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Gravitational solitons in Levi-Cività spacetime

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Applying the Pomeransky inverse scattering method to the four-dimensional vacuum Einstein equation and using the Levi-Cività solution for a seed, we construct a cylindrically symmetric single-soliton solution. Although the Levi-Cività spacetime generally includes singularities on its axis of symmetry, it is shown that for the obtained single-soliton solution, such singularities can be removed by choice of certain special parameters. This single-soliton solution describes propagation of nonlinear cylindrical gravitational shock wave pulses rather than solitonic waves. By analyzing wave amplitudes and time dependence of polarization angles, we provide a physical description of the single-soliton solution.

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I. INTRODUCTION

Time-dependent gravitational soliton solutions in general relativity are interpreted as gravitational solitonic waves propagating in background spacetimes. The so-called inverse scattering method, which was established by Belinski and Zakharov, has been used as one of the powerful tools to construct such soliton solutions [1,2]. In fact, many soliton solutions describing nonlinear gravitational waves have been subsequently found by using such a systematic method [3,4]. It is a noteworthy fact that, in a stationary and axisymmetric case, the application of the method to the four-dimensional vacuum Einstein equation can generate exact solutions of black holes. However, the simple generalization of the method to higher dimensions generally leads to singular solutions. Under these circumstances, Pomeransky [5] modified the original inverse scattering method so that it could generate regular black hole solutions even in higher dimensions. Thereafter, this improved method has played an important role in solution generation for five-dimensional black holes [6,7].

Gravitational solitons with cylindrical symmetry are of special interest to many relativists since they give us the simplest treatment of gravitational waves in an exact form. While Piran *et al.* [8] numerically studied nonlinear interactions of cylindrical gravitational waves with two polarization modes, Tomimatsu [9] first analytically studied such nonlinear phenomenon as the gravitational Faraday rotation for the cylindrical gravitational solitons generated by the Belinski-Zakharav inverse scattering technique. Moreover, the interactions of gravitational soliton waves with a cosmic string were also discussed in Refs. [10–13]. Recently, one of the authors [14,15] analyzed one- and two-soliton solutions constructed by the Pomeransky improved inverse scattering method, and studied nonlinear effects of gravitational waves such as the gravitational Faraday rotation and time shift phenomenon.

Most soliton solutions generated by the inverse scattering method can be obtained by the soliton transformation from seeds with a diagonal form. In particular, in a cylindrically symmetric case, one example of a diagonal metric is the Levi-Cività family, which describes the static and cylindrically symmetric spacetime labeled by two parameters. Therefore this solution can be regarded as the exterior field of an infinite cylinder with uniform mass, and in general it has naked singularities on its axis of symmetry. However, within a certain parameter range, these kinds of singularities can be considered as a line source with infinite length that yields a cylindrically symmetric gravitational field.

In this paper, applying the Pomeransky inverse scattering method and using the Levi-Cività metric for a seed, we generate a single-soliton solution that does not admit staticity but cylindrical symmetry. It is a generalization of the solution obtained from the Minkowski seed [14], since the Levi-Cività spacetime includes the Minkowski spacetime as a special case. Although the Levi-Cività spacetime has singularities on the axis except for the Minkowski spacetime, for the single-soliton solution, such singularities disappear by a certain choice of parameters. It is shown that the solution we present in this paper describes a shock wave pulse of nonlinear outgoing gravitational waves.

This paper is organized as follows. In the following section, we construct a single-soliton solution with a real pole by using the Pomeransky inverse scattering method from the Levi-Cività metric as a seed. In Sec. III, for the single-soliton solution, we calculate the amplitudes and polarization angles for ingoing and outgoing gravitational

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waves. Moreover, we analyze asymptotic behaviors of the nonlinear gravitational waves at the spacetime boundary depending on each choice of parameters. In Sec. IV, we devote ourselves to the summary and discussion on our results. Furthermore, we consider the difference from the single-soliton solution in Ref. [14], which was obtained from the Minkowski seed. In the Appendix, we review definitions of nonlinear cylindrical gravitational waves such as amplitudes and polarization angles, which were first introduced by Piran *et al.* [8] and Tomimatsu [9].

II. SINGLE-SOLITON SOLUTION

In this section, starting from the Levi-Cività solution, we derive a single-soliton solution by the Pomeransky inverse scattering method. The Levi-Cività solution is a static and cylindrically symmetric solution to the four-dimensional vacuum Einstein equation. The metric is given in the following form,

$$ds^2 = \rho^{1+d}d\phi^2 + \rho^{1-d}dz^2 + b^2\rho^{(d^2-1)/2}(d\rho^2 - dt^2), \tag{1}$$

where b and d are independent parameters, and both of them are assumed to be positive without loss of generality. For $d = 1$, this metric recovers the Minkowski metric (with a deficit angle related to b) written in the cylindrical coordinates. In addition, the symmetry of the Levi-Cività spacetime is enhanced due to the existence of an additional Killing vector field $\phi(\partial/\partial z) - z(\partial/\partial\phi)$ for $d = 0$ and $\phi(\partial/\partial t) + t(\partial/\partial\phi)$ for $d = 3$ [16]. It should be noted that this two-parameter family possesses naked curvature singularities on its axis of symmetry $\rho = 0$ except for $d = 1$. For $d > 1$, these singularities can be interpreted as a physical gravitational line source because a test particle is subjected to an attractive force [16]. In particular, for $d \simeq 1$, it can be regarded as the exterior field of the infinitely extended cylinder whose mass per unit length is

$$\lambda = \frac{d-1}{2(d+1)}, \tag{2}$$

in the Newtonian limit. Conversely, for $0 \le d < 1$, the singularities on the axis cannot be understood as such a physical line source because a test particle near its axis suffers from the repulsive force by the line source. The obtained single-soliton solution, however, does not necessarily have a source of repulsive force even if the corresponding Levi-Cività seed includes an unphysical source. Therefore, in what follows, we use the Levi-Cività metric within the range $0 \le d < \infty$ as a seed to generate a single-soliton solution.

Now let us assume that a four-dimensional spacetime admits cylindrical symmetry, namely, that there exist two commuting Killing vector fields, an axisymmetric Killing vector field $\partial/\partial\phi$ and a spatially translational Killing vector field $\partial/\partial z$, where the polar angle coordinate ϕ and the coordinate z have the ranges $0 \le \phi < \Delta\phi$ and $-\infty < z < \infty$, respectively. Under the symmetry assumption, the most general metric that is the solution to the four-dimensional vacuum Einstein equation can be described in the Kompaneets-Jordan-Ehlers form:

$$ds^2 = e^{2\psi}(dz + \omega d\phi)^2 + \rho^2 e^{-2\psi}d\phi^2 + e^{2(\gamma-\psi)}(d\rho^2 - dt^2), \tag{3}$$

where the functions ψ, ω, and γ depend on the time coordinate t and radial coordinate ρ only. Let us define a 2×2 metric g and a metric function f by

$$g = \begin{pmatrix} e^{2\psi} & \omega e^{2\psi} \\ \omega e^{2\psi} & \rho^2 e^{-2\psi} + \omega^2 e^{2\psi} \end{pmatrix}, \tag{4}$$

$$f = e^{2(\gamma-\psi)}, \tag{5}$$

respectively.

For the Levi-Cività metric, the 2×2 metric g_0 and the metric function f_0 are written as

$$g_0 = \mathrm{diag}(\rho^{1-d}, \rho^{1+d}), \tag{6}$$

$$f_0 = b^2\rho^{(d^2-1)/2}, \tag{7}$$

respectively. Following the Pomeransky method [5], let us remove a trivial soliton at $t = t_1$ with a trivial BZ vector (1,0), and then we have the metric

$$g_0' = \mathrm{diag}(\rho^{-1-d}\mu^2, \rho^{1+d}) = \mathrm{diag}\left(\frac{\rho^{3-d}}{\tilde{\mu}^2}, \rho^{1+d}\right), \tag{8}$$

where the functions μ and $\tilde{\mu}$ are defined by

$$\mu = \sqrt{(t-t_1)^2 - \rho^2} - (t - t_1), \tag{9}$$

$$\tilde{\mu} = \frac{\rho^2}{\mu} = -\sqrt{(t-t_1)^2 - \rho^2} - (t - t_1), \tag{10}$$

respectively.

Next, add back a nontrivial soliton with a BZ vector $m_0 = (1, a)$, and then we obtain a single-soliton solution as

$$g_{ab} = g'_{0ab} - \frac{g'_{0ac}m_c\Gamma^{-1}m_d g'_{0db}}{\mu^2}, \tag{11}$$

$$f = f_0\frac{\Gamma}{\Gamma_0}, \tag{12}$$

where

$$\Gamma = \frac{m_a g'_{0ab} m_b}{-\rho^2 + \mu^2}, \tag{13}$$

$$m_a = m_{0b}[\Psi_0^{-1}(\mu,\rho,t)]_{ba}. \tag{14}$$

Here, Γ_0 is Γ evaluated at $a=0$, and $\Psi_0(\lambda,\rho,t)$ is the generating matrix for the metric g_0' in the following form,

$$\Psi_0(\lambda,\rho,t) = \mathrm{diag}\left(\frac{(\rho^2+2t\lambda+\lambda^2)^{(3-d)/2}}{(\tilde{\mu}-\lambda)^2}, (\rho^2+2t\lambda+\lambda^2)^{(1+d)/2}\right), \tag{15}$$

where λ is the spectral parameter. Finally, we reparametrize a as $a(-2t_1)^{1-d}\to a$, and shift the time coordinate t as $t\to t+t_1$.

Thus, from Eq. (11), we can read off the functions ψ, γ, and ω for the single-soliton solution as

$$e^{2\psi} = \frac{1+a^2Fw^2}{1+a^2F}\rho^{1-d}, \tag{16}$$

$$\omega = \frac{aw^{1-d}}{(1-w^2)(1+a^2Fw^2)}\rho^{d-1}, \tag{17}$$

$$e^{2\gamma} = b^2(1+a^2Fw^2)\rho^{(d-1)^2/2}, \tag{18}$$

respectively, with

$$F = \frac{w^{2(2-d)}}{\rho^2(1-w^2)^4}, \tag{19}$$

$$w = -\frac{\sqrt{t^2-\rho^2}-t}{\rho}. \tag{20}$$

This single-soliton solution includes three parameters b, d, and a, where b and d are assumed to be positive. There are no physical restrictions on the parameter a. The parameter a controls all physical behaviors of gravitational waves such as the amplitudes and the polarizations. Note that for $a=0$ this metric recovers the Levi-Cività metric. Except for $a=0$ the metric depends on the time coordinate t as well as the radial coordinate ρ. Furthermore, for $b=d=1$, it completely coincides with the single-soliton solution in Ref. [14], which was generated from the Minkowski seed.

III. ANALYSIS FOR THE SINGLE-SOLITON SOLUTION

In this section, we analyze physical properties of the nonlinear cylindrically symmetric gravitational waves described by the obtained single-soliton solution by seeing wave amplitudes, polarization angles, and C-energy density (see the Appendix for their definitions). In particular, we investigate the dependence of their asymptotic behaviors on d in the neighborhood of its spacetime boundaries.

From the definitions given in Eqs. (A1) and (A2), the amplitudes of ingoing and outgoing waves with the + mode, A_+ and B_+, are calculated, respectively, as

$$A_+ = \frac{2a^2F}{\rho}\sqrt{\frac{u}{v}}\frac{1+a^2Fw^2-(1-w)^2+(d-1)(1-w^2)}{(1+a^2F)(1+a^2Fw^2)} - \frac{d-1}{\rho}, \tag{21}$$

$$B_+ = -\frac{2a^2F}{\rho}\sqrt{\frac{v}{u}}\frac{1+a^2Fw^2-(1+w)^2+(d-1)(1-w^2)}{(1+a^2F)(1+a^2Fw^2)} + \frac{d-1}{\rho}, \tag{22}$$

and, from Eqs. (A3)–(A4), the amplitudes of ingoing and outgoing waves with the $\times$ mode, $A_\times$ and $B_\times$, are obtained as

$$A_\times = \frac{2a\sqrt{F}}{\rho}\sqrt{\frac{u}{v}}\frac{w(1-a^2F+2a^2Fw)+(d-1)(1+w)(1+a^2Fw)}{(1+a^2F)(1+a^2Fw^2)}, \tag{23}$$

$$B_\times = -\frac{2a\sqrt{F}}{\rho}\sqrt{\frac{v}{u}}\frac{w(1-a^2F-2a^2Fw)-(d-1)(1-w)(1-a^2Fw)}{(1+a^2F)(1+a^2Fw^2)}, \tag{24}$$

respectively. From Eqs. (A5) and (A6), the total amplitudes for ingoing and outgoing waves, A and B, are calculated, respectively, as

$$A = \frac{1}{\rho}\sqrt{\frac{a^2Fw^2(2\sqrt{u/v}+1-d)^2+(d-1)^2}{1+a^2Fw^2}}, \tag{25}$$

$$B = \frac{1}{\rho}\sqrt{\frac{a^2Fw^2(2\sqrt{v/u}+1-d)^2+(d-1)^2}{1+a^2Fw^2}}. \tag{26}$$

For $a=0$, namely, (static) Levi-Cività spacetime, A_+ and B_+ do not vanish. However, this should not be surprising because both amplitudes $A_+ = -B_+ = (d-1)/\rho$ are constant for rest observers staying on the world lines $\rho =$ constant. Therefore, the nonvanishing amplitudes do not necessarily mean usual propagation of gravitational waves.

The polarization angles for ingoing and outgoing waves, θ_A and θ_B, given in Eqs. (A7) and (A8) are written as

$$\sin 2\theta_A = 2a\sqrt{F}\sqrt{\frac{u}{v}}\frac{w(1-a^2F+2a^2Fw)+(d-1)(1+w)(1+a^2Fw)}{(1+a^2F)\sqrt{1+a^2Fw^2}\sqrt{a^2Fw^2(2\sqrt{u/v}+1-d)^2+(d-1)^2}}, \tag{27}$$

$$\sin 2\theta_B = -2a\sqrt{F}\sqrt{\frac{v}{u}}\frac{w(1-a^2F-2a^2Fw)-(d-1)(1-w)(1-a^2Fw)}{(1+a^2F)\sqrt{1+a^2Fw^2}\sqrt{a^2Fw^2(2\sqrt{v/u}+1-d)^2+(d-1)^2}}, \tag{28}$$

respectively. The C-energy density is proportional to $\gamma_{,\rho}$, which is related to the amplitudes as Eq. (A13), and is given by

$$\gamma_{,\rho} = \frac{(d-1)^2}{4\rho} + \frac{a^2Fw^2}{\rho}\frac{2(1+3w^2)-d(1-w^4)}{(1-w^2)^2(1+a^2Fw^2)}. \tag{29}$$

In what follows, we focus only on the portion $t \geq 0$ in the spacetime because our interest here is to understand how shock wave pulses propagate throughout the spacetime as time passes. In the following subsections, we analyze the asymptotic behaviors of the above quantities near the spacetime boundaries: the axis of symmetry $\rho = 0$, the light cone $u = 0$, timelike infinity $t \to \infty$, and null infinity $v \to \infty$.

A. Axis of symmetry

Let us see the asymptotic behaviors of wave amplitudes, polarization angles, and C-energy density near the axis $\rho = 0$. In the limit $\rho \to 0$ with the time coordinate fixed at $t = t_0$, the C-energy density (29) behaves as

$$\gamma_{,\rho} = O(\rho^{-1}), \tag{30}$$

for all d except for $d = 1, 3$. Hence, for $d \neq 1, 3$, this spacetime has a singular gravitational source on the axis.

As shown in Ref. [14], in contrast, for $d = 1$, it behaves as $\gamma_{,\rho} = O(\rho)$, which implies that there is no singular source on the axis. This may not be surprising because the corresponding seed (Minkowski spacetime) has no singular source on the axis. However, it should be a surprising fact that Eq. (29) for $d = 3$ asymptotically behaves as

$$\gamma_{,\rho} \simeq \frac{t_0^2 + a^2}{a^2 t_0^2}\rho, \tag{31}$$

near $\rho = 0$ since, as mentioned in the previous section, the Levi-Cività spacetime with $d = 3$ has singularities on the axis. As a result, such singularities on the axis are completely removed after the soliton transformation. Therefore, for $d = 3$, nonexistence of curvature singularities on the axis allows us to evaluate a deficit angle as a meaningful physical quantity. The deficit angle Δ on the axis at arbitrary time is calculated as

$$\Delta = 2\pi - \lim_{\rho\to 0}\frac{\int_0^{\Delta\phi}\sqrt{g_{\phi\phi}}d\phi}{\int_0^{\rho}\sqrt{g_{\rho\rho}}d\rho} \tag{32}$$

$$= 2\pi - \frac{\Delta\phi}{b|a|}. \tag{33}$$

Hence, the deficit angle can be adjusted to be zero by choosing the periodicity of ϕ as $\Delta\phi = 2\pi b|a|$.

Table I shows the asymptotic behaviors of the amplitudes near the axis for each d. While for $0 \leq d \leq 3$ except $d = 1, 3$, the wave amplitudes diverge on the axis, for $d = 1, 3$ all of the amplitudes take finite values there. Note that, for $d = 0, 2$, the wave amplitudes with the $\times$ mode take finite values, and hence the $+$ mode dominates over the $\times$ mode on the axis. For $d > 3$, the wave amplitudes with the $\times$ mode vanish on the axis.

B. Light cone

We turn our attention to the asymptotic behaviors of the gravitational waves near the light cone $t = \rho (u = 0)$ or, equivalently, $w = 1$. Regardless of d, the C-energy density (29) diverges there as

$$\gamma_{,\rho} \simeq \frac{2}{\rho(1-w)^2}. \tag{34}$$

TABLE I. Asymptotic behaviors of the amplitudes near the axis of symmetry $\rho = 0$.

d	A_+	B_+	$A_\times$	$B_\times$	A	B
$d=0$	$\frac{1}{\rho}$	$-\frac{1}{\rho}$	$-\frac{a}{2t_0^2}$	$-\frac{a}{2t_0^2}$	$\frac{1}{\rho}$	$\frac{1}{\rho}$
$0<d<1$	$\frac{1-d}{\rho}$	$-\frac{1-d}{\rho}$	$-\frac{2a(1-d)}{(2t_0)^{2-d}\rho^d}$	$-\frac{2a(1-d)}{(2t_0)^{2-d}\rho^d}$	$\frac{1-d}{\rho}$	$\frac{1-d}{\rho}$
$d=1$	$\frac{2a^2}{t_0(4t_0^2+a^2)}$	$\frac{2a^2}{t_0(4t_0^2+a^2)}$	$\frac{a}{2t_0^2}\frac{4t_0^2-a^2}{4t_0^2+a^2}$	$-\frac{a}{2t_0^2}\frac{4t_0^2-a^2}{4t_0^2+a^2}$	$\frac{\lvert a\rvert}{2t_0^2}$	$\frac{\lvert a\rvert}{2t_0^2}$
$1<d<\frac{3}{2}$	$\frac{d-1}{\rho}$	$-\frac{d-1}{\rho}$	$\frac{2(d-1)(2t_0)^{2-d}}{a\rho^{2-d}}$	$\frac{2(d-1)(2t_0)^{2-d}}{a\rho^{2-d}}$	$\frac{d-1}{\rho}$	$\frac{d-1}{\rho}$
$d=\frac{3}{2}$	$\frac{1}{2\rho}$	$-\frac{1}{2\rho}$	$\frac{4t_0^2-a^2}{2\sqrt{2}at_0^{3/2}\sqrt{\rho}}$	$\frac{4t_0^2+a^2}{2\sqrt{2}at_0^{3/2}\sqrt{\rho}}$	$\frac{1}{2\rho}$	$\frac{1}{2\rho}$
$\frac{3}{2}<d<2$	$\frac{d-1}{\rho}$	$-\frac{d-1}{\rho}$	$-\frac{2a(2-d)}{(2t_0)^{3-d}\rho^{d-1}}$	$\frac{2a(2-d)}{(2t_0)^{3-d}\rho^{d-1}}$	$\frac{d-1}{\rho}$	$\frac{d-1}{\rho}$
$d=2$	$\frac{1}{\rho}$	$-\frac{1}{\rho}$	$\frac{2}{a}+\frac{4a}{4t_0^2+a^2}$	$\frac{2}{a}+\frac{4a}{4t_0^2+a^2}$	$\frac{1}{\rho}$	$\frac{1}{\rho}$
$2<d(d\neq 3)$	$\frac{3-d}{\rho}$	$-\frac{3-d}{\rho}$	$\frac{2(d-2)(2t_0)^{3-d}}{a\rho^{3-d}}$	$-\frac{2(d-2)(2t_0)^{3-d}}{a\rho^{3-d}}$	$\frac{\lvert 3-d\rvert}{\rho}$	$\frac{\lvert 3-d\rvert}{\rho}$
$d=3$	$-\frac{2}{t_0}$	$-\frac{2}{t_0}$	$\frac{2}{a}$	$-\frac{2}{a}$	$\frac{2}{\lvert a\rvert}\sqrt{1+\frac{a^2}{t_0^2}}$	$\frac{2}{\lvert a\rvert}\sqrt{1+\frac{a^2}{t_0^2}}$

The divergence of the C-energy density leads to curvature singularities, whose appearance is commonly unavoidable for the single-soliton solutions. Hence, the spacetime region cannot be analytically extended to the exterior region over $w = 1$. We can interpret the (curvature) singularities on the light cone as the gravitational shock wave propagating at the light velocity from the axis $\rho = 0$ at the moment of $t = 0$ to null infinity $v \to \infty$.

As shown in Table II, since the amplitudes of the $\times$ mode waves asymptotically approach zero on the light cone, the $\times$ mode does not contribute to the shock waves. The main ingredient of the shock waves is made from the outgoing wave with the $+$ mode, whose amplitude diverges there. While for $d = 1$, no ingoing wave crosses the light cone, for $d \neq 1$, the ingoing waves with the $+$ mode exist on the light cone.

C. Timelike infinity

Let us consider the asymptotic behaviors of the gravitational waves at timelike infinity. Table III shows the asymptotic behaviors of the amplitudes (21)–(26) at timelike infinity $t \to \infty$ with the radial coordinate ρ kept constant as $\rho = \rho_0 > 0$. As shown in Ref. [14], for $d = 1$, the spacetime is asymptotically flat at $t \to \infty$, and hence both amplitudes A and B vanish. Since the polarization angles θ_A and θ_B behave as $\theta_A = -\theta_B \simeq \pi/4$, the $\times$ mode for the ingoing and outgoing waves becomes dominant at late time. For $d = 2$, both amplitudes A and B asymptotically approach a nonzero constant at $t \to \infty$, and the angles θ_A and θ_B behave as

$$\sin 2\theta_A \simeq \sin 2\theta_B \simeq \frac{2a\rho_0}{\rho_0^2 + a^2}. \tag{35}$$

For $d = 3$, both amplitudes A and B become constant at $t \to \infty$, and the polarization angles behave as

$$\sin 2\theta_A \simeq -\sin 2\theta_B \simeq \frac{a}{\sqrt{\rho_0^2 + a^2}}. \tag{36}$$

For $d \neq 1, 2, 3$, the amplitudes A and B also become constant at $t \to \infty$, and the polarization angles θ_A and θ_B vanish. This means that the $+$ mode for ingoing and outgoing waves dominates over the $\times$ mode at late time, whose asymptotic behaviors are considerably similar to those that the Tomimatsu solution [9] shows.

TABLE II. Asymptotic behaviors of the amplitudes near the light cone $u = 0$.

d	A_+	B_+	$A_\times$	$B_\times$	A	B
$d\neq 1,\frac{1}{2}$	$\frac{1-d}{v_0}$	$-\frac{2}{\sqrt{v_0 u}}$	$\frac{32(2d-1)u^{3/2}}{av_0^{3/2}}$	$\frac{96}{a}\sqrt{\frac{u}{v_0}}$	$\frac{\lvert 1-d\rvert}{v_0}$	$\frac{2}{\sqrt{v_0 u}}$
$d=\frac{1}{2}$	$\frac{1}{2v_0}$	$-\frac{2}{\sqrt{v_0 u}}$	$-\frac{96u^2}{av_0^2}$	$\frac{96}{a}\sqrt{\frac{u}{v_0}}$	$\frac{1}{2v_0}$	$\frac{2}{\sqrt{v_0 u}}$
$d=1$	$\frac{2\sqrt{u}}{v_0^{3/2}}$	$-\frac{2}{\sqrt{v_0 u}}$	$\frac{32u^{3/2}}{av_0^{3/2}}$	$\frac{96}{a}\sqrt{\frac{u}{v_0}}$	$\frac{2\sqrt{u}}{v_0^{3/2}}$	$\frac{2}{\sqrt{v_0 u}}$

D. Null infinity

Let us focus on the asymptotic behaviors of the gravitational waves at null infinity. Table IV shows the asymptotic behaviors of Eqs. (21)–(26) at null infinity $v \to \infty$ as $u = u_0$ (u_0 is a positive constant). As discussed in Ref. [14], for $d = 1$, because the spacetime is asymptotically flat at $v \to \infty$, both amplitudes A and B vanish. Then the polarization angles approach constant values.[1] For

[1]For $d = 1$, the polarization angles at null infinity behave as

$$\sin 2\theta_A \simeq \mathrm{sgn}(a)\frac{16u_0}{\sqrt{a^2 + 16^2 u_0^2}}, \tag{37}$$

$$\sin 2\theta_B \simeq \mathrm{sgn}(a)\frac{16u_0(3a^2 - 16^2 u_0^2)}{(a^2 + 16^2 u_0^2)^{3/2}}. \tag{38}$$

TABLE III. Asymptotic behaviors of the amplitudes at timelike infinity $t \to \infty$.

d	A_+	B_+	$A_\times$	$B_\times$	A	B
$0 \le d < 1$	$\frac{1-d}{\rho_0}$	$-\frac{1-d}{\rho_0}$	$-\frac{2a(1-d)}{\rho_0^d (2t)^{2-d}}$	$-\frac{2a(1-d)}{\rho_0^d (2t)^{2-d}}$	$\frac{1-d}{\rho_0}$	$\frac{1-d}{\rho_0}$
$d = 1$	$\frac{a^2}{2t^3}$	$\frac{a^2}{2t^3}$	$\frac{a}{2t^2}$	$-\frac{a}{2t^2}$	$\frac{\lvert a\rvert}{2t^2}$	$\frac{\lvert a\rvert}{2t^2}$
$1 < d < 2$	$\frac{1-d}{\rho_0}$	$-\frac{1-d}{\rho_0}$	$\frac{2a(d-1)}{\rho_0^d (2t)^{2-d}}$	$\frac{2a(d-1)}{\rho_0^d (2t)^{2-d}}$	$\frac{d-1}{\rho_0}$	$\frac{d-1}{\rho_0}$
$d = 2$	$\frac{1}{\rho_0} - \frac{2\rho_0}{\rho_0^2+a^2}$	$-\frac{1}{\rho_0} + \frac{2\rho_0}{\rho_0^2+a^2}$	$\frac{2a}{\rho_0^2+a^2}$	$\frac{2a}{\rho_0^2+a^2}$	$\frac{1}{\rho_0}$	$\frac{1}{\rho_0}$
$2 < d < 5/2$	$\frac{d-1}{\rho_0}$	$-\frac{d-1}{\rho_0}$	$\frac{2(d-1)}{a\rho_0^{2-d}(2t)^{d-2}}$	$\frac{2(d-1)}{a\rho_0^{2-d}(2t)^{d-2}}$	$\frac{d-1}{\rho_0}$	$\frac{d-1}{\rho_0}$
$d = 5/2$	$\frac{3}{2\rho_0}$	$-\frac{3}{2\rho_0}$	$\frac{3\rho_0^2+a^2}{a\rho_0^{3/2}\sqrt{2t}}$	$\frac{3\rho_0^2-a^2}{a\rho_0^{3/2}\sqrt{2t}}$	$\frac{3}{2\rho_0}$	$\frac{3}{2\rho_0}$
$\frac{5}{2} < d < 3$	$\frac{d-1}{\rho_0}$	$-\frac{d-1}{\rho_0}$	$\frac{2a(d-2)}{\rho_0^{d-1}(2t)^{3-d}}$	$-\frac{2a(d-2)}{\rho_0^{d-1}(2t)^{3-d}}$	$\frac{d-1}{\rho_0}$	$\frac{d-1}{\rho_0}$
$d = 3$	$\frac{2\rho_0}{\rho_0^2+a^2}$	$-\frac{2\rho_0}{\rho_0^2+a^2}$	$\frac{2a}{\rho_0^2+a^2}$	$-\frac{2a}{\rho_0^2+a^2}$	$\frac{2}{\sqrt{\rho_0^2+a^2}}$	$\frac{2}{\sqrt{\rho_0^2+a^2}}$
$3 < d$	$-\frac{d-3}{\rho_0}$	$\frac{d-3}{\rho_0}$	$\frac{2(d-2)\rho_0^{d-3}}{a(2t)^{d-3}}$	$-\frac{2(d-2)\rho_0^{d-3}}{a(2t)^{d-3}}$	$\frac{d-3}{\rho_0}$	$\frac{d-3}{\rho_0}$

TABLE IV. Asymptotic behaviors of the amplitudes at null infinity $v \to \infty$.

d	A_+	B_+	$A_\times$	$B_\times$	A	B
$d \neq 1, \frac{1}{2}$	$\frac{1-d}{v}$	$-\frac{2a^2(a^2-3\times 16^2u_0^2)}{(a^2+16^2u_0^2)^2\sqrt{u_0 v}}$	$\frac{32a(2d-1)u_0^{3/2}}{(a^2+16^2u_0^2)v^{3/2}}$	$\frac{32a(3a^2-16^2u_0^2)}{(a^2+16^2u_0^2)^2}\sqrt{\frac{u_0}{v}}$	$\frac{\lvert d-1\rvert}{v}$	$\frac{2\lvert a\rvert}{\sqrt{(a^2+16^2u_0^2)u_0 v}}$
$d = \frac{1}{2}$	$\frac{1}{2v}$	$-\frac{2a^2(a^2-3\times 16^2u_0^2)}{(a^2+16^2u_0^2)^2\sqrt{u_0 v}}$	$-\frac{32au_0^2(3a^2+16^2u_0^2)}{(a^2+16^2u_0^2)^2 v^2}$	$\frac{32a(3a^2-16^2u_0^2)}{(a^2+16^2u_0^2)^2}\sqrt{\frac{u_0}{v}}$	$\frac{1}{2v}$	$\frac{2\lvert a\rvert}{\sqrt{(a^2+16^2u_0^2)u_0 v}}$
$d = 1$	$\frac{2a^2\sqrt{u_0}}{(a^2+16^2u_0^2)v^{3/2}}$	$-\frac{2a^2(a^2-3\times 16^2u_0^2)}{(a^2+16^2u_0^2)^2\sqrt{u_0 v}}$	$\frac{32au_0^{3/2}}{(a^2+16^2u_0^2)v^{3/2}}$	$\frac{32a(3a^2-16^2u_0^2)}{(a^2+16^2u_0^2)^2}\sqrt{\frac{u_0}{v}}$	$\frac{2\lvert a\rvert\sqrt{u_0}}{\sqrt{a^2+16^2u_0^2}v^{3/2}}$	$\frac{2\lvert a\rvert}{\sqrt{(a^2+16^2u_0^2)u_0 v}}$

$d = 1/2$, the amplitudes A and B go to zero as $v \to \infty$ as shown in Table IV, and then the polarization angles behave as

$$\sin 2\theta_A \simeq -\frac{64au_0^2(3a^2+16^2u_0^2)}{(a^2+16^2u_0^2)^2 v}, \tag{39}$$

$$\sin 2\theta_B \simeq \mathrm{sgn}(a)\frac{16u_0(3a^2-16^2u_0^2)}{(a^2+16^2u_0^2)^{3/2}}. \tag{40}$$

While for ingoing waves, the $+$ mode dominates over the $\times$ mode at null infinity since $\theta_A \to 0$, for outgoing waves, the polarization angle θ_B approaches constant as $v \to \infty$. For $u_0 = \sqrt{3}\lvert a\rvert/16$, in particular, θ_B asymptotically vanishes.

For $d \neq 1, 1/2$, both amplitudes A and B also vanish at null infinity (see Table IV). From Eqs. (27) and (28), we obtain the asymptotic form of the polarization angles as

$$\sin 2\theta_A \simeq \frac{32(2d-1)au_0^{3/2}}{\lvert d-1\rvert(a^2+16^2u_0^2)^{3/2}\sqrt{v}}, \tag{41}$$

$$\sin 2\theta_B \simeq \mathrm{sgn}(a)\frac{16u_0(3a^2-16^2u_0)}{(a^2+16^2u_0^2)^{3/2}}. \tag{42}$$

Similarly to the case of $d = 1/2$, the polarization angle θ_A vanishes at null infinity while θ_B approaches a constant value.

Thus we find that independently of d, $\tan\theta_B$ asymptotically approaches

$$\tan 2\theta_B \simeq -\frac{16u_0(3a^2-16^2u_0^2)}{a^2-3\times 16^2u_0^2}. \tag{43}$$

Therefore, for $v \to \infty$ with $u_0 = \sqrt{3}\lvert a\rvert/16$, there only exist $+$ mode outgoing waves, while for $v \to \infty$ with $u_0 = \lvert a\rvert/(16\sqrt{3})$, there only exist $\times$ mode outgoing waves.

IV. SUMMARY AND DISCUSSION

In this paper, applying the Pomeransky inverse scattering method to the four-dimensional vacuum Einstein equation, we have constructed the cylindrically symmetric single-soliton solution with a real pole from the Levi-Cività seed metric. The solution obtained in this work has three independent parameters, where two of them, b and d, are originated from the seed and the remaining one a is the BZ-parameter appearing in the soliton transformation, and the Levi-Cività metric is recovered by setting $a = 0$. The single-soliton spacetime is interpreted as propagation of nonlinear gravitational shock waves with cylindrical symmetry. In order to understand physical properties of these gravitational waves, we have classified and analyzed the wave amplitudes and polarization angles for each value of $d \geq 0$.

We summarize the behaviors of gravitational waves near the spacetime boundaries:

(i) Axis of symmetry ($\rho = 0$):

Except for the Minkowski spacetime corresponding to $d = 1$, the Levi-Cività seed metric has singularities on the axis of symmetry $\rho = 0$. In the same way, for the single-soliton solution with $d \neq 1, 3$, the C-energy density diverges on the axis due to the existence of such singularities, and then the wave amplitudes A and B become infinitely large. For $d = 1, 3$, however, the singularities disappear so long as $a \neq 0$. For $d = 3$, the polarization angles of ingoing and outgoing waves have finite and nonzero values on the axis, and approach $\pi/4$ as time passes.

(ii) Light cone ($u = 0$):

Regardless of d, the outgoing wave amplitude becomes infinitely large on the null surface $u = 0$. The spacetime has null curvature singularities, and the C-energy diverges there. This itself is not special but common to all known single-soliton solutions with cylindrical symmetry. The polarization angles for ingoing and outgoing waves vanish on the surface. Thus, we can find that an outgoing shock wave pulse with the $+$ mode is initially emitted from the origin of the spacetime.

(iii) Timelike infinity ($t \to \infty$):

At $t \to \infty$ (with ρ constant), the spacetime described by the obtained single-soliton solution does not asymptotically approach the Minkowski spacetime except for $d = 1$ [14], in which case simultaneously both ingoing and outgoing gravitational waves decay. While for $d = 1$ the $\times$ mode for the ingoing and outgoing waves becomes dominant at late time, for $d = 2, 3$ the $+$ and $\times$ modes of the ingoing and outgoing waves have a comparable order, and for $d \neq 1, 2, 3$ the $+$ mode for both waves comes to be dominant at late time.

(iv) Null infinity ($v \to \infty$ with $u = u_0 > 0$):

Regardless of d, the amplitudes for ingoing and outgoing waves decay at null infinity $v \to \infty$ with $u = u_0$ (u_0 is a positive constant). The polarization angle for ingoing waves takes a constant value, which depends on d at null infinity. In contrast, the one for outgoing waves commonly approaches a constant value, independently of d.

It is well known that although the appearance of singularities on the light cone is commonly unavoidable for the single-soliton solutions with a real pole, such a problem can be resolved for two-soliton solutions with two complex conjugate poles (for instance, see Refs. [9,15]). Therefore, for $d = 3$, it may be interesting to construct such a two-soliton solution with complex conjugate poles because it is expected to be entirely regular everywhere. This issue deserves further study.

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APPENDIX: FORMULAS

In this section, we give definitions of amplitudes and polarization angles of nonlinear cylindrically symmetric gravitational waves. Following Refs. [8,9], we introduce their amplitudes as

$$A_+ = 2\psi_{,v}, \tag{A1}$$

$$B_+ = 2\psi_{,u}, \tag{A2}$$

$$A_\times = \frac{e^{2\psi}\omega_{,v}}{\rho}, \tag{A3}$$

$$B_\times = \frac{e^{2\psi}\omega_{,u}}{\rho}, \tag{A4}$$

where A_+ and B_+ describe ingoing and outgoing waves in the $+$ mode, respectively, and $A_\times$ and $B_\times$ denote ingoing and outgoing waves in the $\times$ mode, respectively. Now the advanced ingoing and outgoing null coordinates u and v are defined by $u = (t - \rho)/2$ and $v = (t + \rho)/2$, respectively. Total amplitudes of ingoing and outgoing waves are defined by

$$A = \sqrt{A_+^2 + A_\times^2}, \tag{A5}$$

$$B = \sqrt{B_+^2 + B_\times^2}, \tag{A6}$$

respectively, and polarization angles θ_A and θ_B for the respective wave amplitudes are given by

$$\sin 2\theta_A = \frac{A_\times}{A}, \tag{A7}$$

$$\sin 2\theta_B = \frac{B_\times}{B}. \tag{A8}$$

Thus, the vacuum Einstein equation can be written in terms of these quantities. Actually, the nonlinear differential equations for the functions ψ and ω are replaced by

$$A_{+,u} = \frac{A_+ - B_+}{2\rho} + A_\times B_\times, \tag{A9}$$

$$B_{+,v} = \frac{A_+ - B_+}{2\rho} + A_\times B_\times, \tag{A10}$$

$$A_{\times,u} = \frac{A_\times + B_\times}{2\rho} - A_+ B_\times, \tag{A11}$$

$$B_{\times,v} = -\frac{A_\times + B_\times}{2\rho} - A_\times B_+, \tag{A12}$$

and the function γ is determined by

$$\gamma_{,\rho} = \frac{\rho}{8}(A^2 + B^2), \tag{A13}$$

$$\gamma_{,t} = \frac{\rho}{8}(A^2 - B^2). \tag{A14}$$

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