

Accurate decay-constant ratios f_{B^*}/f_B and $f_{B_s^*}/f_{B_s}$ from Borel QCD sum rules

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We present our analysis of the decay constants of the beauty vector mesons B^* and B_s^* within the framework of dispersive sum rules for the two-point correlator of vector currents in QCD. While the decay constants of the vector mesons f_{B^*} and $f_{B_s^*}$ —similar to the decay constants of the pseudoscalar mesons f_B and f_{B_s} —individually have large uncertainties induced by theory parameters not known with a satisfactory precision, these uncertainties almost entirely cancel out in the ratios of vector over pseudoscalar decay constants. These ratios, thus, may be predicted with very high accuracy due to the good control over the systematic uncertainties of the decay constants gained upon application of our hadron-parameter extraction algorithm. Our final results read $f_{B^*}/f_B = 0.944 \pm 0.011_{\text{OPE}} \pm 0.018_{\text{syst}}$ and $f_{B_s^*}/f_{B_s} = 0.947 \pm 0.023_{\text{OPE}} \pm 0.020_{\text{syst}}$. Thus, both f_{B^*}/f_B and $f_{B_s^*}/f_{B_s}$ are less than unity at 2.5σ and 2σ level, respectively.

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I. INTRODUCTION

The QCD sum-rule approach [1–3], based on the application of Wilson's operator product expansion (OPE) to the properties of individual hadrons, has been extensively used for predicting heavy-meson decay constants. An important finding of these analyses was the strong sensitivity of the decay constants to the values of the input OPE parameters and to the prescription of fixing the effective continuum threshold [4]. The latter governs the accuracy of the quark-hadron duality approximation and, to a large extent, determines the extracted value of the decay constant. Even if the parameters of the truncated OPE are known with arbitrarily high precision, the decay constants may be predicted with only limited accuracy, which we refer to as their systematic uncertainty. In a series of papers [5], we have formulated a new algorithm for fixing the effective threshold within Borel QCD sum rules and for obtaining reliable estimates for the systematic uncertainties. This procedure opened the possibility to provide predictions for the decay constants with a controlled accuracy [6,7].

Here we study the decay constants of the vector beauty mesons f_{B^*} and $f_{B_s^*}$. As is already known from the analysis of the decay constants of the pseudoscalar mesons B and B_s [6], the OPE uncertainties in the obtained predictions are rather large. The same occurs also for the B^* and B_s^* mesons. However, the OPE uncertainties to a great extent cancel out in the ratios of the decay constants of vector and pseudoscalar beauty mesons. An important result reported

here is that the systematic uncertainties of the decay constants are rather small and well under control. Therefore, these ratios are predicted with a very good accuracy. It should be taken into account that we address a rather subtle effect at a few-percent level; *a priori*, it is not clear whether QCD sum rules are, in principle, capable to provide theoretical predictions at this level of accuracy. Obviously, the control over the systematics is becoming crucial.

The ratio of the decay constants of vector over pseudo-scalar heavy mesons is an interesting quantity: it is known to be unity in the heavy-quark limit and to approach this limit from below because of the radiative corrections [8]. For beauty mesons, the few existing sum-rule analyses (which, however, could not gain good control over the systematic uncertainties, see a detailed discussion in [4,5]) reported f_{B^*}/f_B slightly above unity [9,10]. Constituent-quark models typically also yield $f_{B^*}/f_B > 1$ [11]. A similar conclusion has been reached by interpolation of the lattice data from the charm-quark mass region to the beauty-quark mass [12].

The first indication that this ratio for beauty mesons is below unity was given in our papers [13]. Recently, HPQCD [14] also reported an accurate value of $f_{B^*}/f_B < 1$, in excellent agreement with the results of [13]. The analysis of [13], although conclusively indicating $f_{B^*}/f_B < 1$, observed an unpleasant dependence of the extracted decay constants of the vector beauty mesons on the renormalization scale μ chosen for the evaluation of the vector correlation function. This analysis solves the

problem of the sensitivity to the choice of the scale μ by improving the extraction procedures for the decay constants and arrives at new predictions stable with respect to the choice of μ . Our detailed results read

$$\begin{aligned} f_{B^*}/f_B &= 0.944 \pm 0.011_{\text{OPE}} \pm 0.018_{\text{sys}}, \\ f_{B_s^*}/f_{B_s} &= 0.947 \pm 0.023_{\text{OPE}} \pm 0.020_{\text{sys}} \end{aligned} \quad (1.1)$$

in more than excellent agreement with the latest results from lattice QCD [14]. Let us emphasize once more that the OPE uncertainties cancel to a large extent in the above ratios. Thus, decisive for obtaining an accurate sum-rule result is our capability to control the systematic uncertainties of the QCD sum-rule method.

II. QCD VECTOR CORRELATOR AND SUM RULE FOR VECTOR-MESON DECAY CONSTANT f_V

The decay constants of ground-state vector mesons may be extracted by analyzing the two-point correlation function

$$\begin{aligned} & i \int d^4x e^{ipx} \langle 0 | T(j_\mu(x) j_\nu^\dagger(0)) | 0 \rangle \\ &= \left(-g_{\mu\nu} + \frac{P_\mu P_\nu}{p^2} \right) \Pi(p^2) + \frac{P_\mu P_\nu}{p^2} \Pi_L(p^2) \end{aligned} \quad (2.1)$$

of the heavy-light vector currents for a heavy quark Q of mass m_Q and a light quark q of mass m ,

$$j_\mu(x) = \bar{q}(x) \gamma_\mu Q(x), \quad (2.2)$$

or, more precisely, the Borel transform of its transverse structure $\Pi(p^2)$ to the Borel variable τ , $\Pi(\tau)$. Equating $\Pi(\tau)$ as calculated within QCD and the expression obtained by insertion of a complete set of hadron states yields the sum rule

$$\begin{aligned} \Pi(\tau) &= f_V^2 M_V^2 e^{-M_V^2 \tau} + \int_{s_{\text{phys}}}^{\infty} ds e^{-s\tau} \rho_{\text{hadr}}(s) \\ &= \int_{(m_Q+m)^2}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu). \end{aligned} \quad (2.3)$$

Here, M_V labels the mass, f_V the decay constant, and $\varepsilon_\mu(p)$ the polarization vector of the vector meson V under study:

$$\langle 0 | \bar{q} \gamma_\mu Q | V(p) \rangle = f_V M_V \varepsilon_\mu(p). \quad (2.4)$$

For the correlator (2.1), $s_{\text{phys}} = (M_P + M_\pi)^2$ is the physical continuum threshold, wherein M_P denotes the mass of the lightest pseudoscalar meson containing Q . For large values of τ , the ground state dominates the correlator and

thus its properties may be extracted from the correlation function (2.1).

In perturbation theory, the correlation function is found as an expansion in powers of the strong coupling ‘‘constant’’ $\alpha_s(\mu)$. The best known three-loop perturbative spectral density has been calculated in [15] in terms of the pole mass of the heavy quark Q (that is, in the present case, M_b) and for a massless second quark [hereafter, we use the abbreviation $a(\nu) = \alpha_s(\nu)/\pi$, where $\alpha_s(\nu)$ is the running coupling at renormalization scale ν in the $\overline{\text{MS}}$ scheme]:

$$\begin{aligned} \rho_{\text{pert}}(s) &= \rho^{(0)}(s, M_b) + a(\nu) \rho^{(1)}(s, M_b) \\ &+ a^2(\nu) \rho^{(2)}(s, M_b, \mu) + \dots \end{aligned} \quad (2.5)$$

For both quarks having nonzero masses, the two-loop spectral density in terms of their pole masses was obtained in [3].

The power corrections are also separately scale independent; their explicit expressions can be found in [9,16]. For instance, for pseudoscalar (P) and vector (V) currents the quark-condensate contributions may be written in the form

$$\begin{aligned} \Pi_{\text{power}}^P(\tau) &= -\bar{m}_b(\nu) \langle \bar{q} q(\nu) \rangle M_b^2 \left[\exp(-M_b^2 \tau) \left(1 + \frac{3}{2} C_F a \right) \right. \\ &\quad \left. - \frac{3}{2} C_F a \Gamma(0, M_b^2 \tau) \right], \end{aligned} \quad (2.6)$$

$$\begin{aligned} \Pi_{\text{power}}^V(\tau) &= -\bar{m}_b(\nu) \langle \bar{q} q(\nu) \rangle \\ &\times \left[\exp(-M_b^2 \tau) \left(1 + \frac{1}{2} C_F a \right) \right. \\ &\quad \left. + \frac{1}{2} C_F a M_b^2 \tau \Gamma(-1, M_b^2 \tau) \right], \end{aligned} \quad (2.7)$$

where $\bar{m}_b(\nu)$ is the b -quark $\overline{\text{MS}}$ mass at renormalization scale ν , $\bar{m}_b(\nu) \langle \bar{q} q(\nu) \rangle$ is a scale-independent combination, and $\Gamma(n, z)$ is the incomplete gamma function [17].

However, even if the lowest-order contributions to the perturbative expansion and the vacuum condensates of lowest dimensions are known to good accuracy, a truncated OPE does not allow one to calculate the correlator for sufficiently large τ , such that the continuum states give a negligible contribution to $\Pi(\tau)$ in the corresponding range of τ . In order to get rid of the continuum contribution, the concept of duality is invoked: Perturbative-QCD spectral density $\rho_{\text{pert}}(s)$ and hadron spectral density $\rho_{\text{hadr}}(s)$ resemble each other at large values of s ; thus, for values of the integration lower limit \bar{s} chosen sufficiently large, that is to say, (far) above the resonance region, one arrives at the duality relation

$$\int_{\bar{s}}^{\infty} ds e^{-s\tau} \rho_{\text{hadr}}(s) = \int_{\bar{s}}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s). \quad (2.8)$$

Now, in order to express the hadron continuum contribution in terms of the perturbative contribution, the relation (2.8) should be extended down to the hadronic or physical threshold s_{phys} . However, since the spectral densities $\rho_{\text{pert}}(s)$ and $\rho_{\text{hadr}}(s)$ obviously differ in the region near s_{phys} , one can reasonably only expect to obtain a relationship of the form

$$\int_{s_{\text{phys}}}^{\infty} ds e^{-s\tau} \rho_{\text{hadr}}(s) = \int_{s_{\text{eff}}(\tau)}^{\infty} ds e^{-s\tau} \rho_{\text{pert}}(s), \quad (2.9)$$

where the effective threshold $s_{\text{eff}}(\tau)$ is clearly different from the physical threshold s_{phys} , $s_{\text{eff}}(\tau) \neq s_{\text{phys}}$, and, moreover, must be a function of the Borel parameter τ [4,5]. By virtue of (2.9), we may hence rewrite the QCD sum rule (2.3) as

$$f_V^2 M_V^2 e^{-M_V^2 \tau} = \int_{(m_Q+m)^2}^{s_{\text{eff}}(\tau)} ds e^{-s\tau} \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu) \equiv \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)). \quad (2.10)$$

We refer to the right-hand side of this relation as the ‘‘dual correlator,’’ and to the masses and decay constants extracted from this expression as the corresponding ‘‘dual’’ quantities. In addition to $\rho_{\text{pert}}(s, \mu)$ and $\Pi_{\text{power}}(\tau, \mu)$, the extraction of f_V requires, as further input, a criterion that fixes the functional behaviour of the effective continuum threshold $s_{\text{eff}}(\tau)$.

We shall demonstrate that QCD sum rules allow a very satisfactory extraction of the vector-meson decay constants, with an accuracy that is certainly competitive to that found within the framework of lattice QCD.

III. OPE AND CHOICE OF RENORMALIZATION SCHEME AND SCALE FOR HEAVY-QUARK MASS

The starting point of our discussion is the OPE for the correlator (2.1). The three-loop perturbative spectral density $\rho_{\text{pert}}(s, M)$ was calculated in [15] in terms of the pole mass of the heavy quark. A nice feature of the pole-mass OPE is that each of the known perturbative contributions to the dual correlator is positive. Unfortunately, the pole-mass OPE does not provide a visible hierarchy of the perturbative contributions to the extracted predictions, which raises doubts whether the $O(\alpha_s^2)$ -truncated pole-mass OPE is indeed a good starting point for a reliable analysis of decay constants.

A well-known remedy is to reorganize the perturbative expansion in terms of the b -quark running $\overline{\text{MS}}$ mass $\bar{m}_b(\mu)$, related (in the notations of [16]) to the corresponding pole mass M_b by

$$M_b = \bar{m}_b(\mu) / (1 + a(\mu)r_m^{(1)} + a^2(\mu)r_m^{(2)} + O(a^3)). \quad (3.1)$$

The spectral densities in the $\overline{\text{MS}}$ scheme are found by expanding the pole-mass spectral densities in powers of $a(\mu)$ and omitting terms of order $O(a^3)$ and higher; starting at order $O(a)$, they contain two parts: the ‘‘genuine’’ part from [15] and the part induced by the lower perturbative orders when expanding the pole mass in terms of the running mass. By this, however, due to the truncation of the perturbative series, one gets an explicit (unphysical) dependence of the dual correlator and of the extracted decay constant on the scale μ . In principle, any scale should be equivalently good. In practice, however, the distinctness of the hierarchy of the perturbative contributions to the dual correlator depends on the precise choice of the scale. This opens a possibility of choosing the scale μ such that the hierarchy of the new perturbative expansion is improved.

Figures 1 and 2 depict the dual decay constants of the B^* and B mesons, respectively. For the b -quark $\overline{\text{MS}}$ mass, we use the value determined in [18] by matching our QCD sum-rule results for f_B to those of lattice QCD¹:

$$\bar{m}_b(\bar{m}_b) = (4.247 \pm 0.034) \text{ GeV}. \quad (3.2)$$

The numerical values adopted for other relevant OPE parameters are [7,16,18,21]

$$\begin{aligned} m(2 \text{ GeV}) &= (3.42 \pm 0.09) \text{ MeV}, \\ m_s(2 \text{ GeV}) &= (93.8 \pm 2.4) \text{ MeV}, \\ \alpha_s(M_Z) &= 0.1184 \pm 0.0020, \\ \left\langle \frac{\alpha_s}{\pi} GG \right\rangle &= (0.024 \pm 0.012) \text{ GeV}^4, \\ \langle \bar{q}q \rangle(2 \text{ GeV}) &= -[(267 \pm 17) \text{ MeV}]^3, \\ \frac{\langle \bar{s}s \rangle(2 \text{ GeV})}{\langle \bar{q}q \rangle(2 \text{ GeV})} &= 0.8 \pm 0.3. \end{aligned} \quad (3.3)$$

The purpose of Figs. 1 and 2 is the illustration of the main features of the dual correlators (2.10), therefore the QCD sum-rule estimates shown here are obtained for a τ -independent effective threshold: $s_{\text{eff}} = \text{const}$. The numerical value of the latter is, in each case, found by requiring maximal stability of the extracted decay constant in the Borel window. We emphasize that our results for the decay constants reported in the next Sections are obtained using the τ -dependent effective thresholds.

From Figs. 1 and 2, we conclude that the $O(\alpha_s^2)$ -truncated pole-mass OPE exhibits no hierarchy of the perturbative expansion and should not be used.

¹As shown in [18], the PDG average $\bar{m}_b(\bar{m}_b) = (4.18 \pm 0.030) \text{ GeV}$ [19] (see also [20] for a recent overview of the b -quark mass results) leads to a considerably larger value of f_B , incompatible with the latest lattice-QCD results. However, the precise value of $\bar{m}_b(\bar{m}_b)$ has negligible impact on the ratio of the decay constants of vector and pseudoscalar mesons.

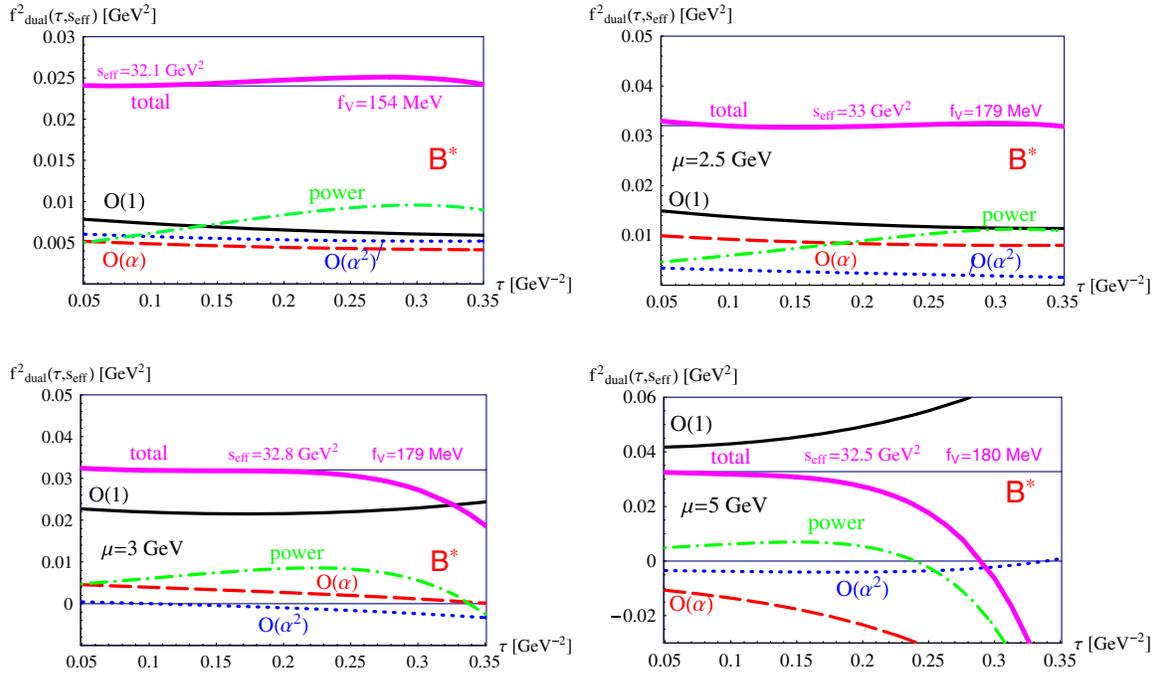


FIG. 1 (color online). QCD sum-rule estimates of the B^* -meson decay constant using the pole-mass OPE (a) and the running-mass OPE at the renormalization scales $\mu = 2.5$ (b), $\mu = 3$ (c), and $\mu = 5 \text{ GeV}$ (d). The running-mass OPE for $\bar{m}_b(\bar{m}_b) = 4.247 \text{ GeV}$ is shown. The pole-mass OPE employs the corresponding two-loop pole mass $M_b = 4.87 \text{ GeV}$. For each case, separately, a constant effective continuum threshold s_{eff} is determined by requiring maximal stability of the predicted decay constant in a Borel window of the maximal width $0.05 \leq \tau \text{ (GeV}^{-2}\text{)} \leq 0.15$. Bold lines (lilac)—total findings, solid lines (black)— $O(1)$ contributions; dashed lines (red)— $O(\alpha_s)$ contributions; dotted lines (blue)— $O(\alpha_s^2)$ contributions; dot-dashed lines (green)—power contributions.

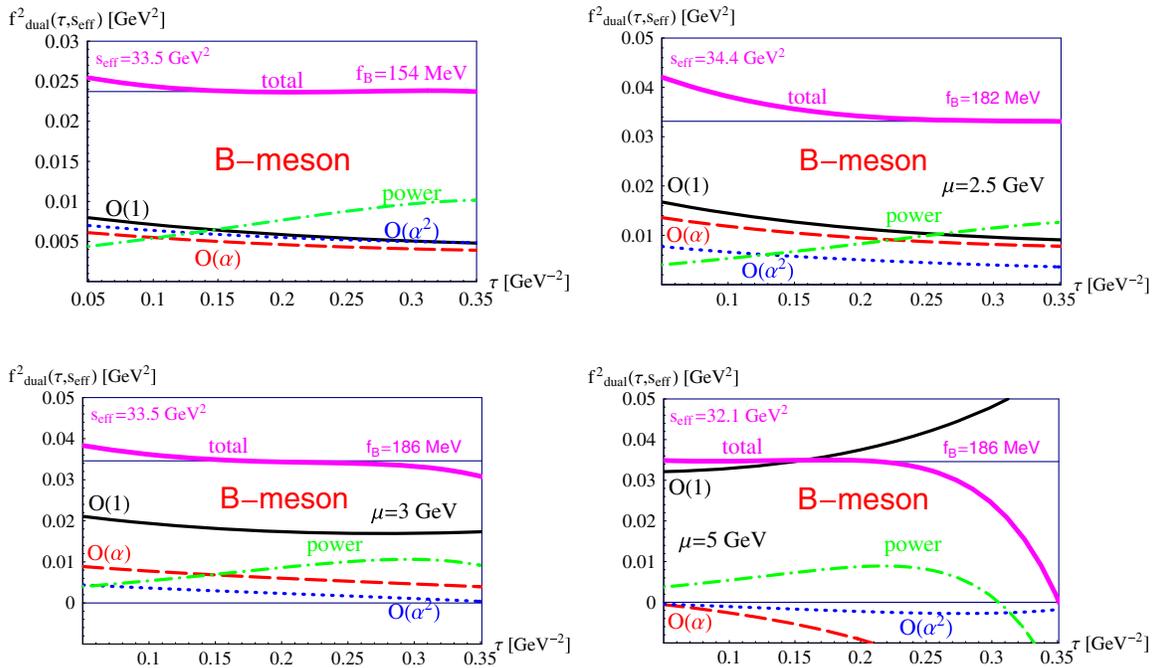


FIG. 2 (color online). Same as Fig. 1 but for the B meson.

Unfortunately, the hierarchy of the running-mass OPE is also not guaranteed automatically and depends strongly on the scale μ .

Let us define a scale $\hat{\mu}$ by demanding $M_b = \bar{m}_b(\hat{\mu})$. From the $O(a^2)$ relation between $\overline{\text{MS}}$ and pole mass, we find $\hat{\mu} \approx 2.23$ GeV. At this scale, the perturbative hierarchy of the $\overline{\text{MS}}$ expansion is worse than that of the pole-mass expansion because the $O(1)$ spectral densities coincide, whereas the $O(\alpha_s)$ spectral density in the $\overline{\text{MS}}$ scheme receives a positive contribution compared to the pole-mass scheme. For lower scales $\mu < \hat{\mu}$, the hierarchy of the $\overline{\text{MS}}$ expansion gets worse with decreasing μ . For higher scales $\mu > \hat{\mu}$, first the hierarchy of the $\overline{\text{MS}}$ expansion improves with rising μ (Figs. 1 and 2). However, as the scale μ becomes sufficiently larger than $\hat{\mu}$, the ‘‘induced’’ contributions, which mainly reflect the poorly behaved expansion of the pole mass in terms of the running mass, start to dominate over the ‘‘genuine’’ contributions. This is evident in Figs. 1 and 2: at $\mu = 5$ GeV, the $O(1)$ contribution to the dual correlator rises steeply with τ , whereas the $O(a)$ contribution becomes negative in order to compensate the rising $O(1)$ contribution. Finally, for large values of μ we mainly observe a compensation between the ‘‘induced’’ contributions. We may expect in this case the accuracy of the expansion to deteriorate.

Figures 1 and 2 also reveal an essential difference between pseudoscalar and vector correlators: at the same scale μ , the good reproduction of the observed mass of the vector meson requires lower values of τ compared to its pseudoscalar partner. This implies that the Borel window for the vector correlator should be chosen at lower values of τ than the corresponding window for the pseudoscalar correlator. Moreover, for $\mu \gtrsim 5\text{--}6$ GeV the vector-meson mass cannot be reproduced in a reasonably broad τ window and so the QCD sum rule cannot predict the vector-meson decay constant.

For the present analysis, we thus choose as range of scales $\mu = 3\text{--}5$ GeV: On the one hand, in this range we observe a reasonable hierarchy of the perturbative contributions to the correlator. On the other hand, we shall see that for this range of scales one can find sufficiently broad τ windows where the decay constants may be reliably extracted by our algorithm. For the vector mesons, the upper bound of this window depends on μ .

IV. EXTRACTION OF THE BEAUTY-MESON DECAY CONSTANTS FROM OUR QCD SUM RULE

In order to extract the decay constants, we first have to find a τ window such that the OPE provides a sufficiently accurate description of the exact correlator (i.e., all higher-order radiative and power corrections are under control). Next, we must determine the τ dependence of the effective

threshold $s_{\text{eff}}(\tau)$.² The appropriate algorithm was developed and verified within quantum-mechanical potential models [5] and shown to work successfully for the decay constants of heavy pseudoscalar and vector charmed mesons [6,7]. We introduce a ‘‘dual invariant mass’’ M_{dual} and a ‘‘dual decay constant’’ f_{dual} by defining

$$\begin{aligned} M_{\text{dual}}^2(\tau) &\equiv -\frac{d}{d\tau} \log \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)), \\ f_{\text{dual}}^2(\tau) &\equiv M_V^{-2} e^{M_V^2 \tau} \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)). \end{aligned} \quad (4.1)$$

For a properly constructed $\Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau))$, the dual mass coincides with the actual ground-state mass M_V . Therefore, any deviation of the dual mass from M_V is an indication of the contamination of the dual correlator by excited states.

For any trial function for the effective threshold, we derive a variational solution by minimizing the difference between the dual mass (4.1) and the actual (i.e., experimentally measured) mass in the Borel window. This variational solution provides the decay constant then via (4.1). We consider a set of polynomial Ansätze for the effective threshold, viz.,

$$s_{\text{eff}}^{(n)}(\tau) = \sum_{j=0}^n s_j^{(n)} \tau^j, \quad (4.2)$$

and fix the coefficients $s_j^{(n)}$ (the knowledge of which then allows us to compute the decay constant f_V) by minimizing

$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [M_{\text{dual}}^2(\tau_i) - M_V^2]^2 \quad (4.3)$$

over the Borel window. Still, different Ansätze for $s_{\text{eff}}(\tau)$ yield different sum-rule predictions for the decay constants.

Careful studies of quantum-mechanical potential models indicate that it suffices to allow for polynomials up to third order: In this case, the band delimited by the predictions arising from linear, quadratic, and cubic Ansätze for $s_{\text{eff}}(\tau)$ encompasses the true value of the decay constant. Even a good knowledge of the truncated OPE does not allow us to determine the decay constant precisely, but it enables us to provide a range of values containing the true value of this decay constant. The width of this range may then be regarded as the ‘‘systematic error’’ related to the principally limited accuracy of QCD sum rules. Presently, we are not aware of any other possibility to acquire a more reliable estimate for the systematic error. It is noteworthy that

²We point out that the τ -dependent effective threshold enables us to describe the hadron excited states and the continuum exactly, without any approximation. The corresponding τ -dependent effective threshold is referred to as the exact effective continuum threshold in [4,5]. The approximations emerge when one models the unknown exact effective threshold—which is expected to be a smooth slowly varying function of τ in the Borel window—by some trial function of τ .

considering a τ -independent threshold would not allow us to probe the accuracy of the obtained estimate for f_V .

On top of the systematic error comes the ‘‘OPE-related error’’ of the decay constant: the OPE parameters are known only with some errors, which induce a corresponding error of f_V . We determine this OPE-related (statistical) error by averaging the results for the decay constant assuming for the OPE parameters Gaussian distributions with the central values and standard deviations quoted in (3.3) and a flat distribution over the scale μ in the range $3 < \mu$ (GeV) < 5 .

A. Decay constant of the B^* meson

1. Choice of renormalization scale

In principle, the decay constant should be independent of the scale μ at which the correlation function is evaluated. In practice, however, due to the truncations of the perturbative expansion and the series of power corrections, and the necessity to isolate the ground-state contribution from the hadron continuum states, a reliable extraction of the decay constant may be performed in only a limited range of the scale μ . For the vector beauty meson, the suitable range of μ is found to be $\mu = 3\text{--}5$ GeV: For $\mu \leq 3$ GeV, the perturbative expansion for the vector correlator does not exhibit a satisfactory perturbative convergence and therefore gives

no reason to believe that the unknown higher-order radiative corrections both in the perturbative part of the correlation function and in the radiative corrections to the condensates are negligible. At higher scales $\mu \geq 5$ GeV, the B^* mass cannot be reproduced with the required accuracy, signalling that there the contamination of the excited states cannot be cleaned out.

2. Choice of Borel-parameter window

We require that the B^* – B mass splitting and the masses of B^* and B mesons are reproduced, separately, with an accuracy not worse than 5 MeV for any τ value within the selected ranges. As follows from the properties of the dual correlators, this requirement provides two constraints on the choice of the τ window for B^* :

- (i) The τ window for B^* should be chosen at lower values of τ compared to the B -meson case.
- (ii) The precise choice of the τ window for B^* should correlate with the scale μ at which the correlator is evaluated.

To satisfy the above criteria for B^* , we set the lower boundary at $\tau_{\min}(\text{GeV}^{-2}) = 0.01$ and choose a μ -dependent upper boundary of the form $\tau_{\max}(\text{GeV}^{-2}) = 0.31 - 0.05 \mu(\text{GeV})$, which choice enables us to extract f_{B^*} with

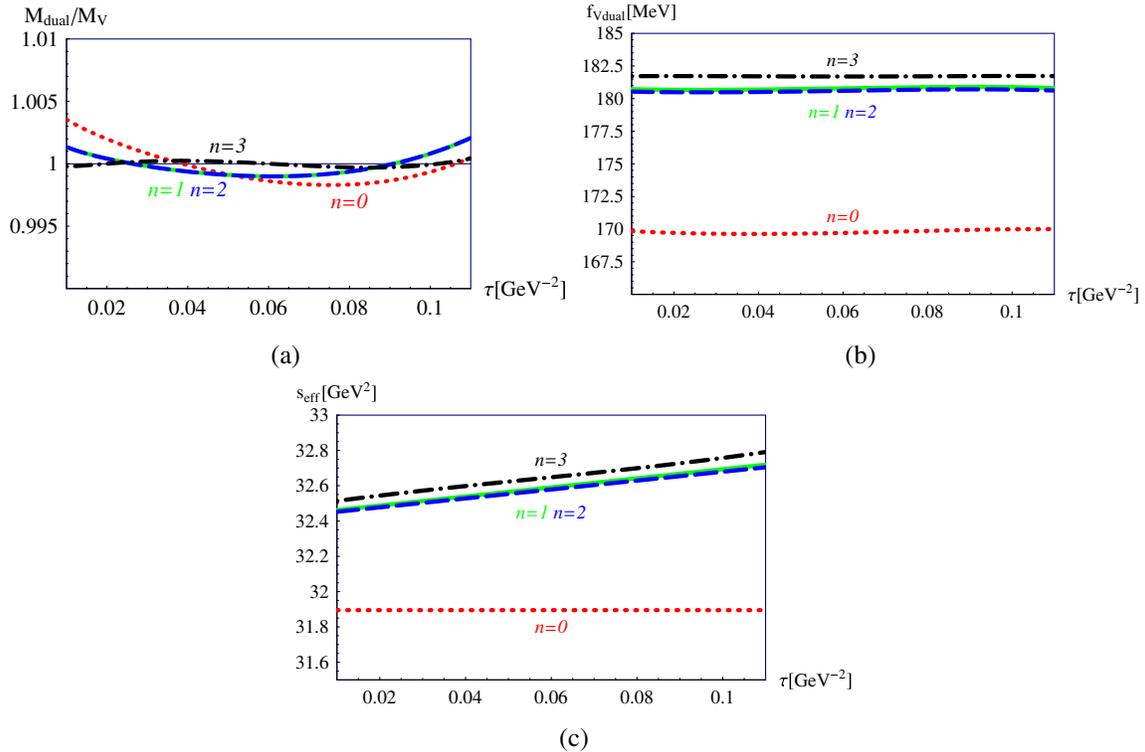


FIG. 3 (color online). Dependence on the Borel parameter τ of the dual mass (a) and the dual decay constant (b) of the B^* meson, obtained by adopting different Ansätze (4.2) for the effective threshold $s_{\text{eff}}(\tau)$ and fixing these thresholds by minimizing (4.3); the results are presented for the central values of all OPE parameters. (c) The τ -dependent effective thresholds as obtained by our algorithm. The integer $n = 0, 1, 2, 3$ is the degree of the polynomial in our Ansatz (4.2) for $s_{\text{eff}}(\tau)$: dotted lines (red)— $n = 0$; solid lines (green)— $n = 1$; dashed lines (blue)— $n = 2$; dot-dashed lines (black)— $n = 3$.

a systematic uncertainty not worse than 5 MeV and strongly diminishes the unphysical scale dependence of the decay constant f_{B^*} .

Figure 3 shows the application of our procedure for fixing the effective threshold and extracting the resulting f_{B^*} . The dependence of our QCD sum-rule result on the relevant OPE parameters, i.e., the b -quark mass $m_b \equiv \bar{m}_b(\bar{m}_b)$, the quark condensate $\langle \bar{q}q \rangle \equiv \langle \bar{q}q(2 \text{ GeV}) \rangle$ and the gluon condensate $\langle aGG \rangle$, proves to be well described by a linear relation:

$$f_{B^*}^{\text{dual}}(m_b, \langle \bar{q}q \rangle, \langle aGG \rangle) = (181.8 \pm 4_{\text{sys}}) \left(1 - \frac{11}{181.8} \delta_{m_b} \right) \left(1 + \frac{7}{181.8} \delta_{\langle qq \rangle} \right) \times \left(1 - \frac{1}{181.8} \delta_{\langle aGG \rangle} \right) \text{ MeV}, \quad (4.4)$$

with

$$\begin{aligned} \delta_{m_b} &= \frac{m_b - 4.247 \text{ GeV}}{0.034 \text{ GeV}}, \\ \delta_{\langle qq \rangle} &= \frac{|\langle \bar{q}q \rangle|^{1/3} - 0.267 \text{ GeV}}{0.017 \text{ GeV}}, \\ \delta_{\langle aGG \rangle} &= \frac{\langle aGG \rangle - 0.024 \text{ GeV}^4}{0.012 \text{ GeV}^4}. \end{aligned} \quad (4.5)$$

The above parameters δ take values between -1 and $+1$ when the corresponding OPE quantity varies in its 1σ interval. Varying all other OPE parameters in their 1σ ranges leads to an effect on f_{B^*} of less than 1 MeV and is not shown here. Trusting in our experience from exactly solvable examples, we assume that the systematic uncertainty interval contains the true value of the decay constant

and that inside this interval the true decay-constant value has a flat distribution.

As evident from Fig. 3(a), using a constant threshold leads to a contamination of the dual correlator by excited states (beyond the acceptable level), while this contamination is strongly reduced for $n > 0$: the values of the decay constant in Fig. 3(b) resulting for $n > 0$ are nicely grouped together, whereas the $n = 0$ prediction emerges some 10 MeV below.

A particularly convincing feature of the presented extraction procedure is the insensitivity of the extracted value of f_{B^*} (as well as that of f_B) to scale variations in the interval $\mu = 3\text{--}5$ GeV (Fig. 4), achieved by demanding an accurate reproduction of the B^* mass in the full τ window, which requires a specific choice of the τ window correlated with the scale μ at which the correlator is evaluated. Such choice of the τ window allows us to keep the systematic uncertainty, estimated by the half width of the band encompassing the results for the linear, quadratic, and cubic thresholds, at a level below 4 MeV in the full μ range. Therefore, (4.4) describes well the result for any μ from the range $\mu = 3\text{--}5$ GeV.

Assuming Gaussian distributions for all OPE parameters collected in (3.3), we get the distribution of f_{B^*} depicted in Fig. 5. For the average and the standard deviation of the B^* -meson decay constant, we obtain

$$f_{B^*} = (181.8 \pm 13.1_{\text{OPE}} \pm 4_{\text{sys}}) \text{ MeV}. \quad (4.6)$$

The OPE uncertainty is composed as follows: 11 MeV are due to the variation of m_b and 6 MeV arise from the quark condensate. The uncertainties of all other OPE parameters contribute less than 1 MeV to the OPE uncertainty of f_{B^*} . The corresponding QCD sum-rule outcome for the B -meson decay constant f_B from our earlier investigation [18] reads

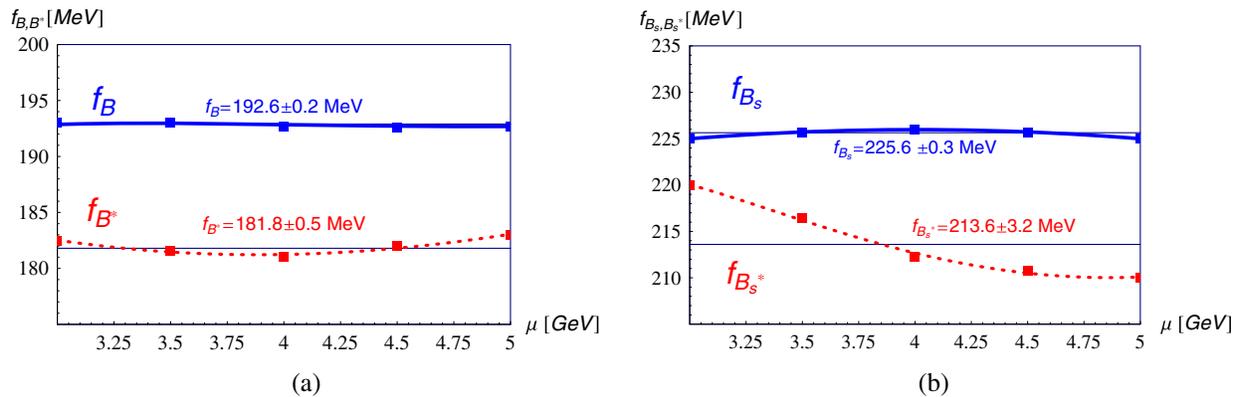


FIG. 4 (color online). Renormalization-scale dependence of the predicted decay constants: (a) $f_B^{\text{dual}}(\mu)$ and $f_{B^*}^{\text{dual}}(\mu)$, (b) $f_{B_s}^{\text{dual}}(\mu)$ and $f_{B_s^*}^{\text{dual}}(\mu)$. For each decay constant, we depict the μ -related uncertainty, i.e., the standard deviation calculated assuming a flat μ distribution in the range $\mu = 3\text{--}5$ GeV. Dotted lines (red)—vector beauty mesons; solid lines (blue)—pseudoscalar beauty mesons.

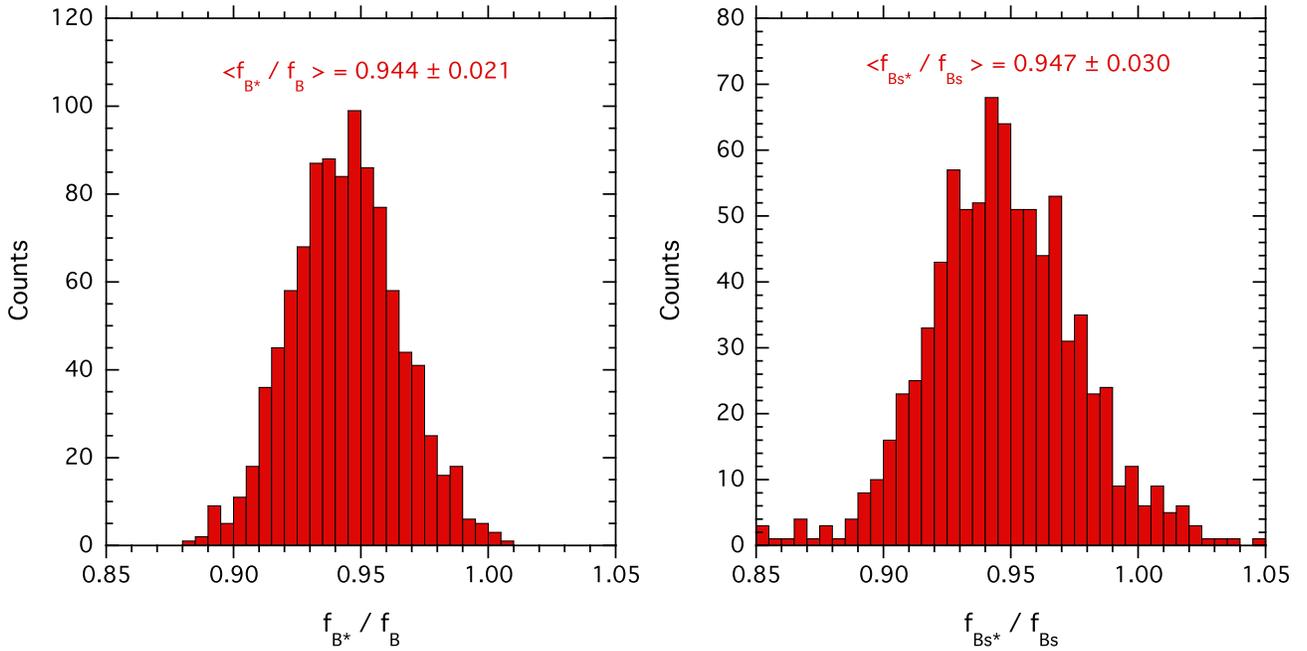


FIG. 5 (color online). Distributions of the ratios f_{B^*}/f_B and $f_{B_s^*}/f_{B_s}$ of beauty-meson decay constants, obtained by generating 1000 bootstrap events. For both ratios, their final distributions possess Gaussian-like shapes, with the standard deviations quoted in the plots.

$$\begin{aligned}
 f_B^{\text{dual}}(m_b, \langle \bar{q}q \rangle, \langle aGG \rangle) \\
 &= (192.6 \pm 3_{\text{sys}}) \left(1 - \frac{12.6}{192.6} \delta_{m_b} \right) \left(1 + \frac{6.8}{192.6} \delta_{\langle \bar{q}q \rangle} \right) \\
 &\quad \times \left(1 + \frac{1}{192.6} \delta_{\langle aGG \rangle} \right) \text{ MeV}.
 \end{aligned} \tag{4.7}$$

As is obvious from (4.4) and (4.7), the OPE uncertainties cancel out, to a great extent, in the ratio, which, consequently, can be predicted with a rather high accuracy:

$$f_{B^*}/f_B = 0.944 \pm 0.011_{\text{OPE}} \pm 0.018_{\text{sys}}. \tag{4.8}$$

The main contribution to the OPE error in the ratio arises from the gluon condensate, which enters with different sign in the pseudoscalar and the vector correlator (in detail: $\pm 0.01_{\langle aGG \rangle} \pm 0.005_{m_b} \pm 0.001_{\langle \bar{q}q \rangle}$). The total uncertainty of the ratio is dominated by the *systematic* uncertainties of the decay constants. Figure 5 shows the distribution of the ratio as obtained by a bootstrap analysis.

B. Decay constant of the B_s^* meson

For B_s^* , we choose the same Borel-parameter window as for B^* and again require that the deviation of the dual mass from the known B_s^* mass does not exceed 10 MeV in the full τ window. Our findings for the B_s^* -meson decay constant may be cast in the form

$$\begin{aligned}
 f_{B_s^*}^{\text{dual}}(\mu = \bar{\mu}, m_b, \langle \bar{s}s \rangle, \langle aGG \rangle) \\
 &= (213.6 \pm 6) \left(1 - \frac{13.2}{213.6} \delta_{m_b} \right) \left(1 + \frac{11.8}{213.6} \delta_{\langle \bar{s}s \rangle} \right) \\
 &\quad \times \left(1 - \frac{1}{213.6} \delta_{\langle aGG \rangle} \right) \text{ MeV},
 \end{aligned} \tag{4.9}$$

where $\bar{\mu}$ is defined in (4.11) and

$$\delta_{\langle \bar{s}s \rangle} = \frac{|\langle \bar{s}s \rangle|^{1/3} - 0.248 \text{ GeV}}{0.033 \text{ GeV}}. \tag{4.10}$$

Unfortunately, the sensitivity of $f_{B_s^*}$ to the choice of the scale μ at which the vector correlator is evaluated turns out to be rather pronounced. This dependence on the choice of μ may be parametrized by a series in powers of $\log(\mu/\bar{\mu})$:

$$\begin{aligned}
 f_{B_s^*}^{\text{dual}}(\mu) &= 213.6 \text{ MeV} [1 - 0.12 \log(\mu/\bar{\mu}) + 0.11 \log^2(\mu/\bar{\mu}) \\
 &\quad + 0.43 \log^3(\mu/\bar{\mu})], \\
 \bar{\mu} &= 3.86 \text{ GeV}.
 \end{aligned} \tag{4.11}$$

Averaging over the OPE parameters (using Gaussian distributions of all OPE parameters except for μ , for which a flat distribution in the range $\mu = 3\text{--}5$ GeV is assumed) yields

$$f_{B_s^*} = (213.6 \pm 18.2_{\text{OPE}} \pm 6_{\text{sys}}) \text{ MeV}, \tag{4.12}$$

with the following main contributions to the OPE error: 11.5 MeV from the s -quark condensate and 14.1 MeV from

m_b ; an uncertainty of 3.2 MeV arises from the μ dependence of $f_{B_s^*}$.

For the pseudoscalar B_s meson, our corresponding estimates read

$$\begin{aligned}
 f_{B_s}^{\text{dual}}(m_b, \langle \bar{s}s \rangle, \langle aGG \rangle) \\
 &= (225.6 \pm 3_{\text{sys}}) \left(1 - \frac{14.1}{225.6} \delta_{m_b} \right) \\
 &\times \left(1 + \frac{11.5}{225.6} \delta_{\langle \bar{s}s \rangle} \right) \left(1 + \frac{1}{225.6} \delta_{\langle aGG \rangle} \right) \text{ MeV}.
 \end{aligned} \tag{4.13}$$

As seen in Fig. 4, the sensitivity of f_{B_s} to the choice of μ is negligible. The total OPE uncertainty of f_{B_s} is rather large:

$$f_{B_s} = (225.6 \pm 18.3_{\text{OPE}} \pm 3_{\text{sys}}) \text{ MeV}. \tag{4.14}$$

The decomposition of the OPE error reads 11.5 MeV due to the error of the s -quark condensate and 14.1 MeV due to the error of $\bar{m}_b(\bar{m}_b)$; the uncertainties of the other OPE parameters contribute at the level of 1 MeV.

Similar to the f_{B^*}/f_B case, except for the gluon-condensate contribution the OPE uncertainties cancel, to a great extent, in the ratio of the decay constants, which may thus be predicted rather accurately:

$$f_{B_s^*}/f_{B_s} = 0.947 \pm 0.023_{\text{OPE}} \pm 0.020_{\text{sys}}. \tag{4.15}$$

The OPE uncertainty in the ratio is dominated by the sensitivity of $f_{B_s^*}$ to the choice of the scale μ . The (obligatory) bootstrap analysis gives for the ratio $f_{B_s^*}/f_{B_s}$ the nearly Gaussian distribution shown in Fig. 5.

V. SUMMARY AND CONCLUSIONS

Exploiting the tools offered by QCD sum rules, we analyzed in great detail the decay constants of the beauty vector mesons, paying special attention to the uncertainties arising in our predictions for the decay constants: the OPE error, related to the precision with which the QCD parameters are known, and the systematic error, intrinsic to the QCD sum-rule approach as a whole, reflecting the limited accuracy of the extraction procedure. Our findings are as follows:

- (i) As was already noted in the case of heavy pseudoscalar mesons [7], also for the vector correlator the perturbative expansion in terms of the heavy-quark pole mass does not exhibit good convergence. Reorganizing the OPE in terms of the corresponding running mass allows us to choose a range of scales for which, upon evaluation of the correlator, the

perturbative hierarchy becomes explicit. For scales $\mu \leq 2.5\text{--}3$ GeV, also the running-mass OPE does not exhibit any hierarchy of perturbative contributions; at too large scales $\mu \gtrsim 5\text{--}6$ GeV, we observe a strong cancellation between the large positive zero-order and the large negative first-order contributions, thus signalling that the accuracy of the OPE may deteriorate. There is, however, a sizeable interval of scales, $3 \leq \mu$ (GeV) ≤ 5 , where the $O(a^2)$ -truncated OPE provides a good description of the dual correlation function.

- (ii) Requiring the known value of the meson mass to be well reproduced in a relatively broad τ window leads, in the case of the vector mesons, to some correlation between the scale μ at which the correlator is evaluated and the upper boundary of the τ window: for $\mu \gtrsim 5$ GeV, the Borel window for the vector correlator shrinks and thus no meaningful extraction of the decay constants of B^* and B_s^* from sum rules is possible. The observed correlation between the parameters of the Borel window and the value of μ strongly reduces the (unphysical) μ dependence of the extracted beauty-meson decay constants.
- (iii) The τ dependence of the effective threshold and the details of the algorithm for fixing this quantity are crucial for obtaining realistic estimates of the systematic uncertainty of the extracted decay constant. For the analysis of the ratios of the decay constants of vector to pseudoscalar beauty mesons, where the mass splitting between the vector and the pseudoscalar partners amounts to some 45 MeV only, the stringent requirement to reproduce this splitting and the individual masses of vector and pseudoscalar beauty mesons with an accuracy not worse than 5 MeV in the full τ range is crucial for obtaining the low systematic uncertainty of the extracted decay constants.
- (iv) The decay constants of pseudoscalar and vector beauty mesons exhibit a strong dependence on the precise value of $\bar{m}_b(\bar{m}_b)$. Therefore, the $B_{(s)}$ and $B_{(s)}^*$ decay constants suffer from large OPE uncertainties. The systematic uncertainties of the extracted decay constants are of the level of a few MeV and remain under good control.
- (v) The ratios f_{B^*}/f_B and $f_{B_s^*}/f_{B_s}$ can be predicted with very good accuracy because of large cancellations between the OPE uncertainties in the ratios and a good control over the systematic uncertainties of the decay constants. Our final results read

$$f_{B^*}/f_B = 0.944 \pm 0.021,$$

$$f_{B_s^*}/f_{B_s} = 0.947 \pm 0.030,$$

where the error given is the total uncertainty, including the systematic and the OPE uncertainty. The resulting distributions are close to normal distributions (Fig. 5), thus the quoted errors are Gaussian standard deviations.

These results correspond to the m_b value (3.2) obtained by matching the QCD sum-rule analysis to the lattice results for f_B . Given Eqs. (4.4), (4.7), (4.9), and (4.13), one easily obtains the ratio for other input m_b values. For instance, using the PDG value $m_b = 4.18$ GeV [19] instead of the value (3.2) leads to a reduction by -0.005 for f_{B^*}/f_B and by -0.002 for $f_{B_s^*}/f_{B_s}$, which is well within the quoted uncertainties.

- (vi) Our results are in excellent agreement with and have a precision comparable to the recent lattice QCD values [14]

$$f_{B^*}/f_B = 0.941 \pm 0.026,$$

$$f_{B_s^*}/f_{B_s} = 0.953 \pm 0.023.$$

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