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Time travel in transformation optics: Metamaterials with closed null geodesics

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We apply the methods of transformation optics to theoretical descriptions of spacetimes that support closed null geodesic curves. The metric used is based on frame dragging spacetimes, such as the van Stockum dust or the Kerr black hole. Through transformation optics, this metric is analogous to a material that in theory should allow for communication between past and future. Presented herein is a derivation and description of the spacetime and the resulting permeability, permittivity, and magnetoelectric couplings that a material would need in order for light in the material to follow closed null geodesics. We also address the paradoxical implications of such a material and demonstrate why such a material would not actually result in a violation of causality. A full derivation of the Plebanski equations is also included.

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I. INTRODUCTION

Since the first papers by Pendry [1] and Leonhardt [2], the subject of invisibility cloaks has garnered much attention in the literature. For the design of these cloaks, the recent field of transformation optics (TO) was developed. Though TO bestows on the scientist near-unlimited control of the movement of light, most research efforts have been directed to perfecting the invisibility devices that initially drew attention to the field.

TO uses the coordinate invariance of Maxwell's equations to set up an analogy between Maxwell's equations in curved vacuum spacetime to Maxwell's equations in a flat spacetime with a particular medium. Therefore, so long as we have a mathematical description of a curved spacetime, an analogous material can be constructed within which light will behave similarly to the curved spacetime. For a fuller introduction to the subject of TO, the reader is referred to the excellent paper by Leonhardt and Philbin [3].

Due to the direct parallels between TO and general relativity (GR), some interest has been directed toward the design of materials that would simulate various models of spacetime, hopefully allowing astronomers to study systems such as black holes in the laboratory. Proposals for materials mimicking the effects of de Sitter space [4], Schwarzschild black holes [5], Kerr black holes [6], spatial wormholes [7], Alcubierre warp drive geometries [8], and so-called optical black holes [9,10] have been put forward. The possibilities for materials are nearly endless, and any kind of spacetime geometry, no matter how bizarre—including those which do not solve the Einstein equations—can, at least in theory, be modeled using the formalism of TO.

Herein, we propose a material that will use transformation optics to simulate one of the more imaginative and hotly debated aspects of curved spacetimes; namely, the possibility of time travel to the past along closed causal curves. In the literature, most interest in time travel metrics has been directed toward curves that allow matter to move back in time, often called closed timelike curves (CTCs). In TO, we are interested in the movement of light, which in GR is known to follow null geodesics. Of particular interest is the case of closed null geodesics (CNGs), which would form a continuous loop in lab time from the future to the past. In Sec. III we construct a metric that produces CNGs, without regard to the usual physical constraints of the energy conditions.

Using the same procedure that led to the invisibility cloak design and the dielectric black hole design, we use the metric tensor of our spacetime that supports CNGs to generate material parameter tensors (MPs), namely permittivity ϵ, permeability μ, and magnetoelectric tensors γ_1, γ_2, the effect of which on the fields is expressed by

$$\vec{D} = \epsilon\vec{E} + \gamma_1\vec{H}, \qquad \vec{B} = \mu\vec{H} + \gamma_2\vec{E}. \tag{1}$$

We will show that within this resultant material there exist CNGs that can span a finite lab time difference T, allowing signaling from the future to the past along the CNGs. This would, in theory, allow information from the future to have an impact on the past.

In Sec. II we derive the Plebanski equations [11], which relate curved spacetimes to material parameters, following the Minkowski formalism of electromagnetism. After this, in Sec. III we propose a simplified spacetime metric that should produce such curves. In Sec. IV we combine the results of the previous sections and explicitly solve for the MPs based on the formalism presented. We also provide a sketch showing how such a material could be used to communicate with the past. Finally, in Sec. V, we point out technical limitations which should impede the actual functioning of any such device as that proposed here.

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II. TRANSFORMATION OPTICS IN SPACETIME

Here we derive the Plebanski equations, which connect curved spacetimes with bianisotropic media. These equations were predicted by many, including Landau and Lifschitz [12], but saw their first practical use by Plebanski [11] in 1959. We begin our derivation following Thompson *et al.* [13], with the field-strength tensor $\mathbf{F} = d\mathbf{A}$. In Cartesian components,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \tag{2}$$

and its dual tensor is $\mathbf{G} = \star\mathbf{F}$, with Cartesian components

$$G_{\mu\nu} = \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & D_z & -D_y \\ -H_y & -D_z & 0 & D_x \\ -H_z & D_y & -D_x & 0 \end{pmatrix}, \tag{3}$$

both given in Minkowski spacetime using $c = 1$ and the $(-+++)$ sign convention for the metric.

In terms of these two tensors, the four Maxwell equations simplify to the two equations

$$d\mathbf{F} = 0 \Leftrightarrow \nabla_{[\alpha}F_{\beta\gamma]} = 0 \tag{4a}$$

$$d\mathbf{G} = \mathbf{J} \Leftrightarrow \nabla_{[\alpha}G_{\beta\gamma]} = \sqrt{|g|}\epsilon_{\alpha\beta\gamma\delta}j^{\delta}, \tag{4b}$$

where $g = \det(\mathbf{g})$ and $\mathbf{g}$ is the metric tensor of spacetime.

The defining equation $\mathbf{G} = \star\mathbf{F}$ states that $\mathbf{G}$ is the Hodge dual of $\mathbf{F}$, where $\star$ is the Hodge star operator, the effect of which on $\mathbf{F}$ can be expressed in component form by

$$\mathbf{G} = \star\mathbf{F} \Leftrightarrow G_{\alpha\beta} = \frac{1}{2}\sqrt{|g|}\epsilon_{\alpha\beta\mu\nu}g^{\mu\lambda}g^{\nu\kappa}F_{\lambda\kappa}. \tag{5}$$

From here, Ref. [13] goes on to derive a covariant expression of TO in spacetime. The authors propose a slightly more general relationship, $\mathbf{G} = \boldsymbol{\chi}(\star\mathbf{F})$, where $\boldsymbol{\chi}$ is an antisymmetric 4-tensor that is meant to contain all information about media in the spacetime. They then apply coordinate transformations to express the effect of a curved spacetime with metric $\mathbf{g}$ in terms of a flat spacetime with a medium given by $\boldsymbol{\chi}$. Their result proves TO to be a covariant theory applicable to any coordinate system.

Having noted their result, and the concomitant assurance that TO works in any spacetime, we take a slightly simpler approach, following that of Plebanski [11]. We will work only in Cartesian coordinates and assume our medium to be stationary relative to the laboratory coordinates. Let us start with (5),

$$G_{\mu\nu} = (\star\mathbf{F})_{\mu\nu} = \frac{1}{2}\sqrt{|g|}\epsilon_{\mu\nu\alpha\beta}g^{\alpha\lambda}g^{\beta\kappa}F_{\lambda\kappa}. \tag{6}$$

From the forms given by (2) and (3), clearly $E_a = F_{a0}$, $H_a = G_{0a}$, $D_a = \frac{1}{2}\epsilon_{abc}G_{bc}$, and $F_{ab} = \epsilon_{abc}B_c$. Let us first consider specifically the $D_1 = G_{23}$ component. Then

$$\begin{aligned} D_1 &= \frac{\sqrt{|g|}}{2}\epsilon_{23\alpha\beta}g^{\alpha\lambda}g^{\beta\kappa}F_{\lambda\kappa} \\ &= \frac{\sqrt{|g|}}{2}\epsilon_{23\alpha\beta}[g^{\alpha 0}g^{\beta a}F_{0a} + g^{\alpha a}g^{\beta 0}F_{a0} + g^{\alpha a}g^{\beta b}F_{ab}] \\ &= \frac{\sqrt{|g|}}{2}[2g^{0a}g^{10}F_{a0} - 2g^{1a}g^{00}F_{a0} + 2g^{0a}g^{1b}F_{ab}] \\ &= \sqrt{|g|}(g^{10}g^{0a} - g^{00}g^{1a})E_a + \sqrt{|g|}g^{0a}g^{1b}\epsilon_{abc}B_c. \end{aligned}$$

Performing the same evaluation on the other two components, we find

$$D_a = \sqrt{|g|}(g^{a0}g^{b0} - g^{00}g^{ab})E_b + \sqrt{|g|}g^{ad}g^{0c}\epsilon_{dcb}B_b. \tag{7}$$

Next we look at $H_1 = G_{01}$. Here

$$\begin{aligned} H_1 &= \frac{\sqrt{|g|}}{2}\epsilon_{01\alpha\beta}g^{\alpha\lambda}g^{\beta\kappa}F_{\lambda\kappa} \\ &= \frac{\sqrt{|g|}}{2}\epsilon_{1ab}[g^{a0}g^{bc}F_{0c} + g^{ac}g^{b0}F_{c0} + g^{ac}g^{bd}F_{cd}] \\ &= \sqrt{|g|}\epsilon_{1ab}g^{a0}g^{bc}E_c + \frac{1}{2}\sqrt{|g|}\epsilon_{1ab}g^{ac}g^{bd}\epsilon_{cde}B_e, \end{aligned}$$

with the general result

$$H_a = \sqrt{|g|}\epsilon_{abc}g^{b0}g^{cd}E_d + \frac{\sqrt{|g|}}{2}\epsilon_{abc}g^{bd}g^{ce}\epsilon_{def}B_f. \tag{8}$$

These two equations can be written more simply as

$$D_a = e^{ab}E_b + f^{ab}B_b, \qquad H_a = h^{ab}B_b + k^{ab}E_b, \tag{9}$$

and, upon rearrangement and simplification to the standard form, we arrive at the result of Plebanski,

$$D_a = -\frac{\sqrt{|g|}}{g_{00}}g^{ab}E_b + \epsilon_{abc}\frac{g_{0b}}{g_{00}}H_c \tag{10a}$$

$$B_a = -\frac{\sqrt{|g|}}{g_{00}}g^{ab}H_b - \epsilon_{abc}\frac{g_{0b}}{g_{00}}E_c. \tag{10b}$$

In terms of the MPs, this gives

$$\epsilon^{ab} = \mu^{ab} = -\frac{\sqrt{|g|}}{g_{00}}g^{ab} \tag{11a}$$

$$\gamma_1^{ab} = (\gamma_2^T)^{ab} = \epsilon_{acb}\frac{g_{0c}}{g_{00}}, \tag{11b}$$

where

$$D_a = \epsilon^{ab}E_b + \gamma_1^{ab}H_b, \qquad B_a = \mu^{ab}H_n + \gamma_2^{ab}E_b. \tag{12}$$

Thus, the curved spacetime of $g_{\alpha\beta}$ is equivalent to a flat spacetime with MPs $\epsilon^{ab}, \mu^{ab}, \gamma_1^{ab}, \gamma_2^{ab}$.

As Thompson and Plebanski both note [11,13], the above equations do not conserve index type, nor are they covariant, and they are only applicable to stationary media within a locally flat lab frame in Cartesian coordinates. Despite this, the above equations are completely equivalent to the more general, covariant approach in Ref. [13] for the case of a stationary medium and can be extended to the covariant equations. These equations should therefore be applicable to a medium intended to emulate the effects of a curved spacetime.

III. CONSTRUCTING A METRIC WITH CLOSED NULL GEODESICS

There has been much debate in scientific circles about the possibility of time travel to the past. Forward time travel is, of course, trivially simple to achieve; it is the reverse situation that gives us such trouble. Most proposals require either particular models of the entire Universe that are empirically false (for instance the Gödel metric [14]) or else highly idealized systems that cannot be physically realized (such as the negative energy densities of wormholes or the infinite rotating systems of van Stockum spacetimes [15]). For this reason, many physicists are comfortable dismissing the predicted causality violations in these contrived spacetimes as purely mathematical and unphysical—as they say, "garbage in/garbage out." More damning, Stephen Hawking has proposed a mechanism dubbed the Chronology Protection Conjecture [16], whereby quantum propagators approaching CTCs (should any exist) are shown to be unstable—leading to divergent stress-energies—and hence break whatever time-travel metric they come from, prohibiting time travel. For a fuller discussion of the possibilities of time travel in a general relativistic framework, the reader is referred to a lecture by Thorne [17] on the topic of the possibility of CTCs in GR.

Whether such a spacetime can or cannot be realized physically through various arrangements of stress-energy is not of interest of this present work; we make our curved spacetimes with metamaterials, not stress-energy densities. We are interested, however, that a spacetime with CNGs can be described mathematically in terms of a metric tensor.

As in Eq. (10) above, if we have a spacetime with a metric $g_{\alpha\beta}$, it is possible to use this metric to calculate a related material with MPs $\epsilon^{ab}, \mu^{ab}, \gamma_1^{ab}, \gamma_2^{ab}$ so that light inside the material emulates light in the curved spacetime. We will now explicitly construct a metric that allows for CNGs. Our calculation is intended as a proof of concept, and hence we will keep the result as theoretically simple as possible.

We begin with a simple metric with two unspecified functions,

$$ds^2 = -dt^2 + dr^2 + Bd\phi^2 + dz^2 + 2Fd\phi dt, \tag{13}$$

where $c = 1$ and B and F are functions of r and ϕ only. We will use the most general case for now and apply some reasonable restrictions later. The metric tensor and its inverse are then

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & F & 0 \\ 0 & 1 & 0 & 0 \\ F & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$g^{\alpha\beta} = \begin{pmatrix} \frac{-B}{B+F^2} & 0 & \frac{F}{B+f^2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{F}{B+F^2} & 0 & \frac{1}{B+F^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

The condition for a curve to be a null curve is

$$0 = ds^2 = -dt^2 + dr^2 + Bd\phi^2 + dz^2 + 2Fdtd\phi. \tag{15}$$

In addition to the null condition, a null geodesic also satisfies the geodesic equations

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0, \tag{16}$$

for any affine parameter λ. This requires knowledge of the Christoffel symbols, which we can find using the formula

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}). \tag{17}$$

Solving, all vanish, except for

$$\Gamma^0_{01} = \frac{1}{2}\frac{F}{B+F^2}\frac{\partial F}{\partial r}, \qquad \Gamma^2_{01}\frac{1}{2}\frac{1}{B+F^2}\frac{\partial F}{\partial r}$$
$$\Gamma^1_{02} = -\frac{1}{2}\frac{\partial F}{\partial r}, \qquad \Gamma^1_{22} = -\frac{1}{2}\frac{\partial B}{\partial r}$$
$$\Gamma^0_{12} = \frac{1}{2}\frac{F\frac{\partial B}{\partial r} - B\frac{\partial F}{\partial r}}{B+F^2}, \qquad \Gamma^2_{12} = \frac{1}{2}\frac{F\frac{\partial F}{\partial r} + \frac{\partial B}{\partial r}}{B+F^2}$$
$$\Gamma^0_{22} = \frac{-B\frac{\partial F}{\partial \phi} + \frac{1}{2}F\frac{\partial B}{\partial \phi}}{B+F^2}, \qquad \Gamma^2_{22} = \frac{F\frac{\partial F}{\partial \phi} + \frac{1}{2}\frac{\partial B}{\partial \phi}}{B+F^2}.$$

We now impose a specific form for our curve, having constant radius and height; that is, $t = t(\phi)$, $r = \text{const}, z = \text{const}$, and we are using ϕ to parameterize.

To enforce closure of the curve, we require that $t(\phi) = t(\phi + 2\pi)$. For simplicity, we will use $u(\phi) = \frac{dt}{d\phi}$ in all future equations.

There are two first integrals of note. The null condition (15) becomes

$$0 = -u^2 + 2Fu + B. \tag{18}$$

Also, since our spacetime has no time dependence, we have the Killing equation (see Ref. [18]) with Killing vector $\xi^\alpha = (1, 0, 0, 0)$, which leads to

$$-K = g_{\alpha\beta}\xi^\alpha u^\beta = -u + F, \tag{19}$$

for constant K. Further, the geodesic equations give us

$$0 = \frac{du}{d\phi} - \frac{B}{B + F^2}\frac{\partial F}{\partial \phi} + \frac{1}{2}\frac{F}{B + F^2}\frac{\partial B}{\partial \phi} \tag{20}$$

$$0 = 2F\frac{\partial F}{\partial \phi} + \frac{\partial B}{\partial \phi} \tag{21}$$

$$0 = -2\frac{\partial F}{\partial r}u - \frac{\partial B}{\partial r}. \tag{22}$$

We now attempt to solve this system of coupled equations for F and B.

Consider Eq. (19); here $u = u(\phi)$ and $F = F(r, \phi)$. However, the two added together equal a constant, K. Therefore, it must be the case that $F(r, \phi) = f(\phi)$. We can then cross out all $\frac{\partial F}{\partial r}$ terms in Eqs. (20)–(22). Now let us combine (18) with (19); this gives us

$$B = K^2 - f^2, \tag{23}$$

which likewise implies that $B(r, \phi) = b(\phi)$, and we can remove all $\frac{\partial B}{\partial r}$ terms. Note that in this case, with $K^2 = f^2 + b$, Eqs. (21) and (22) are trivially satisfied. The last step is to check (20). Note that in terms of u

$$F(r, \phi) = f(\phi) = u - K \tag{24}$$

$$B(r, \phi) = b(\phi) = 2Ku - u^2. \tag{25}$$

From this, we find

$$\frac{\partial F}{\partial \phi} = \frac{du}{d\phi}, \qquad \frac{\partial B}{\partial \phi} = 2K\frac{du}{d\phi} - 2u\frac{du}{d\phi} = -2f\frac{du}{d\phi}$$

such that

$$\begin{aligned}\frac{du}{d\phi} - \frac{b}{K^2}\frac{\partial F}{\partial \phi} + \frac{1}{2}\frac{f}{K^2}\frac{\partial B}{\partial \phi} &= \frac{du}{d\phi} - \frac{b}{K^2}\frac{du}{d\phi} - \frac{f^2}{K^2}\frac{db}{d\phi} \\ &= \frac{du}{d\phi}\left(1 - \frac{b}{K^2} - \frac{f^2}{K^2}\right) \\ &= \frac{du}{d\phi}\left[1 - \frac{K^2}{K^2}\right] = 0.\end{aligned}$$

Therefore, all four geodesic equations are satisfied for a curve $u^\alpha = (u(\phi), 0, 1, 0)$, in a spacetime with metric components defined as in Eqs. (25) and (24). Note that here $u(\phi)$ is unspecified. We are free to pick u however we would like, subject to the restriction that $u(\phi) = u(\phi + 2\pi)$. Here we will choose the path

$$t(\phi) = T\sin^2\frac{\phi}{2} \tag{26}$$

$$u(\phi) = h(\phi) = \frac{T}{2}\sin\phi, \tag{27}$$

which will loop up by T in time, before looping back to the origin.

We can write the metric, in terms of $h(\phi) = \frac{T}{2}\sin\phi$, as

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & h - K & 0 \\ 0 & 1 & 0 & 0 \\ h - K & 0 & h(2K - h) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

Note, it is crucial that $K \neq 0$, for then the metric will be singular.

Suppose we consider a new geodesic, parametrized by affine parameter λ with 4-velocity $u^\alpha = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$. This geodesic satisfies the geodesic equations, which are now simplified to

$$0 = \ddot{t} + \frac{\dot{\phi}^2}{K^2}\left(\frac{1}{2}f\frac{\partial b}{\partial \phi} - b\frac{\partial f}{\partial \phi}\right) \tag{29}$$

$$0 = \ddot{r} \tag{30}$$

$$0 = \ddot{\phi} + \frac{\dot{\phi}^2}{K^2}\left(f\frac{\partial f}{\partial \phi} + \frac{1}{2}\frac{\partial b}{\partial \phi}\right) \tag{31}$$

$$0 = \ddot{z}, \tag{32}$$

where $f(\phi) = h(\phi) - K$ and $b(\phi) = 2Kh(\phi) - h(\phi)^2$. These solve as

$$\dot{t}(\lambda) = \dot{\phi}_o h(\phi) + \dot{t}_o \tag{33}$$

$$r(\lambda) = \dot{r}_o\lambda + r_o \tag{34}$$

$$\phi(\lambda) = \dot{\phi}_o\lambda + \phi_o \tag{35}$$

$$z(\lambda) = \dot{z}_o\lambda + z_o, \tag{36}$$

giving the requirements for a geodesic in this spacetime. If we choose $\dot{t}_o = \dot{r}_o = \dot{z}_o = 0$, then $\dot{t}(\phi) = \dot{\phi}_o h(\phi)$, returning our desired curve, up to the choice of affine parametrization in $\dot{\phi}_o$.

As a check of the physicality of (28), we follow Landau and Lifschitz [12] and ensure that the sign of the subdeterminants of the metric is negative. Most of these are trivially satisfied, with the exception of

$$\begin{vmatrix} g_{00} & g_{02} \\ g_{20} & g_{22} \end{vmatrix} = \begin{vmatrix} -1 & f \\ f & b \end{vmatrix},$$
$$\begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} -1 & 0 & f \\ 0 & 1 & 0 \\ f & 0 & b \end{vmatrix},$$
$$\begin{vmatrix} g_{00} & g_{02} & g_{03} \\ g_{20} & g_{22} & g_{23} \\ g_{30} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} -1 & f & 0 \\ f & b & 0 \\ 0 & 0 & 1 \end{vmatrix}, \tag{37}$$

all of which reduce to $-b - f^2 = -K^2 < 0$. Thus, as per Landau and Lifschitz, the metric in (28) corresponds to a physical spacetime.

We have thus proved that our spacetime permits a CNG. We will now move on to combining this equation with the Plebanski equations, to solve for the analogous medium.

IV. ANTITELEPHONIC MEDIUM

The metric above has been shown to support closed null geodesics—paths along which light is known to move backward relative to lab time. TO allows us to use the spacetime metric (28) to formulate a material within which light should exhibit the same behavior as in the curved spacetime above; namely, light in the material should also move in CNGs. We propose a device that uses this material to communicate with the past or the future, which we henceforth refer to as the TO antitelephone for its purported ability of anticausal signaling, after the famous thought experiment by Einstein using tachyons to the same purpose [19].

Our metric tensor in Cartesian components is

$$g_{\alpha\beta} = \begin{pmatrix} -1 & -\frac{fy}{r^2} & \frac{fx}{r^2} & 0 \\ -\frac{fy}{r^2} & 1-(1-\frac{b}{r^2})\frac{y^2}{r^2} & (1-\frac{b}{r^2})\frac{xy}{r^2} & 0 \\ \frac{fx}{r^2} & (1-\frac{b}{r^2})\frac{xy}{r^2} & 1-(1-\frac{b}{r^2})\frac{x^2}{r^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

still using $c = 1$. We calculate the MPs as in (11). These we find to be

$$\epsilon^{ab} = \mu^{ab} = \begin{pmatrix} \frac{r}{K}\frac{y^2}{r^2} + \frac{K}{r}\frac{x^2}{r^2} & (\frac{K}{r} - \frac{r}{K})\frac{xy}{r^2} & 0 \\ (\frac{K}{r} - \frac{r}{K})\frac{xy}{r^2} & \frac{r}{K}\frac{x^2}{r^2} + \frac{K}{r}\frac{y^2}{r^2} & 0 \\ 0 & 0 & \frac{K}{r} \end{pmatrix} \tag{39a}$$

$$\gamma_1^{ab} = (\gamma_2^T)^{ab} = \begin{pmatrix} 0 & 0 & -\frac{f}{r}\frac{x}{r} \\ 0 & 0 & -\frac{f}{r}\frac{y}{r} \\ \frac{f}{r}\frac{x}{r} & \frac{f}{r}\frac{y}{r} & 0 \end{pmatrix}, \tag{39b}$$

listed in Cartesian components. Due to symmetries, the MPs take the simplest form in (orthonormal) cylindrical components,

$$\epsilon^{ab} = \mu^{ab} = \begin{pmatrix} \frac{K}{r} & 0 & 0 \\ 0 & \frac{r}{K} & 0 \\ 0 & 0 & \frac{K}{r} \end{pmatrix},$$
$$\gamma_1^{ab} = (\gamma_2^T)^{ab} = \begin{pmatrix} 0 & 0 & -\frac{f}{r} \\ 0 & 0 & 0 \\ \frac{f}{r} & 0 & 0 \end{pmatrix}. \tag{40}$$

The magnetoelectric couplings γ_1, γ_2 can both be reexpressed in terms of cross products with the vector $\vec{w} = -\frac{f}{r}\hat{\phi}$, such that $\gamma_1\vec{v} = \vec{w} \times \vec{v}$ and $\gamma_2\vec{v} = -\vec{w} \times \vec{v}$, for any vector $\vec{v}$. Following Leonhardt and Philbin [3], this vector $\vec{w}$ can be interpreted as the velocity of a moving medium. Our $\vec{w}$ corresponds to a material rotating axially with varying angular velocity, parallel to the rotating sources that give rise to frame dragging in GR. Since $f(\phi)$ is oscillatory in ϕ, a physical material moving with velocity $\vec{w} = -\frac{f}{r}\hat{\phi}$ will be axially compressed and stretched during its rotation.

The material parameters in (39), if realized, should lead to the formation of closed causal curves, as discussed already in Sec. III. To further solidify this connection, in the Appendix we solve the Maxwell equations in the antitelephonic medium using the eikonal approximation; in (A21) we find that in the limit of geometric optics the light rays obey

$$0 = g^{\alpha\beta}S_{,\alpha}S_{,\beta} = b(\phi)(S_{,0})^2 - 2f(\phi)S_{,0}S_{,2} - (S_{,2})^2, \tag{41}$$

in the usual interpretation of $S_{,0} = \omega$ as the frequency and $\nabla S = \vec{k}$ as the wave vector of the electromagnetic fields. That is, the wave 4-vector of the fields moving in (39) is a null vector in the corresponding curved spacetime of (28). This further establishes the connection between the spacetime and the medium.

We now sketch a rough picture of how the antitelephonic medium might be used to send signals to the past.

We suppose, first of all, some cylinder with MPs as in (39). The cylinder need not be solid, to avoid the singularities when $r = 0$. We also assume that $r \leq K$, which will let ϵ, μ retain non-negative values. Outside of the cylinder, there needs to be a method of shunting light into and out of the constant-radius CNGs; this might be accomplished by means of fiber optic cables pointed tangentially to the cylinder. At any time between $t = t_o$ and $t = t_f$ (with $T = t_f - t_o$), the light can be extracted at a suitable angular position around the antitelephonic medium, according to $t = t(\phi)$ as in (26). By whatever means light is inserted and extracted, it will be important that the future recipients receive the message it encodes sometime before $t = t_f$, to give them time to respond.

In Fig. 1 we provide a visual illustration for how information from the future can be sent to the past. Here, Alice and Bob are communicating by light signals propagating along a CNG within the TO antitelephone. The optical spacetime geometry of the antitelephonic medium allows for Bob's message to reach into the past, enabling Alice to know the future. This presents an apparent violation of causality. We now look at possible solutions to this quandary.

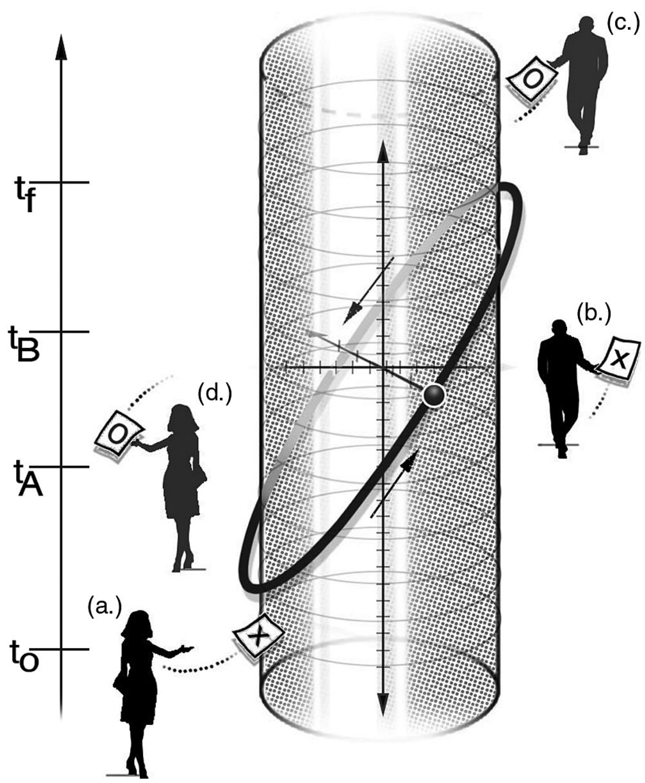

FIG. 1. Alice and Bob, separated by total time $T = t_f - t_o$, communicate using the TO antitelephone. The axis of the cylinder is along the time axis. (a.) Alice encodes her message ("X") at time $t = t_o$ and places it in the antitelephone. (d.) Alice receives Bob's message ("O") at time $t = t_A$. (b.) Bob receives Alice's message of "X" at time $t = t_B$. (c.) At any time before $t = t_f$, Bob replies to Alice with his own message of "O." After t_f in the lab frame of reference, the light in the antitelephone vanishes.

V. LIMITATIONS TO CAUSALITY VIOLATION

The preservation of causality is essential in the classical understanding of physics. Therefore, it is prudent to examine a number of possible mechanisms that will prevent a device like the TO antitelephone from working as outlined above.

Within the context of GR, there exist situations where the standard understanding of causality can become muddled. As an example, supposing a laboratory frame moving along a CTC, the causal order inside the laboratory will proceed as normal, and all experiments performed therein will function properly. However, in a frame of reference outside of the CTC, an inertial observer may see such bizarre occurrences as shards of glass collecting themselves in to a beaker and rising in to the air. If the laboratory is caused to move into and out of the CTC, an experimenter inside the lab may find herself arriving before she left, as measured by exterior clocks, even if her own clocks show an increase in time [20].

While this violation of causality is allowable in the classical understanding of GR, Stephen Hawking has proposed the Chronology Protection Conjecture in order to, as he says, "make the universe safe for historians" [16]. In his paper, Hawking claims that any curved spacetime allowing CTCs is impossible to construct in GR because it will require unphysical distributions of stress energy, such as negative energy density. To go further, Hawking then demonstrates that even if such a spacetime were to exist, it would be impossible for an object to move in to the region containing CTCs due to divergences in its quantum mechanical propagator and the resulting recurvature of spacetime from the energy needed to reach that region. Kim and Thorne [21] likewise find this problem of diverging stress-energy but suggest that perhaps quantum gravitation effects will limit it; whether this is or is not the case will depend on the form of the eventual theory of quantum gravitation, which is still being debated. In this present work, we are not dealing with actual gravity, and hence quantum gravitational effects are not of interest. Further, we do not need to approach the noncausal region directly; as mentioned earlier, we could also use fiber optic cables, completely bypassing this objection.

In pursuit of chronology protection, Hawking briefly considers a metric very similar to our own, namely

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - f d\phi^2 + 2f d\phi dt,$$

for $f = r^2 t^2 \sin^4\theta \sin^2\frac{\pi r}{a}$, to demonstrate the inability of causality violation. This spacetime differs from ours mostly in the form of the frame-dragging coefficients. Hawking's principal objection to such a spacetime is that, based on the Einstein equations, such a metric must come from negative energy density; otherwise, Hawking accepts that such a spacetime will in fact lead to violations of causality if objects could reach the noncausal regions. This issue of negative energy density is not a problem for us in TO, since we intend to construct our optical spacetime from a metamaterial, which does not have to follow any sort of energy condition. Therefore, while Hawking's conjecture offers a strong argument against general relativistic time machines to the past, neither of his arguments is of interest to the TO antitelephone.

From a general relativistic point of view, light moving in a spacetime with a metric such as (28) will violate causality. While effects of GR are predicted to stop this, these effects have no bearing in TO. Therefore, if the material can be built, then theory predicts that it can be used to violate causality. Since a violation of causality has never been observed, we should look closer at the material.

Looking at the MPs in (39), they are seen to be anisotropic, inhomogeneous, and to take values not possible in natural materials. This is analogous to other systems studied with TO. For instance, the invisibility cloak is anisotropic and requires an infinite light speed near the boundary of the cloaked region, yet has been subject to much study and lately constructed in reduced models [22]. These pathologies are usually physically achieved by means of metamaterials, such as those made of lattices of split-ring resonators. Each lattice site can be modeled as an RLC circuit, where the loop of the ring itself is the inductor and the tiny gap in the ring serves as the capacitor. From this, we find the permittivity and permeability follow a Lorentz model [23]. Looking at $\epsilon_\phi = r/K$ from (40), in the Lorentz model this becomes [24]

$$\epsilon_\phi(\omega) = 1 - \frac{\omega_P^2}{\omega_o^2 - \omega^2 - i\omega\gamma}, \tag{42}$$

where ω_P is a function of the capacitance and inductance, ω_o is the resonant frequency of the circuit, and γ is related to the resistivity, specifying the loss in the circuit. This kind of model will allow us to physically implement even the pathological values of the components of ϵ, μ called for in (39).

The MPs in (39) also include magnetoelectric couplings. These are not present in most simple TO applications, such as the invisibility cloak, though they do appear in the TO model of the Alcubierre warp drive presented by Smolyaninov in Ref. [8], where much attention is given to ensure that the MPs of the warp drive conform to physically reasonable standards. In particular, a theoretical study by Brown *et al.* [25] (paralleling an earlier study by O'Dell [26]) shows that the magnetoelectric coupling has an upper bound placed on it by thermodynamic free energy considerations, namely

$$(\gamma_1^{ab})^2 \le \epsilon^{aa}\mu^{bb}. \tag{43}$$

A typical component for us is $\gamma_1^{31} = \frac{f}{r}\cos\phi$. We also have $\epsilon^{33} = \frac{K}{r}$ and

$$\mu^{11} = \frac{r}{K}\sin^2\phi + \frac{K}{r}\cos^2\phi.$$

Therefore,

$$\begin{aligned} \frac{f^2}{r^2}\cos^2\phi = (\gamma_1^{31})^2 \le \epsilon^{33}\mu^{11} &= \sin^2\phi + \frac{K^2}{r^2}\cos^2\phi \\ &= 1 + \frac{f^2}{r^2}\cos^2\phi + \left(\frac{b-r^2}{r^2}\right)\cos^2\phi, \end{aligned} \tag{44}$$

or

$$0 \le 1 + \left(\frac{b-r^2}{r^2}\right)\cos^2\phi, \tag{45}$$

so that thermodynamic stability of γ_1 reduces to the condition that $b(\phi) \ge 0$ for all ϕ. However, it is not possible for $b(\phi)$ to be strictly non-negative, as the negativity of $b(\phi)$ is crucial to the reversal of the time direction in the metric (28) that allows for the closure of the null geodesic.

The antitelephonic medium considered in Sec. IV is therefore thermodynamically prohibited. While it may be true that a material such as (39) would violate causality if actually constructed, it is not actually possible to construct a medium with the necessary γ_1, γ_2. Thus, the TO antitelephone represents another instance of garbage in/garbage out—we put in the requirement of causality violation, and out come unphysical magnetoelectric tensors.

Having defeated the specific TO antitelephone of Sec. IV, we note a more general principle that should defeat any TO antitelephone. Consider light in an isotropic medium that follows the Lorentz model (42). We have

$$\vec{D}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\epsilon(\omega)\vec{E}(\omega)e^{-i\omega t}. \tag{46}$$

As is well known, if we write $\vec{E}(\omega)$ as a Fourier transform of the time domain, insert into (46), and exchange the order of integration, the above can be simplified to

$$\vec{D}(t) = \vec{E}(t) + \int_{-\infty}^{\infty} d\tau\vec{E}(t-\tau)\chi(\tau), \tag{47}$$

where $\chi(\tau)$ is a response function telling us how strongly the electric field $\vec{E}(t-\tau)$ at time $t-\tau$ contributes to the field $\vec{E}(t)$ at the present. Notice, for $\tau < 0$, it is not specifically ruled out that the future fields contribute to the present fields; if $\chi(\tau)$ is nonzero for $\tau < 0$, then we will have the future electric fields influencing the present.

However, for the Lorentz model, the response function takes the form of a contour integral,

$$\chi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \frac{\omega_P^2 e^{-i\omega\tau}}{\omega_o^2 - \omega^2 - i\gamma\omega}. \tag{48}$$

This integrand has poles at $\omega = -i\frac{\gamma}{2} \pm \sqrt{\omega_o^2 - \frac{\gamma^2}{4}}$, which are in the lower half-plane. When $\tau > 0$ (meaning we are considering past contributions), our contour is in the bottom half-plane, and we pick up contributions from these two poles. When $\tau < 0$ (meaning we are considering future contributions), our contour is in the top half-plane, which has no zeros, so $\chi(\tau) = 0$ for all $\tau < 0$. Therefore, in the Lorentz model, future fields do not affect the present [24]. Though we have applied this thought to a much simpler situation than that of the bianisotropic materials in TO antitelephones, it still serves to illustrate the problem.

We further notice that this effect is due to a nonzero resistivity γ; for if $\gamma = 0$ above, then the poles of the contour integral in (48) fall on the real axis, which will contribute for both half-planes (that is, from the future and the past). This is a curious point. The resistivity γ represents in a certain sense the loss of energy in the system due to heat as current passes through a resistive element. This loss in resistive elements is connected to heat, and this suggests entropy.

Any real material—whether it follows the Lorentz model or not—must undergo loss to heat as electric fields move through it. Imagine, then, light within an antitelephonic medium from the future event t_n moving to a slightly earlier event t_{n-1} as it travels along a CNG. To do this, the electric fields of the light must interact with the elements of the material, which produces some amount of heat δQ over the time $t_{n-1} - t_n$, which gets transmitted as waste heat to the air in the room. In the lab frame, this means that the overall heat of the system is *decreasing* over time $t_n - t_{n-1}$ and that waste heat δQ is being absorbed *by* the element *from* the air to go into diverting the path of light. This clearly entails a decrease in entropy in a process that does nothing more than convert heat to useful work (the work in the response of the material to the light fields). Hence, for any real material, no matter what its material parameters may do on paper, it is impossible for the future to communicate with the past. We find this consideration the strongest, as the Second Law has thus far proven unassailable.

VI. CONCLUSIONS AND FUTURE WORK

We have seen then that the theoretical framework behind TO allows for materials that support CNGs. We have explicitly derived just such a material ourselves. These materials, if possible to physically construct, have the predicted effect of allowing for the violation of causality, with information from the future being able to reach the past. This behavior follows from nothing more than the macroscopic material properties of the medium, in terms of how it responds to incoming electromagnetic (E&M) fields.

However, it is ultimately impossible to actually construct such a material. This raises an interesting limitation to transformation optics that suggests the need for a set of equations—analogous to the energy conditions from general relativity—that can relate the metric tensor to definite physical limits of the material parameters. Such equations would be able to quickly tell researchers the limits of their transformation media. These might be able to determine, for instance, whether it is the case that any material coming from a spacetime with CNGs (and not just that considered here) will be impossible to actually build.

Apart from such equations, it would be of interest to investigate the propagators corresponding to the advanced and retarded solutions in the spacetime/medium considered here. In particular, future work may wish to investigate the behavior of the advanced and retarded solutions at the turning point of $t(\phi)$ in (26), their possible symmetries, and what this says about the causal behavior of the system. This behavior could be studied in both the curved space and the bianistropic material, to see if there is any discrepancy in the two which may shed additional light on the failure of the material to actually violate causality.

In conclusion, thermodynamic limitations to the magnetoelectric effect will prevent the TO antitelephone from being built, and further thermodynamic considerations prevent any such device from violating causality, even if they could be built. Thus, even for transformation optics, history will indeed remain safe for historians.

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APPENDIX: EIKONAL APPROXIMATION IN THE ANTITELEPHONIC MEDIUM

Consider electromagnetic fields $\vec{E}(t,\phi) = E_o e^{iS(t,\phi)/\varepsilon}\hat{z}$ and $\vec{H}(t,\phi) = H_o e^{iS(t,\phi)/\varepsilon}\hat{r}$, initially propagating in the medium (40) in the axial direction. Note that because the MPs in (40) are diagonal in cylindrical components $\epsilon\vec{E} = \mu\vec{E} = \frac{K}{r}\vec{E}$, and similarly $\epsilon\vec{H} = \mu\vec{H} = \frac{K}{r}\vec{H}$. We will insert this ansatz for the electromagnetic fields into Maxwell's equations and in the end take the eikonal approximation by collecting only terms of highest order in $\frac{1}{\varepsilon}$; this will return us to the picture of geometric optics.

Beginning with Maxwell's equation for $\nabla\times\vec{H}=\frac{\partial}{\partial t}\vec{D}$, we have

$$\nabla\times\vec{H}=\frac{\partial}{\partial t}\vec{D}=\frac{\partial}{\partial t}(\epsilon\vec{E}+\vec{w}\times\vec{H}) \tag{A1a}$$

$$=\frac{K}{r}\frac{\partial}{\partial t}\vec{E}+\frac{\partial}{\partial t}(\vec{w}\times\vec{H}). \tag{A1b}$$

Proceeding in the usual way, taking the curl and using Maxwell's equation $\nabla\times\vec{E}=-\frac{\partial}{\partial t}\vec{B}$, we find

$$\nabla\times(\nabla\times\vec{H})$$
$$=\frac{K}{r}\frac{\partial}{\partial t}(\nabla\times\vec{E})+\frac{\partial}{\partial t}\nabla\times(\vec{w}\times\vec{H}) \tag{A2a}$$

$$=\frac{K}{r}\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\vec{B}\right)+\frac{\partial}{\partial t}\nabla\times(\vec{w}\times\vec{H}) \tag{A2b}$$

$$=-\frac{K}{r}\frac{\partial^2}{\partial t^2}(\mu\vec{H}-\vec{w}\times\vec{E})+\frac{\partial}{\partial t}\nabla\times(\vec{w}\times\vec{H}) \tag{A2c}$$

$$=-\frac{K}{r}\frac{\partial^2}{\partial t^2}\left(\frac{K}{r}\vec{H}-\vec{w}\times\vec{E}\right)+\frac{\partial}{\partial t}\nabla\times(\vec{w}\times\vec{H}) \tag{A2d}$$

$$=-\frac{K^2}{r^2}\frac{\partial^2}{\partial t^2}\vec{H}+\frac{K}{r}\frac{\partial^2}{\partial t^2}(\vec{w}\times\vec{E})+\frac{\partial}{\partial t}\nabla\times(\vec{w}\times\vec{H}). \tag{A2e}$$

Evaluating the cross products, we find

$$\vec{w}\times\vec{E}=-\frac{f}{r}E_z\hat{\phi}\times\hat{z}=-\frac{f}{r}E_z\hat{r} \tag{A3}$$

$$\nabla\times(\nabla\times\vec{H})=\frac{1}{r}\frac{\partial}{\partial\phi}\left(-\frac{1}{r}\frac{\partial}{\partial\phi}H_r\right)\hat{r}-\frac{\partial}{\partial r}\left(-\frac{1}{r}\frac{\partial}{\partial\phi}H_r\right)\hat{\phi} \tag{A4}$$

$$\nabla\times(\vec{w}\times\vec{H})=\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{f}{r}H_r\right)\hat{r}-\frac{\partial}{\partial r}\left(\frac{f}{r}H_r\right)\hat{\phi}. \tag{A5}$$

Further, evaluating the derivatives with $H_r=H_oe^{iS/\varepsilon}, E_z=E_oe^{iS/\varepsilon}$,

$$\frac{\partial^2}{\partial t^2}\vec{H}=\left(i\frac{S_{,00}}{\varepsilon}-\frac{(S_{,0})^2}{\varepsilon^2}\right)H_r\hat{r} \tag{A6}$$

$$\frac{\partial^2}{\partial t^2}(\vec{w}\times\vec{E})=-\frac{f}{r}\left(i\frac{S_{,00}}{\varepsilon}-\frac{(S_{,0})^2}{\varepsilon^2}\right)E_z\hat{r} \tag{A7}$$

$$\nabla\times(\nabla\times\vec{H})=-\frac{1}{r^2}\left(i\frac{S_{,22i}}{\varepsilon}-\frac{(S_{,2})^2}{\varepsilon^2}\right)H_r\hat{r}-\frac{1}{r^2}i\frac{S_{,2}}{\varepsilon}H_r\hat{\phi} \tag{A8}$$

$$\frac{\partial}{\partial t}\nabla\times(\vec{w}\times\vec{H})$$
$$=\left(i\frac{f'}{r^2}\frac{S_{,0}}{\varepsilon}+i\frac{f}{r^2}\frac{S_{,02}}{\varepsilon}-\frac{f}{r^2}\frac{S_{,2}S_{,0}}{\varepsilon^2}\right)H_r\hat{r}-i\frac{f}{r^2}\frac{S_{,0}}{\varepsilon}H_r\hat{\phi}. \tag{A9}$$

Taking the eikonal approximation, ignoring terms of first or lower order in $\frac{1}{\varepsilon}$, these simplify to

$$\frac{\partial^2}{\partial t^2}\vec{H}=-\frac{(S_{,0})^2}{\varepsilon^2}H_r\hat{r} \tag{A10}$$

$$\frac{\partial^2}{\partial t^2}(\vec{w}\times\vec{E})=\frac{f}{r}\frac{(S_{,0})^2}{\varepsilon^2}E_z\hat{r} \tag{A11}$$

$$\nabla\times(\nabla\times\vec{H})=\frac{1}{r^2}\frac{(S_{,2})^2}{\varepsilon^2}H_r\hat{r} \tag{A12}$$

$$\frac{\partial}{\partial t}\nabla\times(\vec{w}\times\vec{H})=-\frac{f}{r^2}\frac{S_{,2}S_{,0}}{\varepsilon^2}H_r\hat{r}. \tag{A13}$$

The right-hand side of (A2e) is

$$\mathrm{RHS}=\left(\frac{1}{\varepsilon^2}\right)\left[\left(K^2+Kf\frac{E_o}{H_o}\right)(S_{,0})^2-fS_{,2}S_{,0}\right]\hat{r}\frac{H_oe^{iS}}{r^2}, \tag{A14}$$

while the left-hand side of (A2e) is just

$$\mathrm{LHS}=\left(\frac{1}{\varepsilon^2}\right)(S_{,2})^2\hat{r}\frac{H_oe^{iS}}{r^2}. \tag{A15}$$

Putting both together and canceling common terms on both sides,

$$(S_{,2})^2=\left(K^2+Kf\frac{E_o}{H_o}\right)(S_{,0})^2-fS_{,2}S_{,0}. \tag{A16}$$

Finally, in Eq. (A16), we need a way to express $\frac{E_o}{H_o}$. Note that from (A1), we have

$$\nabla\times\vec{H}=\frac{K}{r}\frac{\partial}{\partial t}\vec{E}+\frac{\partial}{\partial t}(\vec{w}\times\vec{H}) \tag{A17}$$

or

$$-\frac{1}{r}iS_{,2}H_o=\frac{K}{r}iS_{,0}E_o+\frac{f}{r}iS_{,0}H_o. \tag{A18}$$

Rearranging, then

$$\frac{E_o}{H_o} = -\frac{fS_{,0} + S_{,2}}{KS_{,0}}, \tag{A19}$$

so that

$$K^2 + Kf\frac{E_o}{H_o} = K^2 - Kf\frac{fS_{,0} + S_{,2}}{KS_{,0}} = K^2 - f^2 - f\frac{S_{,2}}{S_{,0}}. \tag{A20}$$

Plugging in to (A16),

$$0 = (S_{,0})^2\left(K^2 - f^2 - f\frac{S_{,2}}{S_{,0}}\right) - S_{,2}^2 - fS_{,2}S_{,0} \tag{A21a}$$

$$= b(\phi)(S_{,0})^2 - 2f(\phi)S_{,0}S_{,2} - (S_{,2})^2. \tag{A21b}$$

This is the eikonal equation for light in a curved spacetime,

$$0 = g^{\alpha\beta}S_{,\alpha}S_{,\beta}, \tag{A22}$$

where we take $g^{\alpha\beta}$ as given above in (28). Thus, in the usual interpretation of geometric optics, the light rays in the medium will follow the same paths as light rays in our curved space and thus should be expected to move along CNGs.

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