Fermionic propagator in an intense background

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(Received 5 March 2015; published 26 May 2015)

New results for the fermion propagator in a laser background are presented. We show that the all orders electron propagator can be written in a compact and appealing form as a sum of sideband poles with a matrix wave function renormalization and a matrix valued mass shift. This last result is essential in the fermionic theory if we are to maintain that both the mass and its square pick up a correction only at order e^2. A perturbative verification of our results is carried out.

I. INTRODUCTION

In this paper we will discuss fermionic propagation in background fields. Specifically we are interested in electrons propagating in a plane wave background [1]. This is important for understanding recent work with high power laser facilities [2,3]. The result we will obtain is simple and has a clear physical interpretation. A surprise is that all the renormalization factors are matrices.

The propagator is one of the basic building blocks linking physical processes with the structure of quantum field theory. We recall that for a scalar field, $\phi(x)$, the Feynman propagator is given by the time ordered product

$$\langle 0|\mathrm{T}\phi(x)\phi^\dagger(y)|0\rangle = \int đ^4 p\, e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\epsilon}, \quad (1)$$

where the fields are taken to be free and the bar in the measure indicates appropriate factors of 2π. In the presence of loop corrections the form of the propagator is preserved and typically becomes

$$\langle 0|\mathrm{T}\phi(x)\phi^\dagger(y)|0\rangle = \int đ^4 p\, e^{-ip\cdot(x-y)} \frac{iZ_2}{p^2 - (m^2 + \delta m^2) + i\epsilon}, \quad (2)$$

which can be interpreted as the propagation of renormalized fields with a renormalized mass.

A similar, but richer, structure in the propagator is found when the scalar field interacts with a laser plane wave background [4–14]

$$\sum_{n=-\infty}^{\infty} \int đ^4 p\, e^{-ip\cdot(x-y)} \frac{iZ_2^{(n)}}{(p+nk)^2 - (m^2 + \delta m^2) + i\epsilon}, \quad (3)$$

where we see a sum over so-called sideband states [15–17] and note that the mass shift is common to all sidebands. The exact form of the mass and the momentum dependent wave function renormalization factors depend on the details of

the laser background. For a linearly polarized background, the propagator has the form

$$\sum_{n=-\infty}^{\infty} \int đ^4 p\, e^{-ip\cdot(x-y)} \frac{i\mathrm{J}_n^2(eu_p, e^2 v_p)}{(p+nk)^2 - m_*^2 + i\epsilon}, \quad (4)$$

where u_p and v_p are laser parameters that we define in (20). The mass shift is here given by $m_*^2 = m^2 - \frac{1}{2}e^2 a^2$ where a_μ is the (spacelike) laser amplitude and this mass shift does not acquire higher order corrections. The wave function renormalization for the various sidebands is written in terms of a product of generalized Bessel functions which can be expressed in terms of normal Bessel functions as

$$\mathrm{J}_n(eu_p, e^2 v_p) = \sum_{r=-\infty}^{\infty} J_{n-2r}(eu_p) J_r(e^2 v_p). \quad (5)$$

The fine details in (4) are polarization dependent but the broad structures for a plane wave laser background, including the sum over sidebands and the mass shift being at order e^2 only, are common.

Note that in the literature, the two point function in a laser background is widely referred to as the propagator. However, through interactions with the background, this two point function includes diagrams where the initial and final momenta of the matter field are distinct. In this paper, and in our summary above, the focus is on the diagonal part of the two point function which does not include momentum altering vertex effects. This is what we call the propagator in this paper and this is given for the scalar theory in (4).

As in our previous work, we are guided below by experience from infrared physics. A charge propagating in a laser is indistinguishable from a charge which absorbs and emits the same number of laser photons. These laser induced degeneracies parallel the soft and collinear degeneracies of the infrared regime in QED and QCD [18–21]. Understanding the mass shift in a laser background may also help to clarify the current versus constituent mass distinction in QCD [22].

Turning now to fermionic QED, the perturbative propagator without a background has the form

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \mathrm{d}^4 p\, e^{-ip\cdot(x-y)} \frac{iZ_2}{\not{p} - (m+\delta m) + i\epsilon},$$
$$= \int \mathrm{d}^4 p\, e^{-ip\cdot(x-y)} \frac{iZ_2(\not{p} + m + \delta m)}{p^2 - (m+\delta m)^2 + i\epsilon}, \quad (6)$$

where we note the presence of single powers of m.

The extension of this theory to plane wave laser backgrounds has been previously considered [4,5,23,24]. The results obtained also display the sideband structure of the scalar theory but the two point function is quite involved, see, e.g., (5.7) and (5.8) in Ref. [4], and does not clearly resemble a renormalized propagator. In particular in the numerator of the propagator obtained by Reiss and Eberly [4] there are two gamma matrix structures, one proportional to $\not{p} - e\not{A} + m$ and one proportional to the laser momentum, $\not{k}$, each multiplied by combinations of Bessel functions. Similarly, additional gamma matrix structures are obtained in Appendix A of Brown and Kibble [5] and by Ritus [11].

The mass shift in the fermionic theory also satisfies the spin-independent relation [4,5]

$$m_*^2 = (m+\delta m)^2 = m^2 - \frac{1}{2}e^2 a^2, \quad (7)$$

to all orders in the coupling. This might naturally suggest, see for example Sec. 40 of [25] and more recently [26], that

$$m_* = m + \delta m = m\sqrt{1 - \frac{e^2 a^2}{2m^2}}, \quad (8)$$

which would, however, mean that δm would acquire corrections to all orders in the coupling through an expansion of the above.

Below we will first show that an explicit perturbative calculation disagrees with this expectation for the mass shift. We will then present an all orders calculation of the fermion propagator which will reveal a simple form which can be interpreted, as for the scalar theory, in terms of mass and wave function renormalization factors. An unexpected feature is that all the renormalization factors we find, including the mass shift, are matrices. The mass shift will itself only be corrected at order e^2, but will still satisfy (7) to all orders. As this is surprising we will independently verify all of the structures we obtain via a low order perturbative calculation. Finally we will present some conclusions. Some technical details are presented in the appendixes.

II. PERTURBATIVE EXPANSION

The Lagrangian density describing the interactions of a fermion with the laser is

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi. \quad (9)$$

In this the covariant derivative is given by $D_\mu = \partial_\mu + ieA_\mu$ and, in a linearly polarized background,

$$A_\mu(x) = a_\mu \cos(k \cdot x), \quad (10)$$

where $k = k_0(1,0,0,1)$ is the null vector characterizing the laser. Note that the gauge fixing condition used is $k \cdot a = 0$.

The Feynman rules for the vertices in this linearly polarized theory are given in Fig. 1. For this polarization, the rules are identical for the absorption and the emission of a laser photon. The two diagrams that contribute at lowest order in the coupling are shown in Fig. 2.

The first diagram yields after a partial fraction expansion

$$\frac{ie^2}{16(p \cdot k)^2}(\not{p} + m)\not{a}(\not{p} + \not{k} + m)\not{a}(\not{p} + m)$$
$$\times \left(-\frac{1}{p^2 - m^2} + \frac{2p \cdot k}{(p^2 - m^2)^2} + \frac{1}{(p+k)^2 - m^2}\right), \quad (11)$$

and the second diagram is obtained by replacing $k \to -k$. In our perturbative calculations we suppress the $i\epsilon$ prescription. The gamma matrix structures are now expanded depending upon whether the denominator contains poles in $p^2 - m^2$ or is in an adjacent sideband. In this initial discussion, we shall restrict ourselves to the central sideband alone. In the double pole coefficient the $\not{k}$ term alone survives when the two diagrams are added, while in the single pole coefficient the $\not{k}$ term cancels. Thus the two diagrams contribute to the central sideband

$$\frac{ie^2}{8(p \cdot k)^2}\left(-(\not{p} + m)\not{a}(\not{p} + m)\not{a}(\not{p} + m)\frac{1}{p^2 - m^2}\right.$$
$$\left. + (\not{p} + m)\not{a}\not{k}\not{a}(\not{p} + m)\frac{2p \cdot k}{(p^2 - m^2)^2}\right). \quad (12)$$

FIG. 1. The three point Feynman rules for linear polarization.

FIG. 2. Leading order corrections to the propagator.

In the first term we rewrite

$$(\not{p} + m)\not{a}(\not{p} + m)\not{a}(\not{p} + m)$$
$$= (\not{p} + m)\not{a}(-(p^2 - m^2)\not{a} + 2p \cdot a(\not{p} + m)). \quad (13)$$

The first part here will not contribute any pole to the central sideband and we have verified that the sum of all such terms without poles cancel when the contributions from all sidebands are summed. We will therefore just drop them below. The double pole term in (12) will also generate single pole terms in a similar fashion. The sum of all the terms containing poles in the central sideband is found to be

$$ie^2\left(-\frac{(p \cdot a)^2}{2(p \cdot k)^2}\frac{\not{p} + m}{p^2 - m^2} + \frac{a^2 \not{k}}{4p \cdot k}\frac{1}{p^2 - m^2}\right.$$
$$\left. - \frac{1}{2}a^2\frac{\not{p} + m}{(p^2 - m^2)^2}\right). \quad (14)$$

We should now compare this with the general form of (6). At this leading order we expect three contributions. Writing $Z_2^{(0)} = 1 + \delta Z_2^{(0)}$ [where $\delta Z_2^{(0)}$ is expected to have a matrix form] this would give

$$ie^2\left(\frac{\delta Z_2^{(0)}(\not{p} + m)}{p^2 - m^2} + \delta m\frac{1}{p^2 - m^2} - \frac{1}{2}a^2\frac{1}{(p^2 - m^2)^2}\right), \quad (15)$$

where we have used (7) in the last term.

In (14) the double pole term is the expected mass shift from the expansion of the denominator of the central sideband. The first term gives us information about the wave function renormalization factor. However, the second term in (14), which involves a^2 as would be expected of a mass shift, contains a $\not{k}$ factor and is not obtained by simply expanding the numerator in (6). We will therefore now carry out an all orders calculation of the propagator to resolve this puzzle and also to determine the form of $Z_2^{(n)}$.

III. VOLKOV FERMION

We now want to construct the spinor field $\psi_V(x)$ which satisfies

$$(i\not{D} - m)\psi_V(x) = 0. \quad (16)$$

This Volkov equation (16) is solved by

$$\psi_V(x) = \int d^3p\,\frac{m}{E_p^*}\left(\mathcal{F}_\ell(x, \bar{p})D_\ell(x, \bar{p})\mathcal{U}_V^{(\alpha)}(p)a_V^{(\alpha)}(p)\right.$$
$$\left. + \mathcal{F}_\ell(x, -\bar{p})D_\ell(x, -\bar{p})\mathcal{V}_V^{(\alpha)}(p)b_V^{\dagger(\alpha)}(p)\right). \quad (17)$$

There is a lot of structure to this solution that we need to unpack carefully.

The on-shell momentum $\bar{p}_\mu = (E_p^*, \underline{p})$ satisfies the shifted on-shell condition of the quasimomentum:

$$\bar{p}^2 - m_*^2 = E_p^{*2} - \underline{p}^2 - m_*^2 = 0. \quad (18)$$

Just as for the scalar case, plane waves are distorted in this background and become equal to

$$D_\ell(x, \bar{p}) = e^{-i\bar{p}\cdot x}e^{i(eu_{\bar{p}}\sin(k\cdot x) + e^2 v_{\bar{p}}\sin(2k\cdot x))},$$
$$= e^{-i\bar{p}\cdot x}\sum_{n=-\infty}^{\infty}e^{ink\cdot x}\mathrm{J}_n(eu_{\bar{p}}, e^2 v_{\bar{p}}). \quad (19)$$

In this expression we have defined the laser variables $u_{\bar{p}}$ and $v_{\bar{p}}$ by

$$u_{\bar{p}} = -\frac{\bar{p} \cdot a}{\bar{p} \cdot k} \quad \text{and} \quad v_{\bar{p}} = \frac{a^2}{8\bar{p} \cdot k}. \quad (20)$$

The rest of Eq. (17) reflects the fermionic nature of QED. Focusing initially on the first term in $\psi_V(x)$, we see that in the Volkov equation the derivative will only act on the factors $\mathcal{F}_\ell(x, \bar{p})D_\ell(x, \bar{p})$. The prefactor $\mathcal{F}_\ell(x, \bar{p})$ is a matrix given by

$$\mathcal{F}_\ell(x, \bar{p}) = \left(1 + e\frac{\not{k}A(x)}{2\bar{p} \cdot k}\right). \quad (21)$$

The action of the covariant derivative is then given by

$$i\not{D}\mathcal{F}_\ell(x, \bar{p})D_\ell(x, \bar{p}) = \mathcal{F}_\ell(x, \bar{p})D_\ell(x, \bar{p})(\not{\bar{p}} + 2e^2 v_{\bar{p}}\not{k}). \quad (22)$$

Similarly, in the second part of (17) we get

$$i\not{D}\mathcal{F}_\ell(x, -\bar{p})D_\ell(x, -\bar{p})$$
$$= \mathcal{F}_\ell(x, -\bar{p})D_\ell(x, -\bar{p})(-\not{\bar{p}} - 2e^2 v_{\bar{p}}\not{k}). \quad (23)$$

The Volkov spinors are then introduced to satisfy

$$(\not{\bar{p}} + 2e^2 v_{\bar{p}}\not{k})\mathcal{U}_V^{(\alpha)}(p) = m\mathcal{U}_V^{(\alpha)}(p) \quad (24)$$

and

$$(\not{\bar{p}} + 2e^2 v_{\bar{p}}\not{k})\mathcal{V}_V^{(\alpha)}(p) = -m\mathcal{V}_V^{(\alpha)}(p). \quad (25)$$

The precise form for these Volkov spinors is presented in Appendix A.

Combining these results we see that $\psi_V(x)$ as defined by Eq. (17) does indeed satisfy the Volkov equation (16).

Having constructed the Volkov field, $\psi_V(x)$, we now simply state the expression for the Dirac adjoint

$$\bar\psi_{\rm V}(y) = \int {\rm d}^3 q\, \frac{m}{E_q^*}\, (\bar{\mathcal{U}}_{\rm V}^{(\beta)}(q)\mathcal{F}_\ell(y, -\bar q)D_\ell^\dagger(y, \bar q)a_{\rm V}^{\dagger(\beta)}(q)$$
$$+ \bar{\mathcal{V}}_{\rm V}^{(\beta)}(q)\mathcal{F}_\ell(y, \bar q)D_\ell^\dagger(y, -\bar q)b_{\rm V}^{(\beta)}(q)). \tag{26}$$

Returning to the on-shell condition (24) for the Volkov spinors, we can directly reinterpret this as

$$\bar{\slashed p}\mathcal{U}_{\rm V}^{(\alpha)}(p) = (m - 2e^2 v_{\bar p}\slashed k)\mathcal{U}_{\rm V}^{(\alpha)}(p), \tag{27}$$

that is, in the spinor theory, the mass shift has a matrix form with $m_* = m + \delta\bar m$ with

$$\delta\bar m = -2e^2 v_{\bar p}\slashed k = -\frac{e^2 a^2}{4\bar p \cdot k}\slashed k. \tag{28}$$

Let us now show that this is consistent with the result familiar from the scalar theory. We multiply both sides of (27) by $\bar{\slashed p}$ and anticommute the $\bar{\slashed p}$ factor through the $\slashed k$ to obtain

$$\bar p^2\mathcal{U}_{\rm V}^{(\alpha)}(p) = (m\bar{\slashed p} - 4e^2 v_{\bar p}\bar p \cdot k + 2e^2 v_{\bar p}\slashed k\bar{\slashed p})\mathcal{U}_{\rm V}^{(\alpha)}(p). \tag{29}$$

We now reuse (27) in the final term and, recalling that $k^2 = 0$, obtain the spin-independent on-shell result

$$\bar p^2\mathcal{U}_{\rm V}^{(\alpha)}(p) = m_*^2\mathcal{U}_{\rm V}^{(\alpha)}(p) = \left(m^2 - \frac{1}{2}e^2 a^2\right)\mathcal{U}_{\rm V}^{(\alpha)}(p). \tag{30}$$

Note that this requires $\delta\bar m$ to be a matrix, as one would otherwise generate corrections to m^2 at order e^4.

However, we cannot yet identify δm in (15) from this, as if we were to replace δm by $\delta\bar m$ we would obtain (14) but with the wrong sign for the $\slashed k$ term. We shall though see later that the result for $\delta\bar m$ is correct and that the naive expectation (15) is wrong.

IV. THE PROPAGATOR

We are now in a position to calculate the Volkov propagator

$${}_{\rm V}\langle 0|{\rm T}\psi_{\rm V}(x)\bar\psi_{\rm V}(y)|0\rangle_{\rm V} = \theta(x^0 - y^0)_{\rm V}\langle 0|\psi_{\rm V}(x)\bar\psi_{\rm V}(y)|0\rangle_{\rm V}$$
$$- \theta(y^0 - x^0)_{\rm V}\langle 0|\bar\psi_{\rm V}(y)\psi_{\rm V}(x)|0\rangle_{\rm V}, \tag{31}$$

where the Volkov vacuum is identified through the modes in (17) and (26). Note that the creation and annihilation operators satisfy

$$\{a_{\rm V}^{(\alpha)}(p), a_{\rm V}^{\dagger(\beta)}(q)\} = (2\pi)^3\frac{E_p^*}{m}\delta^{\alpha\beta}\delta^{(3)}(p - q). \tag{32}$$

Looking at the first vacuum expectation value we have

$${}_{\rm V}\langle 0|\psi_{\rm V}(x)\bar\psi_{\rm V}(y)|0\rangle$$
$$= \int {\rm d}^3 p\, \frac{1}{2E_p^*}\mathcal{F}_\ell(x, \bar p)(\bar{\slashed p} + m - \delta\bar m)$$
$$\times \mathcal{F}_\ell(y, -\bar p)D_\ell(x, \bar p)D_\ell^\dagger(y, \bar p). \tag{33}$$

In a similar way we obtain

$${}_{\rm V}\langle 0|\bar\psi_{\rm V}(y)\psi_{\rm V}(x)|0\rangle_{\rm V}$$
$$= \int {\rm d}^3 p\, \frac{1}{2E_p^*}\mathcal{F}_\ell(x, -\bar p)(\bar{\slashed p} - m - \delta\bar m)\mathcal{F}_\ell(y, \bar p)$$
$$\times D_\ell(x, -\bar p)D_\ell^\dagger(y, -\bar p). \tag{34}$$

We now use these to build up the propagator. To this end we require the following results which follow from (19)

$$\mathcal{F}_\ell(x, \bar p)D_\ell(x, \bar p)$$
$$= \sum_{n=-\infty}^{\infty} e^{-i\bar p\cdot x}e^{ink\cdot x}\left(\mathcal{J}_n(\bar p) + e\frac{\slashed k\slashed a}{4\bar p \cdot k}\mathcal{J}_n^\Delta(\bar p)\right), \tag{35}$$

where we have condensed our notation for the generalized Bessel function so that

$$\mathcal{J}_n(\bar p) = {\rm J}_n(eu_{\bar p}, e^2 v_{\bar p}) \tag{36}$$

and

$$\mathcal{J}_n^\Delta(\bar p) = \mathcal{J}_{n-1}(\bar p) + \mathcal{J}_{n+1}(\bar p). \tag{37}$$

In a similar way we obtain

$$\mathcal{F}_\ell(x, -\bar p)D_\ell(x, -\bar p)$$
$$= \sum_{n=-\infty}^{\infty} e^{i\bar p\cdot x}e^{ink\cdot x}\left(\mathcal{J}_n(-\bar p) + e\frac{\slashed a\slashed k}{4\bar p \cdot k}\mathcal{J}_n^\Delta(-\bar p)\right), \tag{38}$$

while

$$\mathcal{F}_\ell(y, -\bar p)D_\ell^\dagger(y, \bar p)$$
$$= \sum_{n=-\infty}^{\infty} e^{i\bar p\cdot y}e^{-ink\cdot y}\left(\mathcal{J}_n(\bar p) + e\frac{\slashed a\slashed k}{4\bar p \cdot k}\mathcal{J}_n^\Delta(\bar p)\right) \tag{39}$$

and

$$\mathcal{F}_\ell(y, \bar p)D_\ell^\dagger(y, -\bar p)$$
$$= \sum_{n=-\infty}^{\infty} e^{-i\bar p\cdot y}e^{-ink\cdot y}\left(\mathcal{J}_n(-\bar p) + e\frac{\slashed k\slashed a}{4\bar p \cdot k}\mathcal{J}_n^\Delta(-\bar p)\right). \tag{40}$$

Inserting these expansions into (33) and (34) and using the decomposition

$$\psi_V(x) = \sum_{n=-\infty}^{\infty} \psi_n(x), \qquad (41)$$

gives the propagator defined through the diagonal contributions to the two point function

$$
{}_V\langle 0|\psi_n(x)\bar{\psi}_n(y)|0\rangle_V
$$
$$
= \int \mathrm{d}^3 p \frac{1}{2E_p^*} e^{-i\bar{p}\cdot(x-y)} e^{ink\cdot(x-y)} \left(\mathcal{J}_n(\bar{p}) + e\frac{k\slashed{a}}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(\bar{p}) \right)
$$
$$
\times (\slashed{\bar{p}}+m-\delta\bar{m}) \left(\mathcal{J}_n(\bar{p}) + e\frac{\slashed{a}k}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(\bar{p}) \right), \qquad (42)
$$

and

$$
{}_V\langle 0|\bar{\psi}_n(y)\psi_n(x)|0\rangle_V
$$
$$
= \int \mathrm{d}^3 p \frac{1}{2E_p^*} e^{i\bar{p}\cdot(x-y)} e^{ink\cdot(x-y)} \left(\mathcal{J}_n(-\bar{p}) + e\frac{\slashed{a}k}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(-\bar{p}) \right)
$$
$$
\times (\slashed{\bar{p}}-m-\delta\bar{m}) \left(\mathcal{J}_n(-\bar{p}) + e\frac{k\slashed{a}}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(-\bar{p}) \right). \qquad (43)
$$

The Volkov propagator then becomes equal to the sum over n of

$$
-\int \frac{dp^0}{2\pi}\frac{\mathrm{d}^3 p}{2E_p^*}\frac{i}{p^0 - i\epsilon} \left(e^{i(p^0-E_p^*+nk^0)(x^0-y^0)} e^{i(\underline{p}-n\underline{k})\cdot(\underline{x}-\underline{y})} \right.
$$
$$
\times \left(\mathcal{J}_n(\bar{p}) + e\frac{k\slashed{a}}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(\bar{p}) \right)(\slashed{\bar{p}}+m-\delta\bar{m})
$$
$$
\times \left(\mathcal{J}_n(\bar{p}) + e\frac{\slashed{a}k}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(\bar{p}) \right)
$$
$$
- e^{-i(p^0-E_p^*-nk^0)(x^0-y^0)} e^{-i(\underline{p}+n\underline{k})\cdot(\underline{x}-\underline{y})}
$$
$$
\times \left(\mathcal{J}_n(-\bar{p}) + e\frac{\slashed{a}k}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(-\bar{p}) \right)(\slashed{\bar{p}}-m-\delta\bar{m})
$$
$$
\left. \times \left(\mathcal{J}_n(-\bar{p}) + e\frac{k\slashed{a}}{4\bar{p}\cdot k}\mathcal{J}_n^\Delta(-\bar{p}) \right) \right). \qquad (44)
$$

Combining the variables p^0 and p^i into an off shell four moment p^μ (details of which are given in Appendix B) we can write the fermionic Volkov propagator as

$$
\sum_{n=-\infty}^{\infty} \int \mathrm{d}^4 p\, e^{-ip\cdot(x-y)} \left(\mathcal{J}_n(p+nk) + e\frac{k\slashed{a}}{4p\cdot k}\mathcal{J}_n^\Delta(p+nk) \right)
$$
$$
\times \frac{i(\slashed{p}+n\slashed{k}+m-\delta\slashed{m})}{(p+nk)^2 - m_*^2 + i\epsilon}
$$
$$
\times \left(\mathcal{J}_n(p+nk) + e\frac{\slashed{a}k}{4p\cdot k}\mathcal{J}_n^\Delta(p+nk) \right). \qquad (45)
$$

Note that here we have removed the bar over the mass shift matrix defined earlier in (28) since the momentum in (45) is now off shell.

V. INTERPRETATION

The naive expectation might have been that we would find a result of the form

$$
\sum_{n=-\infty}^{\infty} \int \mathrm{d}^4 p\, e^{-ip\cdot(x-y)} Z_2^{(n)} \frac{i(\slashed{p}+n\slashed{k}+m+\delta m)}{(p+nk)^2 - m_*^2 + i\epsilon}, \qquad (46)
$$

however, what we have found in (45) looks very different. It contains matrix dependent wave function renormalization constants on both sides of the usual gamma matrix structure expected in a fermionic propagator as well as a minus sign in the mass shift, $\delta\slashed{m}$ in the numerator of (45), which we will now explain.

Setting $n = 0$ for simplicity and ignoring the $i\epsilon$, we make the simple observation that because the mass shift is a matrix, then it can naturally be regrouped as

$$
\frac{1}{\slashed{p}-(m+\delta\slashed{m})} = \frac{1}{(\slashed{p}-\delta\slashed{m})-m}, \qquad (47)
$$

which means when we carry out the standard algebraic rearrangement we obtain

$$
\frac{1}{(\slashed{p}-\delta\slashed{m})-m} = \frac{\slashed{p}-\delta\slashed{m}+m}{(\slashed{p}-\delta\slashed{m})^2 - m^2},
$$
$$
= \frac{\slashed{p}+m-\delta\slashed{m}}{p^2 + e^2a^2/2 - m^2},
$$
$$
= \frac{\slashed{p}+m-\delta\slashed{m}}{p^2 - m_*^2}. \qquad (48)
$$

Here we recognize the standard mass shift in the denominator and the minus sign present in the numerator of (45).

We finally define the wave function renormalization matrix by

$$
\sqrt{\mathbf{Z}_2^{(n)}} = \mathcal{J}_n(p+nk) + e\frac{\slashed{a}k}{4p\cdot k}\mathcal{J}_n^\Delta(p+nk), \qquad (49)
$$

and note that

$$
\sqrt{\bar{\mathbf{Z}}_2^{(n)}} := \gamma^0 \sqrt{\mathbf{Z}_2^{(n)}}^\dagger \gamma^0
$$
$$
= \mathcal{J}_n(p+nk) + e\frac{k\slashed{a}}{4p\cdot k}\mathcal{J}_n^\Delta(p+nk). \qquad (50)
$$

This yields a very appealing way to write the sideband propagator. If we reinstate the sum over sidebands, we see that the Volkov propagator for fermions can be written as

$$\sum_{n=-\infty}^{\infty} {}_{\mathrm{V}}\langle 0|\mathrm{T}\psi_n(x)\bar{\psi}_n(y)|0\rangle_{\mathrm{V}}$$

$$= \sum_{n=-\infty}^{\infty} \int \d^4 p \, e^{-ip\cdot(x-y)} \sqrt{\bar{\mathbf{Z}}_2^{(n)}}$$

$$\times \frac{i}{\not{p} + n\not{k} - (m + \delta\not{m}) + i\epsilon} \sqrt{\mathbf{Z}_2^{(n)}}. \qquad (51)$$

VI. DOES THIS AGREE WITH PERTURBATION THEORY?

We will now check that our result for the propagator (51) agrees with perturbation theory by calculating the lowest order corrections to the free propagator extending our earlier result (14) to the adjacent sidebands.

We will require the expansions of the generalized Bessel functions

$$\mathcal{J}_0(p) = \mathrm{J}_0(eu_p, e^2 v_p) = 1 - \frac{e^2 u_p^2}{4} + \mathcal{O}(e^4),$$

$$\mathcal{J}_{\pm 1}(p) = \mathrm{J}_{\pm 1}(eu_p, e^2 v_p) = \pm \frac{eu_p}{2} + \mathcal{O}(e^3). \qquad (52)$$

We should immediately note that (51) is indeed an expansion in e^2. In (45) we see that for any value of n the bracketed terms containing the combinations of generalized Bessel functions are either both odd or both even in e due to the property $\mathrm{J}_n(-x, y) = (-1)^n \mathrm{J}_n(x, y)$. As there are two bracketed terms, the overall propagator will be an expansion in e^2.

Let us first consider the central sideband at leading order. The wave function renormalization factors are

$$\sqrt{\mathbf{Z}_2^{(0)}} = \sqrt{\bar{\mathbf{Z}}_2^{(0)}} = 1 - e^2 \left(\frac{p\cdot a}{p\cdot k}\right)^2, \qquad (53)$$

as the matrix factors cancel at this order here. Using $\delta\not{m}$ it is now simple to reproduce all of (14) from an expansion of (51). This calculation is sensitive to both the matrix form of the mass shift and the sign in the numerator of (48).

In the first upper sideband we proceed similarly to (13) but adding and subtracting $\not{k}$ terms as follows

$$(\not{p} + m)\not{a}(\not{p} + \not{k} + m)\not{a}(\not{p} + m)$$
$$= (\not{p} + m)\not{a}(\not{p} + \not{k} + m)\not{a}(\not{p} + \not{k} + m - \not{k}), \qquad (54)$$

which may also be repeated on the left. This generates zero pole terms that cancel with those found in the central sideband and from the first lower sideband. The terms containing poles in the first upper sideband are thus found to be

$$ie^2 \left(\frac{1}{4}\left(\frac{p\cdot a}{p\cdot k}\right)^2 \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} + \frac{1}{4}\frac{p\cdot a}{p\cdot k}\not{a}\frac{1}{(p+k)^2 - m^2} \right.$$
$$\left. - \frac{1}{4}\left(\frac{p\cdot a}{p\cdot k}\right)^2 \not{k}\frac{1}{(p+k)^2 - m^2} - \frac{1}{8}\frac{p\cdot a}{p\cdot k}a^2\not{k}\frac{1}{(p+k)^2 - m^2} \right), \qquad (55)$$

and similarly with $k \to -k$ for the first lower sideband. (The adjacent sidebands will first acquire mass shifts at the next order in the coupling.)

This calculation also agrees with (51) at this order. In the first upper sideband, we note that

$$\sqrt{\bar{\mathbf{Z}}_2^{(1)}} = e\left(-\frac{1}{2}\frac{p\cdot a}{p\cdot k} + \frac{\not{k}\not{a}}{4p\cdot k} \right), \qquad (56)$$

at leading order in e. There is a similar expression for $\sqrt{\mathbf{Z}_2^{(1)}}$. Substituting this into (51) yields (55). The final a^2-dependent term in (55) may appear surprising in this sideband as there is no contribution from the mass shift in the perturbative expansion at this order. It arises via a product of two $\not{a}$ factors in the gamma matrix structures. We note that this calculation is sensitive to the matrix structure of the wave function renormalization factors. The first lower sideband is again simply reproduced by changing $k \to -k$ everywhere in (55).

As the agreement across these three sidebands depends upon the form of the various structures in (51), it verifies the emergence of a matrix mass shift and matrix wave function structures ordered to the left and right.

We also comment here that the propagator in the laser background cannot be written in the form

$$\frac{i}{\not{p} - m - \Sigma}, \qquad (57)$$

with $-i\Sigma$ being the self-energy. This is because the usual derivation of (57) involves the identification of the perturbative expansion

$$\frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}(-i\Sigma)\frac{i}{\not{p} - m} + \cdots, \qquad (58)$$

as a Taylor expansion of (57). However, this requires that $\not{p}$ and Σ commute. This is true in normal perturbative QED since the self-energy then only depends on $\not{p}$ and scalars. However, in the laser context, there are $\not{k}$ and $\not{a}$ factors which do not commute with $\not{p}$ and hence obstruct an expression of the form (57).

VII. CONCLUSIONS

In this paper we have derived an expression for the fermion propagator in a linearly polarized plane wave. In QED, as well as in the scalar theory, all of the laser induced structures in the propagator can be understood as

multiplicative renormalizations of the free propagator. The distinctive feature highlighted in this paper is that for fermionic QED these multiplicative factors are matrices. Matrix structures have already been seen in earlier work [4,5,11], but here we have seen that the electron mass shift is a matrix such that both the mass and also the mass squared only pick up corrections at leading order in the coupling. Although we have focused in this paper on linearly polarized lasers, our approach can be generalized to other polarizations, see [13]. We expect that such a matrix valued mass shift will persist in more realistic models of laser pulses [27]. We note that matrix wave function renormalization constants have been seen previously in studies of the infrared [28] although the mass shift did not there involve a matrix. We have also checked our all orders calculations through a first order perturbative expansion of the propagator which tested all of the key features of our overall result (51).

The results presented in this paper will be important for calculating scattering processes in QED in laser backgrounds, see for example [29] where the propagator is used to calculate double Compton scattering, and also [30,31] for this process. More generally see [32–35] for other processes. It is intriguing to speculate how the matrix renormalization constants impact upon the calculation of S-matrix elements and the physical predictions of QED in a laser background.

ACKNOWLEDGMENTS

APPENDIX A: VOLKOV SPINORS

The defining equations (24) and (25) allow us to construct the Volkov spinors by suitably boosting the static spinors

$$
\mathcal{U}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathcal{U}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},
$$

$$
\mathcal{V}^{(1)}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathcal{V}^{(2)}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{A1}
$$

To this end, we define

$$
\mathcal{U}_{\mathrm{V}}^{(\alpha)}(p) = \frac{1}{\sqrt{2m(m + E_p^* + 2e^2 v_{\bar{p}} k_0)}}
$$
$$
\times (\bar{\slashed{p}} + 2e^2 v_{\bar{p}} \slashed{k} + m)\mathcal{U}^{(\alpha)}(0) \tag{A2}
$$

and

$$
\mathcal{V}_{\mathrm{V}}^{(\alpha)}(p) = \frac{1}{\sqrt{2m(m + E_p^* + 2e^2 v_{\bar{p}} k_0)}}
$$
$$
\times (-\bar{\slashed{p}} - 2e^2 v_{\bar{p}} \slashed{k} + m)\mathcal{V}^{(\alpha)}(0). \tag{A3}
$$

With these conventions we have the inner products

$$
\bar{\mathcal{U}}_{\mathrm{V}}^{(\alpha)}(p)\mathcal{U}_{\mathrm{V}}^{(\beta)}(p) = \delta^{\alpha\beta} \quad \text{and} \quad \bar{\mathcal{V}}_{\mathrm{V}}^{(\alpha)}(p)\mathcal{V}_{\mathrm{V}}^{(\beta)}(p) = -\delta^{\alpha\beta}, \tag{A4}
$$

as well as the tensor products

$$
\mathcal{U}_{\mathrm{V}}^{(\alpha)}(p)\bar{\mathcal{U}}_{\mathrm{V}}^{(\alpha)}(p) = \frac{\bar{\slashed{p}} + 2e^2 v_{\bar{p}} \slashed{k} + m}{2m} \tag{A5}
$$

and

$$
\mathcal{V}_{\mathrm{V}}^{(\alpha)}(p)\bar{\mathcal{V}}_{\mathrm{V}}^{(\alpha)}(p) = \frac{\bar{\slashed{p}} + 2e^2 v_{\bar{p}} \slashed{k} - m}{2m}. \tag{A6}
$$

APPENDIX B: VOLKOV PROPAGATOR DETAILS

The argument taking us from the initial expression for the Volkov propagator in Eq. (44) to the final covariant expression (45) involves several key steps that we outline below.

Starting from (44) we make the substitution $p^0 \to E_p^* - p^0 - nk^0$ in the first part and $p^0 \to p^0 + E_p^* + nk^0$ in the second to give

$$
-\int \frac{\text{đ}^4 p}{2E_p^*} \left(\frac{i}{E_p^* - p^0 - nk^0 - i\epsilon} e^{-ip^0(x^0 - y^0)} e^{i(\underline{p} - n\underline{k})\cdot(\underline{x} - \underline{y})} \right.
$$
$$
\times \left(\mathcal{J}_n(\bar{p}) + e\frac{\slashed{k}\slashed{a}}{4\bar{p}\cdot k}\mathcal{J}_n^{\Delta}(\bar{p}) \right)(\bar{\slashed{p}} + m - \delta\bar{m})
$$
$$
\times \left(\mathcal{J}_n(\bar{p}) + e\frac{\slashed{a}\slashed{k}}{4\bar{p}\cdot k}\mathcal{J}_n^{\Delta}(\bar{p}) \right)
$$
$$
- \frac{i}{E_p^* + p^0 + nk^0 - i\epsilon} e^{-ip^0(x^0 - y^0)} e^{-i(\underline{p} + n\underline{k})\cdot(\underline{x} - \underline{y})}
$$
$$
\times \left(\mathcal{J}_n(-\bar{p}) + e\frac{\slashed{a}\slashed{k}}{4\bar{p}\cdot k}\mathcal{J}_n^{\Delta}(-\bar{p}) \right)(\bar{\slashed{p}} - m - \delta\bar{m})
$$
$$
\left. \times \left(\mathcal{J}_n(-\bar{p}) + e\frac{\slashed{k}\slashed{a}}{4\bar{p}\cdot k}\mathcal{J}_n^{\Delta}(-\bar{p}) \right) \right). \tag{B1}
$$

If we now shift the three-momentum in the first part by $\underline{p} \to \underline{p} + n\underline{k}$ and by $\underline{p} \to -\underline{p} - n\underline{k}$ in the second (noting a change in sign in $\delta\bar{m}$) then we have

$$-\int \frac{\mathrm{d}^4 p}{2E^*_{p+nk}} e^{-ip\cdot(x-y)} \left(\frac{i}{E^*_{p+nk} - p^0 - nk^0 - i\epsilon} \left(\mathcal{J}_n(\overline{p+nk}) + e \frac{k\!\!\!/ a\!\!\!/}{4\overline{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\overline{p+nk}) \right) \right.$$

$$\times (\overline{p\!\!\!/ + nk\!\!\!/} + m - \delta\bar{m}) \left(\mathcal{J}_n(\overline{p+nk}) + e \frac{a\!\!\!/ k\!\!\!/}{4\overline{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\overline{p+nk}) \right)$$

$$-\frac{i}{E^*_{p+nk} + p^0 + nk^0 - i\epsilon} \left(\mathcal{J}_n(\widetilde{p+nk}) + e \frac{a\!\!\!/ k\!\!\!/}{4\widetilde{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\widetilde{p+nk}) \right)$$

$$\left. \times (-\widetilde{p\!\!\!/ + nk\!\!\!/} - m + \delta\tilde{m}) \left(\mathcal{J}_n(\widetilde{p+nk}) + e \frac{k\!\!\!/ a\!\!\!/}{4\widetilde{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\widetilde{p+nk}) \right) \right). \tag{B2}$$

In this we have simply shifted on-shell momenta in the first part but in the second part we have noted that under the shift

$$\bar{p} \to (E_{p+nk}, -\underline{p} - n\underline{k}),$$
$$= -(-E_{p+nk}, \underline{p} + n\underline{k}),$$
$$= -\widetilde{p+nk}, \tag{B3}$$

where we denote by the tilde the negative energy on-shell four-momenta.

Combining these terms over a common denominator allows us to extract the covariant propagator and we get

$$\int \frac{\mathrm{d}^4 p}{2E^*_{p+nk}} \frac{ie^{-ip\cdot(x-y)}}{(p+nk)^2 - m^2_* + i\epsilon} \left((E^*_{p+nk} + p^0 + nk^0) \left(\mathcal{J}_n(\overline{p+nk}) + e \frac{k\!\!\!/ a\!\!\!/}{4\overline{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\overline{p+nk}) \right) \right.$$

$$\times (\overline{p\!\!\!/ + nk\!\!\!/} + m - \delta\bar{m}) \left(\mathcal{J}_n(\overline{p+nk}) + e \frac{a\!\!\!/ k\!\!\!/}{4\overline{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\overline{p+nk}) \right)$$

$$-(E^*_{p+nk} - p^0 - nk^0) \left(\mathcal{J}_n(\widetilde{p+nk}) + e \frac{a\!\!\!/ k\!\!\!/}{4\widetilde{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\widetilde{p+nk}) \right)$$

$$\left. \times (-\widetilde{p\!\!\!/ + nk\!\!\!/} - m + \delta\tilde{m}) \left(\mathcal{J}_n(\widetilde{p+nk}) + e \frac{k\!\!\!/ a\!\!\!/}{4\widetilde{(p+nk)}\cdot k} \mathcal{J}^\Delta_n(\widetilde{p+nk}) \right) \right). \tag{B4}$$

We now recognize the two terms as simply the residues associated with the two poles of the covariant denominator. Hence we recover the covariant expression (45).

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