Black hole initial data without elliptic equations

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We explore whether a new method to solve the constraints of Einstein's equations can be applied to provide initial data for black holes. We show that this method, which does not involve elliptic equations, can be successfully applied to a nonlinear perturbation of a Schwarzschild black hole by establishing the well-posedness of the resulting constraint problem. We discuss its possible generalization to the boosted, spinning multiple black hole problem.

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I. INTRODUCTION

The prescription of physically realistic initial data for black holes is a crucial ingredient to the simulation of the inspiral and merger of binary black holes and the computation or the radiated gravitational waveform. Initialization of the simulation is a challenging problem due to the nonlinear constraint equations that the data must satisfy. The traditional solution expresses the constraints in the form of elliptic equations. Here we consider a radically new method of solving the constraints which does not require elliptic solvers [1]. We show, at least for nonlinear perturbations of Schwarzschild black hole data, that the Hamiltonian and momentum constraints lead to a well-posed strongly hyperbolic problem whose solutions satisfy the full constraint system. The possibility of extending this approach to binary black holes offers a simple alternative way to provide boundary conditions for the initialization problem that might prove to be more physically realistic.

The inspiral and merger of a binary black hole is expected to be the strongest possible source of gravitational radiation for the emerging field of gravitational wave astronomy. The details of the gravitational waveform supplied by numerical simulation is a key tool to enhance detection of the gravitational signal and interpret its scientific content. It is thus important that the initial data do not introduce spurious effects into the waveform. Such "junk radiation" is common to all current methods for supplying initial data and appears early in the simulation as a high frequency component of the waveform. This can be a troublesome feature with regard to matching the waveform in the nonlinear regime spanned by the simulation to the post-Newtonian chirp waveform provided by perturbation theory. The initial parameters governing the black hole spins, mass ratio, and ellipticity of the binary orbit have to be adjusted to include the effect of this transitory period. As a result, it becomes difficult to match exactly to the parameters governing the post-Newtonian orbit. In addition, although the high frequency component of the junk radiation appears to dissipate after some early transitory period, there is no quantitative measure of its low frequency component which might affect the ensuing waveform.

All initialization methods presently in use reduce the constraint problem to a system of elliptic equations, which require boundary conditions at inner boundaries in the strong field region surrounding the singularities inside the black holes, as well as at an outer boundary surrounding the system. The new method we consider here only requires data on the outer boundary, which is in the weak field region where the choice of boundary data can be guided by asymptotic flatness. The constraints are then satisfied by an inward "evolution" of the hyperbolic system along radial streamlines.

The initial data for solving Einstein's equations consist of a pair of symmetric tensor fields (h_{ij}, K_{ij}) on a smooth three-dimensional manifold Σ, where h_{ij} is a Riemannian metric and K_{ij} is interpreted as the extrinsic curvature of Σ after its embedding in a 4-dimensional space-time. The constraints on a vacuum solution (see, e.g., Refs. [2,3]) consist of

$$ {}^{(3)}R + (K^j{}_j)^2 - K_{ij}K^{ij} = 0, \tag{1.1} $$

$$ D_j K^j{}_i - D_i K^j{}_j = 0, \tag{1.2} $$

where ${}^{(3)}R$ and D_i denote the scalar curvature and the covariant derivative operator associated with h_{ij}, respectively.

The standard approach to solving the constraints is based upon the conformal method, introduced by Lichnerowicz [4] to recast the Hamiltonian constraint (1.1) as an elliptic equation and later extended by York [5,6] to reduce the momentum constraint (1.2) also to an elliptic system. For a review of the historic implementation of this method in numerical relativity, see [7].

A major obstacle in prescribing black hole initial data is the presence of a singularity inside the black hole. The

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initial strategy for handling the singularity was the excision of the singular region inside the black hole [8]. In this case, an artificial inner boundary condition for the elliptic system is posed on boundaries inside the apparent horizons surrounding the individual black holes. Other strategies have since been proposed. One is the puncture method in which the initial hypersurface extends though a wormhole to an internal asymptotically flat spatial infinity, which is then treated by conformal compactification [9]. Here the freedom in the choice of conformal factor governing the compactification enters as an effective boundary condition. In addition, it is known that the puncture quickly changes its nature. In fact, early attempts to simulate binary black holes failed until it was realized that the punctures must be allowed to move. Studies of this feature in the case of a single black hole revealed that the puncture quickly transits from the internal spatial infinity to an internal timelike infinity [10]. This realization has given rise to the trumpet version of initial data, in which the initial Cauchy hypersurface extends to an internal timelike infinity with asymptotically finite surface area [10,11]. Trumpet data offer a promising alternative to puncture data but their merits have not yet been extensively explored in binary black hole simulations [12].

Coupled to these techniques for avoiding singularities is the choice of initial time slice. For example, there are many ways to prescribe Schwarzschild initial data depending, say, upon whether the initial Cauchy hypersurface is time symmetric or horizon penetrating. Here we will focus on initial data in Kerr-Schild form [13,14], which for the Schwarzschild case corresponds to ingoing Eddington-Finklestein coordinates, which extend from spatial infinity to the singularity and penetrate the horizon. The new approach to solving the constraints that we consider becomes degenerate for a time symmetric initial slice, whose extrinsic curvature vanishes. However, time symmetric space-times contain as much ingoing as outgoing gravitational waves, so they are not the appropriate physical models for studying binary waveforms. Although our focus here is on data in Kerr-Schild form, we do not wish to imply that this approach would not work for puncture or trumpet data.

A very attractive feature of Kerr-Schild initial data is that it provides a preferred Minkowski background to construct boosted black holes by means of a Lorentz transformation. Two independent ways of prescribing Kerr-Schild initial data have been proposed. In one version, the 4-dimensional aspect of the Kerr-Schild ansatz is preserved as much as possible [15]. This leads to a workable scheme for superimposing nonspinning black holes but the generalization to the spinning case remains problematic. In the other case, the Kerr-Schild ansatz is loosened to a 3-dimensional version that allows superposition of multiple spinning black holes [16]. This has been implemented to provide data for boosted, spinning binary black holes and plays an important role in current simulations [17].

There are several variants to the new method of solving the constraints proposed in [1,18–20], depending upon which components of the initial data are assigned freely. They all avoid elliptic equations. Here we apply the simplest of these variants to the initial data problem for black holes. In this variant, the Hamiltonian and momentum constraints constitute a strongly hyperbolic system which only requires data on a 2-surface surrounding the black holes.

In Sec. II, we review this new approach. In Sec. III, we show that the requirements for well-posedness of the underlying algebraic-hyperbolic constraint problem are satisfied by a Schwarzschild black hole described in Kerr-Schild form. In Sec. IV, we present an explicit proof that nonlinear perturbations of Schwarzschild black hole data in Kerr-Schild form lead to a well-posed, strongly hyperbolic problem.

In Sec. V, we conclude with a discussion of the possibility of extending this approach to general data for a system of boosted, spinning multiple black holes. We show how the initial metric data for multiple black holes can be freely prescribed in 4-dimensional superimposed Kerr-Schild form for the individual boosted, spinning black holes. Two pieces of extrinsic curvature data, which represent the two gravitational degrees of freedom, can also be freely prescribed by superimposing the individual black hole data. The remaining extrinsic curvature data are then determined by the algebraic-hyperbolic constraint system. In a linear theory, the superposition of such nonradiative data would lead to a nonradiative solution. This suggests that this new method may offer an alternative approach to suppressing junk radiation and to controlling the effect of initial data on a binary orbit. However, due to the nonlinearity of Einstein's equations, there is no guarantee that, in the strong field region between the individual black holes, these superimposed free data do not introduce spurious radiation. A completely analytic resolution of these issues does not seem possible. A major motivation for this paper is to encourage the numerical experimentation necessary to explore the merit and feasibility of this new approach.

II. A NEW APPROACH TO THE CONSTRAINTS

We assume that the topology of Σ allows a smooth foliation by a one-parameter family of homologous 2-surfaces. In the application to black hole initial data, we assume for simplicity a foliation $\mathcal{S}_\rho$ by topological spheres described by the level surfaces $\rho = \text{const}$ of a smooth function.

Choose now a vector field ρ^i on Σ such that $\rho^i \partial_i \rho = 1$. Then the unit normal $\hat{n}^i$ to $\mathcal{S}_\rho$ has the decomposition

$$\hat{n}^i = \hat{N}^{-1}[\rho^i - \hat{N}^i], \tag{2.1}$$

where the "lapse" $\hat{N}$ and "shift" $\hat{N}^i$ of the vector field ρ^i are determined by $\hat{n}_i = \hat{N}\partial_i\rho$ and $\hat{N}^i = \hat{\gamma}^i{}_j\rho^j$, with $\hat{\gamma}^i{}_j = \delta^i{}_j - \hat{n}^i\hat{n}_j$.

The 3-metric h_{ij} on Σ then has the $2+1$ decomposition

$$h_{ij} = \hat{\gamma}_{ij} + \hat{n}_i \hat{n}_j, \tag{2.2}$$

where $\hat{\gamma}_{ij}$ is the metric induced on the surfaces $\mathcal{S}_\rho$. The extrinsic curvature $\hat{K}_{ij}$ of $\mathcal{S}_\rho$ is given by

$$\hat{K}_{ij} = \hat{\gamma}^l{}_i D_l \hat{n}_j = \frac{1}{2}\mathcal{L}_{\hat{n}}\hat{\gamma}_{ij}. \tag{2.3}$$

The extrinsic curvature K_{ij} of Σ, which forms part of the initial data, has the decomposition

$$K_{ij} = \boldsymbol{\kappa}\, \hat{n}_i \hat{n}_j + [\hat{n}_i \mathbf{k}_j + \hat{n}_j \mathbf{k}_i] + \mathbf{K}_{ij}, \tag{2.4}$$

where $\boldsymbol{\kappa} = \hat{n}^k \hat{n}^l K_{kl}$, $\mathbf{k}_i = \hat{\gamma}^k{}_i \hat{n}^l K_{kl}$, and $\mathbf{K}_{ij} = \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j K_{kl}$. Here we use boldfaced symbols to indicate tensor fields tangent to $\mathcal{S}_\rho$. In addition, we shall denote the trace and trace-free parts of $\hat{K}_{ij}$ and $\mathbf{K}_{ij}$ by $\hat{K}^l{}_l = \hat{\gamma}^{kl}\hat{K}_{kl}$, $\mathbf{K}^l{}_l = \hat{\gamma}^{kl}\mathbf{K}_{kl}$, $\overset{\circ}{\hat{K}}_{ij} = \hat{K}_{ij} - \frac{1}{2}\hat{\gamma}_{ij}\hat{K}^l{}_l$, and $\overset{\circ}{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \frac{1}{2}\hat{\gamma}_{ij}\mathbf{K}^l{}_l$, respectively.

By replacing the initial data set (h_{ij}, K_{ij}) by the seven fields $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \overset{\circ}{\mathbf{K}}_{ij}, \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l)$, the Hamiltonian and momentum constraints (1.1) and (1.2) can be expressed as [1] (see also [18–20])

$$\mathcal{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \hat{D}^l \mathbf{k}_l + 2\dot{\hat{n}}^l \mathbf{k}_l - \left[\boldsymbol{\kappa} - \frac{1}{2}(\mathbf{K}^l{}_l)\right](\hat{K}^l{}_l) + \overset{\circ}{\mathbf{K}}_{kl}\overset{\circ}{\hat{K}}{}^{kl} = 0, \tag{2.5}$$

$$\begin{aligned}\mathcal{L}_{\hat{n}}\mathbf{k}_i &+ (\mathbf{K}^l{}_l)^{-1}[\boldsymbol{\kappa}\hat{D}_i(\mathbf{K}^l{}_l) - 2\mathbf{k}^l \hat{D}_i \mathbf{k}_l] \\ &+ (2\mathbf{K}^l{}_l)^{-1}\hat{D}_i[{}^{(3)}R - \overset{\circ}{\mathbf{K}}_{kl}\overset{\circ}{\mathbf{K}}{}^{kl}] \\ &+ (\hat{K}^l{}_l)\mathbf{k}_i + \left[\boldsymbol{\kappa} - \frac{1}{2}(\mathbf{K}^l{}_l)\right]\dot{\hat{n}}_i - \dot{\hat{n}}^l \overset{\circ}{\mathbf{K}}_{li} + \hat{D}^l \overset{\circ}{\mathbf{K}}_{li} = 0,\end{aligned} \tag{2.6}$$

where $\boldsymbol{\kappa}$ is determined by

$$\boldsymbol{\kappa} = (2\mathbf{K}^l{}_l)^{-1}\left[\overset{\circ}{\mathbf{K}}_{kl}\overset{\circ}{\mathbf{K}}{}^{kl} + 2\mathbf{k}^l \mathbf{k}_l - \frac{1}{2}(\mathbf{K}^l{}_l)^2 - {}^{(3)}R\right], \tag{2.7}$$

$\hat{D}_i$ and $\hat{R}$ denote the covariant derivative operator and scalar curvature associated with $\hat{\gamma}_{ij}$, respectively, and $\dot{\hat{n}}_k = \hat{n}^l D_l \hat{n}_k = -\hat{D}_k(\ln \hat{N})$. Here (2.7) provides an algebraic solution to the Hamiltonian constraint (1.1) (for more details see [1]). The four quantities $(\boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l)$ are subject to the constraints, whereas the remaining eight variables $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \overset{\circ}{\mathbf{K}}_{ij})$ are freely specifiable throughout Σ. Here $\overset{\circ}{\mathbf{K}}_{ij}$ encodes the two free gravitational degrees of freedom.

Given the free data $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \overset{\circ}{\mathbf{K}}_{ij})$, Eqs. (2.6)–(2.7) were shown to comprise a first order strongly hyperbolic system for the vector valued variable $(\mathbf{K}^l{}_l, \mathbf{k}_i)$ provided $\boldsymbol{\kappa}$ and $\mathbf{K}^l{}_l$ are of opposite sign,

$$\boldsymbol{\kappa}\mathbf{K}^l{}_l = -C^2, \quad C \neq 0. \tag{2.8}$$

It was also verified in [1] that, given the values of $(\mathbf{k}_i, \mathbf{K}^l{}_l)$ on some "initial" surface $\mathcal{S}_0$ satisfying (2.8), solutions to the nonlinear system (2.5)–(2.7) exist (at least locally) in a neighborhood of $\mathcal{S}_0$, and that the fields (h_{ij}, K_{ij}) built up from these solutions satisfy the full constraint system (1.1)–(1.2).

III. THE FREE AND CONSTRAINED SCHWARZSCHILD DATA

The successful application of this new approach to the constraint problem depends upon a judicious choice of gauge, determined by the lapse of the initial Cauchy hypersurface Σ, and a judicious choice of foliation $\mathcal{S}_\rho$. We begin by considering data in Kerr-Schild form, in which the space-time metric has the form

$$\begin{aligned} g_{ab} &= \eta_{ab} + 2H\ell_a \ell_b, \\ g^{ab} &= \eta^{ab} - 2H\ell^a \ell^b, \end{aligned} \tag{3.1}$$

where H is a smooth function (except at singularities) on $\mathbb{R}^4$ and ℓ_a is null with respect to both g_{ab} and an implicit background Minkowski metric η_{ab}. In inertial coordinates (t, x^i) adapted to η_{ab},

$$\begin{aligned} g_{ab}dx^a dx^b = (-1 + 2H\ell_t{}^2)dt^2 &+ 4H\ell_t \ell_i dt dx^i \\ &+ (\delta_{ij} + 2H\ell_i \ell_j)dx^i dx^j, \end{aligned} \tag{3.2}$$

where $\ell^a = g^{ab}\ell_b = \eta^{ab}\ell_b$ and $g^{ab}\ell_a\ell_b = \eta^{ab}\ell_a\ell_b = -(\ell_t)^2 + \ell^i\ell_i = 0$. The Kerr-Schild metrics also satisfy the background geodesic condition

$$\eta^{bc}\ell_c \partial_b \ell_a = 0 \tag{3.3}$$

and wave equation

$$\eta^{ab}\partial_a \partial_b H = 0. \tag{3.4}$$

We can relate the Kerr-Schild metric to the $3+1$ decomposition of the space-time metric

$$g_{ab} = h_{ab} - n_a n_b, \tag{3.5}$$

where n^a is the future directed unit normal to the $t = \text{const}$ hypersurfaces. Choose a time evolution field t^a satisfying $t^a \partial_a t = 1$. Then n^a has the decomposition

$$n^a = N^{-1}(t^a - N^a), \tag{3.6}$$

where N and N^a denote the space-time lapse and shift, determined by

$$N = -(t^e n_e), \quad n_a = -N\partial_a t, \quad \text{and} \quad N^a = h^a{}_e t^e, \tag{3.7}$$

respectively.

In the Kerr-Schild space-time coordinates (t, x^i), the metric has components

$$g_{\alpha\beta} = \begin{pmatrix} -N^2 + N_i N^i & N_i \\ N_j & h_{ij} \end{pmatrix}. \tag{3.8}$$

It follows that

$$h_{ij} = \delta_{ij} + 2H\ell_i\ell_j, \qquad h^{ij} = \delta^{ij} - \frac{2H\ell^i\ell^j}{1 + 2H\ell_t^2}, \tag{3.9}$$

$$N = \frac{1}{\sqrt{1 + 2H\ell_t^2}}, \tag{3.10}$$

$$N_i = 2H\ell_t\ell_i, \qquad N^i = 2HN^2\ell_t\ell^i. \tag{3.11}$$

A direct calculation of the extrinsic curvature

$$K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij} = (2N)^{-1}[\partial_t h_{ij} - (D_i N_j + D_j N_i)] \tag{3.12}$$

gives

$$N^{-1}K_{ij} = -\ell_t[\partial_i(H\ell_j) + \partial_j(H\ell_i)] + N^{-2}\partial_t(H\ell_i\ell_j) + 2H\ell^t\ell^k\partial_k(H\ell_i\ell_j) - H(\ell_i\partial_j\ell_t + \ell_j\partial_i\ell_t). \tag{3.13}$$

For a Kerr space-time

$$H = \frac{rM}{r^2 + a^2\cos^2\theta}, \tag{3.14}$$

where the Boyer-Lindquist radial coordinate r is related to the Cartesian inertial spatial coordinates $x^i = (x_1, x_2, x_3)$ according to

$$r^2 = \frac{1}{2}[(\rho^2 - a^2) + \sqrt{(\rho^2 - a^2)^2 + 4a^2x_3^2}], \tag{3.15}$$

with

$$\rho^2 = x_1^2 + x_2^2 + x_3^2 \tag{3.16}$$

and

$$\ell_a = \left(1, \frac{rx_1 + ax_2}{r^2 + a^2}, \frac{rx_2 - ax_1}{r^2 + a^2}, \frac{x_3}{r}\right). \tag{3.17}$$

As H and ℓ_a are t-independent and $\ell_t = 1$, the extrinsic curvature (3.13) simplifies to

$$K_{ij} = -\ell_t N[\partial_i(H\ell_j) + \partial_j(H\ell_i) + 2H\ell_i\ell_j\ell^k\partial_k H].$$

Formally, for the purpose of applying the approach in Sec. II to a generic inspiral and merger, it would be sufficient to show that the required sign condition (2.8) holds for a boosted Kerr black hole. Here we restrict our investigation to the Schwarzschild case, where the choice of foliation $\mathcal{S}_\rho$ is guided by spherical symmetry and the algebraic simplicity allows a clear exposition of the approach.

For a Schwarzschild black hole, the spin parameter $a = 0$ and the Kerr-Schild form of the metric simplifies to

$$H = \frac{M}{r}, \qquad \ell_i = \frac{x_i}{r} = \partial_i r, \qquad r^2 = \delta^{ij}x_1x_j, \tag{3.18}$$

with lapse

$$N = (1 + 2H)^{-1/2} \tag{3.19}$$

and 3-metric

$$h_{ij} = \delta_{ij} + 2H\ell_i\ell_j. \tag{3.20}$$

(Here $-\ell^a$ is a future directed ingoing null vector, which corresponds to the convention for ingoing Eddington-Finklestein coordinates.) Thus

$$\partial_i H = -\frac{M}{r^3}x_i, \qquad \partial_j(H\ell_i) = \frac{M}{r^4}[r^2\delta_{ij} - 2x_ix_j] \tag{3.21}$$

and (3.18) reduces to

$$K_{ij} = -\frac{2M}{r^2\sqrt{1 + 2H}}(\delta_{ij} - [2 + H]\ell_i\ell_j). \tag{3.22}$$

We choose the foliation $\mathcal{S}_\rho$ by setting $\rho = r$, with $\rho^i = \ell^i$, corresponding to the "spatial" lapse and shift

$$\hat{N} = \sqrt{1 + 2H}, \qquad \hat{N}^i = 0, \tag{3.23}$$

unit normal

$$\hat{n}_i = \sqrt{1 + 2H}\ell_i, \quad \hat{n}^i = h^{ij}\hat{n}_j = \frac{1}{\sqrt{1 + 2H}}\ell^i, \tag{3.24}$$

and intrinsic 2-metric

$$\hat{\gamma}_{ij} = h_{ij} - \hat{n}_i\hat{n}_j = \delta_{ij} - \ell_i\ell_j, \qquad \hat{\gamma}^{ij} = \delta^{ij} - \ell^i\ell^j. \tag{3.25}$$

A straightforward calculation gives the extrinsic curvature components of Σ,

$$\boldsymbol{\kappa} = \hat{n}^k\hat{n}^l K_{kl} = \frac{2M(1 + H)}{r^2(1 + 2H)^{3/2}}, \tag{3.26}$$

$$\mathbf{k}_i = \hat{\gamma}^k{}_i \hat{n}^l K_{kl} = 0, \tag{3.27}$$

$$\mathbf{K}_{ij} = \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j K_{kl} = -\frac{2M}{r^2\sqrt{1+2H}} \hat{\gamma}_{ij}, \tag{3.28}$$

$$\mathbf{K}^l{}_l = \hat{\gamma}^{kl} \mathbf{K}_{kl} = -\frac{4M}{r^2\sqrt{1+2H}}, \tag{3.29}$$

and

$$\overset{\circ}{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \frac{1}{2}\hat{\gamma}_{ij}\mathbf{K}^l{}_l = 0. \tag{3.30}$$

Note that $\boldsymbol{\kappa}$ and $\mathbf{K}^l{}_l$ are globally nonvanishing and have opposite sign, in agreement with the condition (2.8) for strong hyperbolicity.

From (2.3) along with

$$\begin{aligned}
\mathcal{L}_{\hat{n}}\hat{\gamma}_{ij} &= \hat{n}^k \partial_k \hat{\gamma}_{ij} + \hat{\gamma}_{kj}(\partial_i \hat{n}^k) + \hat{\gamma}_{ik}(\partial_j \hat{n}^k) \\
&= \frac{1}{\sqrt{1+2H}} \frac{x^k}{r} \partial_k \left[-\frac{x_i x_j}{r^2} \right] \\
&\quad + \left(\delta_{kj} - \frac{x_k x_j}{r^2} \right) \partial_i \left[\frac{1}{\sqrt{1+2H}} \frac{x^k}{r} \right] \\
&\quad + \left(\delta_{ik} - \frac{x_i x_k}{r^2} \right) \partial_j \left[\frac{1}{\sqrt{1+2H}} \frac{x^k}{r} \right] \\
&= \frac{2}{r\sqrt{1+\frac{2M}{r}}} \left(\delta_{kj} - \frac{x_k x_j}{r^2} \right),
\end{aligned} \tag{3.31}$$

the extrinsic curvature of the $\rho = r = \text{const}$ foliated surfaces is given by

$$\hat{K}_{ij} = \frac{1}{r\sqrt{1+2H}} \hat{\gamma}_{ij}, \tag{3.32}$$

so it follows that

$$\hat{K}^l{}_l = \hat{\gamma}^{kl} \hat{K}_{kl} = \frac{2}{r\sqrt{1+2H}} \tag{3.33}$$

and

$$\overset{\circ}{\hat{K}}_{ij} = \hat{K}_{ij} - \frac{1}{2}\hat{\gamma}_{ij}\hat{K}^l{}_l = 0. \tag{3.34}$$

IV. NONLINEAR PERTURBATIONS OF A SCHWARZSCHILD BLACK HOLE

Here we investigate nonlinear perturbations of the Kerr-Schild initial data for a Schwarzschild black hole. In doing so, we simplify the discussion by assigning Schwarzschild values to the freely specifiable variables $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \overset{\circ}{\mathbf{K}}_{ij})$. As a result, the initial 3-metric h_{ij} retains its Schwarzschild value and, in particular, $\overset{\circ}{\mathbf{K}}_{ij} = 0$ and $\hat{N}$ and ${}^{(3)}R$ have no angular dependence. For a more general perturbation, $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \overset{\circ}{\mathbf{K}}_{ij})$ would enter as explicit terms in the resulting system for $(\boldsymbol{\kappa}, \mathbf{K}^l{}_l, \mathbf{k}_i,)$.

In this setting, (2.5)–(2.6) reduce to

$$\mathcal{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \hat{D}^l \mathbf{k}_l - \left[\boldsymbol{\kappa} - \frac{1}{2}(\mathbf{K}^l{}_l) \right] (\hat{K}^l{}_l) = 0, \tag{4.1}$$

$$\mathcal{L}_{\hat{n}}\mathbf{k}_i + (\mathbf{K}^l{}_l)^{-1}[\boldsymbol{\kappa}\hat{D}_i(\mathbf{K}^l{}_l) - 2\mathbf{k}^l \hat{D}_i \mathbf{k}_l] + (\hat{K}^l{}_l)\mathbf{k}_i = 0, \tag{4.2}$$

where $\boldsymbol{\kappa}$, determined by (2.7), reduces to

$$\boldsymbol{\kappa} = (2\mathbf{K}^l{}_l)^{-1} \left[2\mathbf{k}^l \mathbf{k}_l - \frac{1}{2}(\mathbf{K}^l{}_l)^2 - {}^{(3)}R \right]. \tag{4.3}$$

It is easy to check that these equations hold for a Schwarzschild solution, for which ${}^{(3)}R = \frac{8M^2}{r^4(1+2H)^2}$, $\hat{n}^i \partial_i = \frac{1}{\sqrt{1+2H}} \partial_r$, $\mathbf{k}_i = 0$, and neither $\mathbf{K}^l{}_l$ nor $\boldsymbol{\kappa}$ have angular dependence.

In spherical coordinates $x^i = (r, x^A)$, $x^A = (\theta, \phi)$,

$$\hat{\gamma}_{ij} dx^i dx^j = r^2 q_{AB} dx^A dx^B, \tag{4.4}$$

where q_{AB} is the unit sphere metric. Then (4.1)–(4.2) become

$$\frac{1}{\sqrt{1+2H}} \partial_r \mathbf{K}^l{}_l - \hat{D}^B \mathbf{k}_B - \left[\boldsymbol{\kappa} - \frac{1}{2}(\mathbf{K}^l{}_l) \right] (\hat{K}^l{}_l) = 0, \tag{4.5}$$

$$\frac{1}{\sqrt{1+2H}} \partial_r \mathbf{k}_A + (\mathbf{K}^l{}_l)^{-1}[\boldsymbol{\kappa}\partial_A(\mathbf{K}^l{}_l) - 2\mathbf{k}^B \hat{D}_A \mathbf{k}_B] + (\hat{K}^l{}_l)\mathbf{k}_A = 0. \tag{4.6}$$

Now consider nonlinear perturbations of Schwarzschild. We denote by $\delta V = V - V_S$ the deviation of a variable V from its Schwarzschild value V_S. Then (4.5)–(4.6) take the form

$$\frac{1}{\sqrt{1+2H}} \partial_r \delta \mathbf{K}^l{}_l - \frac{q^{BC}}{r^2} \partial_C \delta \mathbf{k}_B = F_1, \tag{4.7}$$

$$\frac{1}{\sqrt{1+2H}} \partial_r \delta \mathbf{k}_A + \frac{\boldsymbol{\kappa}}{\mathbf{K}^l{}_l} \partial_A \delta \mathbf{K}^l{}_l - \frac{2q^{BD}\mathbf{k}_D}{r^2 \mathbf{K}^l{}_l} \partial_A \delta \mathbf{k}_B = F_A, \tag{4.8}$$

where F_1 and F_A represent lower differential order terms. This is a coupled quasilinear system for the vector valued variable $U_\alpha = (u_1, u_A) = (\delta \mathbf{K}^l{}_l, \delta \mathbf{k}_A)$. The system (4.7)–(4.8) has matrix form

$$\partial_\tau U_\alpha = \mathcal{L}_\alpha{}^{\beta C} \partial_C U_\beta + F_\alpha, \tag{4.9}$$

where $\partial_\tau = (1+2H)^{-1/2}\partial_r$, $F_\alpha = (F_1, F_A)$, and

$$\mathcal{L}_1{}^{1C} = 0, \qquad \mathcal{L}_1{}^{BC} = \frac{1}{r^2} q^{BC}, \tag{4.10}$$

$$\mathcal{L}_A{}^{1C} = -\frac{\boldsymbol{\kappa}}{\mathbf{K}^l{}_l}\delta_A^C, \qquad \mathcal{L}_A{}^{BC} = \frac{2}{r^2\mathbf{K}^l{}_l} q^{BD}\mathbf{k}_D\delta_A^C. \tag{4.11}$$

The requirement that (4.9) be a strongly hyperbolic system [21,22] is that there exists a positive bilinear form $H_{\beta\gamma}$ such that $\mathcal{L}(\omega)_{\beta\alpha} = H_{\beta\gamma}\mathcal{L}_\alpha{}^{\gamma C}\omega_C$ is symmetric for each choice of ω_C. It is straightforward to check that such a symmetrizer is given by

$$H_{11} = -\frac{\mathbf{K}^l{}_l}{\boldsymbol{\kappa}}, \qquad H_{1A} = 0, \tag{4.12}$$

$$H_{A1} = \frac{2\mathbf{k}_A}{\boldsymbol{\kappa}}, \qquad H_{AB} = r^2 q_{AB}. \tag{4.13}$$

The positivity of the symmetrizer for perturbations of Schwarzschild,

$$H_{\alpha\beta}v^\alpha v^\beta = -\frac{\mathbf{K}^l{}_l}{\boldsymbol{\kappa}}(v^1)^2 + \frac{2}{\boldsymbol{\kappa}}\mathbf{k}_A v^1 v^A + r^2 q_{AB}v^A v^B > 0, \quad v^\alpha \neq 0, \tag{4.14}$$

follows from the near Schwarzschild approximations

$$-\frac{\mathbf{K}^l{}_l}{\boldsymbol{\kappa}} \approx \frac{2(1+2H)}{1+H}, \qquad \frac{\mathbf{k}_A}{\boldsymbol{\kappa}} \approx 0. \tag{4.15}$$

Furthermore, the ω_A independence of $H_{\alpha\beta}$ implies that the system is symmetric hyperbolic as well as strongly hyperbolic.

Given near Schwarzschild data for $(\mathbf{K}^l{}_l, \mathbf{k}_A)$ on a surface $\mathcal{S}_R$ surrounding a Schwarzschild black hole, strong hyperbolicity is a sufficient condition for the system (4.5)–(4.6) to produce a unique solution of the constraint problem in some neighborhood of $\mathcal{S}_R$. Furthermore, the problem is well posed so that the solution depends continuously on the data. For linearized perturbations the solution extends globally to $r = 0$.

V. FUTURE PROSPECTS

We have shown that the new treatment of the constraints proposed in [1] leads to a well-posed constraint problem for nonlinear perturbations of a Schwarzschild black hole in Kerr-Schild form. As is generally the case for nonlinear problems, the solution is only guaranteed locally in a neighborhood of the outer surface $\mathcal{S}_R$ on which the data are prescribed. The issue of a global solution to the nonlinear problem is best explored by numerical techniques for integrating the hyperbolic system inward along the ρ-streamlines emanating from $\mathcal{S}_R$.

The well-posedness of this problem extends to perturbations representing a Kerr black hole with small spin and boost. The question of whether it extends further to a Kerr black hole with maximal spin and arbitrary boost is more complicated. Its resolution would depend, among other things, upon a judicious choice of the foliation $\mathcal{S}_\rho$ and the ρ-streamlines along which the evolution proceeds. This is akin to choosing the lapse and shift for a timelike Cauchy evolution.

The ultimate utility of this new approach rests upon its extension to multiple black holes. Formally, it can be applied to the multiple black hole problem using a modification of the superimposed Kerr-Schild data proposed in [16,17], which is based upon the ansatz that the initial 3-metric for a binary black hole is given by

$$h_{ij} = \delta_{ij} + 2H^{[1]}\ell_i{}^{[1]}\ell_j{}^{[1]} + 2H^{[2]}\ell_i{}^{[2]}\ell_j{}^{[2]}, \tag{5.1}$$

where $H^{[n]}$ and $\ell_i{}^{[n]}$ correspond to the Kerr-Schild data for individual boosted, spinning black holes. In [16,17], the actual 3-metric data are only conformal to (5.1), with the conformal factor chosen to satisfy the Hamiltonian constraint.

In our new approach to the constraints, it is possible to retain the superimposed Kerr-Schild initial data in their strict 4-dimensional form

$$g_{ab} = \eta_{ab} + 2H^{[1]}\ell_a{}^{[1]}\ell_b{}^{[1]} + 2H^{[2]}\ell_a{}^{[2]}\ell_b{}^{[2]}, \tag{5.2}$$

where $\ell_a{}^{[n]}$ are null with respect to the background Minkowski metric. This determines the initial lapse and shift as well as the initial 3-metric (5.1) for an evolution along the ρ-streamlines. It is an attractive strategy because it retains much of the algebraic simplicity of the Kerr-Schild metric; e.g., $\ell_a{}^{[1]}$ and $\ell_a{}^{[2]}$ satisfy the background geodesic equation (3.1), $H^{[1]}$ and $H^{[2]}$ satisfy the background wave equation (3.3), and the metric can be explicitly inverted, although in a more complicated form than (3.4).

Given the background metric (5.2), the Hamiltonian constraint can be imposed to express the extrinsic curvature component $\boldsymbol{\kappa}$ algebraically in terms of $\mathbf{K}^l{}_l$ and explicitly known terms via (2.7). The extrinsic curvature components $\mathbf{K}_{ij} = \mathbf{K}_{ij} - \frac{1}{2}\hat{\gamma}_{ij}\mathbf{K}^l{}_l$ can be freely prescribed, say, by superposition of their individual Kerr-Schild values. Given a suitable foliation of the initial hypersurface $\mathcal{S}_\rho$ and vector field ρ^i, the remaining components of the extrinsic curvature data, $\mathbf{K}^l{}_l$ and $\mathbf{k}_i$, could then be determined from the hyperbolic system (2.5)–(2.6) obtained from the momentum constraint. The only data necessary are the values of $\mathbf{K}^l{}_l$ and $\mathbf{k}_i$ on a large surface $\mathcal{S}_R$ surrounding the system. The surface data for $\mathbf{K}^l{}_l$ and $\mathbf{k}_i$ could be prescribed (again tentatively) by the superposition of their individual Kerr-Schild values.

A major concern in such a scheme is the effect of caustics, where the ingoing ρ-streamlines focus, or a crossover surface $\mathcal{S}_X$ where these streamlines from opposing points of $\mathcal{S}_R$ meet. For a single black hole, the streamlines can be chosen so that the caustics and crossovers are inside the apparent horizon, where the interior can be excised. However, for binary black hole data, although the caustics can be arranged to lie inside the black holes, the crossover surface $\mathcal{S}_X$ will in general span the region between them. In that case, unless $\mathcal{S}_X$ can be chosen to be a surface of reflection symmetry, as in the case of data for an axisymmetric head-on collision, the inward evolution from $\mathcal{S}_R$ may produce a discontinuity on $\mathcal{S}_X$; i.e., the data induced on $\mathcal{S}_X$ may not be single-valued.

Considerable numerical experimentation might be necessary to deal with this issue. The following strategy, which puts the flexibility of symmetric hyperbolic systems to use, is only schematic. Unlike the iterative global nature of elliptic solvers, hyperbolic evolution proceeds locally along the ρ-streamlines and can be stopped freely. This can be utilized to adjust the crossover surface, by numerical experimentation, so that it minimizes the discontinuity on $\mathcal{S}_X$ along each pair of intersecting ρ-streamlines. Then any discontinuity of the solution on $\mathcal{S}_X$ might be removed by averaging. Since the hyperbolic evolution of the constraint system can also proceed in the outward ρ-direction, a smooth solution, using this averaged data on $\mathcal{S}_X$, can then be extended outward to $\mathcal{S}_R$.

The simplicity of such a scheme for binary black hole initial data is extremely attractive. Whether it can be successfully implemented is again a matter for numerical study. If such studies were indeed successful they would lead to questions of the utmost physical importance: Does the resulting binary black hole initial data suppress junk radiation? Does it give better control over the orbital and spin parameters of a binary system? The only data needed on a single large surface in the asymptotic region surrounding the system distinguishes this approach from other solutions to the constraint problem which rely on elliptic equations. Whether this feature improves the physical content and control of the initial data is again a matter for numerical investigation.

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