C. \tag{6}$$

III. ROTATION LAWS

The general-relativistic rotation law employed in the literature [3–7] has the form

$$j(\Omega)=A^2(\Omega_c-\Omega), \tag{7}$$

where A and Ω_c are parameters. In the Newtonian limit and large A, one arrives at the rigid rotation, $\Omega=\Omega_c$, while for small A one gets the constant angular momentum per unit mass. A three-parameter expression for j is proposed in Ref. [2].

Below we define a new family of rotation laws,

$$j(\Omega)\equiv\frac{w^{1-\delta}\Omega^{\delta}}{1-\frac{\kappa}{c^2}w^{1-\delta}\Omega^{1+\delta}+\frac{\Psi}{c^2}}, \tag{8}$$

where w, δ, κ, and Ψ are parameters. The rotation curves $\Omega(r,z)$ ought to be recovered from the equation

$$\frac{w^{1-\delta}\Omega^{\delta}}{1-\frac{\kappa}{c^2}w^{1-\delta}\Omega^{1+\delta}+\frac{\Psi}{c^2}}=\frac{V^2}{(\Omega-\frac{\omega}{c^2})(1-\frac{V^2}{c^2})}. \tag{9}$$

For $\delta\neq-1$, the general-relativistic Bernoulli equation (6) acquires a simple algebraic form,

$$\left(1+\frac{h}{c^2}\right)e^{\nu/c^2}\sqrt{1-\frac{V^2}{c^2}}\times\left(1-\frac{\kappa}{c^2}w^{1-\delta}\Omega^{1+\delta}+\frac{\Psi}{c^2}\right)^{\frac{-1}{(1+\delta)\kappa}}=C. \tag{10}$$

We shall explain now the meaning and status of the four constants w, δ, κ, and Ψ. Assume that there exists the Newtonian limit [the zeroth order of the post-Newtonian expansion (0PN)] of the rotation law. This yields

$$\Omega_0=\frac{w}{r^{\frac{2}{1-\delta}}}. \tag{11}$$

Thus, w and δ can be obtained from the Newtonian limit. Moreover, the constant w is any real number, while δ is nonpositive—due to the stability requirement [1]—and satisfies the bounds $-\infty\leq\delta\leq0$ and $\delta\neq-1$. These two constants can be given *a priori* within the given range of values. Let us remark at this point that the rotation law (8), and consequently the Newtonian rotation (11), applies primarily to single rotating toroids and toroids rotating around black holes. In the case of rotating stars, one would have to construct a special differentially rotating law, with the aim of avoiding singularity at the rotation axis.

The two limiting cases $\delta=0$ and $\delta=-\infty$ correspond to the constant angular momentum per unit mass ($\Omega_0=w/r^2$) and the rigid rotation ($\Omega=w$), respectively. The Keplerian rotation is related to the choice of $\delta=-1/3$ and $w^2=GM$, where M is a mass [8]. The case with $\delta=-1$ should be considered separately, but we expect that the reasoning will be similar.

The values of κ and Ψ are problematic. One possibility to get them is to apply the post-Newtonian (PN) expansion. The rotation law in the PN expansion scheme should not be given *a priori* but is expected to build up—in the subsequent orders of c^{-2}—from the Newtonian rotation law. The Newtonian rotation curves are specified arbitrarily, but the next PN corrections should be defined uniquely. This is, however, a well-known property of the post-Newtonian expansions, that they are nonunique. Damour, Jaranowski, and Schäfer [9] demand that a test body rotating circularly in a Schwarzschild space-time satisfies exactly the Keplerian rotation law with $\Omega^2=GM/R^3$, where R is the areal radius. Inspired by this we impose a fixing condition (F-condition hereafter)—that a rotating infinitely thin disk made of dust in a Schwarzschild space-time satisfies exactly the Bernoulli equation and the Keplerian rotation law.

Consider a rotating, infinitely thin, and weightless disk of dust in the Schwarzschild geometry. This is textbook knowledge that there exists a stationary solution—each particle of dust can move along a circular trajectory of a radius R with the angular velocity $\Omega=\sqrt{GM/R^3}$. We shall present this solution in conformal coordinates, using our formalism. The conformal Schwarzschild metric reads $ds^2=-\Phi^2/f^2(dx^0)^2+f^4(dr^2+dz^2+r^2d\phi^2)$, where $\Phi=1-GM/(2c^2\sqrt{r^2+z^2})$ and $f=1+GM/(2c^2\sqrt{r^2+z^2})$. The angular velocity is equal to the Keplerian velocity $\Omega^2=GM/(\sqrt{r^2+z^2}^3f^6)$ and $R=\sqrt{r^2+z^2}f^2$. The total energy per unit mass Ψ vanishes for a test dust. Let the disk lie on the $z=0$ plane, and assume the rotation law with $\delta=-1/3$ and $\kappa=3$:

$$\frac{w^{4/3}\Omega^{-1/3}}{1-\frac{3}{c^2}w^{4/3}\Omega^{2/3}}=\frac{V^2}{\Omega(1-\frac{V^2}{c^2})}. \tag{12}$$

Here $V^2=\Omega^2r^2f^6/\Phi^2$. Notice that $h=0$; the enthalpy per unit mass vanishes for dust. This is a simple exercise to show that $w=\sqrt{GM}$ and $\Omega^2=GM/(r^3f^6)$ solve both Eq. (12) and the Bernoulli equation (10); the constant in (10) equals unity.

IV. 1PN CORRECTIONS TO ANGULAR VELOCITY

Taking into account the above, we shall prove that if $\kappa=(1-3\delta)/(1+\delta)+\mathcal{O}(c^{-2})$ and $\Psi=4c_0+\mathcal{O}(c^{-2})$, where c_0 is the Newtonian hydrodynamic energy per unit mass, then the exact solution satisfies the 1PN equations. We shall use the formalism of Ref. [3] and the rotation law (9) and recover most of the results obtained in the 1PN approach employed in Ref. [10]. Notice that if $\delta = -1/3$, then $\kappa = 3$—one recovers the coefficient in front of $w^{4/3}\Omega^{2/3}$ in (12) that is required by the F-Condition.

The 1PN approximation corresponds to the choice of metric exponents $\alpha = \beta = -\nu = -U$ with $|U| \ll c^2$ [11]. Define $\omega \equiv r^{-2}A_\phi$. The spatial part of the metric

$$ds^2 = -\left(1 + \frac{2U}{c^2} + \frac{2U^2}{c^4}\right)(dx^0)^2 - 2c^{-3}A_\phi dx^0 d\phi + \left(1 - \frac{2U}{c^2}\right)(dr^2 + dz^2 + r^2 d\phi^2) \tag{13}$$

is conformally flat.

We split different quantities (ρ, p, h, U, and v^i) into their Newtonian (denoted by subscript 0) and 1PN (denoted by subscript 1) parts. For example, for ρ, Ω, Ψ, and U, this splitting reads

$$\rho = \rho_0 + c^{-2}\rho_1, \tag{14a}$$

$$\Omega = \Omega_0 + c^{-2}v_1^\phi, \tag{14b}$$

$$\Psi = \Psi_0 + \mathcal{O}(c^{-2}), \tag{14c}$$

$$U = U_0 + c^{-2}U_1. \tag{14d}$$

Notice that, up to the 1PN order,

$$\frac{1}{\rho}\partial_i p = \partial_i h_0 + c^{-2}\partial_i h_1 + \mathcal{O}(c^{-4}), \tag{15}$$

where the 1PN correction h_1 to the specific enthalpy can be written as $h_1 = \frac{dh_0}{d\rho_0}\rho_1$. For the polytropic equation of state, this gives $h_1 = (\gamma - 1)h_0\rho_1/\rho_0$.

Making use of the introduced above splitting of quantities into Newtonian 0PN and 1PN parts, one can extract from Eq. (6) the 0PN- and 1PN-level Bernoulli equations. The 0PN equation reads

$$h_0 + U_0 - \frac{\delta - 1}{2(1+\delta)}\Omega^2 r^2 = c_0, \tag{16}$$

where c_0 is a constant that can be interpreted as the energy per unit mass. At the Newtonian level, this is supplemented by the Poisson equation for the gravitational potential

$$\Delta U_0 = 4\pi G\rho_0, \tag{17}$$

where Δ denotes the flat Laplacian. The first correction v_1^ϕ to the angular velocity Ω is obtained from the perturbation expansion of the rotation law (9) up to terms of the order c^{-2}. Assuming that $\Psi_0 = 4c_0$, one arrives at

$$v_1^\phi = -\frac{2}{1-\delta}\Omega_0^3 r^2 + \frac{A_\phi}{r^2(1-\delta)} - \frac{4\Omega_0 h_0}{1-\delta}, \tag{18}$$

where we applied Eqs. (11) and (16).

Remember that in the Newtonian gauge imposed in the line element (13) the geometric distance to the rotation axis is given by $\tilde{r} = r(1 - U_0/c^2) + \mathcal{O}(c^{-4})$. It is enlightening to write down the full expression for the angular velocity, up to the terms $\mathcal{O}(c^{-4})$:

$$\Omega = \Omega_0 + \frac{v_1^\phi}{c^2} = \frac{w}{\tilde{r}^{2/(1-\delta)}} - \frac{2}{c^2(1-\delta)}\Omega_0(U_0 + \Omega_0^2 r^2) + \frac{A_\phi}{r^2c^2(1-\delta)} - \frac{4}{c^2(1-\delta)}\Omega_0 h_0. \tag{19}$$

This expression reduces to

$$\Omega = \Omega_0 + \frac{v_1^\phi}{c^2} = \frac{w}{\tilde{r}^{2/(1-\delta)}} - \frac{4}{rc^2(1-\delta)}\Omega_0 h_0, \tag{20}$$

in the case of test fluids, at the symmetry plane $z = 0$. For the dust, in the Schwarzschild geometry, we get

$$\Omega = \Omega_0 + \frac{v_1^\phi}{c^2} = \frac{w}{\tilde{r}^{3/2}}; \tag{21}$$

the 1PN correction to Ω_0 is equal to $\frac{3U_0}{2c^2}\Omega_0$. Thus, the F-condition is satisfied in the 1PN order.

After these considerations we are able to interpret the meaning of various contributions to the 1PN angular velocity Ω. The first term is simply the Newtonian rotation law rewritten as a function of the geometric distance, as given at the 1PN level of approximation, from the rotation axis. The second term in (19) vanishes at the plane of symmetry, $z = 0$, for circular Keplerian motion of test fluids in the monopole potential $-GM/R$. Thus, it is sensitive both to the contribution of the disk self-gravity at the plane $z = 0$ and the deviation from the strictly Keplerian motion. The third term is responsible for the geometric frame dragging. The last term represents the recently discovered dynamic antidragging effect; it agrees (for the monomial angular velocities $\Omega_0 = r^{-2/(1-\delta)}w$) with the result obtained earlier in Ref. [10].

A. A comment on the term $-\frac{2}{c^2(1-\delta)}\Omega_0^3 r^2$, that has been missing in ref. [10]

The reason for this omission is the following. There is a gauge freedom in choosing an integrability condition for the 1PN hydrodynamic equation, due to the fact that the Bernoulli equation of the 1PN order is specified up to a function $F(r)$. We assumed in Ref. [10], in order to get the

1PN Bernoulli equation as in Ref. [11], that $F(r) = 0$; but that is not consistent with the F-condition. It appears that the right value is $F(r) = -\Omega_0^4 r^4/(1+\delta)$, which leads to the emergence of the term in question.

The vectorial component A_ϕ satisfies the following equation:

$$\Delta A_\phi - 2\frac{\partial_r A_\phi}{r} = -16\pi G r^2 \rho_0 \Omega_0. \qquad (22)$$

The 1PN Bernoulli equation does not influence the 1PN correction to the angular velocity. It has the form

$$c_1 = -h_1 - U_1 - \Omega_0 A_\phi + 2r^2(\Omega_0)^2 h_0 - \frac{3}{2}h_0^2 - 4h_0 U_0 - 2U_0^2 - \frac{\delta - 1}{4(1+\delta)} r^4 \Omega_0^4 + F(r), \qquad (23)$$

where c_1 is a constant. To derive (23) we again used $\Psi_0 = 4c_0$. This result agrees with the 1PN calculation of Ref. [10] up to the term $F(r)$.

The 1PN potential correction U_1 can be obtained from

$$\Delta U_1 = 4\pi G(\rho_1 + 2p_0 + \rho_0(h_0 - 2U_0 + 2r^2(\Omega_0)^2)). \qquad (24)$$

Equations (22) and (24) have been derived in Ref. [10] in the framework of 1PN approximation. They can be also obtained directly from the Einstein equations written for the metric (2), as derived, e.g., in Ref. [3]. Here we recall a version similar to that used in Ref. [6]; it turns out to be more convenient than the original form of Ref. [3]. The relevant equations read

$$\Delta\nu = 4\pi \frac{G}{c^2} e^{2\alpha/c^2}\left[\rho(c^2+h)\frac{1+V^2/c^2}{1-V^2/c^2} + 2p\right] + \frac{1}{2c^4} r^2 e^{2(\beta-\nu)/c^2}\nabla\omega\cdot\nabla\omega - \frac{1}{c^2}\nabla(\beta+\nu)\cdot\nabla\nu$$

and

$$\left(\Delta + \frac{2}{r}\partial_r\right)\omega = -16\pi\frac{G}{c^2}e^{2\alpha/c^2}\rho(c^2+h)\frac{\Omega - \omega/c^2}{1-V^2/c^2} + \frac{1}{c^2}\nabla(\nu - 3\beta)\cdot\nabla\omega,$$

where ∇ denotes the "flat" gradient operator. The remaining Einstein equations yield corrections of higher orders.

In summary, we have shown that—for $-\infty < w < \infty$ and $-\infty \le \delta \le 0$, $\delta \ne -1$—the choice $\kappa = (3-\delta)/(1+\delta) + \mathcal{O}(c^{-2})$ and $\Psi = 4c_0 + \mathcal{O}(c^{-2})$ in the rotation law (9) guarantees that, if there exists an exact solution analytic in powers of c^{-2}, then it satisfies the 0PN and 1PN approximating equations.

One easily finds out that the rotation law (8) satisfies the generalized Rayleigh criterion [12] for stability $\frac{dj}{d\Omega} < 0$ up to 1PN order, assuming that δ is strictly negative.

B. Comments on the 1PN corrections to the angular velocity

In the following considerations, we assume $w > 0$, which means $\Omega_0 > 0$, but the reasoning is symmetric under the parity operation $w \to -w$. The specific enthalpy $h \ge 0$ is nonnegative; thence $-\frac{4\Omega_0 h_0}{1-\delta}$ is nonpositive—the instantaneous 1PN dynamic reaction discovered in Ref. [10] slows the motion: it "antidraggs" a system. In contrast to that, the well-known geometric term with A_ϕ is positive [10], and the contribution $\frac{A_\phi}{r^2(1-\delta)}$ to the angular velocity is positive—it pushes a rotating fluid body forward. Thus, the two terms in (18) counteract.

Dust is special—the specific enthalpy h_0 vanishes, and hence dust test bodies are exposed only to the geometric effect: the frame dragging. Even more special is the rigid (uniform) rotation—the correction terms v_1^ϕ are proportional to $1/(1-\delta)$, and they vanish, because now $\delta = -\infty$. Uniformly rotating disks are already known to minimize the total mass energy for a given baryon number and total angular momentum [13]. The vanishing of the 1PN correction v_1^ϕ is their another distinguishing feature.

It follows from our discussion that, assuming the F-condition, one has three free parameters: w, δ, and Ψ; the parameter κ is a given function of δ. The full system of Einstein–Bernoulli equations can be solved numerically within this class of data, and the resulting solutions are expected to possess 0PN and 1PN limits.

V. CONCLUDING REMARKS

We write down the general-relativistic rotation laws, recover the well-known geometric dragging of frames, and derive a full form of the two other weak-field effects, including the dynamic antidragging effect of Ref. [10]. The latter can be robust according to the numerics of Ref. [10], but the ultimate conclusion requires a fully general-relativistic treatment, that is the use of the new rotation laws. The frame dragging occurs—through the Bardeen–Petterson effect [14]—in some active galactic nuclei (AGNs) [15]. The two other effects can lead to its observable modifications in black hole systems with heavy disks.

In the weak-field approximation of general relativity, the angular velocity of toroids depends primarily on the distance from the rotation axis—as in the Newtonian hydrodynamics—but the weak-field contributions make the rotation curve dependent on the height above the symmetry plane of a toroid.

The new rotation laws would allow the investigation of self-gravitating fluid bodies in the regime of strong gravity for general-relativistic versions of Newtonian rotation curves. In particular, they can be used in order to describe stationary heavy disks in tight accretion systems with central black holes. These highly relativistic systems can be created in the merger of compact binaries consisting of pairs of black holes and neutron stars [16,17], but they might exist in some active galactic nuclei.

The new general-relativistic rotation laws can be applied to the study of various open problems in the post-Newtonian perturbation scheme of general-relativistic hydrodynamics. We demonstrate in this paper that an adaptation of the condition used in Ref. [9] ensures uniqueness up the 1PN order. Further applications include the investigation of convergence of the post-Newtonian pertubation scheme as well as the existence of the Newtonian and post-Newtonian limits of solutions.

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