Generalized energy conditions in extended theories of gravity

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In this work, we consider the further degrees of freedom related to curvature invariants and scalar fields in extended theories of gravity (ETG). These new degrees of freedom can be recast as *"effective fluids"* that differ in nature with respect to the standard matter fluids generally adopted as sources of the field equations. It is, thus, somewhat misleading to apply the standard general relativistic energy conditions to this effective energy-momentum tensor, as the latter contains the matter content and a geometrical quantity, which arises from the specific ETG considered. Here we explore this subtlety, extending our previous work, in particular, to cases with the contracted Bianchi identities with diffeomorphism invariance and to cases with generalized explicit curvature-matter couplings, which imply the nonconservation of the energy-momentum tensor. Furthermore, we apply the analysis to specific ETGs, such as scalar-tensor gravity and $f(R)$ gravity. Thus, in the context of ETGs, interesting results appear such as matter that may exhibit unusual thermodynamical features, for instance, gravity that retains its attractive character in the presence of large negative pressures; or alternatively, we verify that repulsive gravity may occur for standard matter.

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I. INTRODUCTION

Modifications and extensions of general relativity (GR) can be traced back to the early times of GR [1–6]. The first extensions were aimed to unify gravity with electromagnetism while recent interest in such modifications arises from cosmology, astrophysics and quantum gravity [7–10]. In particular, cosmological observations lead to the introduction of additional *ad hoc* concepts like dark energy and dark matter if one restricts the dynamics to the standard Einstein theory. On the other hand, the emergence of such new ingredients of cosmic fluids could be interpreted as a first signal of a breakdown of GR on large, infrared scales [11,12]. In such a way, modifications and extensions of GR become a natural alternative if such "dark" elements are not found out. In particular, several recent works focused on the cosmological implications of alternative gravity since such models may lead to the explanation of the acceleration effect observed in cosmology [13–18] and to the explanation of the missing matter puzzle observed at astrophysical scales [19–27].

While it is very natural to extend Einstein's gravity to theories with additional geometric degrees of freedom [28–30], recent attempts focused on the idea of modifying the gravitational Lagrangian leading to higher-order field equations. Due to the increased complexity of the field equations, a huge amount of work considered some formally equivalent theories, in which a reduction of the order of the field equations can be achieved by considering the metric and the connection as independent objects [31,32]. However, a concern which arises with generic extended and modified gravity theories is linked to the initial value problem and the definition of the energy conditions. It is unclear if standard methods can be used in order to tackle these problems in any theory. Hence, it is doubtful that the full Cauchy problem can be properly addressed if one takes into account the results already obtained in GR. On the other hand, in alternative gravities, such as gauge theories, the initial value formulation and the energy conditions depend on suitable constraints and gauge choices, precisely as in GR [33,34]. A different approach is possible, showing that the Cauchy problem for alternative gravities can be well formulated and well posed in vacuo, while it can be, at least, well formulated for various forms of matter fields like perfect fluids, Klein-Gordon and

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Yang-Mills fields [35]. A similar situation also holds for the energy conditions which can strictly depend on the kind of fluids adopted as sources in the field equations.

In fact, there are serious problems of deep and fundamental principle at the semiclassical level, and certain classical systems exhibit seriously pathological behavior; in particular, the classical energy conditions are typically violated by semiclassical quantum effects [36]. In this context, some effort has gone into finding possible semiclassical replacements for the classical energy conditions [37]. Recently, classical and quantum versions of a "flux energy condition" (FEC and QFEC) were developed based on the notion of constraining the possible fluxes measured by timelike observers [38]. It was shown that the naive classical FEC was satisfied in some situations, and even for some quantum vacuum states, while its quantum analogue (the QFEC) was satisfied under a rather wide range of conditions. Furthermore, several nonlinear energy conditions suitable for use in the semiclassical regime were developed, and it was shown that these nonlinear energy conditions behave much better than the classical linear energy conditions in the presence of semiclassical quantum effects [39].

However, in the context of alternative theories of gravity that in a wide sense *extend* GR, the issue of the energy conditions is extremely delicate. Note that the further degrees of freedom carried by these extended theories of gravity (ETGs) can be recast as generalized *"effective fluids"* that differ in nature with respect to the standard matter fluids generally adopted as sources of the field equations [10]. This approach has been extensively explored in the literature, namely, the energy conditions have been used to constrain $f(R)$ theories of gravity [40–42] and extensions involving nonminimal curvature-matter couplings [43–49]; bounds on modified Gauss-Bonnet $f(G)$ gravity from the energy conditions have also been analyzed [50–52], and with a nonminimal coupling to matter [53]; the recently proposed $f(R,T)$ gravity models [54], where T is the trace of the energy-momentum tensor and R is the curvature scalar, have also been tested using the energy conditions [55–57]; and constraints have also been placed [58] on the $f(R,T,R_{\mu\nu}T^{\mu\nu})$ extension [59,60]; bounds have been placed on modified teleparralel gravity [61–63]; and the null-energy condition violations have been studied in bimetric gravity [64].

However, one should add a cautionary note of the results obtained in the literature, such as the majority of those considered above have recast the further degrees of freedom carried by these ETGs as generalized *effective fluids* that differ in nature with respect to the standard matter fluids generally adopted as sources of the field equations [10]. Note that while standard fluids (e.g., perfect matter fluids), generally obey standard equations of state (and then one can define every thermodynamic quantity such as the adiabatic index, temperature, etc.), these "fictitious" fluids can be related to scalar fields or further gravitational degrees of freedom. In these cases, the physical properties can result ill-defined and the energy conditions could rigorously work as in GR. The consequences of such a situation can be dramatic since the causal and geodesic structures of the theory could present serious shortcomings as well as the energy-momentum tensor could not be consistent with the Bianchi identities and the conservation laws.

This paper is outlined in the following manner. In Sec. II, we briefly review the energy conditions in GR and discuss the geometrical implications of such conditions. Section III is devoted to set the energy conditions in ETGs by considering, in particular, the contracted Bianchi identities, the nonconservation of the energy-momentum tensor, the propagation equations and the role of conformal transformations. In Sec. IV, we take into account some particular theories, i.e., scalar-tensor theories and $f(R)$ gravity, where R is the Ricci scalar. Finally, we discuss our results and draw some conclusions in Sec. V.

II. THE ENERGY CONDITIONS IN GENERAL RELATIVITY

In GR, the Einstein field equation govern the interplay between the geometry of the spacetime and the matter content. More specifically, the field equation is given by

$$G_{ab} = 8\pi G T_{ab}, \tag{1}$$

where the energy-momentum tensor of the matter fields, T_{ab} is related to the Einstein tensor $G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R$, with R_{ab} the Ricci tensor, which is defined as the trace of the Riemann curvature tensor $R^d{}_{adb} = R_{ab}$, and $R = R^a{}_a$. Thus, the imposition of specific conditions on T_{ab} are translated into corresponding conditions on the Einstein tensor G_{ab}. Note that the Einstein equations can also be cast as conditions on the Ricci tensor, that is

$$R_{ab} = 8\pi G\left(T_{ab} - \frac{1}{2} T g_{ab}\right). \tag{2}$$

In this form, the role of energy-matter is more relevant.

In general, in considering the energy conditions, we take into account a congruence of timelike curves whose tangent 4-vector is, for instance, W^a. The latter represents the velocity vector of a family of observers. One may also consider a field of null vectors, k^a, so that $g_{ab}k^a k^b = 0$ implies that $G_{ab}k^a k^b = R_{ab}k^a k^b$.

These choices enable us to identify the physical quantities measured by the observers related to the timelike vector W^a. Indeed, with respect to the latter vector field W^a, the energy-momentum tensor can then be decomposed as

$$T^{ab} = \rho W^a W^b + p(g^{ab} + W^a W^b) + \Pi^{ab} + 2q^{(a}W^{b)}, \tag{3}$$

where ρ and p are the energy density and the (isotropic) pressure measured by the observers moving with velocity W^a, Π^{ab} is the anisotropic stress tensor, and q^a is the current vector of the heat or energy flow. These quantities are given by the following relations,

$$\rho = T_{cd}W^cW^d, \tag{4}$$

$$3p = T_{cd}h^{cd}, \tag{5}$$

$$\Pi^{ab} = \left(h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}\right)T_{cd}, \tag{6}$$

$$q^a = W^cT_{cd}h^{ad}, \tag{7}$$

respectively, where $h^{ab} = g^{ab} + W^aW^b$ is the metric induced on the spatial hypersurfaces orthogonal to W^a. Throughout this work, we adopt the $(-+++)$ signature convention and the speed of light is $c = 1$.

A. The classification of energy conditions

The energy conditions are defined by considering contractions of timelike and null vectors with respect to the Ricci, Einstein and energy-momentum tensors [65]. They can be classified as follows:

(i) The *weak-energy condition* is defined as

$$T_{ab}W^aW^b \geq 0, \tag{8}$$

where W^a is a timelike vector, i.e., $W^aW_a = -1$. From Eq. (3), we verify that this entails that $\rho \geq 0$. As presented by Hawking and Ellis [65], such a condition is equivalent to establishing that the energy density measured by any observer is non-negative. It is straightforward to demonstrate that any standard matter fluid is consistent with such a condition. Through the Einstein field equations where the curvature of spacetime is considered, condition (8) translates into

$$G_{ab}W^aW^b \geq 0 \tag{9}$$

which is equivalent to

$$R_{ab}W^aW^b \geq -\frac{R}{2}. \tag{10}$$

and also to

$$R_{ab}W^aW^b \geq -4\pi G(\rho - 3p). \tag{11}$$

Here we have used the fact that, from Eq. (3), we can recast the Einstein equations as

$$R^{ab} = 8\pi G\left[\frac{\rho + 3p}{2}W^aW^b + \Pi^{ab} + 2q^{(a}W^{b)} + \frac{\rho - p}{2}(g^{ab} + W^aW^b)\right]. \tag{12}$$

(ii) The *dominant-energy condition* states that, in addition to the condition (8), one also has that $T^{ab}W_b$ is a nonspacelike vector, where as before W^a is a timelike vector, so that $W^aW_a = -1$. This corresponds to having a local energy flow vector which is nonspacelike in addition to the non-negativity of the energy density. In this sense, the causal structure of the spacetime is determined.

(iii) The *null-energy condition* states that

$$T_{ab}k^ak^b \geq 0, \tag{13}$$

where k^a is a null vector, i.e., $k^ak_a = 0$. This implies $R_{ab}k^ak^b \geq 0$, through the Einstein field equation. A very useful meaning of this condition is that it's violation implies that the Hamiltonian of the corresponding system is necessarily unbounded from below (we refer the reader to [66] for more details).

(iv) The *strong-energy condition* is given by

$$T_{ab}W^aW^b \geq \frac{1}{2}TW^aW_a, \tag{14}$$

where W^a is a timelike vector. Alternatively, in GR and through the Einstein field equations, the above inequality takes the form

$$R_{ab}W^aW^b \geq 0 \tag{15}$$

which, as we will see in what follows through the Raychaudhuri equation, states that gravity must be attractive.

Summarizing, such conditions define the causal structure, the geodesic structure and the nature of the gravitational field in a spacetime filled by a standard fluid matter endowed with a regular equation of state.

B. Geometrical implications of the energy conditions

The geometrical implications of the energy conditions can be put in evidence as soon as we consider the decomposition [67]

$$\nabla_bW_a = \sigma_{ab} + \frac{\theta}{3}h_{ab} + \omega_{ab} - \dot{u}_aW_b, \tag{16}$$

with the following definitions,

$$h_{ab} = g_{ab} + W_aW_b, \tag{17}$$

$$\sigma_{ab} = h^c_{(a}\nabla_cW_dh^d_{b)} - \frac{h_{ab}}{3}h^c_a\nabla_cW_dh^{ad}, \tag{18}$$

$$\theta = h_a^c \nabla_c W_d h^{ad}, \tag{19}$$

$$\omega_{ab} = h^c_{[a} \nabla_c W_d h^d_{b]}, \tag{20}$$

$$\dot{W}^a = W^b \nabla_b W^a, \tag{21}$$

respectively, where we have considered all possible combinations of the metric tensor and timelike vectors. Here h_{ab} is the *projection tensor*, σ_{ab} is the *shear tensor*, θ is the *expansion scalar*, and ω_{ab} is the vorticity tensor. Note that h_{ab} is orthogonal to W^a, $W^a h_{ab} = 0$ and, hence, it is the metric induced on the 3-hypersurfaces orthogonal to W^a, as mentioned before.

Equipped with the latter kinematical quantities whose contractions give rise to the so-called *optical scalars* [68], we derive, from the Ricci identities,[1] the following relations,

$$\dot{\theta} + \frac{\theta^2}{3} + 2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a} = -R_{ab} W^a W^b, \tag{22}$$

and

$$h_a{}^f h_b{}^g [(\sigma_{fg})^{\cdot} - \dot{W}_{(f;g)}] = \dot{W}_a \dot{W}_b - \omega_a \omega_b - \sigma_{af}\sigma^f{}_b - \frac{2}{3}\theta\sigma_{ab} - h_{ab}\left(-\frac{1}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{1}{3}\dot{W}^c{}_c\right) + \frac{1}{2}\left(h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}\right)\left(R_{cd} - \frac{1}{2}g_{cd}R\right) \tag{23}$$

$$h^a{}_b\left[\exp\left(\frac{2}{3}\int \theta dt\right)\omega^b\right]^{\cdot} = \sigma^a{}_b\left[\exp\left(\frac{2}{3}\int \theta dt\right)\omega^b\right] + \frac{1}{2}\left[\exp\left(\frac{2}{3}\int \theta dt\right)\right]\eta^{abcd} W_b \dot{W}_{(c;d)}, \tag{24}$$

where Eq. (22) is the so-called *Raychaudhuri equation*. It is important to emphasize that Eqs. (22)–(24) only carry a geometrical meaning, as they are directly derived from the Ricci identities. It is only when we choose a particular theory that we establish a relation between quantities that appear in their right-hand sides, such as $R_{ab}W^aW^b$ in Eq. (22), and the energy-momentum tensor describing matter fields.

For instance, let us consider a null congruence k^a and a vanishing vorticity $\omega_{ab} = 0$. The Raychaudhuri equation (22) reduces to

$$\frac{d\theta}{dv} = -\left[\frac{\theta^2}{3} + 2\sigma^2 + R_{ab}k^a k^b\right], \tag{25}$$

where v is an affine parameter along the null geodesics. This means that, in GR, it is possible to associate the null-energy condition with the focusing (attracting) characteristic of the spacetime geometry. Gravitational lensing is a very important application of this feature as widely discussed in [68].

III. THE PROBLEM OF ENERGY CONDITIONS IN EXTENDED THEORIES OF GRAVITY

In the context of ETGs, consider the following generalized gravitational field equations, which encapsulate a large class of interesting cases

$$g_1(\Psi^i)(G_{ab} + H_{ab}) = 8\pi G g_2(\Psi^j) T_{ab}, \tag{26}$$

where the factors $g_1(\Psi^i)$ modifies the coupling with the matter fields in T^{ab} and $g_2(\Psi^i)$ incorporates explicit curvature-matter couplings of the gravitational theory considered [18,25]; Ψ^j generically represents either curvature invariants or other gravitational fields, such as scalar fields, contributing to the dynamics of the theory. The additional tensor H_{ab} represents an additional geometric term with regard to GR that encapsulates the geometrical modifications introduced by the extended theory under consideration.

Note, that GR is immediately recovered by imposing $H_{ab} = 0$, $g_1(\Psi^i) = g_2(\Psi^i) = 1$. In this sense we are dealing with extended theories of gravity, in that the underlying hypothesis is that GR (and its positive results) can be recovered as a particular case in any "extended" theory of gravitation [69].

A. Contracted Bianchi identities and diffeomorphism invariance

Consider the specific case of $g_1(\Psi^i) = g(\Psi^i)$ and $g_2(\Psi^i) = 1$, so that the field equation (26) reduces to

$$g(\Psi^i)(G_{ab} + H_{ab}) = 8\pi G T_{ab}. \tag{27}$$

Taking into account the contracted Bianchi identities and the diffeomorphism invariance of the matter action, which implies the covariant conservation of the energy-momentum tensor, $\nabla_b T^{ab} = 0$, one deduces the following conservation law:

[1]The Ricci identities prescribe that $\nabla_c\nabla_d u^a - \nabla_d\nabla_c u^a = R^a{}_{bcd}u^b$ for any vector field u^a.

$$\nabla_b H^{ab} = -\frac{8\pi G}{g^2} T^{ab} \nabla_b g. \tag{28}$$

Note that from Eq. (27), in order to have an extended Bianchi identity $\nabla_b H^{ab} = 0$, for a nondiverging value of the coupling g, we must have a vacuum and, therefore, $G_{ab} = -H_{ab}$.

Now, an imposition of specific energy conditions on the energy-momentum tensor T^{ab} carries over the conditions to the combination of G_{ab} and H_{ab} and not just for the Einstein tensor. Thus, in the context of ETGs, it is not possible to obtain a simple geometrical implication from the conditions imposed. For instance, in GR, suppose that the strong-energy condition holds. This would mean that $R_{ab}W^a W^b \geq 0$, and consequently through the Einstein field equation we would have $\rho + 3P \geq 0$. On the other hand, this entails gravity with an attractive character, since given Eq. (22), one verifies that the geodesics are focusing [65]. However, in the ETG case under consideration, this condition just states that

$$g(\Psi^i)\left(R_{ab} + H_{ab} - \frac{1}{2} g_{ab} H\right) W^a W^b \geq 0, \tag{29}$$

which does not necessarily entail $R_{ab}W^a W^b \geq 0$, so that one cannot conclude that the attractive nature of gravity is equivalent to the satisfaction of the strong-energy condition, in the particular ETG under consideration [70].

However, in the literature, it is common practice to transport the term H^{ab} to the right-hand side of the gravitational field equation, and write the latter as a modified Einstein field equation, namely,

$$G_{ab} = 8\pi G T^{\rm eff}_{ab}, \tag{30}$$

where $T^{\rm eff}_{ab}$ is considered as an effective energy-momentum tensor, defined by $T^{\rm eff}_{ab} = T_{ab}/g - 8\pi G H_{ab}$. Thus, the meaning which is attributed to the energy conditions is the satisfaction of some inequality by the combined quantity $T^{ab}/g - H^{ab}$. It is, therefore, somewhat misleading to call these impositions as energy conditions since they do not emerge only from T^{ab} but from a combined quantity where we are dealing with a geometrical H^{ab} as an additional stress-energy tensor. Indeed, we emphasize that H^{ab} is a geometrical quantity, in the sense that it can be given by geometrical invariants as R or scalar fields different from ordinary matter fields.

However, if the ETG under consideration allows an equivalent description upon an appropriate conformal transformation, it then becomes justified to associate the transformed H^{ab} to the redefined T^{ab} in the conformally transformed Einstein frame. This is, for instance, the case for scalar-tensor gravity theories, and for instance in $f(R)$ gravity [10]. Indeed, conformal transformations play an extremely relevant role in the discussion of the energy conditions. In particular, they allow us to put in evidence the further degrees of freedom coming from ETGs under the form of curvature invariants and scalar fields. More specifically, several generalized theories of gravity can be redefined as GR plus a number of appropriate fields coupled to matter by means of a conformal transformation in the so-called Einstein frame.

In fact, in scalar-tensor gravity, in the so-called Jordan frame one has a separation between the geometrical terms and the standard matter terms that can be cast as in Eq. (27), where H_{ab} involves a mixture of both the scalar and tensor gravitational fields. A main role in this analysis is played by recasting the theory, by conformal transformations, in the Einstein frame where matter and geometrical quantities can be formally dealt exactly such as in GR. However, the energy conditions can assume a completely different meaning going back to the Jordan frame, and then they could play a crucial role in identifying the physical frame as first pointed out in [71]. Although, it is completely clear that different "frames" just correspond to field redefinitions, all of which are equally physical.

Now, under a suitable conformal transformation the field equations can be recast as

$$\tilde{G}_{ab} = \tilde{T}^M_{ab} + \tilde{T}^\varphi_{ab}, \tag{31}$$

where $\tilde{T}^M_{ab}$ is the transformed energy momentum of matter, and $\tilde{T}^\varphi_{ab}$ is an energy-momentum tensor for the redefined scalar field φ which is coupled to the matter. Thus, it makes sense to consider the whole right-hand side of (31) as an effective energy-momentum tensor. Then one finds results where one draws conclusions about the properties of G_{ab} such whether it focuses geodesics directly from those conditions holding on $T^{\rm eff}_{ab}$, where $T^{\rm eff}_{ab} = \tilde{T}^M_{ab} + \tilde{T}^\varphi_{ab}$. This ignores the fact that H_{ab} originally possesses a geometrical character, and, thus, the conclusions may be too hasty if not supported by the physical analysis of sources. We refer the reader to [70] for a detailed analysis on this issue.

B. Nonconservation of the energy-momentum tensor

A main role in the formulation of the correct energy conditions for ETGs is played by the contracted Bianchi identities that guarantee specific conservation laws. In fact, being $\nabla_b G^{ab} = 0$, the physical features of H^{ab} can be derived. On the other hand, the Bianchi identities guarantee the self-consistency of the theory. However, an interesting class of extended theories of gravity that exhibit an explicit curvature-matter coupling have recently been proposed in the literature [18,25]. The latter coupling imply a general nonconservation of the energy-momentum tensor, and consequently a trademark of these specific ETGs is nongeodesic motion [18,25].

We will briefly analyze these theories in the formalism outlined above. In order to incorporate the explicit curvature-matter coupling, consider the field equation given by

Eq. (26). Note that in ETGs of the form (26) in the presence of the nonconservation of the energy-momentum tensor, the contracted Bianchi identities yield

$$\nabla_b H^{ab} = \nabla_b \left(\frac{T^{ab}}{\bar{g}} \right), \tag{32}$$

where the factor $\bar{g} = g_1/g_2$ is defined, and we have considered that $8\pi G = 1$ for notational simplicity. Now, Eq. (32) implies the following relationship:

$$\begin{aligned} \nabla_b T^{ab} &= \bar{g} \nabla_b H^{ab} + \left(\frac{\nabla_b \bar{g}}{\bar{g}} \right) T^{ab} \\ &= \nabla_b (\bar{g} H^{ab}) + \left(\frac{\nabla_b \bar{g}}{\bar{g}} \right) [T^{ab} - (\bar{g} H^{ab})]. \end{aligned} \tag{33}$$

Thus a trademark of these specific class of ETGs is that the matter fields do not, in general, follow the geodesics of spacetime [72].

Let us introduce the following useful definitions:

$$\tilde{\rho} = (\bar{g} H_{cd}) W^c W^d, \tag{34}$$

$$3\tilde{p} = (\bar{g} H_{cd}) h^{cd}, \tag{35}$$

$$\tilde{\Pi}^{ab} = \left(h^{ac} h^{bd} - \frac{1}{3} h^{ab} h^{cd} \right) (\bar{g} H_{cd}), \tag{36}$$

$$\tilde{q}^a = W^c (\bar{g} H_{cd}) h^{ad}. \tag{37}$$

We derive

$$\dot{\rho} + (\rho + p) \nabla_b W^b + W_a \nabla_b \Pi^{ab} = W_a \nabla_b \bar{g} H^{ab}, \tag{38}$$

so that the departure from the usual conservation equations depends on the term

$$W_a \nabla_b (\bar{g} H^{ab} - \Pi^{ab}). \tag{39}$$

Therefore in what regards this balance equation, the term $\bar{g} H_{ab}$ plays a role which is analogous to that of the anistropic stress tensor Π_{ab}, given by Eq. (6). We can recast the latter equations as

$$\begin{aligned} &\dot{\rho} + (\rho + p)\theta + \Pi^{ab}\sigma^{ab} + \nabla_b q^b + \dot{W}_a q^a \\ &\quad = [\dot{\tilde{\rho}} + (\tilde{\rho} + \tilde{p})\theta + \tilde{\Pi}^{ab}\sigma^{ab} + \nabla_b \tilde{q}^b + \dot{W}_a \tilde{q}^a] \\ &\qquad + \left(\frac{\dot{\bar{g}}}{\bar{g}} \right) (\tilde{\rho} - \rho) + \left(\frac{\nabla_b \bar{g}}{\bar{g}} \right) (\tilde{q}^b - q^b), \end{aligned} \tag{40}$$

or as

$$\dot{\rho} - \dot{\tilde{\rho}} + [(\rho - \tilde{\rho}) + (p - \tilde{p})]\theta = -(\Pi^{ab} - \tilde{\Pi}^{ab})\sigma^{ab} - \nabla_b (q^b - \tilde{q}^b) - \dot{W}_a (q^a - \tilde{q}^b) + \left(\frac{\dot{\bar{g}}}{\bar{g}} \right) (\tilde{\rho} - \rho) + \left(\frac{\nabla_b \bar{g}}{\bar{g}} \right) (\tilde{q}^b - q^b). \tag{41}$$

Analogously, we derive an equation for the acceleration $\dot{W}^a$. We obtain the following relationships:

$$\begin{aligned} [(\rho - \tilde{\rho}) + (p - \tilde{p})]\dot{W}^a + h_a^b [\nabla_b (p - \tilde{p})] &= -h_a^c \nabla_b (\Pi_c^b - \tilde{\Pi}_c^b) - h_a^c (\dot{q}_c - \dot{\tilde{q}}_c) + \left(\frac{\nabla_b \bar{g}}{\bar{g}} \right) \\ &\quad \times \{ (p - \tilde{p}) h_a^b + [(\Pi_a^b - \tilde{\Pi}_a^b) - (\tilde{q}_a - q_a) W^b] \}. \end{aligned} \tag{42}$$

These equations show how the $\bar{g} H^{ab}$ term modifies the standard energy-density conservation equation and the generalized Navier-Stokes equation for the acceleration, both derived from the contracted Bianchi identities. It is important to emphasize that, although the contracted Bianchi identities are geometrical relations in their essence and, hence, do not depend on the specific gravitational theory under consideration, when we translate them into equations governing the behavior of the matter fields, the choice of the theory intervenes. This happens in association with the $\bar{g} H^{ab}$ terms, that is with the tilded quantities that we have defined in the Einstein frame. In summary, the validity of the contracted Bianchi identities selects suitable theories and may allow the definition of self-consistent energy conditions.

C. Propagation equations and extended theories of gravity

In the present subsection, we consider the specific case of $\bar{g} = g$, i.e., $g_2 = 1$, and consequently the covariant conservation of the energy-momentum tensor. The role of propagation equations deserve a particular discussion in this context. We have already written the propagation equations for the expansion θ, for the shear σ_{ab} and for the vorticity ω_{ab}, that is Eqs. (22)–(24), and have pointed out that these equations do not reflect the particular gravitational theory under consideration since they are derived directly from the $3 + 1$ decomposition of the Ricci identities that come from the Riemann tensor.

The prescription for a given gravitational theory enters into play when we replace quantities such as $R_{ab}W^aW^b$ into the Raychaudhuri Eq. (22). For the theories under consideration here, the latter geometrical quantity is replaced by the inequality (29), which, according to the definition (34) (recall that in the present context we have $\bar{g} = g$, i.e., $g_2 = 1$, and the covariant conservation of the energy-momentum tensor), only involves the energy density of matter and that given by the latter equation. However, when we consider the shear propagation equation, the role of the particular ETG comes out by replacing $\frac{1}{2}(h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd})(R_{cd} - \frac{1}{2}g_{cd}R)$. We, thus, have

$$\begin{aligned}\frac{1}{2}\left(h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}\right)\left(R_{cd} - \frac{1}{2}g_{cd}R\right) &= \frac{1}{2}\left(h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}\right)\left(-H_{ab} + \frac{T_{ab}}{g}\right)\\ &= \frac{1}{g}(-\tilde{\Pi}^{ab} + \Pi^{ab}). \end{aligned}\tag{43}$$

In general, the discussion of the energy conditions in ETGs is made in relation to the spatially homogeneous and isotropic FLRW universes, which implies that $\sigma_{ab} = 0$ and $\omega_{ab} = 0$.[2] One question which is then of interest is to assess the possible role of the ETG theories in perturbing the universe away from its Friedmann state. Clearly this depends on the term $\tilde{\Pi}^{ab}$ being nonvanishing. The interesting result that we want to put forward is that in theories like $f(R)$ gravity and scalar-tensor gravity, the quantity $\tilde{\Pi}^{ab}$ is vanishing and so they do not introduce any modification with respect to GR in the shear propagation equation. If the shear starts vanishing, it remains so. Indeed, theories where $\tilde{\Pi}^{ab} \neq 0$ exist (e.g. inhomogeneous cosmologies [73]) but we do not consider them in the present context.

IV. EXAMPLES OF EXTENDED THEORIES OF GRAVITY

Taking into account the above discussion, the correct identification of the function $g_i(\Psi^j)$ $(i = 1, 2)$, and the tensor H_{ab} defined in Sec. III enables one to formulate the energy conditions for any ETG. Recall that the functions $g_i(\Psi^j)$ are related to the gravitational coupling that can be nonminimal, and the tensor H_{ab} is the contribution to the effective energy-momentum tensor containing the further degrees of freedom of the ETG. Below, we give some specific examples of theories that fit well in the context of the above discussion.

[2]One also has the vanishing of the electric and magnetic parts of the Weyl tensor C_{abcd}, $E_{ab} = C_{acbd}W^cW^d$ and $H^*_{ab} = \frac{1}{2}\eta_{ac}{}^{gh}C_{ghbd}W^cW^d$, respectively, where η^{abcd} is the totally-skew symmetric pseudotensor.

A. Scalar-tensor gravity

In this subsection, we extend and complement the analysis outlined in [70]. The scalar-tensor gravity [74], to which Brans-Dicke is the archetype, can be based on the action

$$S = \frac{1}{16\pi}\int \sqrt{-g}d^4x\left[\phi R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu} + 2\phi\lambda(\phi)\right] + S_M, \tag{44}$$

where the gravitational coupling is assumed variable and a self-interaction potential is present; S_M is the standard matter part. Varying this action with respect to the metric g_{ab} and the scalar field ϕ yields the field equations

$$\begin{aligned} R_{ab} - \frac{1}{2}g_{ab}R - \lambda(\phi)g_{ab} &= \frac{\omega(\phi)}{\phi^2}\left[\phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}\phi_{;c}\phi^{;c}\right]\\ &\quad + \frac{1}{\phi}[\phi_{;ab} - g_{ab}\phi_{;c}{}^{;c}] + 8\pi G\frac{T_{ab}}{\phi}, \end{aligned}\tag{45}$$

and

$$\Box\phi + \frac{2\phi^2\lambda'(\phi) - 2\phi\lambda(\phi)}{2\omega(\phi)+3} = \frac{1}{2\omega(\phi)+3}[8\pi GT - \omega'(\phi)\phi_{;c}\phi^{;c}], \tag{46}$$

where $T \equiv T^a{}_a$ is the trace of the matter energy-momentum tensor and $G \equiv (2\omega + 4)/(2\omega + 3)$ is the gravitational constant normalized to the Newton value. Additional to these equations, one also requires diffeomorphism invariance and consequently the conservation of the matter content $\nabla^b T_{ab} = 0$. The latter also preserves the equivalence principle. Brans-Dicke theory is characterized by the restriction of $\omega(\phi)$ being a constant, and of $\lambda = \lambda' = 0$.

According to the discussion in the previous section, for the general class of scalar-tensor theories, the tensor term H_{ab} is defined by

$$\begin{aligned} H_{ab} = &-\frac{\omega(\phi)}{\phi^2}\left[\phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}\phi_{;c}\phi^{;c}\right]\\ &- \frac{1}{\phi}[\phi_{;ab} - g_{ab}\phi_{;c}{}^{;c}] - \lambda(\phi)g_{ab}, \end{aligned}\tag{47}$$

and the coupling functions are given by $g_1(\Psi^i) = \phi$, which we shall assume positive, and $g_2(\Psi^i) = 1$. The above considerations on the energy conditions straightforwardly apply. In particular Eq. (29) is easily recovered like the other energy conditions. Taking into account the assumption $\phi > 0$, the condition $R_{ab}W^aW^b \geq 0$, that yields the focusing of the timelike congruence becomes

$$\left(T_{ab}-\frac{1}{2}g_{ab}T\right)W^aW^b \geq \phi\left(H_{ab}-\frac{1}{2}g_{ab}H\right)W^aW^b. \quad (48)$$

Notice that even in the presence of a mild violation of the energy condition, the satisfaction of the above condition allows for the focusing of the timelike paths. This is an interesting result since matter may exhibit unusual thermodynamical features, for instance, the presence of negative pressures, and yet gravity retains its attractive character. Alternatively, we see that repulsive gravity may occur for common matter, i.e., for matter that satisfies all the energy conditions. This happens when H_{ab} has the reverse sign in (48). The energy conditions in the Jordan frame was considered in [75], where the null-energy condition, in its usual form, can appear to be violated by transformations in the conformal frame of the metric.

The decomposition (34)–(37) of the tensor H_{ab} into components parallel to the timelike vector flow W^a and orthogonal to it, is given by the following relationship,

$$\begin{aligned} H^{ab} &= H_{\|}W^aW^b + H_{\perp}h^{ab} + 2H_{\perp}^{(a}W^{b)} + H_{\perp}^{\langle ab\rangle} \\ &= \frac{1}{\phi}[\tilde{\rho}W^aW^b + \tilde{p}h^{ab} + 2\tilde{q}^{(a}W^{b)} + \tilde{\pi}^{ab}], \end{aligned} \quad (49)$$

where $H_{\|}$ and $H_{\perp}$ are scalars, $H_{\perp}^a$ is a vector and $H_{\perp}^{\langle ab\rangle}$ is a projected trace-free symmetric tensor (PSTF). This decomposition permits to translate the condition (48) into

$$\frac{1}{\phi}(\rho+3p) - (H_{\|}+3H_{\perp}) \geq 0. \quad (50)$$

In the latter expression we have used

$$\begin{aligned} H_{\|} = &-\frac{\omega(\phi)}{2\phi^2}(3\dot{\phi}^2 - h^{cd}\nabla_c\phi\nabla_c\phi) \\ &-\frac{1}{\phi}h^{cd}\nabla_c\nabla_d\phi + \lambda(\phi), \end{aligned} \quad (51)$$

$$\begin{aligned} H_{\perp} = &-\frac{\omega(\phi)}{3\phi^2}\left(\frac{\dot{\phi}^2}{2} - \frac{1}{2}h^{cd}\nabla_c\phi\nabla_c\phi\right) \\ &-\frac{1}{2\phi}\left(W^aW^b\nabla_c\nabla_d\phi - \frac{1}{3}h^{cd}\nabla_c\nabla_d\phi\right) - \lambda(\phi). \end{aligned} \quad (52)$$

Clearly, gravity is repulsive or attractive depending on the functions $\omega(\phi)$ and $\lambda(\phi)$. Indeed, Eq. (29) reads

$$W^aW^bR_{ab} - \frac{\omega(\phi)}{\phi^2}\left(\phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}\phi_{;c}\phi^{;c}\right) - \frac{1}{\phi}(\phi_{;ab} - g_{ab}\phi_{;c}{}^{;c}) - \lambda(\phi)g_{ab} = W^aW^b\frac{8\pi}{\phi}\left(T_{ab} - \frac{1}{2}g_{ab}T\right) \geq 0, \quad (53)$$

which amounts to

$$W^aW^b\left[\frac{8\pi}{\phi}\left(T_{ab} - \frac{\omega+1}{2\omega+3}g_{ab}T\right) + \frac{\omega}{\phi^2}\nabla_a\phi\nabla_b\phi + \frac{\nabla_a\nabla_b\phi}{\phi} - \frac{1}{2\phi}\frac{\omega'}{2\omega+3}g_{ab}\nabla_c\nabla^c\phi + g_{ab}\frac{\phi\lambda' - (\omega+1)\lambda}{2\omega+3}\right] \geq 0. \quad (54)$$

Considering a Friedmann-Lemaître-Robertson-Walker (FLRW) universe, we derive the following inequality:

$$\begin{aligned} &\frac{8\pi G}{\phi}\frac{(\omega+3)\rho + 3\omega p}{2\omega+3} + \frac{\lambda}{3} + \frac{\omega}{3}\frac{\dot{\phi}^2}{\phi^2} + \frac{\dot{\omega}}{2(2\omega+3)}\frac{\dot{\phi}}{\phi} \\ &\quad + H\frac{\dot{\phi}}{\phi} \geq 0. \end{aligned} \quad (55)$$

This result shows how the functions $\omega(\phi)$ and $\lambda(\phi)$ define whether gravity is attractive or repulsive in the scalar-tensor cosmological models.

Furthermore, upon a conformal transformation of the theory into the so-called Einstein frame, using $g_{ab} \to \bar{g}_{ab} = (\phi/\phi_*)g_{ab}$, the condition for gravity to be attractive with the redefined Ricci tensor becomes

$$\tilde{R}_{ab}u^au^b = \frac{4\pi}{\phi_*}(\bar{\rho} + 3\bar{p}) + \frac{8\pi}{\phi_*}[\dot{\varphi}^2 - \tilde{V}(\varphi)] \geq 0. \quad (56)$$

Here $\varphi = \int\sqrt{(2\omega+3)/2}\,\mathrm{d}\ln\phi$ is the redefined scalar field, $V(\varphi) = \lambda(\phi(\varphi))/\phi(\varphi)$ is the rescaled potential, $\bar{\rho} = \rho/\phi^2$, $\bar{p} = p/\phi^2$, and ϕ_* is an arbitrary value of ϕ that guarantees, on the one hand, that the conformal factor is dimensionless, and, on the other hand, that it might be related to Newton's gravitational constant G_N by setting $\phi_* = G_N^{-1}$. Despite the fact that the inequality (56) adopts the familiar form found in general relativistic models endowed with a combination of matter and a scalar field, the role of the functions $\omega(\phi)$ and $\lambda(\phi)$ underlies the result because the definitions of φ and $V(\varphi)$ depend on them. Another interesting feature, in the Einstein frame, is that the matter and the scalar field are interacting with each other as revealed by the scalar field equation

$$\ddot{\varphi} + \bar{\theta}\,\dot{\varphi} = -\frac{\partial V(\varphi)}{\partial\varphi} - \frac{\partial\bar{\rho}(\varphi,\bar{a})}{\partial\varphi}. \quad (57)$$

So the dependence on the parameters that underlie, on the one, the shape of the self-interacting potential $V(\varphi)$, and on the other hand, the coupling $\partial_\varphi \bar{\rho} \propto \alpha(\varphi) a^{-3\gamma}$, where $\alpha = (\sqrt{2\omega+3})$, when considering a perfect fluid with $\bar{p} = (\gamma - 1)\bar{\rho}$.

In a cosmological setting, gravity may exhibit a transition from being attractive into becoming repulsive when the interplay between the intervening components is such that those which violate the strong-energy condition become dominating. The typical case is provided when $V(\varphi)$ has a nonvanishing minimum [69].

B. $f(R)$ gravity

The action in this case is

$$S = \frac{1}{16\pi} \int \sqrt{-g} f(R) d^4x + S_M, \tag{58}$$

where R is the Ricci scalar (we refer the reader to [76] for further details). The H_{ab} term includes nonlinear combinations of the curvature invariants built from the Riemann and Ricci tensors as well as from derivatives of these tensors, and the couplings $g_1(\Psi^i) = F(R) = f'(R)$ and $g_2(\Psi^i) = 1$, where the prime is the derivative with respect to R. In fact, the gravitational field equation is given by

$$F(R)G_{ab} + \frac{1}{2}[RF(R) - f(R)]g_{ab} - \nabla_a \nabla_b F(R) + g_{ab}\Box F(R) = 8\pi G T_{ab}, \tag{59}$$

which can be recast as

$$G_{ab} = 8\pi G \left(\frac{T_{ab}}{F(R)}\right) - \frac{1}{F(R)}\left[\frac{1}{2}(RF(R) - f(R))g_{ab} - \nabla_a \nabla_b F(R) + g_{ab}\Box F(R)\right], \tag{60}$$

so that we identify

$$H_{ab} = \frac{1}{F(R)}\left\{\frac{1}{2}[RF(R) - f(R)]g_{ab} - \nabla_a \nabla_b F(R) + g_{ab}\Box F(R)\right\}. \tag{61}$$

Note that, as before, ∇_a is the covariant derivative operator associated with g_{ab}, $\Box \equiv g^{ab}\nabla_a \nabla_b$ is the covariant d'Alembertian, and T^M_{ab} is the contribution to the stress-energy tensor from ordinary matter. Clearly, the above considerations hold completely, and gravity is attractive or repulsive depending on the form of $f(R)$.

In the present case we have

$$H_\parallel = -\frac{1}{F}\left[\frac{1}{2}(RF - f) - h^{cd}\nabla_c \nabla_d F\right], \tag{62}$$

$$H_\perp = \frac{1}{F}\left[\frac{1}{2}(RF - f) - \frac{1}{3}h^{cd}\nabla_c \nabla_c F + \Box F\right], \tag{63}$$

so that gravity is attractive when

$$8\pi G(\rho + 3p) \geq [(RF - f) - 2h^{cd}\nabla_c \nabla_d F + 3\Box F]. \tag{64}$$

Note, however, that this latter condition is still not a condition on any initial data or on matter $T_{\mu\nu}$. Indeed, the higher derivatives may still be eliminated using the equations of motion; thus, it is not an energy condition.

This condition reduces to the usual $(\rho + 3p) \geq 0$ when $f \propto R$ and, hence, GR is recovered. More importantly, it reveals how the nonlinear terms in the action induce attractive or repulsive effects. If there were no matter, i.e., in a vacuum setting, gravity would become repulsive if

$$(RF - f) - 2h^{cd}\nabla_c \nabla_d F + 3\Box F \leq 0. \tag{65}$$

We refer the reader to [77,78] for considerations on the nonattractive character of gravity in $f(R)$ theories.

If instead of the strong-energy condition we evaluate the null-energy condition $R_{ab}k^a k^b \geq 0$, there is once again a considerable simplification of the equations, and we obtain focusing of light bundles when

$$T_{ab}k^a k^b + k^a k^b \frac{\nabla_a \nabla_b F}{F(R)} \geq 0. \tag{66}$$

This is a kind of Poisson-like inequality which effectively yields the lensing effect.

We emphasize that in a cosmological setting, the above considerations are particularly important, as in GR the presence of dark energy implies the violation of specific energy conditions. However, in the generalized approach outlined above, there is no violation but just a reinterpretation of the further degrees of freedom emerging from dynamics.

For instance, consider a flat FRW metric given by $ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$, so that Eq. (60) immediately yields the following the field equations:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3F(R)}\left\{\frac{1}{2}[f(R) - RF(R)] - 3\left(\frac{\dot{a}}{a}\right)\dot{R}F'(R)\right\} = \frac{\kappa}{3}\rho, \tag{67}$$

$$\left(\frac{\ddot{a}}{a}\right) + \frac{1}{2F(R)}\left\{\frac{\dot{a}}{a}\dot{R}F'(R) + \ddot{R}F'(R) + \dot{R}^2 F''(R) - \frac{1}{3}[f(R) - RF(R)]\right\} = -\frac{\kappa}{6}(\rho + 3p). \tag{68}$$

Indeed, in the literature, these field equations are usually written as effective Friedman equations, in the following form,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3}\rho_{\rm tot}, \tag{69}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{\kappa}{6}(\rho_{\rm tot} + 3p_{\rm tot}), \tag{70}$$

where $\rho_{\rm tot} = \rho + \rho_{(c)}$ and $p_{\rm tot} = p + p_{(c)}$, and the quantities $\rho_{(c)}$ and $p_{(c)}$, are defined as

$$\rho_{(c)} = \frac{1}{\kappa F(R)}\left\{\frac{1}{2}[f(R) - RF(R)] - 3\left(\frac{\dot{a}}{a}\right)\dot{R}F'(R)\right\},$$

$$p_{(c)} = \frac{1}{\kappa F(R)}\left\{2\left(\frac{\dot{a}}{a}\right)\dot{R}F'(R) + \ddot{R}F'(R) + \dot{R}^2F''(R) - \frac{1}{2}[f(R) - RF(R)]\right\},$$

respectively. However, one should always bear in mind that these quantities have a geometrical origin and should not be interpreted as a fluid.

Now, from Eq. (70), it is transparent that an accelerated expansion can be obtained by imposing the condition $\rho_{\rm tot} + 3p_{\rm tot} < 0$. Note that, in principle, one may impose that normal matter obeys all of the energy conditions, and the acceleration $\ddot{a} \geq 0$ is attained by considering an appropriate functional form for $f(R)$. For simplicity, consider a vacuum, $\rho = p = 0$, so that the energy conditions are borderline satisfied. Now appropriately defining a parameter $\omega_{\rm eff} = p_{(c)}/\rho_{(c)}$, one may impose a function $f(R)$. For instance, consider the model $f(R) = R - \mu^{2(n+1)}/R^n$ analyzed in [79]. By choosing a generic power law for the scale factor, the parameter can be written as

$$\omega_{\rm eff} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}, \tag{71}$$

and the desired value of $\omega_{\rm eff} < -1/3$ may be attained, by appropriately choosing the value of the parameter n.

We emphasize that the message that one obtains from this analysis is precisely that in the generalized approach outlined in this work, there are no violation of the GR energy conditions, but just a reinterpretation of the further degrees of freedom emerging from the dynamics.

V. SUMMARY AND DISCUSSION

In this work, we have considered the further degrees of freedom related to curvature invariants and scalar fields in extended theories of gravity (ETG). These new degrees of freedom can be recast as *effective fluids* that carry different meanings with respect to the standard matter fluids generally adopted as sources of the field equations. It is, thus, somewhat misleading to apply the standard general relativistic energy conditions to this effective energy momentum, as the latter contains the matter content and geometrical quantities which arise from the particular ETG considered. It can be shown, as in Sec. III, that the further dynamical content of ETG can be summed up into two coupling functions g_1 and g_2 and an additional tensor H_{ab} where all the geometrical modifications are present. Clearly, GR is immediately recovered as soon as $g_1 = g_2 = 1$ and $H_{ab} = 0$. Here we explored these features in cases with the contracted Bianchi identities with diffeomorphism invariance and in cases with generalized explicit curvature-matter couplings, which imply the nonconservation of the energy-momentum tensor. Furthermore, we applied the analysis to specific ETGs, such as scalar-tensor gravity and $f(R)$ gravity. The main outcomes are that matter can exhibit further thermodynamical features and gravity can retain its attractive character in the presence of large negative pressures. On the other hand, repulsive gravity may occur for standard matter.

As a general result, the fact that further degrees of freedom, related to ETG, can be handled under the standard of effective fluids allows us, in principle, to set consistent energy conditions for large classes of theories. In this sense, the formulation of the Cauchy problem can be considered a standard feature for several theories of gravity. From a cosmological point of view, these considerations are crucial. For example, the presence of dark energy can be considered a straightforward violation of energy conditions in the standard sense of GR. In our generalized approach, there is no violation but just a reinterpretation of the further degrees of freedom emerging from dynamics.

Furthermore, a few considerations in the context of scalar-tensor theories are in order. Note that in the Einstein frame one verifies that the energy conditions are satisfied but may be violated in the Jordan frame [71,80,81]. This fact does not eliminate the presence of singularities when both frames are considered equivalent (see below for a discussion on the latter issue) [82]. Thus, in order to avoid these ambiguities, one may wonder that due to the fact that the energy conditions essentially hold in relativity, why not restrict oneself to the Einstein frame formulation (31) and not bother with the geometrical or matter nature of the appropriate quantities? However, it is important to mention that in some specific situations it is also possible that the weak-energy condition is satisfied in the Jordan frame [80] and, thus, evades the problems mentioned above. In addition to this, there are situations where it is useful to work in the Jordan frame. For instance, if one uses the equivalence principle (EP) as a guide in constructing one's theory, then it is useful to work in the Jordan frame, as here the EP is satisfied, and the latter is violated in the Einstein due to the fifth force arising as a result of the anomalous coupling of the scalar field to matter. Nevertheless, one may argue that this may be misleading as the EP could indeed be violated in nature, provided that the violations are extremely small in order to evade detection from current measurements and, thus, serve to place stringent constraints on theories that imply the nonconservation of the energy-momentum tensor and that consequently manifest nongeodesic motion. One may also mention that if one only restricts attention to the Einstein frame, one may also lose sight of the original motivations and modifications of gravity in the geometrical sector. Indeed, the conformal transformation mixes the geometric and matter degrees of freedom, which results in many interpretational ambiguities [83]. Furthermore, note that Dicke's argument is purely classical and, in this respect, at the quantum level the equivalence of both frames is not proven. In fact, when the metric is quantized, one can find inequivalent quantum theories [84]. In addition to this, considering the semiclassical regime, in which gravity is classical and the matter fields are quantized, one would also expect that the conformal frames are inequivalent, and we refer the reader to [82] (and references therein) for more details.

The viewpoint that the Einstein and Jordan frames are physically equivalent is correct and consistent and can be traced back to Dicke's original paper [85], where the conformal transformation technique was introduced. Indeed, in the spirit of Dicke's paper, both conformal frames are equivalent provided that in the Einstein frame the units of mass, time and space scale as appropriate powers of the scalar field and are, thus, varying. More specifically, physics must be conformally invariant and the symmetry group of gravity should be enlarged to incorporate conformal conformations, in addition to the group of diffeomorphisms [80]. However, it is common practice in the literature to consider that in the Einstein frame, measurements are referred to in a rigid system of units, instead of units varying with the conformal factor and, consequently, resulting in the nonequivalence of the Jordan and the Einstein frames [80]. Although this approach is perfectly legitimate from a mathematical point of view, one should keep in mind that both theories are physically inequivalent, for instance, when one considers cosmological or black hole solutions. The question then becomes, "which of the two conformal frames is physical?" In the context of the energy conditions, these are satisfied in the Einstein frame and violated in the Jordan frame. For instance, in this context, the violations of the weak-energy condition in the Jordan frame are also responsible for the violation of the second law of black hole thermodynamics [80] (and references therein). In fact, if the weak-energy condition is violated, the Hawking-Penrose singularity theorems [65] also do not apply in the original Jordan frame. In order to circumvent this difficulty, one may consider the approach outlined in [75], in that the second law of black hole thermodynamics is taken as fundamental, and then one modifies the null-energy condition in a given theory of gravity to ensure that the classical black hole solution has an entropy that increases with time. This approach seems appealing as the null-energy condition does not seem to rest on any fundamental principle of physics, unlike the second law of black hole thermodynamics.

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