Quantum gravity and causal structures:
Second quantization of conformal Dirac algebras

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It is postulated that quantum gravity is a sum over causal structures coupled to matter via scale evolution. Quantized causal structures can be described by studying simple matrix models where matrices are replaced by an algebra of quantum mechanical observables. In particular, previous studies constructed quantum gravity models by quantizing the moduli of Laplace, weight, and defining-function operators on Fefferman–Graham ambient spaces. The algebra of these operators underlies conformal geometries. We extend those results to include fermions by taking an $\mathfrak{osp}(1|2)$ "Dirac square root" of these algebras. The theory is a simple, Grassmann, two-matrix model. Its quantum action is a Chern–Simons theory whose differential is a first-quantized, quantum mechanical Becchi-Rouet-Stora-Tyutin operator. The theory is a basic ingredient for building fundamental theories of physical observables.

I. INTRODUCTION

Our aim is to construct quantum gravity models based on two key premises:

(i) Physics is the art of pre- and post-dicting the future and past—the first step to construct quantum gravity is to average over all possible causal structures.

(ii) Physics is observation-based—the basic data for a theory of quantum gravity should be an algebra of observables.

A first concrete step in this direction was taken in seminal work of Bars and collaborators: They wrote down equations that predicted the Hamiltonian of quantum mechanics and constructed an action principle that could be used to quantize these equations [1]. Remarkably, they found that the moduli space of their equations was labeled by Fefferman–Graham (FG) metrics. These are $(d+2)$-dimensional ambient metrics that are in correspondence with d-dimensional conformal geometries and hence spacetime conformal structures [2].

Of course, it is hard to imagine that quantum gravity could be based only on conformal geometries *alias* Weyl invariant systems. Here, a second crucial observation was made by Bailey, Eastwood, and Gover (BEG): The FG construction realizes conformal geometries as the curved analog of the conical space of ambient lightlike rays. Solutions to Einstein's equations amount to curved analogs of conical sections [3].

The BEG description of Einstein's equations amounts to finding a parallel ambient vector field known as a parallel scale tractor while Bars' approach amounts to quantizing algebras generalizing the Laplace, weight, and defining function operators on an FG ambient space. [These $\mathfrak{sp}(2)$ "GJMS algebras" first arose in a conformal geometry context in a study of invariant Laplacians by Graham, Jennes, Mason, and Sparling [4].] Recently, these two approaches were melded in a study of quantum gravities obtained by coupling scale in the BEG sense to the Bars quantized conformal geometries [5].

The Klein–Gordon operator is fundamental for physics but is underpinned by the Dirac operator. In this paper, we

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quantize a certain "square root" of the GJMS algebra. This leads to the simplest of this class of quantum gravities, namely an infinite-dimensional two-matrix model. The model enjoys a gigantic gauge symmetry, since in some sense it contains infinite towers of interacting higher spins. However, its quantum action is a simple Chern–Simons theory obtained using the geometry of Batalin–Vilkovisky (BV) quantization [6] discovered by Alexandrov, Kontsevich, Schwarz, and Zaboronsky (AKSZ) [7].

II. MASSLESS DIRAC EQUATION

The massless Dirac equation

$$\gamma^\mu \nabla_\mu \psi = 0$$

is Weyl invariant in any dimension d and curved spacetime, under local metric and spinor transformations

$$g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}, \qquad \psi \mapsto \Omega^{\frac{1-d}{2}}\psi.$$

In other words, the Dirac operator is conformally invariant [8]. As observed by Dirac, this equation can be rewritten in a $(d+2)$-dimensional conformal space [9], the curved analog (M, g_{MN}) of which was discovered by Fefferman and Graham by demanding that the ambient, signature $(d, 2)$, metric obeyed [10]

$$g_{MN} = \nabla_M X_N, \qquad (1)$$

where here ∇ is the ambient Levi-Civita connection [2]. The closed homothety X_M generates dilations whose eigenvalues are conformal weights, while the zero locus of its square X^2 defines the curved conformal cone. The massless Dirac equation then corresponds to ambient spinors $\Psi \in \mathbf{S}M$ subject to

$$S^+\Psi = 0 = S^-\Psi,$$

where $\{\Gamma_M, \Gamma_N\} = 2g_{MN}$ and

$$S^+ := \slashed{X}, \qquad S^- := \slashed{\nabla}. \qquad (2)$$

Our goal is not to reformulate the Dirac equation [11] but rather to explore quantum gravity by probing the space of *all* possible Dirac operators. We begin with a maneuver reminiscent of string theory's nascent, first-quantized steps.

III. FIRST QUANTIZED DIRAC EQUATION

To describe the Dirac equation ambiently in first quantization, one first notes that the operators $S^\pm$ generate a first class, $\mathfrak{osp}(1|2) := \{S^\pm, Q^{\pm\pm}, Q^{+-}\}$ constraint algebra [15] where [16]

$$Q^{++} := (S^+)^2 = X^2, \qquad Q^{--} := (S^-)^2 = \Delta - \frac{1}{4}R,$$

$$Q^{+-} := \{S^+, S^-\} = 2\nabla_X + d + 2. \qquad (3)$$

We shall call this a *conformal Dirac algebra* (CDA).

The worldline particle model [17–19]

$$S = \int d\tau (P_M \dot{X}^M - [\lambda_i S^i + \lambda_{ij} Q^{ij}])$$

imposes the CDA constraints in Dirac quantization. By making differing gauge choices, it describes various models. These include relativistic and constant curvature spinning particles, the hydrogen atom with spin and other $SO(d, 2)$ invariant conformal models [22]. Its worldline Becchi-Rouet-Stora-Tyutin (BRST) operator can be treated using the detour methods of Ref. [23]. This yields an equivalent, reduced BRST operator

$$Q_{\mathrm{BRST}} = \mathbf{d} + q(2\nabla_X + d + 2) + z\slashed{X} + \slashed{\nabla}\partial_p,$$

which acts on ambient spinor, worldline (z, p, q)-ghost-polynomial wave functions living in the subspace $\mathrm{coker}(z^2) \cap \ker(\partial_p^2) =: \mathcal{H}_{\mathrm{BRST}}$. The reduced Lie algebra differential

$$\mathbf{d} := q(1 - z\partial_z - p\partial_p) - z\partial_p\partial_q \qquad (4)$$

is separately nilpotent acting on $\mathcal{H}_{\mathrm{BRST}}$. The BRST cohomology is that of the ghost-number-graded complex

$$0 \to \mathbf{S}M \xrightarrow{\binom{S^+}{S^-}} \mathbf{S}M^{\otimes 2} \xrightarrow{\left(\begin{smallmatrix} S^-S^+ - 2 & -S^+S^+ \\ S^-S^- & -S^+S^- - 2 \end{smallmatrix}\right)} \mathbf{S}M^{\otimes 2} \xrightarrow{(-S^- \ \ S^+)} \mathbf{S}M \to 0.$$

Massless spinors form the ghost number zero cohomology.

IV. SECOND QUANTIZATION

In second quantization the operators $S^\pm$ are off shell and obey equations of motion. These are exactly the integrability conditions required for the above sequence of maps to be a complex, namely

$$[S^-, S^+S^+] - 2S^+ = 0 = [S^-S^-, S^+] - 2S^-.$$

These equations follow from the action principle

$$S_{\mathrm{cl}} = \mathrm{tr}\left(S^+S^- + \frac{1}{2}S^+S^+S^-S^-\right). \qquad (5)$$

This is a simple, Grassmann, two-matrix model except that the trace is over spinor bundle $\mathbf{S}M$ operators.

The above integrability conditions ensure that the composite operators $(S^{\pm}, S^{\pm}S^{\pm}, \{S^+, S^-\})$ obey an $\mathfrak{osp}(1|2)$ algebra. Indeed, one can "integrate in" new "fields" Q^{ij} and finds an equivalent cubic action principle

$$S_{\text{cl}} = \text{tr}\left(\frac{1}{2} S^i S_i + \frac{1}{2} Q^{ij} Q_{ij} - \frac{1}{3} S^i Q_{ij} S^j\right).$$

This is our classical action principle; nontrivial solutions include the Dirac operator multiplet given in Eqs. (2) and (3). These solutions rely on the FG metric condition (1). This shows that the moduli space is parameterized, in part, by conformal geometries and hence causal structures. Moreover our space of observables is the algebra of operators acting on ambient spinors.

V. QUANTUM ACTION

The classical action (5) enjoys a huge gauge invariance

$$\delta S^{\pm} = [S^{\pm}, \varepsilon].$$

For example, expanding the operator valued parameter

$$\varepsilon = \epsilon + \xi^M \nabla_M + \zeta^{MN} \nabla_M \nabla_N + \cdots,$$

the ambient fields (ϵ, ξ^M) parameterize Maxwell and ambient diffeomorphism invariances while ζ^{MN} is the parameter for the first of an infinite tower of higher spin gauge symmetries [1].

To quantize the theory, we must construct its quantum action. We start by second quantizing the worldline BRST Hilbert space $\mathcal{H}_{\text{BRST}} \ni \mathcal{A}$, which means that a wave function $\mathcal{A}$ becomes a field and thus, in this context, an operator on $\mathbf{S}M$. In modern BV language, $\mathcal{A}$ is a coordinate for an infinite dimensional Q-manifold [24] whose differential is given by $\mathbf{d}$ of Eq. (4).

Expanding the polynomial $\mathcal{A}$ in powers of (z, p, q) determines the BV field content:

$$\mathcal{A} := S^+ + z\lambda^* + pC^* + zpS^- + q(S_+^* + zC + p\lambda + zpS_-^*).$$

The quantum action is a Chern–Simons theory [25],

$$S_{\text{qu}} = \text{tr}\int\left(\mathcal{A}\mathbf{d}\mathcal{A} + \frac{2}{3}\frac{\partial}{\partial p}\mathcal{A}^3\right). \tag{6}$$

Here, the integral denotes the ghost measure given by projection onto monomials proportional to zpq. The partial p-derivative is thus *not* a total derivative but rather defines a cyclic triple product. By construction, the above action enjoys an enhanced gauge invariance,

$$\delta\mathcal{A} = \mathbf{d}_{\mathcal{A}}\mathcal{E} := \mathbf{d}\mathcal{E} + \frac{\partial}{\partial p}[\mathcal{A}, \mathcal{E}],$$

which is fixed by path integrating over any (odd) Lagrangian submanifold of the underlying Q-manifold [26]. In the above formula, the operator ∂_p is required both on grounds

of ghost number and that the commutator of field and parameters lives in $\ker(\partial_p^2)$.

The quantum action (6) is a sum of a classical action

$$S_{\text{cl}} = \text{tr}\left(S^+ S^- - \frac{1}{2}\lambda^2 - \lambda\{S^+, S^-\}\right),$$

plus the standard minimal BV terms built from antifields multiplying the corresponding BRST variations of fields:

$$sS^{\pm} = \{C, S^{\pm}\}, \qquad s\lambda = [C, \lambda], \qquad sC = C^2.$$

Notice that there is an additional "gauge field" λ. This is an auxiliary field corresponding to the operator Q^{+-}; when λ is integrated out classically, we recover the action (5).

At the quantum level, the model's path integral sums over all possible Dirac operators whose on-shell moduli space corresponds to conformal geometries, and so as promised, the model is a weighted average over causal structures.

VI. CONCLUSIONS

Our model is closely related to other leading candidates for quantum gravity theories, in particular string field theory [27] Vasiliev's higher spin theory [28]. String theory is finite, anomaly- and tachyon-free. However, it is far from clear that this holds for either the present model or the Vasiliev theory (see, however, Ref. [29]), although at least the field content of the latter model is already well understood (see, for example, Ref. [30]). Addressing these gaps is an obvious future research direction.

The presence of a graviton is at least easier to understand using the parallel scale tractor of Ref. [3]; the key is to couple conformal geometry to scale. For example, the action principle $S = \int \lambda^{ij} Q_{ij}\psi$ is known to be gauge equivalent to the Einstein–Hilbert action [31,32]. In Ref. [5], coupling to scale was achieved by supersymmetrizing the algebra of observables. The Hilbert space trace became an $\mathcal{N} = 2$ supertrace, and in turn the model there was found to have a graviton in its spectrum. Coupling our model to scale should similarly yield a propagating graviton.

Another key question is the computation of physical correlators. Here, the advantage of the BV and AKSZ methods comes to the fore; observables can be viewed as the homology of Lagrangian submanifolds of the Q-manifold [26,33]. In a similar vein, perturbation theory is also (in principle) simple because the BV propagator is just $\frac{\delta}{\Delta}$ where δ and Δ are the (worldline) anti-BRST differential and BRST Laplacian, respectively [26] (see also Ref. [5]).

Despite the unanswered questions listed above, the model has some compelling features. Albeit infinite dimensional, it is a simple matrix model and thus conceivably finite—the theory can be regulated using well-studied (see Ref. [34]) matrix models. Moreover, unlike string and Vasiliev theories, the cubic product describing the joining

and splitting of Hilbert spaces is very simple, which may augur well for model building. In particular, the extension to form fields based on an $\mathfrak{osp}(2|2)$ algebra is immediate.

ACKNOWLEDGMENTS

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