PHYSICAL REVIEW D **91,** 105028 (2015)

Nontopological solitons in the model of the self-interacting complex vector field

A. Yu. Loginov*

Tomsk Polytechnic University, 634050 Tomsk, Russia
(Received 8 March 2015; published 27 May 2015)

The model of the self-interacting complex vector field is considered. It is shown that there are nontopological solitons in this model, and research into their properties is undertaken. The asymptotic dependences on a phase frequency are derived for the energy and the Noether charge of the soliton in the thick-wall regime. The asymptotic expressions are obtained for the energy density, the Noether charge density, and the phase frequency of the soliton in the thin-wall regime. The soliton solutions of the model field equations are obtained numerically. The dependences on the phase frequency are presented for the energy and the Noether charge of the soliton. The dependence of the soliton energy on the soliton Noether charge is obtained numerically. It follows from this dependence that the nontopological soliton is unstable to the decay in the free massive vector bosons in the thick-wall regime but is stable to this decay in the thin-wall regime.

DOI: 10.1103/PhysRevD.91.105028 PACS numbers: 11.10.Lm, 11.27.+d

I. INTRODUCTION

A nontopological soliton is a spatially localized field configuration that is an extremum of the energy functional at a fixed value of the Noether charge. The distinguishing feature of nontopological solitons is the time dependence $\propto \exp(-i\omega t)$ of their fields. It is known that nontopological solitons exist in many models of field theory with global symmetries and conserved Noether charges [1,2]. The Q ball [3] that emerges in $U(1)$-invariant models of the self-interacting complex scalar field is the simplest nontopological soliton. Q balls also exist in scalar field models with the spontaneous breaking of global Abelian symmetry [4,5] and in scalar field models with global non-Abelian symmetry [6,7]. Q balls are present in supersymmetric extensions of the Standard Model with flat directions in the scalar field potential. Specifically, it was shown in Refs. [8,9] that the Q balls exist in the minimal supersymmetric Standard Model. In these supersymmetric extensions of the Standard Model, the Q balls are formed from scalar field condensates (s leptons or s quarks) carrying nonzero leptonic or baryonic quantum numbers. The Q balls are of great interest in cosmological models describing the evolution of the early Universe [10,11]. Within the framework of these models, the Q balls are the places where dark matter is concentrated; this fact can help explain the observed baryonic asymmetry.

The different types of nontopological solitons arise in global-symmetric models of several interacting scalar fields. The classic example is the nontopological soliton of the Friedberg-Lee-Sirlin model [12]. This model describes the system of two interacting scalar fields, one of which is real and the other is complex. The model has global $U(1)$ symmetry and renormalizable interaction potential.

All the solitons enumerated above are formed from condensates of scalar fields. The next type of quantum bosonic fields forming condensates is a four-vector field that describes particles of spin $s = 1$. These particles can be either massless gauge bosons (γ quantum, gluon, W boson), or massive vector particles (ρ, ω meson). The presence of massless gauge bosons means a local gauge invariance of a model. Let us note that, in all of the examples given above, the existence of nontopological solitons is caused by a global invariance of the corresponding Lagrangians. Therefore, a Noether charge of such solitons is not the source of a gauge field. Nontopological solitons exist also in models with a local gauge invariance, both Abelian [13–15] and non-Abelian [16,17]. In particular, the model of the self-interacting scalar Higgs doublet with non-Abelian $SU(2)$ gauge invariance was considered in Ref. [16]. It was shown that the existence of the nontopological solitons is possible in this model, and their main properties were studied. The vacuum of this model completely breaks the local gauge symmetry, and so the existence of the soliton becomes possible only due to the additional global Lagrangian symmetry. As a result of the complete $SU(2)$ gauge group breaking, the gauge fields decay exponentially at spatial infinity, and there are no long-range gauge fields of the soliton [16]. Let us remark in this connection that the nontopological solitons of Abelian gauge models [13–15] have a long-range gauge field.

The model of the self-interacting scalar Higgs doublet with non-Abelian $SU(2) \times U(1)$ gauge invariance was considered in Ref. [18]. This model is the bosonic sector of the Standard Model of the electroweak interactions [19,20]. It was shown that the nontopological solitons exist in this model. In contrast to the model considered in Ref. [16], the vacuum of the Standard Model does notbreak

*aloginov@tpu.ru

the $SU(2)\times U(1)$ gauge group completely but remains invariant with respect to its electromagnetic subgroup $U_{\mathrm{em}}(1)$. Therefore, the nontopological soliton of the Standard Model will have a long-range gauge field. This long-range gauge field is the electric field, and the Noether charge of the soliton is the electric charge. In comparison with Ref. [16], the presence of the long-range electric field changes the properties of the nontopological soliton significantly.

The solitons [16–18] are formed from condensates of Higgs scalar and gauge vector fields. In this paper, we investigate the nontopological soliton in the model of the self-interacting complex four-vector field. This model does not contain scalar fields, and its self-interacting complex four-vector field is not a gauge field. The Lagrangian of the model has global $U(1)$ symmetry, and the model's self-interaction potential is a cubic polynomial in the squared absolute value of the four-vector field. The properties of the nontopological soliton in this model appear similar in many ways to those of Q balls.

This paper is structured as follows. In Sec. II, we describe briefly the Lagrangian and the field equations for the model of the self-interacting complex vector field. It is shown, with the help of Hamilton formalism, that the number of independent field variables corresponds to the massive charged particle of spin $s=1$. By means of Hamilton formalism and the Lagrange multipliers method, the time dependence is established for the four-vector field of the nontopological soliton. In Sec. III, we give the ansatz used for solving the model field equations and investigate some of its properties. We derive the system of nonlinear differential equations for the functions of the ansatz and the expressions for the energy and Noether charge functionals in terms of these functions. The asymptotic properties of the solution as $r\to 0$ and $r\to\infty$ are investigated. The problem of soliton stability with respect to fluctuations is considered. In Sec. IV, we study the properties of the nontopological soliton in the thick-wall and thin-wall regimes. The stability of the nontopological soliton to the decay in the free massive vector bosons is considered for these extreme regimes. In Sec. V, we describe the procedure for numerically solving the system of nonlinear differential equations for the functions of the ansatz. We present the results of our numerical soliton solution for the functions of the ansatz, the energy density, and the Noether charge density for some set of the model's parameters. The dependences on the phase frequency are presented for the energy and Noether charge of the soliton. The dependence of the soliton energy on the soliton Noether charge is obtained numerically. Finally, in Sec. VI, we compare the properties for the nontopological soliton of the complex scalar field and the nontopological soliton of the complex vector field.

Throughout the paper, the natural units $c=1$, $\hbar=1$ are used.

II. THE LAGRANGIAN AND THE FIELD EQUATIONS

The Lagrangian density for the model of the self-interacting complex vector field is

$$\mathcal{L}=-\frac{1}{2}F_{\mu\nu}F^{*\mu\nu}-U(W_{\mu}W^{*\mu}), \tag{1}$$

where $F_{\mu\nu}=\partial_{\mu}W_{\nu}-\partial_{\nu}W_{\mu}$, W_{μ} is the complex four-vector field, and $U(W_{\mu}W^{*\mu})$ is a self-interaction potential for the complex four-vector field. In this paper, the following self-interaction potential is used:

$$U(W_{\mu}W^{*\mu})=-m^{2}W_{\mu}W^{*\mu}-\frac{g}{2}(W_{\mu}W^{*\mu})^{2}-\frac{h}{3}(W_{\mu}W^{*\mu})^{3}. \tag{2}$$

This potential coincides in form with the potential for the model of the self-interacting complex scalar field [1,5]; it is known that there are Q balls in this model. We suppose that the self-interaction constants g, h are positive. Note that model (1), (2) is nonrenormalizable, so it can be considered only as the effective field model.

Lagrangian (1) is invariant under the global Abelian transformations $W_{\mu}\to\exp(i\psi)W_{\mu}$. The corresponding Noether current is

$$j^{\mu}=i(F^{*\mu\nu}W_{\nu}-F^{\mu\nu}W_{\nu}^{*}). \tag{3}$$

The presence of a global symmetry group and the corresponding Noether charge is the necessary condition for the existence of nontopological solitons [1]. By varying Lagrangian (1) in $W^{*\mu}$, we obtain the field equations of the model

$$\partial_{\nu}F^{\nu\mu}-U'(W_{\mu}W^{*\mu})W^{\mu}=0, \tag{4}$$

where

$$U'(W_{\mu}W^{*\mu})=-m^{2}-g(W_{\mu}W^{*\mu})-h(W_{\mu}W^{*\mu})^{2} \tag{5}$$

is the derivative of $U(W_{\mu}W^{*\mu})$ with respect to its argument $W_{\mu}W^{*\mu}$. Using the well-known formula $T_{\mu\nu}=2\partial\mathcal{L}/\partial g^{\mu\nu}-g_{\mu\nu}\mathcal{L}$, we obtain the symmetric energy-momentum tensor of the model:

$$\begin{aligned}T_{\mu\nu}=&-F_{\mu\lambda}F_{\nu}^{*\lambda}-F_{\mu\lambda}^{*}F_{\nu}{}^{\lambda}+\frac{1}{2}g_{\mu\nu}F_{\sigma\lambda}F^{*\sigma\lambda}\\&-U'(W_{\mu}W^{*\mu})(W_{\mu}W_{\nu}^{*}+W_{\mu}^{*}W_{\nu})+U(W_{\mu}W^{*\mu})g_{\mu\nu}.\end{aligned} \tag{6}$$

From (6), we obtain the expression for the energy density of the self-interacting complex vector field:

$$\mathcal{E} \equiv T_{00} = F_{0i}F^{*}_{0i} + \frac{1}{2}F_{ij}F^{*}_{ij} - 2U'(W_{\mu}W^{*\mu})|W_0|^2 + U(W_{\mu}W^{*\mu}). \tag{7}$$

If the self-interaction constants g, h satisfy the condition

$$h > \frac{1}{4}\frac{g^2}{m^2}, \tag{8}$$

then energy density (7) is positive on arbitrary nontrivial field configurations and vanishes only as $W^{\mu} = 0$. Also, the energy density has no local extrema on spatially homogeneous and time-independent field configurations, and only one global minimum exists at $W^{\mu} = 0$. If condition (8) is not satisfied, then there are spatially homogeneous and time-independent field configurations on which energy density (7) is negative and unbounded from below (see Appendix B). Therefore model (1), (2) is stable if condition (8) holds and unstable otherwise. In the following, we shall assume that condition (8) is fulfilled. Thus, trivial field configuration $W^{\mu} = 0$ is the unique classical vacuum of the model. The vacuum $W^{\mu} = 0$ is invariant under the global gauge and Lorentz transformations, so there is no spontaneously broken symmetry in the model.

As $g \to 0$, $h \to 0$ field equations (4) go into well-known Proca equations [21] for the free massive vector field: $\partial_{\nu}F^{\nu\mu} + m^2 W^{\mu} = 0$. From these equations, it follows that the free massive vector field satisfies the additional condition $\partial_{\mu}W^{\mu} = 0$. This condition reduces the number of independent components for the free massive vector field from four to three. The three independent complex components correspond to the six spin-charge degrees of freedom for the charged massive particle of spin $s = 1$.

We shall use Hamiltonian formalism to find the number of independent canonical variables for the self-interacting complex vector field. From (1) and (2), we obtain the expressions for the generalized momenta:

$$\pi_0 = 0, \quad \pi_i = F^{*}_{i0}, \quad \pi^{*}_0 = 0, \quad \pi^{*}_i = F_{i0}, \tag{9}$$

and the Hamiltonian density

$$\mathcal{H} = \pi_i\partial_0 W^i + \pi^{*}_i\partial_0 W^{*i} - \mathcal{L} = \pi_i\pi^{*}_i + \frac{1}{2}F_{ij}F^{*}_{ij} - \pi_i\partial_i W_0 - \pi^{*}_i\partial_i W^{*}_0 + U(W_{\mu}W^{*\mu}). \tag{10}$$

From (9), it follows that the primary constraints are imposed on the generalized momenta: $\pi_0 = 0$, $\pi^{*}_0 = 0$. The primary constraints must hold at any instant of time; therefore, Poisson brackets of the generalized momenta π_0, π^{*}_0 with Hamiltonian $H = \int \mathcal{H} d^3x$ must vanish. This condition leads us to the secondary constraints among the canonical variables:

$$\partial_i\pi_i + U'(W_{\mu}W^{*\mu})W^{*}_0 = 0,$$
$$\partial_i\pi^{*}_i + U'(W_{\mu}W^{*\mu})W_0 = 0. \tag{11}$$

Note that secondary constraints (11) are identical to field equation (4) with the index $\mu = 0$. This equation does not contain the second time derivative of the fields, and so it is an analog of Gauss's law for gauge theories. The four constraints reduce the number of independent canonical variables for the massive complex vector field from 16 to 12. The 12 independent canonical variables π_i, π^{*}_i, W^i, W^{*i} correspond to the six spin-charge degrees of freedom as it should be for the charged massive particle of spin $s = 1$.

It can be shown, with the help of integration by parts and Eqs. (11), that expressions (7) and (10) lead to the same value for the field energy. By definition, the nontopological soliton is an extremum of the energy functional $E = \int \mathcal{E} d^3x = \int \mathcal{H} d^3x$ at the fixed value of the Noether charge $Q = \int j^0 d^3x$. From the method of Lagrange multipliers, it follows that the nontopological soliton is an unconditional extremum of the functional

$$F = \int \mathcal{H} d^3x - \omega \int j_0 d^3x = E - \omega Q, \tag{12}$$

where ω is the Lagrange multiplier. From (3) and (9), we obtain the expression for the Noether charge density in terms of the canonically conjugated variables: $j^0 = i(\pi^{*}_i W^{*i} - \pi_i W^i)$. In the following, we shall assume that the time components W_0, W^{*}_0 satisfy constraints (11). By varying (12) with respect to the independent canonically conjugated variables π_i, π^{*}_i, W^i, W^{*i}, and using the Hamilton field equations

$$\partial_0 W^i = \frac{\delta H}{\delta \pi_i}, \qquad \partial_0 \pi_i = -\frac{\delta H}{\delta W^i}, \tag{13}$$

we obtain

$$\begin{aligned}\delta F = &- \int (\partial_0\pi_i - i\omega\pi_i)\delta W^i d^3x \\ &- \int (\partial_0\pi^{*}_i + i\omega\pi^{*}_i)\delta W^{*i} d^3x \\ &+ \int (\partial_0 W^i + i\omega W^i)\delta\pi_i d^3x \\ &+ \int (\partial_0 W^{*i} - i\omega W^{*i})\delta\pi^{*}_i d^3x = 0.\end{aligned} \tag{14}$$

From (11) and (14), we get the time dependence for the components of the soliton four-vector field:

$$W_0(\mathbf{x}, t) = \exp(-i\omega t)w_0(\mathbf{x}),$$
$$\mathbf{W}(\mathbf{x}, t) = \exp(-i\omega t)\mathbf{w}(\mathbf{x}). \tag{15}$$

The extremum condition for F can be written in the general form $\delta F = \delta E - \omega \delta Q = 0$. Then it follows from this condition that the important relation occurs for the nontopological soliton

$$\frac{dE}{dQ} = \omega, \tag{16}$$

where ω is some function of Q.

III. THE ANSATZ AND SOME PROPERTIES OF THE SOLUTION

We use the spherically symmetric radial ansatz for the functions $w_0(\mathbf{x})$, $\mathbf{w}(\mathbf{x})$ in (15) to find the solutions to field equations (4):

$$w_0(\mathbf{x}) = iu(r), \qquad \mathbf{w}(\mathbf{x}) = \frac{\mathbf{x}}{r} v(r). \tag{17}$$

A similar ansatz was used in Ref. [16] to find the soliton solutions of the non-Abelian gauge model; it was also used in Ref. [18] to find the soliton solutions in the bosonic sector of the Standard Model. The feature of ansatz (17) is that the spatial components of the corresponding field strength tensor $F_{\mu\nu}$ vanish:

$$F^{00} \equiv 0, \quad F^{0i} = i\exp(-i\omega t)\frac{x^i}{r}(u'(r) - \omega v(r)), \quad F^{ij} = 0. \tag{18}$$

Substituting (15), (17), and (18) into field equations (4), we shall obtain the system of ordinary differential equations for the functions $u(r)$, $v(r)$:

$$u''(r) + \frac{2}{r}u'(r) - \omega\left(v'(r) + \frac{2}{r}v(r)\right) - m^2 u(r) - gu(r) \times (u(r)^2 - v(r)^2) - hu(r)(u(r)^2 - v(r)^2)^2 = 0, \tag{19}$$

$$\omega u'(r) + (m^2 - \omega^2)v(r) + gv(r)(u(r)^2 - v(r)^2) + hv(r)(u(r)^2 - v(r)^2)^2 = 0. \tag{20}$$

Let us consider some properties of this system. First, note that Eq. (20) contains the quadric term in ω, while Eq. (19) is linear in ω. It follows from this that field equations (4) with $\mu = 1, 2, 3$ containing the second time derivatives of the fields reduce to Eq. (20). Field equation (4) with $\mu = 0$ containing only the first time derivatives of the fields reduces to Eq. (19). Therefore, Eq. (20) is dynamic, while Eq. (19) is an analog of Gauss's law for gauge models.

Lagrangian (1) is invariant under the C, P, and T transformations of the four-vector field W^μ. The consequence of this is the invariance of system (19), (20) under the discrete transformations:

$$\omega, u, v \to -\omega, u, -v, \tag{21}$$

$$\omega, u, v \to \omega, -u, -v, \tag{22}$$

$$\omega, u, v \to -\omega, -u, v. \tag{23}$$

Substituting (15), (17), and (18) into (3) and (7), we shall obtain the expressions for the Noether current density and the energy density in terms of the functions $u(r)$, $v(r)$:

$$j^\mu(r) = (2v(r)(\omega v(r) - u'(r)), 0, 0, 0), \tag{24}$$

$$\mathcal{E}(r) = (\omega v(r) - u'(r))^2 + m^2(u(r)^2 + v(r)^2) + \frac{g}{2}(u(r)^2 - v(r)^2)(3u(r)^2 + v(r)^2) + \frac{h}{3}(u(r)^2 - v(r)^2)^2(5u(r)^2 + v(r)^2). \tag{25}$$

Then it follows from (21) and (23) that $j^0(\omega)$ is an odd function and $\mathcal{E}(\omega)$ is an even function of the phase frequency

$$j^0(-\omega) = -j^0(\omega), \qquad \mathcal{E}(-\omega) = \mathcal{E}(\omega). \tag{26}$$

It can be shown by direct substitution that M^{0ij} components of the orbital angular momentum tensor $M^{\mu\nu\rho} = x^\rho T^{\mu\nu} - x^\nu T^{\mu\rho}$ and S^{0ij} components of the spin angular momentum tensor $S^{\mu\nu\rho} = W^{*\nu}F^{\rho\mu} - W^{*\rho}F^{\nu\mu} + W^\nu F^{*\rho\mu} - W^\rho F^{*\nu\mu}$ vanish on field configuration (15), (17). Therefore the full angular momentum of the nontopological soliton is equal to zero. It is the consequence of the rotational symmetry for field configuration (15), (17).

It follows from the regularity condition of the solution at $r = 0$ and from the finiteness of the soliton energy that the functions $u(r)$, $v(r)$ satisfy the boundary conditions:

$$u'(0) = 0, \qquad v(0) = 0, \qquad u(\infty) = 0. \tag{27}$$

Substituting the power expansions for $u(r)$, $v(r)$ in Eqs. (19) and (20), we get the behavior of the solution as $r \to 0$:

$$u(r) = a_0 + \frac{a_2}{2!}r^2 + O(r^4), \quad v(r) = b_1 r + \frac{b_3}{3!}r^3 + O(r^5), \tag{28}$$

where

$$a_2 = \frac{a_0}{3}(a_0^4 h + a_0^2 g + m^2 - \omega^2), \qquad b_1 = -\frac{a_0\omega}{3},$$
$$b_3 = -\frac{a_0}{5\omega}(a_0^4 h + a_0^2 g + m^2)(5a_0^4 h + 3a_0^2 g + m^2) + \frac{a_0\omega}{15}(19a_0^4 h + 11a_0^2 g + 3m^2). \tag{29}$$

It follows from (28) that $u(r)$ is an even function of r, while $v(r)$ is an odd function of r. From (24), (25), (28), and (29), we obtain the power expansions for the Noether charge and energy densities as $r \to 0$:

$$j^0(r) = \frac{d_2}{2!} r^2 + O(r^4), \tag{30}$$

$$\mathcal{E}(r) = c_0 + \frac{c_2}{2!} r^2 + O(r^4), \tag{31}$$

where

$$\begin{aligned} c_0 &= a_0^2 m^2 + \frac{3}{2} a_0^4 g + \frac{5}{3} a_0^6 h, \\ c_2 &= \frac{2}{3} a_0 a_2 (16 a_0^4 h + 10 a_0^2 g + 4m^2 + \omega^2) \\ &\quad + \frac{2}{9} a_0^2 \omega^2 (-3 a_0^4 h - a_0^2 g + m^2 + \omega^2), \\ d_2 &= \frac{4}{9} a_0^2 \omega (a_0^4 h + a_0^2 g + m^2). \end{aligned} \tag{32}$$

It follows from (30) and (31) that the Noether charge density vanishes at the center of the nontopological soliton, while the energy density is nonzero.

The finiteness of the soliton energy leads us to the following asymptotics of $u(r)$, $v(r)$ as $r \to \infty$:

$$u(r) \sim c_\infty \frac{e^{-\Delta r}}{r}, \quad v(r) \sim c_\infty \frac{\omega}{\Delta} \left(1 + \frac{1}{\Delta r}\right) \frac{e^{-\Delta r}}{r}, \tag{33}$$

where $\Delta = \sqrt{m^2 - \omega^2}$. It follows from (33) that the phase frequency must satisfy the inequality $|\omega| < m$. Using (24), (25), and (33), we get the asymptotic expressions for the energy and Noether charge densities of the soliton as $r \to \infty$:

$$\mathcal{E}(r) \sim c_\infty^2 m^2 \frac{(\Delta^4 r^2 + (1 + \Delta r)^2 (m^2 + \omega^2))}{(\Delta r)^4} \times \exp(-2\Delta r), \tag{34}$$

$$j^0(r) \sim 2 c_\infty^2 m^2 \omega \frac{(1 + \Delta r)^2}{(\Delta r)^4} \exp(-2\Delta r). \tag{35}$$

If the values of the parameters ω, m, g, and h are fixed, then the behavior of the solution $u(r)$, $v(r)$ as $r \to 0$ is determined by the single parameter a_0. The behavior of the solution $u(r)$, $v(r)$ as $r \to \infty$ is also determined by the single parameter c_∞. Thus, we have the two free parameters. With the help of Eq. (20) and its derivative with respect to r, we can exclude $v(r)$ and $v'(r)$ from Eq. (19). In the result, we obtain a differential equation of the second order for $u(r)$. The continuity condition for $u(r)$ and its derivative $u'(r)$ at arbitrary r give us two equations. So we shall have the two equations for determining the two parameters a_0 and c_∞. According to Ref. [22], this fact is an argument in favor of the existence of the solution for the boundary value problem (19), (20), and (27) in some range of the parameters ω, m, g, and h.

Integrating constraint equations (11) over all space, using Gauss's theorem, and taking into account asymptotics (33), we obtain the integral relation $\int U'(W_\mu(\mathbf{x}) W^{*\mu}(\mathbf{x})) W_0(\mathbf{x}) d^3x = 0$. On field configuration (15), (17), this relation takes the form

$$\int_0^\infty U'(W_\mu W^{*\mu}) u(r) r^2 dr = 0, \tag{36}$$

where $W_\mu W^{*\mu} = u(r)^2 - v(r)^2$. Note that the function $U(W_\mu W^{*\mu})$ decreases monotonically when condition (8) is satisfied. Its derivative $U'(W_\mu W^{*\mu})$ will be negative at any value of the argument $W_\mu W^{*\mu}$. Therefore, it follows from (36) that the function $u(r)$ is alternating, and so it must have at least one node.

It is obvious that soliton solution (15), (17) has three translational zero modes $\delta_i^{(0)\mu} = \partial_i w^\mu$. The rotational zero modes are absent due to the spherical symmetry of the soliton. Following Ref. [5], we wish to show that the extra zero mode appears when the condition $dQ/d\omega = 0$ holds. Let us consider fluctuations in the functional subspace of the spherically symmetric field configuration (15), (17). The first variation of the energy functional for the fixed Noether charge vanishes on the soliton solution. The second variation can be written in the form

$$\delta^2 E_Q = 4\pi \int_0^\infty \tilde{\psi} \mathcal{D} \psi r^2 dr + \frac{1}{I} \left(4\pi \int_0^\infty \tilde{b} \psi r^2 dr\right)^2, \tag{37}$$

where

$$\mathcal{D} = \begin{pmatrix} \Delta_r + U' + 2u^2 U'' & -\omega d/dr - 2\omega/r - 2uvU'' \\ \omega d/dr - 2uvU'' & -U' + 2v^2 U'' - \omega^2 \end{pmatrix}, \tag{38}$$

$$\tilde{b} = \left(-v' - 2v/r, \quad u' - 2\omega v\right), \tag{39}$$

$$I = 4\pi \int_0^\infty v^2 r^2 dr, \tag{40}$$

$U'' = -g - 2h(u^2 - v^2)$ is the second derivative of U with respect to its argument $W_\mu W^{*\mu} = u^2 - v^2$, $\tilde{\psi} = (\delta u, \delta v)$ is the transposed column ψ of u, v fluctuations, and $\Delta_r = d^2/dr^2 + 2/r$ is the radial part of the Laplacian. It follows from (37) that the spherically symmetric zero mode must satisfy the linear integro-differential equation

$$\Lambda\psi \equiv \mathcal{D}\psi + b\frac{4\pi}{I}\int_0^\infty \tilde{b}\psi r^2 dr = 0. \tag{41}$$

With $A \equiv 4\pi I^{-1}\int_0^\infty \tilde{b}\psi r^2 dr$ from Eq. (41), we obtain

$$\mathcal{D}\psi + Ab = 0. \tag{42}$$

Differentiating Eqs. (19) and (20) with respect to ω, one sees that Eq. (42) is satisfied by $\tilde{\psi} = \tilde{w}_\omega \equiv (u_\omega, v_\omega)$ with $A = 1$:

$$\mathcal{D}w_\omega + b = 0, \tag{43}$$

where the subscript ω means the differentiation in ω in the following. Now, if w_ω is the solution to Eq. (42), then from the condition $A = 4\pi I^{-1}\int_0^\infty \tilde{b}w_\omega r^2 dr = 1$ we get the relation

$$\int_0^\infty (\tilde{b}w_\omega - v^2)r^2 dr = 0. \tag{44}$$

Differentiating (24) with respect to ω, we obtain the expression

$$j^0_\omega = 2v^2 + 4\omega v v_\omega - 2u' v_\omega - 2u'_\omega v. \tag{45}$$

It is readily seen that the derivative of the Noether charge with respect to ω,

$$\frac{dQ}{d\omega} = 4\pi\int_0^\infty j^0_\omega r^2 dr, \tag{46}$$

vanishes if the relation (44) holds, and conversely. Thus, the vanishing of $dQ/d\omega$ is the indication of the extra zero mode.

We shall now show that the appearance of the extra zero mode when $dQ/d\omega = 0$ really signals the change of sign for an eigenvalue of the integro-differential operator Λ. That is, if $\lambda(\omega)$ is the eigenvalue of Λ, which is zero when $dQ/d\omega = 0$, then $\lambda(\omega) > 0$ when $dQ/d\omega < 0$, and conversely. Indeed, if $\psi(\omega)$ is the eigenvector corresponding to $\lambda(\omega)$, and $\bar{\omega}$ is defined by $\lambda(\bar{\omega}) = 0$, then we have $\psi(\bar{\omega}) = w_\omega(\bar{\omega})$. Differentiating the equation $\Lambda(\omega)\psi(\omega) = \lambda(\omega)\psi(\omega)$ with respect to ω, we obtain at $\omega = \bar{\omega}$

$$\Lambda_\omega(\bar{\omega})w_\omega(\bar{\omega}) + \Lambda(\bar{\omega})\psi_\omega(\bar{\omega}) = \lambda_\omega(\bar{\omega})w_\omega(\bar{\omega}). \tag{47}$$

By multiplying this equation on the left with $\tilde{w}_\omega$ and integrating over all space, we obtain

$$\int \tilde{w}_\omega(\bar{\omega})\Lambda_\omega(\bar{\omega})w_\omega(\bar{\omega})d^3x = \lambda_\omega(\bar{\omega})\int \tilde{w}_\omega(\bar{\omega})w_\omega(\bar{\omega})d^3x, \tag{48}$$

where the Hermiticity of the matrix differential operator $\mathcal{D}$ and the relation $\Lambda(\bar{\omega})w_\omega(\bar{\omega}) = 0$ were used. However, the left-hand side of Eq. (48) can be written as

$$\begin{aligned}\frac{d}{d\omega}\int \tilde{w}_\omega(\bar{\omega})\Lambda(\bar{\omega})w_\omega(\bar{\omega})d^3x \\ - 2\int \tilde{w}_{\omega\omega}(\bar{\omega})\Lambda(\bar{\omega})w_\omega(\bar{\omega})d^3x \\ = \frac{d}{d\omega}\int \tilde{w}_\omega(\bar{\omega})\Lambda(\bar{\omega})w_\omega(\bar{\omega})d^3x \equiv G_\omega(\bar{\omega}),\end{aligned} \tag{49}$$

where

$$G(\omega) = \int \tilde{w}_\omega(\omega)\Lambda(\omega)w_\omega(\omega)d^3x. \tag{50}$$

Using the explicit form (41) of $\Lambda(\omega)$ and expressions (43) and (45), it can be shown that $G(\omega)$ can be represented as

$$\begin{aligned}G(\omega) &= \left(\int \tilde{b}w_\omega d^3x\right)\frac{1}{I}\left(\int \tilde{b}w_\omega d^3x - I\right) \\ &= \frac{1}{2}\left(\frac{1}{2I}\frac{dQ}{d\omega} - 1\right)\frac{dQ}{d\omega}.\end{aligned} \tag{51}$$

Since $dQ/d\omega = 0$ at $\omega = \bar{\omega}$, it follows from (48)–(51) that

$$\mathrm{sgn}(G(\omega)) = \mathrm{sgn}(\lambda(\omega)) = -\mathrm{sgn}(dQ/d\omega) \tag{52}$$

in a small neighborhood of $\bar{\omega}$. But this is enough to prove that $\mathrm{sgn}(\lambda(\omega)) = -\mathrm{sgn}(dQ/d\omega)$ for any allowable values of ω. Thus we have shown that if $dQ/d\omega > 0$, then the soliton solution has at least one negative mode. This implies that the nontopological soliton is unstable if the condition $dQ/d\omega > 0$ holds. Note that this condition can also be written in the form $(\omega/Q)dQ/d\omega > 0$. In this form, it is valid for an arbitrary choice of sign in the definition of the Noether charge Q and the phase frequency ω.

IV. THE THICK-WALL AND THIN-WALL REGIMES OF THE SOLITON

Let us investigate the properties of the nontopological soliton in the two extreme regimes. In the thick-wall regime, the absolute value of the phase frequency $|\omega|$ tends to its maximum value: $|\omega| \to m$. In this case, the dumping coefficient $\Delta = \sqrt{m^2 - \omega^2}$ in asymptotics (33) tends to zero that leads to the spatial spreading of the nontopological soliton. Let us consider the functional F, which was defined in (12). This functional is related to the energy functional by means of Legendre transformation: $F(\omega) = E(Q) - \omega Q$. On field configuration (15), (17), the functional F is of the form

$$\begin{aligned}F = -4\pi\int_0^\infty ((u'(r) - \omega v(r))^2 + m^2(u(r)^2 - v(r)^2) \\ + \frac{g}{2}(u(r)^2 - v(r)^2)^2 + \frac{h}{3}(u(r)^2 - v(r)^2)^3)r^2 dr.\end{aligned} \tag{53}$$

In (53), we undertake scale transformation of the fields and coordinates:

$$u = \Delta^2 \bar{u}, \qquad v = \Delta \bar{v}, \qquad r = \Delta^{-1} \bar{r}. \tag{54}$$

As a result, the functional (53) can be written as

$$F = \Delta \bar{F} + O(\Delta^3), \tag{55}$$

where

$$\bar{F} = 4\pi \int_0^\infty (2m\bar{v}(\bar{r})\bar{u}'(\bar{r}) + \bar{v}(\bar{r})^2 - m^2\bar{u}(\bar{r})^2 - \frac{g}{2}\bar{v}(\bar{r})^4)\bar{r}^2 d\bar{r}. \tag{56}$$

Notice that the functional $\bar{F}$ does not depend on Δ. In the limit $\Delta \to 0$, it is possible to ignore the higher-order terms in Δ in expression (55). Using known properties of Legendre transformation, we obtain consecutively

$$Q(\omega) = -\frac{dF(\omega)}{d\omega} = \frac{\omega}{\sqrt{m^2 - \omega^2}} \bar{F}, \tag{57}$$

$$E(\omega) = F(\omega) - \omega \frac{dF(\omega)}{d\omega} = \frac{m^2}{\sqrt{m^2 - \omega^2}} \bar{F}. \tag{58}$$

From expressions (57) and (58), we obtain the dependence of the soliton energy on the soliton Noether charge as $|\omega| \to m$:

$$E(Q) = m|Q| + \frac{m\bar{F}^2}{2}|Q|^{-1} + O(|Q|^{-3}). \tag{59}$$

It follows from (57) and (58) that the energy and Noether charge of the nontopological soliton tend to infinity as $|\omega| \to m$.

We suppose that the dimensionless functions $m\bar{u}(\bar{r})$ and $\bar{v}(\bar{r})$ have values of the same order in the thick-wall regime. Then it follows from (54) that the relation holds in the thick-wall regime:

$$|u/m| \ll |v/m| \ll 1. \tag{60}$$

According to (28) and (29), this relation is not valid in the range $r \lesssim \omega^{-1} \approx m^{-1}$. Note, however, that the range $r \lesssim m^{-1}$ is much less than the soliton size in the thick-wall regime. From (24), (25), and (60), we obtain the approximate expressions for the energy and Noether charge densities in the thick-wall regime as $r \gtrsim m^{-1}$:

$$\mathcal{E}(r) \approx (\omega^2 + m^2)v(r)^2 \approx 2m^2 v(r)^2, \tag{61}$$

$$j^0(r) \approx 2\omega v(r)^2. \tag{62}$$

From (61) and (62), it follows that the relation holds between the energy and Noether charge densities in the thick-wall regime as $r \gtrsim m^{-1}$:

$$\mathcal{E}(r) \approx \omega j^0(r). \tag{63}$$

The next extremal regime of the nontopological soliton is the thin-wall regime in which the absolute value of the phase frequency $|\omega|$ tends to some minimum value $\omega_{\min}$. In this regime, the absolute value of the three-vector field $|\mathbf{w}(r)| = v(r)$ tends to some constant value v in the soliton interior. Along with this, the spatial size of the soliton increases indefinitely, and its energy and Noether charge tend to infinity. Throughout the main bulk of the soliton, when its volume $V \to \infty$, the gradient operator gives a factor proportional to $V^{-1/3}$. Therefore, we can ignore the gradient term u' in comparison with v in the expression for the generalized momentum $\pi_i = i\exp(i\omega t)n_i(u'(r) - \omega v(r))$. Then we obtain from constraint equations (11)

$$U'(W_\mu W^{*\mu})W_0 = -\partial_i \pi_i^* = i\exp(-i\omega t)\frac{2\omega v}{r}. \tag{64}$$

If condition (8) has been satisfied, then $U(W_\mu W^{*\mu})$ is the monotonically decreasing function, and so $U'(W_\mu W^{*\mu})$ does not vanish. Therefore, it follows from (64) that in the soliton interior at sufficiently large r the time component W_0 becomes negligibly small. Thus in the interior, which determine the energy and the Noether charge of the soliton in the thin-wall regime, the functions of ansatz (17) satisfy the relation $v(r) \approx v$, $u(r) \approx 0$. On such field configurations, Noether charge density (24) and energy density (25) take the form

$$j^0 \equiv q = 2\omega_{\min} v^2, \qquad \mathcal{E} = (m^2 + \omega_{\min}^2)v^2 - \frac{g}{2}v^4 + \frac{h}{3}v^6. \tag{65}$$

From (65), we get the ratio $\mathcal{E}/q$:

$$\frac{\mathcal{E}}{q} = \frac{q}{4v^2} + \frac{m^2 v^2}{q} - \frac{g}{2}\frac{v^4}{q} + \frac{h}{3}\frac{v^6}{q}. \tag{66}$$

Ratio (66) reaches the minimum value

$$\frac{\mathcal{E}}{q} = \frac{1}{4}\sqrt{16m^2 - 3\frac{g^2}{h}}, \tag{67}$$

when

$$v = \frac{\sqrt{3}}{2}\sqrt{\frac{g}{h}}, \tag{68}$$

$$q = \frac{3}{8} g h^{-\frac{3}{2}} \sqrt{16hm^2 - 3g^2}. \tag{69}$$

From (67) and (69), we obtain the value of the energy density

$$\mathcal{E} = \frac{3}{2}\frac{gm^2}{h} - \frac{9}{32}\frac{g^3}{h^2}. \tag{70}$$

Expressions (68)–(70) are the thin-wall asymptotic values of $|\mathbf{w}| = v$, the Noether charge density, and the energy density in the soliton interior, respectively. Note that the relation holds at these values of v and q:

$$\omega_{\min} = \frac{q}{2v^2} = \frac{\mathcal{E}}{q} = \frac{1}{4}\sqrt{16m^2 - 3\frac{g^2}{h}}. \tag{71}$$

It follows from (71) that in the thin-wall regime the energy and the Noether charge of the soliton satisfy the asymptotic relation

$$\lim_{\omega \to \omega_{\min}} \frac{E}{Q} = \omega_{\min}. \tag{72}$$

From the condition $0 < |\omega_{\min}| < m$, it follows that the self-interacting constants in (2) must satisfy the inequality

$$h > \frac{3}{16}\frac{g^2}{m^2}. \tag{73}$$

This inequality will obviously be valid if condition (8) holds.

The expression (65) for the energy density in the thin-wall regime can be represented as the sum of the kinetic and potential terms

$$\mathcal{E} = \mathcal{T} + \mathcal{V}, \tag{74}$$

where

$$\mathcal{T} = \omega_{\min}^2 v^2, \qquad \mathcal{V} = m^2 v^2 - \frac{g}{2}v^4 + \frac{h}{3}v^6. \tag{75}$$

For asymptotic values (68) and (71), the contributions of these terms are equal to each other:

$$\mathcal{T} = \mathcal{V}, \qquad \mathcal{E} = 2\mathcal{T} = 2\mathcal{V}. \tag{76}$$

Let us consider the question of the soliton stability with respect to the decay in the free massive vector bosons. Using the standard methods [12], it can be shown that the energy and the Noether charge of the lowest-energy free-boson solution satisfy the usual relation

$$E = m|Q|. \tag{77}$$

The lowest-energy free-boson solution corresponds to an ensemble of the free massive vector bosons at rest. From (59) and (77), it follows that, in the thick-wall regime, the energy of the nontopological soliton with the given Q tends from above to the energy of the free-boson solution with the same Q. From (72), (73), and (77), it follows that, in the thin-wall regime, the energy of the nontopological soliton with the given Q is less than the energy of the free-boson solution with the same Q. Hence, in the thin-wall regime, the nontopological soliton is stable to the decay in the free massive vector bosons. On the other hand, in the thick-wall regime, the nontopological soliton is unstable to such decay. This instability can be either classical or quantum mechanical. If the quadratic fluctuation operator has one or more negative eigenvalues in the soliton neighborhood, then the soliton is unstable to the decay on the classical level. If the quadratic fluctuation operator has no negative eigenvalues in the soliton neighborhood, then the soliton decay is a result of tunneling.

It is shown in Sec. III that the quadratic fluctuation operator Λ has at least one negative eigenvalue in the soliton neighborhood if $(\omega/Q)dQ/d\omega > 0$. From (57), it follows that $(\omega/Q)dQ/d\omega = m^2(m^2 - \omega^2)^{-1} > 0$ in the thick-wall regime. We could, therefore, be sure that the nontopological soliton is classically unstable for $|\omega| \in (\bar{\omega}, m)$, where $\bar{\omega}$ is the phase frequency at which $dQ/d\omega = 0$. The research of the soliton stability as $(\omega/Q)dQ/d\omega < 0$ is a rather complicated task and lies beyond the scope of this paper. Let us remark in this regard that if $(\omega/Q)dQ/d\omega < 0$, then the nontopological soliton of the self-interacting complex scalar field (Q ball) is classically stable [5].

V. NUMERICAL RESULTS

The system of differential equations (19) and (20) with boundary condition (27) is the mixed boundary value problem on the semi-infinite interval $r \in [0, \infty)$. This boundary value problem can be solved only by numerical methods. In this paper, the boundary value problem (19), (20), and (27) was solved by using the MAPLE package [23] by the method of finite differences and subsequent Newtonian iterations. The point $r = 0$ is the regular singular point of system (19), (20), and so we applied difference schemes that do not use the boundary values of the functions. Richardson extrapolation was used to accelerate the convergence of the numerical procedure to the exact solution. Formulas (16) and (36) were used to check the correctness of numerical solutions.

The system of differential equations (19) and (20) depends on the four parameters ω, m, g, and h. It is readily seen that a solution of this system has the following dependence on the mass parameter m:

$$\begin{aligned} u(r, m, \omega, g, h) &= m\tilde{u}(\rho, \tilde{\omega}, g, \tilde{h}), \\ v(r, m, \omega, g, h) &= m\tilde{v}(\rho, \tilde{\omega}, g, \tilde{h}), \end{aligned} \tag{78}$$

where $\rho = mr$, $\tilde{\omega} = \omega/m$, and $\tilde{h} = m^2 h$. Note that the dimensionless functions $\tilde{u}$ and $\tilde{v}$ depend only on the three dimensionless parameters $\tilde{\omega}$, g, and $\tilde{h}$. Let us consider a

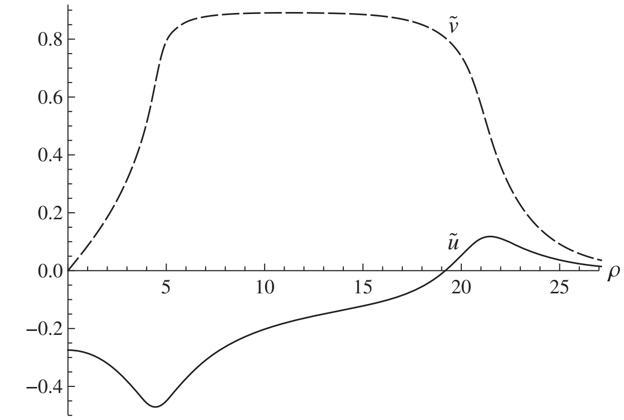

FIG. 1. The numerical solution $\tilde{u}(\rho)$, $\tilde{v}(\rho)$ for $\tilde{\omega} = 0.915$, $g = 1$, $\tilde{h} = 1$. The solid curve is for $\tilde{u}(\rho)$, and the dashed curve is for $\tilde{v}(\rho)$.

general case in which the parameters g and $\tilde{h}$ are values of the same order and take them equal to unity: $g = 1$, $\tilde{h} = 1$. We shall use the dimensionless functions $\tilde{u}$, $\tilde{v}$ and the dimensionless combinations E/m, $\mathcal{E}/m^4$, q/m^3 for presenting numerical results, so the mass parameter m can be an arbitrary positive value.

Figure 1 presents the numerical solution for $\tilde{u}(\rho)$, $\tilde{v}(\rho)$, corresponding to the dimensionless phase frequency $\tilde{\omega} = 0.915$. Figure 2 shows the energy density $\mathcal{E}(\rho)$ divided by m^4 and the Noether charge density $q(\rho)$ divided by m^3, corresponding to the solution in Fig. 1. Notice that $\tilde{\omega} = 0.915$ is the minimum value for which we managed to obtain the numerical solution of the boundary value problem (19), (20), and (27) at $g = 1$, $\tilde{h} = 1$. The difficulty of the numerical solution under a further decrease in $\tilde{\omega}$ is associated with the fact that the boundary value problem (19), (20), and (27) becomes stiff; i.e., small variations of $\tilde{\omega}$ lead to large variations in the solution $\tilde{u}(\rho)$, $\tilde{v}(\rho)$ of the boundary value problem. Thus, the solution in Fig. 1 most closely matches the thin-wall regime, described in Sec. IV. The characteristic features of this solution are the presence of a transition region at small ρ, the vanishing of the Noether charge density $q(\rho)$ at $\rho = 0$, and a significant decrease in the energy density $\mathcal{E}(\rho)$ at $\rho = 0$. The presence of the transition region at small ρ and the vanishing of the Noether charge density at $\rho = 0$ follow from boundary conditions (27) and the behavior of $v(r)$ at small r in (28). Such behavior of $v(r)$ is due to the fact that the spatial components of the regular spherically symmetric vector field $\mathbf{w}(\mathbf{x})$ in (17) must vanish at $r = 0$. Note, in this connection, that nontopological solitons of the scalar field models do not have a central transition region, and their Noether charge density does not vanish at the center [1,5,13]. Between the central and edge transition regions, the energy and Noether charge densities are approximately constant, that is the attribute of the thin-wall regime. Note also that the values of v, q, and $\mathcal{E}$ in the soliton interior are close to their asymptotic thin-wall values (68), (69), and (70).

Figures 3 and 4 are analogous to Figs. 1 and 2 but correspond to the dimensionless phase frequency $\tilde{\omega} = 0.9999$. At the given value of $\tilde{\omega}$, the nontopological soliton is in the thick-wall regime. Comparison of Figs. 1 and 2 and Figs. 3 and 4 shows that the shapes of the nontopological soliton in the thin-wall and thick-wall regimes essentially differ. In particular, the values of the functions $\tilde{u}(\rho)$ and $\tilde{v}(\rho)$ in the thick-wall regime are significantly less than those in the thin-wall regime. From Fig. 3, it follows that, except for the small central region, the relation $|\tilde{u}| \ll |\tilde{v}| \ll 1$ holds in accordance with (60). Note that Fig. 4 shows only the dependence of $\mathcal{E}/m^4$ on ρ. This is because the dependence of q/m^3 on ρ is visually indistinguishable from that of $\mathcal{E}/m^4$. This fact is in accordance with (63) by noting that $|\omega| \to m$ in the thick-wall regime. Let us remark that the dimensionless

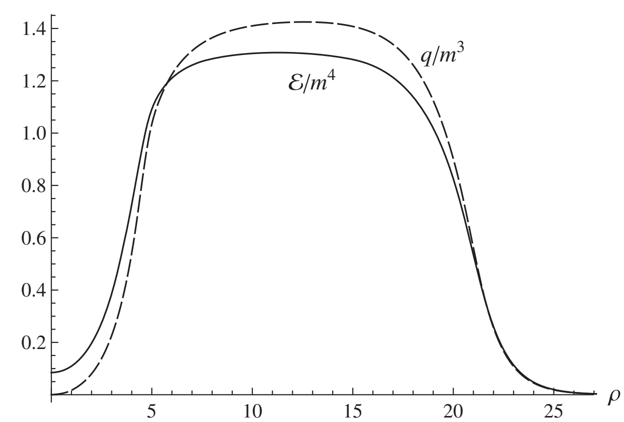

FIG. 2. The dependence of $\mathcal{E}/m^4$ and q/m^3 on ρ. The solid curve is for $\mathcal{E}/m^4$, and the dashed curve is for q/m^3. The parameters $\tilde{\omega}$, g, $\tilde{h}$ are the same as in Fig. 1.

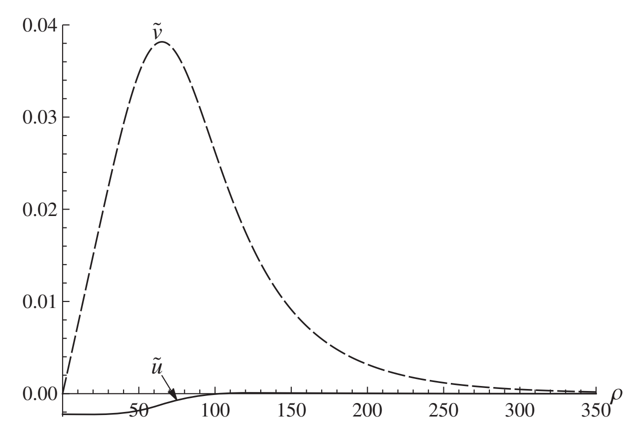

FIG. 3. The numerical solution $\tilde{u}(\rho)$, $\tilde{v}(\rho)$ for $\tilde{\omega} = 0.9999$, $g = 1$, $\tilde{h} = 1$. The solid curve is for $\tilde{u}(\rho)$, and the dashed curve is for $\tilde{v}(\rho)$.

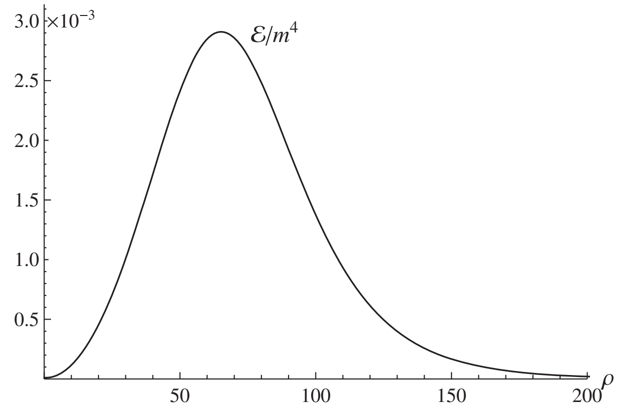

FIG. 4. The dependence of $\mathcal{E}/m^4$ on ρ. The parameters $\tilde{\omega}$, g, $\tilde{h}$ are the same as in Fig. 3.

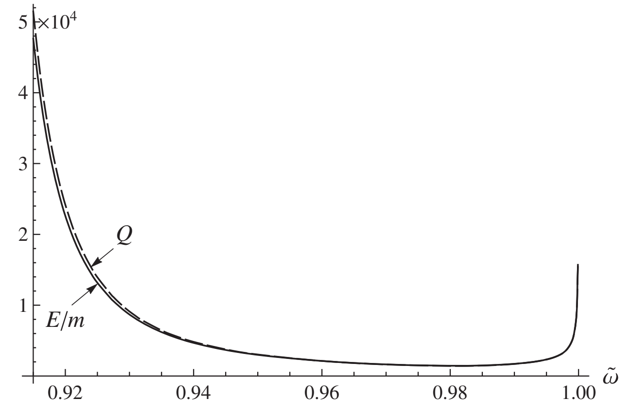

FIG. 6. The soliton energy divided by m (solid curve) and the soliton Noether charge (dashed curve) as functions of the dimensionless phase frequency $\tilde{\omega}$.

combination $\mathcal{E}/m^4$ in Fig. 4 does not vanish at $\rho = 0$. It is equal to the small value $\tilde{u}(0)^2 + (3/2)g\tilde{u}(0)^4 + (5/3)\tilde{h}\,\tilde{u}\,(0)^6 = 4.995 \times 10^{-6}$ in accordance with (28), (31), and (32). From Figs. 3 and 4, it follows that in the thick-wall regime the soliton has no interior with the approximately constant values of the energy and Noether charge densities. Note also that there is no sharp exterior boundary of the soliton in the thick-wall regime.

Let us remark that the function $\tilde{u}(\rho)$ in Figs. 1 and 3 is alternating sign and has exactly one node. This fact is in accordance with (36). On the other hand, the function $\tilde{v}(\rho)$ in Figs. 1 and 3 has no nodes. This suggests that the numerical solutions presented in Figs. 1 and 3 are unexcited.

Figure 5 presents the dependence of the decimal logarithm of the soliton energy divided by m on the decimal logarithm of $\Delta/m = \sqrt{1-\tilde{\omega}^2}$ in the thick-wall regime $|\tilde{\omega}| \to 1$. From this figure, it follows that the relation $\log_{10}(E/m) = \mathrm{const} - \log_{10}(\Delta/m)$ holds in the thick-wall regime in accordance with expression (58).

Figure 6 shows the dependences of the soliton energy E divided by m and the soliton Noether charge Q on the dimensionless phase frequency $\tilde{\omega}$ for $g = 1$, $\tilde{h} = 1$. The dependences $E(\tilde{\omega})/m$ and $Q(\tilde{\omega})$ are presented in the range from the minimum value of $\tilde{\omega}$ to its maximum value that we managed to reach by numerical methods. From Fig. 6, it follows that the energy and the Noether charge of the soliton tend to infinity as $|\tilde{\omega}| \to \tilde{\omega}_{\mathrm{min}}$ (thin-wall regime) and as $|\tilde{\omega}| \to 1$ (thick-wall regime). Such behavior of the energy and the Noether charge of the soliton is in accordance with the conclusions of Sec. IV.

Finally, the dependence of the difference between the soliton energy and the free-boson solution energy $E(Q) - mQ$ divided by m on the Noether charge Q is shown in Fig. 7. The curve $E(Q) - mQ$ consists of two branches and develops a spike at the point of their junction. At this point, the energy and the Noether charge of the soliton attain their minimum values. From Fig. 7, it follows that the solitons corresponding to the upper branch are unstable to the decay in the massive vector bosons. When moving along the upper branch upwards Q, we attain the region of

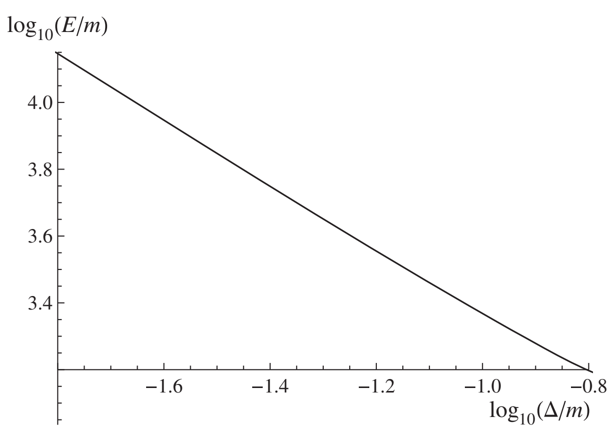

FIG. 5. The dependence of the decimal logarithm of the soliton energy divided by m on the decimal logarithm of Δ/m in the thick-wall regime.

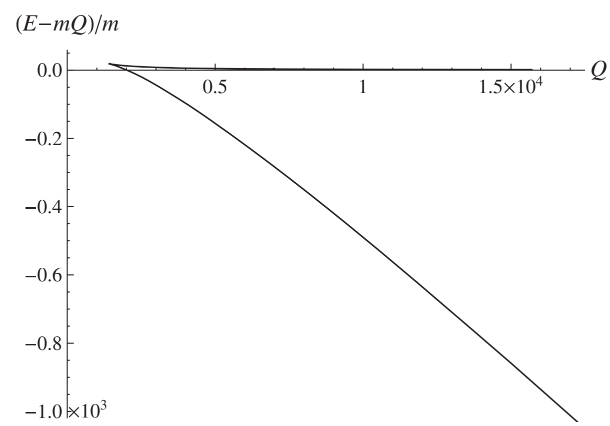

FIG. 7. The difference $E(Q) - mQ$ divided by m as a function of the Noether charge Q.

the thick-wall regime. In accordance with formula (59), in this regime the soliton energy E tends to the value mQ from above. When moving along the lower branch upwards Q, we attain the region of the thin-wall regime. From Fig. 7, it follows that in the thin-wall regime the soliton is stable to the decay in the massive vector bosons. Note also that the curve $E(Q) - mQ$ for the soliton of the self-interacting vector field coincides in form with the similar curve for the soliton of the self-interacting scalar field (Q ball).

We can see from Figs. 6 and 7 that $(\omega/Q)dQ/d\omega > 0$ on the upper branch of the $E(Q) - mQ$ curve. Thus, the nontopological soliton is classically unstable on this branch. From Figs. 6 and 7, it also follows that $d^2E/dQ^2 = d\omega/dQ < 0$ on the lower branch of the $E(Q) - mQ$ curve. Hence, the lower branch is concave, and so the inequality holds:

$$E(Q) < E(Q_1) + E(Q_2), \tag{79}$$

where $Q = Q_1 + Q_2$. From this inequality, it follows that on the lower branch the nontopological soliton is stable to the fission in the several nontopological solitons with the smaller Noether charges.

VI. CONCLUSIONS

In conclusion, we shall undertake a comparison of the properties of the nontopological soliton of the self-interacting vector field and the nontopological soliton of the self-interacting scalar field. First, let us enumerate the common properties of the solitons. Both solitons exist in the limited range of phase frequencies $|\omega| \in (\omega_{\min}, m)$, where m is the mass of the elementary boson of a field model. As $|\omega| \to \omega_{\min}$ ($|\omega| \to m$), the solitons pass into the thin-wall (thick-wall) regime, while their energy and Noether charge tend to infinity. Both solitons have the same shape curve $E(Q) - mQ$. In particular, the curve $E(Q) - mQ$ consists of two branches and has a cuspidal point. The region of high Q on the upper (lower) branch of the curve corresponds to the thick-wall (thin-wall) regime. The analysis of the curve $E(Q) - mQ$ leads us to the conclusion that both solitons are unstable to the decay in the free bosons in the thick-wall regime but are stable to this decay in the thin-wall regime. On the upper branch of the curve, the solitons of both types are classically unstable. Note also that the solitons of both types are spherically symmetric and have no angular momentum.

In contrast to the soliton of the self-interacting scalar field, the soliton of the self-interacting vector field has a central transition region in the thin-wall regime. In particular, at $r = 0$ the Noether charge density of the vector field soliton vanishes, and the energy density attains a minimum value. It is known [1,12] that the field configuration of the scalar field nontopological soliton can be described in terms of a mechanical analogy. It corresponds to a one-dimensional motion of a particle with a unit mass in the "time" r in a viscous medium in the force field of a certain potential. Using this analogy, one can easily explain the behavior of the scalar field nontopological soliton both in the thin-wall and in the thick-wall regimes. Moreover, one can easily determine whether there is a soliton solution for any given values of model parameters. At the same time, the system of differential equations (19) and (20) describing the vector field nontopological soliton has no interpretation in terms of any mechanical analogy. For this reason, the existence of the vector field nontopological soliton should be established for any given m, g, h by means of numerical methods.

A nontopological soliton can be quantized by several alternative methods [24–26]. All these methods in one way or another require knowing the spectrum of the quadratic fluctuation operator in a functional neighborhood of the soliton. Furthermore, the fluctuations should occur in the field sector of the fixed Noether charge. This spectrum can be found only numerically for the specific values of the model's parameters ω, m, g, and h. We can conclude from the numerical results of Sec. IV that at $g \sim m^2 h \sim 1$ the nontopological soliton of the self-interacting vector field is a classical object, because its action over the period $S_T = \int_0^{2\pi/\omega} dt \int d^3x \mathcal{L}$ is $\sim 10^3$–$10^4 \gg 1$, while the Compton wavelength $\lambda = 1/E \sim 10^{-2}$–$10^{-4}$ m^{-1} is much less than its linear size $R \sim 10^1$–10^2 m^{-1}.

ACKNOWLEDGMENTS

This work was supported by the Ministry of Education and Science of the Russian Federation (Program for the Growth of the Competitiveness of National Research Tomsk Polytechnic University) and by the Russian Foundation for Basic Research (Grant No. 15-02-00570-a).

APPENDIX A

Lagrangian (1) contains the constants m, h with the dimensions of mass and mass^{-2}, respectively, and the dimensionless constant g. Let us undertake the scaling transformation of the four-vector field

$$W_\mu = g^{-\frac{1}{2}} \bar{W}_\mu. \tag{A1}$$

Then Lagrangian (1) and energy density (7) can be written as

$$\mathcal{L} = g^{-1}\bar{\mathcal{L}}, \qquad \mathcal{E} = g^{-1}\bar{\mathcal{E}}, \tag{A2}$$

where

$$\bar{\mathcal{L}} = -\frac{1}{2}\bar{F}_{\mu\nu}\bar{F}^{*\mu\nu} - \bar{U}(\bar{W}_\mu \bar{W}^{*\mu}), \tag{A3}$$

$$\bar{\mathcal{E}} = \bar{F}_{0i}\bar{F}^*_{0i} + \frac{1}{2}\bar{F}_{ij}\bar{F}^*_{ij} + \bar{U}(\bar{W}_\mu\bar{W}^{*\mu}) - 2\bar{U}'(\bar{W}_\mu\bar{W}^{*\mu})|\bar{W}_0|^2, \tag{A4}$$

$$\bar{U}(\bar{W}_\mu\bar{W}^{*\mu}) = -m^2\bar{W}_\mu\bar{W}^{*\mu} - \frac{1}{2}(\bar{W}_\mu\bar{W}^{*\mu})^2 - \frac{1}{3}\frac{h}{g^2}(\bar{W}_\mu\bar{W}^{*\mu})^3. \tag{A5}$$

It follows from (A2), (A3), and (A5) that Lagrangian $\bar{\mathcal{L}}$ depends on g, h only by means of ratio h/g^2; hence, the corresponding four-vector solution $\bar{W}_\mu$ also depends on g, h by means of this ratio. Note also that the numerical coefficients of the potentials U and $\bar{U}$ coincide, and so the relation holds: $\bar{W}_\mu(x, m^2, hg^{-2}) = W_\mu(x, m^2, 1, hg^{-2})$. Thus, we determine the general form of dependence on g for the four-vector solution W_μ:

$$W_\mu(x, m^2, g, h) = g^{-\frac{1}{2}}W_\mu(x, m^2, 1, hg^{-2}). \tag{A6}$$

From (A2) and (A6), it follows that

$$\mathcal{L}(x, m^2, g, h) = g^{-1}\mathcal{L}(x, m^2, 1, hg^{-2}), \tag{A7}$$

$$\mathcal{E}(x, m^2, g, h) = g^{-1}\mathcal{E}(x, m^2, 1, hg^{-2}), \tag{A8}$$

in solutions of the field equations (4). If h/g^2 remain fixed and g tends to zero, then the soliton solution W_μ increases indefinitely. As a result, the energy $E = \int \mathcal{E}d^3x$ and the action over the period $S_T = \int_0^{2\pi/\omega} dt \int d^3x\mathcal{L}$ also increase indefinitely, and the nontopological soliton passes into the quasiclassical regime. In this respect, the role of the dimensionless coupling constant g is analogous to that of the gauge coupling constant of the pure Yang-Mills theory.

APPENDIX B

Let us consider some properties of energy density (7) on spatially homogeneous and time -independent field configurations. On such field configurations, energy density (7) depends only on the squared absolute values of the time component W_0 and the three-vector $\mathbf{W}$:

$$\begin{aligned}\mathcal{E} &= -2U'(W_\mu W^{*\mu})|W_0|^2 + U(W_\mu W^{*\mu}) \\ &= 2|W_0|^2(m^2 + g(|W_0|^2 - |\mathbf{W}|^2) \\ &\quad + h(|W_0|^2 - |\mathbf{W}|^2)^2) - m^2(|W_0|^2 - |\mathbf{W}|^2) \\ &\quad - \frac{g}{2}(|W_0|^2 - |\mathbf{W}|^2)^2 - \frac{h}{3}(|W_0|^2 - |\mathbf{W}|^2)^3.\end{aligned} \tag{B1}$$

It turns out that the properties of $\mathcal{E}$ depend in an essential way on the value of $\Delta \equiv g^2 - 4hm^2$. If condition (8) is satisfied and thereby $\Delta < 0$, then there is the global minimum of $\mathcal{E}$ at $|W_0| = 0$, $|\mathbf{W}| = 0$. In this minimum $\mathcal{E} = 0$, but in all other points $\mathcal{E} > 0$, and so the model is stable. There are no other critical points of $\mathcal{E}$ in this case.

If condition (8) does not hold and $\Delta > 0$, then there is the local minimum at $|W_0| = 0$, $|\mathbf{W}| = 0$ in which $\mathcal{E} = 0$. There are also the two critical points of $\mathcal{E}$ at

$$|W_0| = 0, \qquad |\mathbf{W}| = \frac{\sqrt{g \mp \sqrt{\Delta}}}{\sqrt{2h}}. \tag{B2}$$

This critical points, however, are not the extrema but the saddle points. Thus, there is the single minimum (global or local) of the energy density $\mathcal{E}$ at $|W_0| = 0$, $|\mathbf{W}| = 0$ ($W^\mu = 0$), and there are no other local minima or maxima of $\mathcal{E}$.

It can be shown that if $\Delta > 0$, then there is a region in which the energy density $\mathcal{E}$ is negative and unbounded from below. Let us turn to the polar coordinate system:

$$|W_0| = \varrho\cos(\theta), \qquad |\mathbf{W}| = \varrho\sin(\theta). \tag{B3}$$

In this system, the region of the negative energy density can be written as

$$\varrho_1(\theta) < \varrho < \varrho_2(\theta), \tag{B4}$$

where

$$\varrho_{1,2} = \frac{1}{2}\sqrt{\frac{3g(t+2) \mp \sqrt{9g^2(t+2)^2 - 48hm^2(2t+3)}}{-ht(2t+3)}}, \tag{B5}$$

and $t = \cos(2\theta)$. In (B4), the polar angle θ lies in the range $(\pi/4, \theta_{\max})$, where

$$\theta_{\max} = \frac{1}{2}\arccos\left(\frac{2}{3}g^{-2}\left(8hm^2 - 3g^2 + 2m\sqrt{h(16hm^2 - 3g^2)}\right)\right) \tag{B6}$$

for $(3/16)(g^2/m^2) < h < (1/4)(g^2/m^2)$ and

$$\theta_{\max} = \pi/2 \tag{B7}$$

for $0 < h < (3/16)(g^2/m^2)$.

It can be shown that for sufficiently large ϱ the boundaries of the region $\mathcal{E} < 0$ can be written as

$$\theta_{1,2} = \frac{\pi}{4} + \frac{g \mp \sqrt{\Delta}}{4h\varrho^2} + O(\varrho^{-4}). \tag{B8}$$

It can also be shown that for sufficiently large and fixed ϱ the energy density $\mathcal{E}$ reaches the minimum value

$$\mathcal{E}(\theta_{\mathrm{min}}) = -\frac{\Delta}{4h}\varrho^2 + \frac{g^3}{8h^2} + O(\varrho^{-2}) \tag{B9}$$

at the polar angle

$$\theta_{\mathrm{min}} = \frac{\pi}{4} + \frac{g}{4h\varrho^2} + O(\varrho^{-4}). \tag{B10}$$

Note that (B9) and (B10) are valid for any sign of Δ. From (B9), it follows that $\mathcal{E}(\theta_{\mathrm{min}})$ is negative for large ϱ if condition (8) is not satisfied. Moreover, in this case $\mathcal{E}(\theta_{\mathrm{min}})$ decreases indefinitely in the infinitesimal neighborhood of θ_{min} as $\varrho \to \infty$. Needless to say, $\mathcal{E}$ is also unbounded from below if $h < 0$. Thus, the model is unstable if condition (8) is not satisfied.

[1] T. D. Lee and Y. Pang, Phys. Rep. **221**, 251 (1992).
[2] E. Radu and M. Volkov, Phys. Rep. **468**, 101 (2008).
[3] S. Coleman, Nucl. Phys. **B262**, 263 (1985).
[4] A. Kusenko, Phys. Lett. B **406**, 26 (1997).
[5] F. P. Correia and M. Schmidt, Eur. Phys. J. C **21**, 181 (2001).
[6] A. Safian, S. Coleman, and M. Axenides, Nucl. Phys. **B297**, 498 (1988).
[7] A. Safian, Nucl. Phys. **B304**, 392 (1988).
[8] A. Kusenko, Phys. Lett. B **405**, 108 (1997).
[9] A. Kusenko, M. Shaposhnikov, and P. Tinyakov, Pis'ma Zh. Exp. Teor. Fiz. **67**, 229 (1998) [JETP Lett. **67**, 247 (1998)].
[10] A. Kusenko and M. Shaposhnikov, Phys. Lett. B **418**, 46 (1998).
[11] K. Enquist and A. Mazumdar, Phys. Rep. **380**, 99 (2003).
[12] R. Friedberg, T. D. Lee, and A. Sirlin, Phys. Rev. D **13**, 2739 (1976).
[13] K. Lee, J. A. Stein-Schabes, R. Watkins, and L. M. Widrow, Phys. Rev. D **39**, 1665 (1989).
[14] K. N. Anagnostopoulos, M. Axenides, E. G. Floratos, and N. Tetradis, Phys. Rev. D **64**, 125006 (2001).
[15] T. S. Levi and M. Gleiser, Phys. Rev. D **66**, 087701 (2002).
[16] R. Friedberg, T. D. Lee, and A. Sirlin, Nucl. Phys. **B115**, 1 (1976).
[17] R. Friedberg, T. D. Lee, and A. Sirlin, Nucl. Phys. **B115**, 32 (1976).
[18] A. Yu. Loginov, Zh. Exp. Teor. Fiz., **141**, 56 (2012) [JETP **114**, 48 (2012)].
[19] S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).
[20] A. Salam, in *Elementary Particle Physics*, edited by N. Svartholm (Almqvist and Wiksells, Stockholm, 1968).
[21] A. Proca, J. Phys. Radium **7**, 347 (1936); **8**, 23 (1937).
[22] V. Rubakov, *Classical Theory of Gauge Fields* (Princeton University Press, Princeton, NJ, 2002).
[23] Maple User Manual, Maplesoft, 2014.
[24] N. H. Christ and T. D. Lee, Phys. Rev. D **12**, 1606 (1975).
[25] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10**, 4114 (1974); **10**, 4130 (1974).
[26] R. Rajaraman and E. J. Weinberg, Phys. Rev. D **11**, 2950 (1975).