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Adler-Bardeen theorem and cancellation of gauge anomalies to all orders in nonrenormalizable theories

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We prove the Adler-Bardeen theorem in a large class of general gauge theories, including nonrenormalizable ones. We assume that the gauge symmetries are general covariance, local Lorentz symmetry, and Abelian and non-Abelian Yang-Mills symmetries, and that the local functionals of vanishing ghost numbers satisfy a variant of the Kluberg-Stern–Zuber conjecture. We show that if the gauge anomalies are trivial at one loop, for every truncation of the theory there exists a subtraction scheme where they manifestly vanish to all orders, within the truncation. Outside the truncation the cancellation of gauge anomalies can be enforced by fine-tuning local counterterms. The framework of the proof is worked out by combining a recently formulated chiral dimensional regularization with a gauge invariant higher-derivative regularization. If the higher-derivative regularizing terms are placed well beyond the truncation, and the energy scale Λ associated with them is kept fixed, the theory is superrenormalizable and has the property that, once the gauge anomalies are canceled at one loop, they manifestly vanish from two loops onwards by simple power counting. When the Λ divergences are subtracted away and Λ is sent to infinity, the anomaly cancellation survives in a manifest form within the truncation and in a nonmanifest form outside. The standard model coupled to quantum gravity satisfies all the assumptions, so it is free of gauge anomalies to all orders.

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I. INTRODUCTION

The Adler-Bardeen theorem [1,2] is crucial to prove the consistency of a wide class of perturbative quantum field theories. Its main consequence is that the cancellation of gauge anomalies at one loop ensures the cancellation of gauge anomalies to all orders. Thanks to this result, a finite number of conditions is sufficient to determine when a potentially anomalous theory is actually anomaly free. The cancellation conditions can be worked out rather easily because they just involve simplified divergences of one-loop diagrams. If a similar theorem did not hold, a chiral gauge theory, such as the standard model, would have to satisfy infinitely many independent cancellation conditions, to be consistent. The solutions would be very few or contain infinitely many fields.

So far, the Adler-Bardeen theorem has been proved in Abelian and non-Abelian power counting renormalizable gauge theories, including the standard model, but not in more general classes of theories. In this paper we overcome this limitation by working out a more powerful proof that applies to a large class of nonrenormalizable theories and allows us to infer that the standard model coupled to quantum gravity, which is known to be free of gauge anomalies at one loop [3], is also free of gauge anomalies to all orders, and so are most of its extensions.

In general, we must show that when the gauge anomalies are trivial at one loop, there exists a subtraction scheme where they vanish to all orders. Once we know that the scheme exists, we can build it order by order by fine-tuning finite local counterterms. A more powerful result is to provide the right scheme from the beginning, that is to say, define a framework where all potentially anomalous contributions cancel out at one loop and are automatically zero from two loops onwards. We call a statement identifying such a scheme *manifest* Adler-Bardeen theorem. In perturbatively unitary renormalizable theories the manifest Adler-Bardeen theorem has been proved recently [4]. For reasons that we explain in the paper, in nonrenormalizable theories we are not able to determine the subtraction scheme where anomaly cancellation is manifest from two loops onwards. We have to content ourselves with a weaker, yet powerful enough, result, which we call *almost manifest* Adler-Bardeen theorem: given an appropriate truncation T of the theory, we find a subtraction scheme where the gauge anomalies manifestly vanish from two loops onwards within the truncation.

The most common regularization techniques are not very convenient to work out general proofs of the Adler-Bardeen theorem, because they give us no clue about the right subtraction scheme. In Ref. [4] a better regularization technique was built by merging the dimensional regularization with a suitable gauge invariant higher-derivative (HD) regularization [5] and used to prove the manifest Adler-Bardeen theorem in four-dimensional renormalizable perturbatively unitary gauge theories. Unfortunately, several difficulties of the dimensional regularization make it hard to generalize that proof to nonrenormalizable theories. To overcome those problems, in Ref. [6] a *chiral dimensional*

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(CD) regularization technique was defined. Nevertheless, the CD technique alone does not identify the subtraction scheme where gauge anomalies manifestly cancel and must still be merged with a suitable gauge invariant HD regularization. The resulting technique, called chiral-dimensional/higher-derivative (CDHD) regularization, is the right one to generalize the proof of the Adler-Bardeen theorem to nonrenormalizable theories. It has two regularizing parameters: $\varepsilon = d - D$, where d is the physical spacetime dimension and D is the continued dimension, and the energy scale Λ associated with the higher-derivative terms. The limit $\varepsilon \to 0$ must be studied before the limit $\Lambda \to \infty$.

The CDHD technique is organized so that the higher-derivative regularizing terms fall well beyond the truncation. When Λ is kept fixed, a peculiar superrenormalizable higher-derivative theory is obtained, which we call *HD theory*. The HD theory satisfies the manifest Adler-Bardeen theorem by simple power counting arguments. The limit $\Lambda \to \infty$ on the HD theory defines the *final theory*, which is the one we are interested in. We show that we can renormalize the Λ divergences so as to preserve the cancellation of gauge anomalies to all orders within the truncation.

The proof we provide holds under certain assumptions. First, we assume that the gauge symmetries are general covariance, local Lorentz symmetry, and Abelian and non-Abelian Yang-Mills symmetries. At this stage, we cannot include local supersymmetry. Second, we assume that the local functionals of vanishing ghost numbers satisfy a variant of the Kluberg-Stern–Zuber conjecture [7]. The standard model coupled to quantum gravity does not satisfy the ordinary Kluberg-Stern–Zuber conjecture, but satisfies the variant that we assume in this paper. The other key assumption is of course that the one-loop gauge anomalies $\mathcal{A}^{(1)}$ are trivial. In our approach the functional $\mathcal{A}^{(1)}$ is extremely simple, since it can only depend on the gauge fields, their ghosts, and some matter fields. We call $\mathcal{A}^{(1)}$ trivial if there exists a local functional χ of the fields such that $\mathcal{A}^{(1)} = (S_d, \chi)$, where S_d is the d-dimensional tree-level action and (X, Y) are the Batalin-Vilkovisky (BV) antiparentheses [8], recalled in formula (2.2). Other mild technical assumptions needed for the proof (all of which are satisfied by most common theories of fields of spins ≤ 2) are described along the way.

Here are the main statements that we consider in this paper. The most general Adler-Bardeen theorem for the cancellation of gauge anomalies states that

Theorem 1: If the gauge anomalies are trivial at one loop, the subtraction scheme can be fine-tuned so that they vanish to all orders.

In renormalizable theories we actually have a stronger result, the *manifest* Adler-Bardeen theorem [4], stating that

Theorem 2: If the gauge anomalies are trivial at one loop, there exists a subtraction scheme where they cancel at one loop and manifestly vanish from two loops onwards.

In nonrenormalizable theories, instead, we can prove a result that is stronger than 1, but weaker than 2, the almost manifest Adler-Bardeen theorem, which states that

Theorem 3: If the gauge anomalies are trivial at one loop, for every appropriate truncation of the theory there exists a subtraction scheme where they cancel at one loop and manifestly vanish from two loops onwards within the truncation.

The proper way to truncate a nonrenormalizable theory is specified in the next section. We stress again that in nonrenormalizable theories we are not able to prove statement 2, namely find the right subtraction scheme independently of the truncation. We can just find a good subtraction scheme for every truncation. This result is still satisfactory, because Theorem 3 implies Theorem 1. Indeed, let s_T denote the subtraction scheme associated with the truncation T by the proof of Theorem 3. There, the gauge anomalies $\mathcal{A}$ vanish within the truncation. Let $\mathcal{A}_{>T}$ denote a finite class of contributions to the gauge anomalies that lie outside the truncation T, in the scheme s_T. Clearly, the contributions of class $\mathcal{A}_{>T}$ are fully contained in some truncation $T' > T$. There, however, they must vanish. Since two schemes differ by finite local counterterms, there must exist finite local counterterms that cancel the contributions of class $\mathcal{A}_{>T}$ in the scheme s_T. In conclusion, the scheme s_T satisfies Theorem 3 within the truncation and Theorem 1 outside.

It is worthwhile to compare our approach with other approaches to the Adler-Bardeen theorem that can be found in the literature. The original proof given by Adler and Bardeen [1] was designed to work in QED. Most generalizations to renormalizable non-Abelian gauge theories used arguments based on the renormalization group [9–12]. Those arguments work well unless the first coefficients of the beta functions satisfy peculiar conditions [12] (for example, they should not vanish). If the theory is nonrenormalizable, we can build infinitely many dimensionless couplings and can hardly exclude that the first coefficients of their beta functions satisfy peculiar conditions. Algebraic/geometric derivations [13] based on the Wess-Zumino consistency conditions [14] and the quantization of the Wess-Zumino-Witten action also do not seem suitable to be generalized to nonrenormalizable theories. Another method to prove the Adler-Bardeen theorem in renormalizable theories is obtained by extending the coupling constants to spacetime-dependent fields [15]. A tentative regularization-independent approach in nonrenormalizable theories can be found in Ref. [16].

We stress that the proof provided in this paper is the first proof that the standard model coupled to quantum gravity is free of gauge anomalies to all orders. Our arguments and results also apply to the study of higher-dimensional composite fields in renormalizable and nonrenormalizable theories.

In this paper, the powers of $\hbar$ are merely used as tools to denote the appropriate orders of the loop expansion. They are not written explicitly unless necessary. It is understood that the functionals depend analytically on the parameters that are treated perturbatively.

The paper is organized as follows. In Sec. II we provide the setting of the proof. We specify the truncation, recall the properties of the CD regularization technique, and explain how it can be combined with a suitable higher-derivative regularization to build the CDHD regularized theory. In Sec. III we study the properties of the HD theory. In particular, we show that it is superrenormalizable and study the structures of its counterterms and potential anomalies. In Sec. IV we work out the renormalization of the HD theory. In Sec. V we study its one-loop anomalies. In Sec. VI we prove that the HD theory satisfies the manifest Adler-Bardeen theorem. In Sec. VII we subtract the Λ divergences and prove that the final theory satisfies the almost manifest Adler-Bardeen theorem, as well as Theorem 1. In Sec. VIII we show that the standard model coupled to quantum gravity, as well as most of its extensions, belongs to the class of nonrenormalizable theories to which our results apply. Section IX contains our conclusions.

II. GENERAL SETTING

In this section we give the general setup of the proof and specify most of the assumptions we need. First we recall the properties of the CD regularization and explain how it is merged with the HD regularization to build the CDHD regularization. Then we explain how to truncate the theory. Instead of working directly with the standard model coupled to quantum gravity, we formulate a general approach and give specific examples along the way.

Throughout the paper, d denotes the physical spacetime dimension, and $D = d - \varepsilon$ is the continued complex dimension introduced by the dimensional regularization (see Sec. II A for details). We work in $d > 2$. We use the symbol ϕ to collect the "physical fields," that is to say, the Yang-Mills gauge fields $A^a_{\bar\mu}$, the matter fields, and (if gravity is dynamical) the metric tensor $g_{\bar\mu\bar\nu}$ or the vielbein $e^{\bar a}_{\bar\mu}$. The indices $a, b, \ldots$, refer to the Yang-Mills gauge group, while $\bar a, \bar b, \ldots$, refer to the Lorentz group. The indices $\bar\mu, \bar\nu, \ldots$, refer to the physical d-dimensional spacetime $\mathbb{R}^d$, as opposed to the continued spacetime $\mathbb{R}^D$.

We denote the classical action by $S_c(\phi)$. In the case of the standard model coupled to quantum gravity, we take $S_c = S_{c\mathrm{SMG}} + \Delta S_c$, where

$$S_{c\mathrm{SMG}} = \int \sqrt{|g|}\left[-\frac{1}{2\kappa^2}(R + 2\Lambda_{\mathrm{c}}) - \frac{1}{4}F^a_{\bar\mu\bar\nu}F^{a\bar\mu\bar\nu} + \mathcal{L}_m\right] \tag{2.1}$$

and ΔS_c collects the invariants generated as counterterms by renormalization, multiplied by independent parameters. Here, R is the Ricci curvature, $F^a_{\bar\mu\bar\nu}$ are the Yang-Mills field strengths, $\mathcal{L}_m$ is the matter Lagrangian coupled to the metric tensor, g is the determinant of the metric tensor or vielbein $g_{\bar\mu\bar\nu}$, Λ_{c} is the cosmological constant, and $\kappa^2 = 8\pi G$, where G is Newton's constant.

We use the Batalin-Vilkovisky formalism [8] because it is very efficient to keep track of gauge invariance throughout the renormalization algorithm. An enlarged set of fields $\Phi^\alpha = \{\phi, C, \bar C, B\}$ is introduced to collect the physical fields ϕ, the Fadeev-Popov ghosts C, the antighosts $\bar C$, and the Lagrange multipliers B for the gauge fixing. Next, external sources $K_\alpha = \{K_\phi, K_C, K_{\bar C}, K_B\}$ are coupled to the Φ^α symmetry transformations $R^\alpha(\Phi)$ in a way specified below.

If X and Y are functionals of Φ and K, their *antiparentheses* are defined as

$$(X, Y) \equiv \int \left(\frac{\delta_r X}{\delta\Phi^\alpha}\frac{\delta_l Y}{\delta K_\alpha} - \frac{\delta_r X}{\delta K_\alpha}\frac{\delta_l Y}{\delta\Phi^\alpha}\right), \tag{2.2}$$

where the integral is over spacetime points associated with repeated indices and the subscripts l and r in δ_l and δ_r denote the left and right functional derivatives, respectively. The *master equation* is the condition $(S, S) = 0$ and must be solved in D dimensions with the "boundary condition" $S = S_c$ at $C = \bar C = B = K = 0$. At the practical level, we first solve the equation $(S, S) = 0$ in d dimensions, and then interpret its solution S as a D-dimensional action, according to the rules of the CD regularization (see Sec. II A). We denote the non-gauge-fixed solution of the master equation by $\bar S_d(\Phi, K)$. The subscript d reminds us that, although $\bar S_d$ solves $(\bar S_d, \bar S_d) = 0$ in D dimensions, it is just the d-dimensional action interpreted from the D-dimensional point of view. In particular, it may not be well regularized as a D-dimensional action. Once we regularize it, we may not be able to preserve the master equation exactly in $D \neq d$. The violations of the master equation at $D \neq d$ are the origins of potential anomalies.

(I) We assume that the gauge symmetries are general covariance, local Lorentz symmetry, and Abelian and non-Abelian Yang-Mills symmetries. In particular, the gauge algebra is irreducible and closes off shell. We use the second order formalism for gravity and choose the fields Φ and the sources K so that the non-gauge-fixed solution $\bar S_d(\Phi, K)$ of the master equation reads

$$\begin{aligned}\bar S_d(\Phi, K) &= S_c(\phi) + S_K(\Phi, K),\\ S_K(\Phi, K) &= -\int R^\alpha(\Phi)K_\alpha,\end{aligned} \tag{2.3}$$

where the functional S_K (with left-handed fermions ψ_L and scalars φ, for definiteness) reads

$$S_K = \int (C^{\bar\rho}\partial_{\bar\rho}A^a_{\bar\mu} + A^a_{\bar\rho}\partial_{\bar\mu}C^{\bar\rho} - \partial_{\bar\mu}C^a - gf^{abc}A^b_{\bar\mu}C^c)K^{\bar\mu a}_A + \int \left(C^{\bar\rho}\partial_{\bar\rho}C^a + \frac{g}{2}f^{abc}C^bC^c\right)K^a_C$$
$$+ \int (C^{\bar\rho}\partial_{\bar\rho}e^{\bar a}_{\bar\mu} + e^{\bar a}_{\bar\rho}\partial_{\bar\mu}C^{\bar\rho} + C^{\bar a\,\bar b}e_{\bar\mu\,\bar b})K^{\bar\mu}_{\bar a} + \int C^{\bar\rho}(\partial_{\bar\rho}C^{\bar\mu})K^C_{\bar\mu} + \int (C^{\bar a\,\bar c}\eta_{\bar c\,\bar d}C^{\bar d\,\bar b} + C^{\bar\rho}\partial_{\bar\rho}C^{\bar a\,\bar b})K^C_{\bar a\,\bar b}$$
$$+ \int \left(C^{\bar\rho}\partial_{\bar\rho}\bar\psi_L - \frac{i}{4}\bar\psi_L\sigma^{\bar a\,\bar b}C_{\bar a\,\bar b} + g\bar\psi_L T^aC^a\right)K_\psi + \int K_{\bar\psi}\left(C^{\bar\rho}\partial_{\bar\rho}\psi_L - \frac{i}{4}\sigma^{\bar a\,\bar b}C_{\bar a\,\bar b}\psi_L + gT^aC^a\psi_L\right)$$
$$+ \int (C^{\bar\rho}(\partial_{\bar\rho}\varphi) + g\mathcal{T}^aC^a\varphi)K_\varphi - \int B^aK^a_{\bar C} - \int B_{\bar\mu}K^{\bar\mu}_{\bar C} - \int B_{\bar a\,\bar b}K^{\bar a\,\bar b}_{\bar C}. \tag{2.4}$$

Here, T^a and $\mathcal{T}^a$ are the anti-Hermitian matrices associated with the fermion and scalar representations, respectively. The ghosts of Yang-Mills symmetry are C^a, those of local Lorentz symmetry are $C^{\bar a\,\bar b}$, and those of diffeomorphisms are $C^{\bar\mu}$. The pairs $\bar C^a$-B^a, $\bar C_{\bar a\,\bar b}$-$B^{\bar a\,\bar b}$, and $\bar C_{\bar\mu}$-$B_{\bar\mu}$ collect the antighosts and the Lagrange multipliers of Yang-Mills symmetry, local Lorentz symmetry, and diffeomorphisms, respectively. The functional S_K satisfies $(S_K, S_K) = 0$ in arbitrary D dimensions.

We can gauge fix the theory with the help of a gauge fermion $\Psi(\Phi)$, which is a local functional of ghost number -1 that depends only on the fields Φ and contains the gauge-fixing functions $G(\phi)$. For example, $G(\phi) = \partial^{\bar\mu}A_{\bar\mu}$ for the Lorenz gauge in Yang-Mills theories. The typical form of $\Psi(\Phi)$ is

$$\Psi(\Phi) = \int \sqrt{|g|}\bar C\left(G(\phi,\xi) + \frac{1}{2}P(\phi,\xi',\partial)B\right), \tag{2.5}$$

where ξ, ξ' are gauge-fixing parameters and P is an operator that may contain derivatives acting on B. Typically, if the gauge fields ϕ_g have dominant kinetic terms (which are the quadratic terms that have the largest numbers of derivatives) of the form

$$\sim\frac{1}{2}\int \phi_g\partial^{N_{\phi_g}}\phi_g \tag{2.6}$$

inside S_c, we choose G and P such that

$$G(\phi,\xi) \sim \partial^{N_{\phi_g}-1+a}\phi_g + \text{nonlinear terms},$$
$$P(\phi,\xi',\partial) \sim \xi'\partial^{N_{\phi_g}-2+b} + \mathcal{O}(\phi), \tag{2.7}$$

up to terms with fewer derivatives, where $a = b = 0$ for diffeomorphisms and Yang-Mills symmetries, while $a = 1$, $b = 2$ for local Lorentz symmetry. See formula (2.19) for more details. In the case of three-dimensional Chern-Simons theories ($N_{\phi_g} = 1$) we take $a = 1$ and $P = 0$.

The gauge-fixed action S_d is obtained by adding (S_K, Ψ) to $\bar S_d$:

$$S_d(\Phi, K) = \bar S_d + (S_K, \Psi) = S_c + (S_K, \Psi) + S_K. \tag{2.8}$$

Alternatively, S_d is obtained from $\bar S_d$ by applying the canonical transformation generated by

$$F(\Phi, K') = \int \Phi^\alpha K'_\alpha + \Psi(\Phi). \tag{2.9}$$

We still have $(S_d, S_d) = 0$ in D dimensions, but we stress again that in general the action S_d may not be well regularized.

Let $\{\mathcal{G}_i(\phi)\}$ denote a basis of local gauge invariant functionals of the physical fields ϕ, i.e. local functionals such that $(S_K, \mathcal{G}_i) = 0$. Expand the classical action as

$$S_c(\phi) = \sum_i \lambda_i\mathcal{G}_i(\phi), \tag{2.10}$$

where λ_i are independent constants. We call such constants "physical parameters," since they include, or are related to, the gauge coupling constants, the masses, etc. If the theory is power counting renormalizable, $S_c(\phi)$ is restricted accordingly and contains just a finite number of independent parameters λ_i. If the theory is nonrenormalizable, (2.10) must include all the invariants $\mathcal{G}_i$ required by renormalization, which are typically infinitely many.

In several cases, the set $\{\mathcal{G}_i(\phi)\}$ is restricted to the invariants that are inequivalent, where two functionals are considered equivalent if they differ by terms proportional to the S_c field equations. The reason why such a restriction is meaningful is that the counterterms proportional to the field equations can be subtracted away by means of canonical transformations of the BV type, instead of λ_i redefinitions. However, for some arguments of this paper it is convenient to include the terms proportional to the S_c field equations inside the set $\{\mathcal{G}_i(\phi)\}$, which we assume from now on. We can remove them at the end, by means of a convergent canonical transformation and the procedure of Ref. [17]. There, it is shown that, after the transformation, it is always possible to re-renormalize the theory and re-fine-tune its finite local counterterms so as to preserve the cancellation of gauge anomalies. The renormalized Γ functional of the transformed theory is related to the renormalized Γ functional of the starting theory by a (convergent, nonlocal) canonical transformation. See [17] for more details.

We say that an action $\mathcal{S}$ satisfies the Kluberg-Stern–Zuber assumption [7], if every local functional X of ghost number zero that solves the equation $(\mathcal{S}, X) = 0$ has the form

$$X = \sum_i a_i \mathcal{G}_i + (\mathcal{S}, Y), \tag{2.11}$$

where a_i are constants depending on the parameters of the theory, and Y is a local functional of ghost number -1. The Kluberg-Stern–Zuber assumption is very useful to study the counterterms. It is satisfied, for example, when the Yang-Mills gauge group is semisimple and the action $\mathcal{S}$ meets other mild requirements [18]. Unfortunately, the standard model coupled to quantum gravity does not satisfy it, unless its accidental symmetries are completely broken. This forces us to search for a more general version of the assumption.

The accidental symmetries are the continuous global symmetries unrelated to the gauge transformations. Some of them are anomalous, others are nonanomalous. If the gauge group has $U(1)$ factors, let $G_{\rm nas}$ denote the group of nonanomalous accidental symmetries. If the gauge group has no $U(1)$ factors, we take $G_{\rm nas}$ equal to the identity. We denote the local gauge invariant functionals of ϕ that break the group $G_{\rm nas}$ by $\check{\mathcal{G}}_i(\phi)$. We exclude the invariants $\check{\mathcal{G}}_i$ from the set $\{\mathcal{G}_i(\phi)\}$ and the actions S_c, S_d, but include them in more general actions $\check{S}_c$ and $\check{S}_d = \check{S}_c + (S_K, \Psi) + S_K$, multiplied by independent parameters $\check{\lambda}_i$. The invariants that explicitly break the anomalous accidental symmetries are instead included in the set $\{\mathcal{G}_i(\phi)\}$.

It is consistent to switch the invariants $\check{\mathcal{G}}_i$ off, since, when they are absent, renormalization is unable to generate them back as counterterms. However, for some arguments of the proof it is necessary to temporarily switch them on. For this reason, we need to work with both actions S_d and $\check{S}_d$.

The action $\mathcal{S}$ of (2.11) is assumed to be invariant under the group $G_{\rm nas}$. We say that an action $\check{\mathcal{S}}$ that breaks $G_{\rm nas}$ satisfies the extended Kluberg-Stern–Zuber assumption if every local functional X of ghost number zero that solves the equation $(\check{\mathcal{S}}, X) = 0$ has the form

$$X = \sum_i a_i \mathcal{G}_i + \sum_i b_i \check{\mathcal{G}}_i + (\check{\mathcal{S}}, Y), \tag{2.12}$$

where b_i are other constants and Y is local. We say that the action S_d is *cohomologically complete* if its extension $\check{S}_d$ satisfies the extended Kluberg-Stern–Zuber assumption. In Sec. VIII we prove that the standard model coupled to quantum gravity is cohomologically complete.

The variant of the Kluberg-Stern–Zuber assumption that we need for the proof of the Adler-Bardeen theorem is formulated in Sec. II C. In Sec. VIII we show that it is satisfied by the standard model coupled to quantum gravity, as well as most of its extensions. We also prove that the standard model coupled to quantum gravity satisfies a "physical" variant of the Kluberg-Stern–Zuber assumption.

It is straightforward to show that the results of this paper, which we derive for theories with unbroken $G_{\rm nas}$, also hold when $G_{\rm nas}$ is completely, or partially, broken. In the end, it is our choice to decide which symmetries of $G_{\rm nas}$ should be preserved and which ones should be broken. It should also be noted that it may not be easy to establish which accidental symmetries are anonalous and which ones are nonanomalous *a priori*. We have arranged our statements to make them work in any case, under this respect. In the safest case, we can extend the action S_d till $G_{\rm nas} = \mathbb{1}$ and $S_d = \check{S}_d$.

A. Chiral dimensional regularization

If we want to identify the subtraction scheme where the anomaly cancellation is (almost) manifest, we must provide a regularization and a set of specific prescriptions to handle the counterterms and the potentially anomalous contributions in convenient ways. The best regularization technique is obtained by merging the chiral dimensional regularization recently introduced in Ref. [6] with a suitable gauge invariant higher-derivative regularization.

Going through the derivation of Ref. [4], where the manifest Adler-Bardeen theorem was proved in perturbatively unitary, power counting renormalizable four-dimensional gauge theories, it is easy to spot several crucial arguments that do not generalize to wider classes of models in a straightforward way. The main obstacles are due to the dimensional regularization as it is normally understood [19]. Besides the nuisances associated with the definition of γ_5, the dimensionally continued Dirac algebra is responsible for other serious difficulties. For example, it allows us to build infinitely many inequivalent evanescent terms of the same dimensions, and the Fierz identities involve infinite sums. Moreover, it generates ambiguities that plague the classification of counterterms and make it difficult to extract the divergent parts from the antiparentheses of functionals. The CD regularization overcomes these problems. In this subsection we recall how it works.

As usual, we split the D-dimensional spacetime manifold $\mathbb{R}^D$ into the product $\mathbb{R}^d \times \mathbb{R}^{-\varepsilon}$ of the physical d-dimensional spacetime $\mathbb{R}^d$ times a residual $(-\varepsilon)$-dimensional evanescent space $\mathbb{R}^{-\varepsilon}$, where ε is a complex number. Spacetime indices $\mu, \nu, \ldots$, of vectors and tensors are split into bar indices $\bar{\mu}, \bar{\nu}, \ldots$, which take the values $0, 1, \ldots, d-1$, and formal hat indices $\hat{\mu}, \hat{\nu}, \ldots$, which denote the $\mathbb{R}^{-\varepsilon}$ components. For example, the momenta p^μ are split into the pairs $p^{\bar{\mu}}, p^{\hat{\mu}}$, also denoted by $\bar{p}^\mu$, $\hat{p}^\mu$, and the coordinates x^μ are split into $\bar{x}^\mu$, $\hat{x}^\mu$. The formal flat-space metric $\eta_{\mu\nu}$ is split into the physical $d \times d$ flat-space metric $\eta_{\bar{\mu}\bar{\nu}} = \mathrm{diag}(1, -1, \ldots, -1)$ and the formal evanescent metric $\eta_{\hat{\mu}\hat{\nu}} = -\delta_{\hat{\mu}\hat{\nu}}$ (the off-diagonal components $\eta_{\bar{\mu}\hat{\nu}}$ being equal to zero). When we contract

evanescent components, we use the metric $\eta_{\hat{\mu}\hat{\nu}}$, so for example $\hat{p}^2 = p^{\hat{\mu}}\eta_{\hat{\mu}\hat{\nu}}p^{\hat{\nu}}$.

The fields $\Phi(x)$ have the same components they have in d dimensions, and each of them is a function of $\bar{x}$ and $\hat{x}$. For example, spinors ψ^α have $2^{[d/2]_{\rm int}}$ components, where $[d/2]_{\rm int}$ is the integral part of $d/2$, vectors have d components $A_{\bar{\mu}}$, symmetric tensors with two indices have $d(d+1)/2$ components, and so on. In particular, the metric tensor $g_{\mu\nu}$ is made of the diagonal blocks $g_{\bar{\mu}\bar{\nu}}$ and $\eta_{\hat{\mu}\hat{\nu}}$, while the off-diagonal components $g_{\bar{\mu}\hat{\nu}}$ vanish.

The γ matrices are the usual, d-dimensional ones, and satisfy the Dirac algebra $\{\gamma^{\bar{a}}, \gamma^{\bar{b}}\} = 2\eta^{\bar{a}\,\bar{b}}$. If $d = 2k$ is even, the d-dimensional generalization of γ_5 is

$$\tilde{\gamma} = -i^{k+1}\gamma^0\gamma^1\ldots\gamma^{2k-1},$$

which satisfies $\tilde{\gamma}^\dagger = \tilde{\gamma}$, $\tilde{\gamma}^2 = 1$. Left and right projectors $P_L = (1-\tilde{\gamma})/2$, $P_R = (1+\tilde{\gamma})/2$ are defined as usual. The tensor $\varepsilon^{\bar{a}_1\ldots\bar{a}_d}$ and the charge-conjugation matrix $\mathcal{C}$ also coincide with the usual ones. Full $SO(1, D-1)$ invariance is lost in most expressions, replaced by $SO(1, d-1) \times SO(-\varepsilon)$ invariance.

We endow the fields with well-behaved propagators by adding suitable higher-derivative evanescent kinetic terms to the action. We multiply them by inverse powers of some mass M. For example, the regularized action of (left-handed) chiral fermions in curved space reads

$$\int e\bar{\psi}_L i e^{\bar{\mu}}_{\bar{a}}\gamma^{\bar{a}}\mathcal{D}_{\bar{\mu}}\psi_L + S_{{\rm ev}\psi},$$

where $\mathcal{D}_{\bar{\mu}}$ denotes the covariant derivative and

$$S_{{\rm ev}\psi} = \frac{i}{2M}\int e(\varsigma_\psi\psi_L^T\tilde{\mathcal{C}}\hat{\partial}^2\psi_L - \varsigma_\psi^*\bar{\psi}_L\tilde{\mathcal{C}}\hat{\partial}^2\bar{\psi}_L^T), \qquad (2.13)$$

while e is the determinant of the vielbein $e^{\bar{a}}_{\bar{\mu}}$, ς_ψ are constants, and $\tilde{\mathcal{C}}$ coincides with the matrix $\mathcal{C}$ of charge conjugation if $d = 4 \mod 8$; otherwise $\tilde{\mathcal{C}} = -i\gamma^0\gamma^2$ (in $d > 2$).

In the case of Yang-Mills gauge fields in curved space, we choose the gauge fermion

$$\Psi = \int \sqrt{|g|}\bar{C}^a\left(g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}A^a_{\bar{\nu}} + \frac{\xi'}{2}B^a\right).$$

The regularized gauge-fixed action reads

$$-\frac{1}{4}\int \sqrt{|g|}F^a_{\bar{\mu}\bar{\nu}}F^{\bar{\mu}\bar{\nu}\,a} + \int \sqrt{|g|}B^a\left(g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}A^a_{\bar{\nu}} + \frac{\xi'}{2}B^a\right) - \int \sqrt{|g|}\bar{C}^a g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\mathcal{D}_{\bar{\nu}}C^a + S_{{\rm ev}A} + S_{{\rm ev}C},$$

where

$$S_{{\rm ev}A} = \frac{1}{2}\int \sqrt{|g|}g^{\bar{\mu}\bar{\nu}}\left[\frac{\varsigma_A}{M^2}(\hat{\partial}^2A^a_{\bar{\mu}})(\hat{\partial}^2A^a_{\bar{\nu}}) - \frac{\eta_A}{M}g^{\bar{\mu}\bar{\nu}}(\hat{\partial}_{\hat{\rho}}A^a_{\bar{\mu}})(\hat{\partial}^{\hat{\rho}}A^a_{\bar{\nu}})\right],$$
$$S_{{\rm ev}C} = -\int \sqrt{|g|}\left[\frac{\varsigma_C}{M^2}(\hat{\partial}^2\bar{C}^a)^2(\hat{\partial}^2C^a) - \frac{\eta_C}{M}\int \sqrt{|g|}(\hat{\partial}_{\hat{\rho}}\bar{C}^a)(\hat{\partial}^{\hat{\rho}}C^a)\right], \qquad (2.14)$$

while ς_A, ς_C, η_A, and η_C are constants. Quantum gravity can be dealt with in a similar fashion, both in the metric tensor formalism and in the vielbein formalism [6].

Thanks to the higher-derivative evanescent kinetic terms introduced by the CD regularization, the propagators of all the fields have denominators that are equal to products of polynomials

$$D(\bar{p}, \hat{p}, m, \varsigma, \eta) = \bar{p}^2 - m^2 - \varsigma\frac{(\hat{p}^2)^2}{M^2} + \eta\frac{\hat{p}^2}{M} + i0, \qquad (2.15)$$

where ς is a nonvanishing constant of order one and η is another constant. The propagators fall off in all directions $\bar{p}$, $\hat{p}$ for large momenta p. However, they decrease more rapidly or more slowly depending on whether the evanescent or physical components, $\hat{p}$ or $\bar{p}$, of the momenta become large. The structure (2.15) suggests that $\bar{p}$ and $\hat{p}^2$ should be regarded as equally important in the ultraviolet limit. The key point of the CD regularization is to define "weights" so that $\bar{p}$ and $\hat{p}^2$ are equally weighted, and use the weights to replace the dimensions in units of mass that are normally used for power counting. Doing so, we arrive at a *weighted* power counting [20], which gives us an efficient control over the locality of counterterms when the denominators of propagators are products of polynomials of the form (2.15).

Weights are defined in $D = d$, since the corrections of order ε are not important for the weighted power counting. We conventionally take $\bar{p}$ to have weight 1, so the evanescent components $\hat{p}$ of momenta have weight $1/2$. Call the kinetic terms with the largest number of derivatives $\bar{\partial}$ *dominant* kinetic terms. Once they are diagonalized, we write the dominant kinetic terms of the fields Φ as

$$\frac{1}{2}\int \Phi\bar{\partial}^{N_\Phi}\Phi, \quad \text{or} \quad \int \bar{\Phi}\bar{\partial}^{N_\Phi}\Phi, \qquad (2.16)$$

depending on the case. Clearly, the weight of Φ is equal to $(d - N_\Phi)/2$ and coincides with its dimension in units of mass. Weights can be unambiguously assigned to the parameters of the theory and the sources K, by demanding that the action and the scale M be weightless.

The Φ propagators are rational functions of the momenta, of the form

$$\frac{P'_{2w-N_\Phi}(\bar{p},\hat{p})}{P_{2w}(\bar{p},\hat{p})}, \tag{2.17}$$

where P'_{2w-N_Φ} and P_{2w} are $SO(-\varepsilon)$-scalar polynomials of weighted degrees $2w-N_\Phi$ and $2w$, respectively, such that (a) P_{2w} is a scalar under $SO(1,d-1)$, (b) the parameters contained in P_{2w} admit a nontrivial range of values where P_{2w} is positive definite in the Euclidean framework, and (c) the monomials $(\bar{p}^2)^w$ and $(\hat{p}^2)^{2w}$ of $P_{2w}(\bar{p},\hat{p})$ are multiplied by nonvanishing coefficients. The "weighted degree" of a $SO(-\varepsilon)$-scalar polynomial $Q(\bar{p},\hat{p})$ is its ordinary degree once Q is rewritten as a polynomial $\tilde{Q}(\bar{p},\hat{p}^2)$ of $\bar{p}$ and $\hat{p}^2$.

The theories that contain only parameters of non-negative weights (and are such that the propagators fall off with the correct behaviors in the ultraviolet limit) are renormalizable by weighted power counting. The theories that contain some parameters of strictly negative weights are nonrenormalizable. In all cases, the propagators (2.17) must contain only parameters of non-negative weights.

Weighted power counting also ensures that the scale M does not propagate into the physical sector of the theory. Precisely, M is an arbitrary, renormalization-group invariant parameter that belongs to the evanescent sector of the theory from the beginning to the end, so there is no need to take the limit $M\to\infty$ at any stage.

In Ref. [6] we showed that it is possible to find appropriate higher-derivative evanescent kinetic terms for all most common fields, such as scalars, fermions, Yang-Mills fields, gravity in the metric formalism, gravity in the vielbein formalism, Chern-Simons fields, and so on, and arrange the regularized action so that the requirements listed above are fulfilled. The total action is the one that contains all monomials compatible with weighted power counting, as well as the nonanomalous symmetries of the theory, multiplied by the maximum number of independent coefficients.

Some aspects of the CD regularization are reminiscent of Siegel's dimensional reduction [21], which is a popular modified dimensional regularization taylored for supersymmetric theories. Among other things, both techniques make use of the ordinary d-dimensional Dirac algebra. However, in Siegel's approach it is necessary to think that D is "smaller" than d. Then, it is possible to define a D-dimensional gauge covariant derivative and build gauge invariant schemes for gauge theories. Using the CD technique, on the other hand, only the d-dimensional gauge covariant derivative is consistent. Moreover, in Siegel's framework ordinary vectors and tensors are decomposed into multiplets made of vectors/tensors and extra components that behave like scalars (called ε-scalars). The latter are absent in the CD regularization. Another aspect in common is the important role played by the evanescent couplings, although they have different features in the two cases. The dimensional reduction, in its original formulation, has inconsistencies [22], and the evanescent terms can be used to overcome some of those, in both supersymmetric and nonsupersymmetric theories [23].

The CD technique has several advantages, which we now recall. In the ordinary, as well as chiral dimensional regularization we can distinguish divergent, nonevanescent, and evanescent terms, depending on how they behave in the limit $D\to d$. The nonevanescent terms are those that have a regular limit for $D\to d$ and coincide with the value of that limit. The evanescent terms are those that vanish when $D\to d$. They can be of two types: *formal* or *analytic*. The analytically evanescent terms are those that factorize at least one ε, such as $\varepsilon F_{\bar{\mu}\bar{\nu}}F^{\bar{\mu}\bar{\nu}}$, $\varepsilon\bar{\psi}_L i e^{\bar{\mu}}_{\bar{a}}\gamma^{\bar{a}}\mathcal{D}_{\bar{\mu}}\psi_L$. The formally evanescent terms are those that do not factorize powers of ε, such as $\psi_L^T\hat{\partial}^2\psi_L$. The divergences are poles in ε and can multiply either nonevanescent terms or formally evanescent terms. The former are called *nonevanescent divergences*. The latter are called *evanescent divergences*, or *divergent evanescences*, an example being $\psi_L^T\hat{\partial}^2\psi_L/\varepsilon$. The divergent evanescences must be subtracted away like any other divergences, because the locality of counterterms is much clearer that way.

Using the ordinary dimensional regularization, the classification of divergent evanescences in the nonrenormalizable sector presents several problems [4]. Consider the fermionic bilinears $\bar{\psi}_1\gamma^{\rho_1\cdots\rho_k}\psi_2$, where $\gamma^{\rho_1\cdots\rho_k}$ denotes the completely antisymmetric product of $\gamma^{\rho_1},\ldots,\gamma^{\rho_k}$. The independent bilinears of this type are infinitely many, because they do not vanish for $k>d$. Infinitely many Lagrangian terms of the same dimensions can be built with them, such as the four fermion vertices $(\bar{\psi}_1\gamma^{\rho_1\cdots\rho_k}\psi_2)(\bar{\psi}_3\gamma_{\rho_1\ldots\rho_k}\psi_4)$. The Fierz identities contain infinite sums and can be used to relate certain divergent evanescences to finite terms, which makes the classification of both ambiguous. No such problems are present using the CD regularization, because the γ matrices are just the ordinary d-dimensional ones.

Second, the CD technique simplifies the extraction of divergent parts out of the antiparentheses of functionals, which is a key step in all renormalization algorithms. We have to take some precautions to ensure that this operation can safely cross the antiparentheses, so that, for example, $(S,X)_{\rm div}=(S,X_{\rm div})$. The first thing to do to achieve this goal is define the tree-level action S so that it does not contain analytically evanescent terms, but only nonevanescent and formally evanescent terms, multiplied by ε-independent coefficients. In this way, S does not contain dangerous factors of ε, which could simplify the divergences of X inside (S,X). Moreover, the antiparentheses cannot generate factors of ε. Indeed, since the γ matrices are d dimensional, and the fields Φ and the sources K only have d-dimensional components, the formally evanescent quantities that we have are just $\eta^{\hat{\mu}\hat{\nu}}$ and the evanescent components $\hat{p}$ and $\hat{x}$ of momenta and coordinates. These objects can generate factors of ε only by means of the

contractions $\eta^{\hat{\mu}\hat{\nu}}\eta_{\hat{\mu}\hat{\nu}} = -\varepsilon$, $\partial_{\hat{\mu}}x^{\hat{\mu}} = -\varepsilon$, $\hat{\partial}^2\hat{x}^2 = -2\varepsilon$, etc. However, the functional derivatives $\delta/\delta\Phi^\alpha$ and $\delta/\delta K_\alpha$ due to the antiparentheses cannot generate $\eta^{\hat{\mu}\hat{\nu}}\eta_{\hat{\mu}\hat{\nu}}$ because fields and sources have no evanescent components. At the same time, the antiparentheses just multiply correlation functions in momentum space, which are $SO(-\varepsilon)$-scalar, so they cannot generate factors of ε, poles in ε, or expressions such as $\partial_{\hat{\mu}}x^{\hat{\mu}} = -\varepsilon$, $\hat{\partial}^2\hat{x}^2 = -2\varepsilon$, and cannot convert formal evanescences into analytic ones. Ultimately, we can freely cross the sign of antiparentheses, when we extract the divergent parts of local functionals using the CD regularization.

Third, the CD regularization is compatible with invariance under rigid diffeomorphisms, which are the $GL(d,\mathbb{R})$ coordinate transformations

$$x^{\bar{\mu}\prime} = M^{\bar{\mu}}_{\bar{\nu}}x^{\bar{\nu}}, \qquad x^{\hat{\mu}\prime} = x^{\hat{\mu}}, \tag{2.18}$$

where $M^{\bar{\mu}}_{\bar{\nu}}$ is an arbitrary invertible real constant matrix. We can choose the tree-level action S to be completely invariant under this symmetry, even in the gauge-fixing and regularization sectors. To fulfill this requirement, we write the fields Φ and the derivatives $\bar{\partial}$ using lower spacetime indices $\bar{\mu},\bar{\nu},\ldots$, and the sources K using upper spacetime indices. Then, we contract those indices by means of the metric tensor $g_{\bar{\mu}\bar{\nu}}$, its inverse $g^{\bar{\mu}\bar{\nu}}$, or the Kronecker tensor $\delta^{\bar{\mu}}_{\bar{\nu}}$. Finally, we multiply by an appropriate power of $\sqrt{|g|}$, to obtain a scalar density of weight 1, and integrate over spacetime. The derivatives $\hat{\partial}$ must be contracted by means of $\eta^{\hat{\mu}\hat{\nu}}$, to ensure $SO(-\varepsilon)$ invariance.

We formulate the theory without introducing "second metrics" $h_{\mu\nu}$, i.e. additional metrics besides the metric tensor $g_{\bar{\mu}\bar{\nu}}$ and the background metric $g_{B\bar{\mu}\bar{\nu}}$ around which we expand $g_{\bar{\mu}\bar{\nu}}$ perturbatively. Since field translations leave the functional integral invariant, the correlation functions are independent of $g_{B\bar{\mu}\bar{\nu}}$, so we do not consider $g_{B\bar{\mu}\bar{\nu}}$ a second metric. However, the correlation functions may depend on true second metrics $h_{\mu\nu}$, which may enter the classical action through the gauge fixing or the regularization. Several common gauge-fixing functions $G(\phi)$, such as $\eta^{\rho\nu}\partial_\rho g_{\mu\nu}$, do introduce a second metric, which is often the flat-space metric $\eta_{\mu\nu}$.

When two independent metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ are present, the classifications of counterterms and contributions to anomalies are plagued with unnecessary complications. For example, the divergent parts can contain arbitrary dimensionless functions of $g_{\mu\nu}h^{\mu\nu}$, $g_{\mu\nu}h^{\nu\rho}g_{\rho\sigma}h^{\sigma\mu}$, and similar contractions. If the theory contains a unique metric (and a unique vielbein), these arbitrary functions do not appear.

In the approach of this paper, invariance under rigid diffeomorphisms is not completely preserved. If the action S is invariant, the Γ functional is also invariant, as well as its divergent parts. However, sometimes we need to express certain divergent terms $\Delta\Gamma_{\rm div}$ or potentially anomalous terms $\mathcal{A}_{\rm pot}$ in the form (S,χ), where $\chi(\Phi,K)$ is a local functional. Even when $\Delta\Gamma_{\rm div}$ and $\mathcal{A}_{\rm pot}$ are invariant under rigid diffeomorphisms, χ may be noninvariant. The divergent terms $\Delta\Gamma_{\rm div} = (S,\chi)$ are iteratively subtracted by means of canonical transformations generated by

$$F(\Phi,K') = \int \Phi^\alpha K'_\alpha - \chi(\Phi,K').$$

Instead, the potentially anomalous terms $\mathcal{A}_{\rm pot} = (S,\chi)$ are subtracted by redefining the action S as $S - \chi/2$. In these ways, the violation of invariance under rigid diffeomorphisms can propagate into the renormalized action S_R. When no second metrics are present, such a violation is parametrized by multiplicative functions of the determinant g of the metric tensor, which are relatively easy to handle.

To simplify various arguments, we assume that the gauge fermion $\Psi(\Phi)$ is independent of the matter fields. For example, a good gauge fermion for Yang-Mills symmetries, local Lorentz symmetry, and diffeomorphisms in perturbatively unitary theories [where $N_{\phi_g} = 2$ in formulas (2.6) and (2.7)] is [6]

$$\begin{aligned}\Psi(\Phi) = {} & \int \sqrt{|g|}\bar{C}^a\left(g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}A^a_{\bar{\nu}} + \frac{\xi'}{2}B^a\right) + \int e\bar{C}_{\bar{a}\bar{b}}\left(\frac{1}{\kappa}e^{\bar{\rho}\bar{a}}g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\partial_{\bar{\nu}}e^{\bar{b}}_{\bar{\rho}} + \frac{\xi_L}{2}B^{\bar{a}\bar{b}} + \frac{\xi'_L}{2}g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\partial_{\bar{\nu}}B^{\bar{a}\bar{b}}\right)\\ & - \int \sqrt{|g|}\bar{C}_{\bar{\mu}}\left(\frac{1}{\kappa}\partial_{\bar{\nu}}g^{\bar{\mu}\bar{\nu}} + \frac{\xi_G}{\kappa}g^{\bar{\mu}\bar{\nu}}g_{\bar{\rho}\bar{\sigma}}\partial_{\bar{\nu}}g^{\bar{\rho}\bar{\sigma}} - \frac{\xi'_G}{2}g^{\bar{\mu}\bar{\nu}}B_{\bar{\nu}}\right),\end{aligned} \tag{2.19}$$

where the constants ξ', ξ_L, ξ'_L, ξ_G, and ξ'_G are gauge-fixing parameters. We have arranged $\Psi(\Phi)$ so that it is invariant under rigid diffeomorphisms. The factors $1/\kappa$ are inserted to be consistent with the κ structure (2.24), explained in the next subsection, which becomes manifest once we expand the vielbein around flat space and make the other replacements of formula (2.28). The gauge fixing of local Lorentz symmetry contained in (2.19) takes inspiration from the less common gauge condition $\partial^\mu\omega^{ab}_\mu = 0$, rather than the more common condition of symmetric vielbein, because the latter is not compatible with the requirement of having a unique metric. In higher-derivative theories we choose a gauge fermion with a similar structure, the only difference being that the gauge conditions $G(\phi,\xi)$ and the operators $P(\phi,\xi',\partial)$ of formula (2.5) also include higher-derivative terms, to fulfill the conditions (2.7).

Finally, the CD technique preserves the good properties of the dimensional regularization. The most important ones are that (a) the Batalin-Vilkovisky master equation is simply $(S,S)=0$ in $D=d$ (a correction appears on the right-hand side in most nondimensional regularizations), and (b) the local perturbative changes of field variables have Jacobian determinants identically equal to one. Property (b) follows from the fact that the integrals of polynomials $P(p)$ of the momenta in $\mathrm{d}^D p$ vanish.

Summarizing, when the gauge algebra closes off shell, the CD regularized action has the form

$$S(\Phi,K)=S_c(\phi)+(S_K,\Psi)+S_K+S_{\rm ev}=S_d+S_{\rm ev}$$
$$=\bar{S}_d+(S_K,\Psi)+S_{\rm ev}, \tag{2.20}$$

where $S_c(\phi)$ is given by (2.10) and the evanescent part $S_{\rm ev}$ collects the evanescent terms required by the CD regularization, such as $S_{{\rm ev}\psi}$, $S_{{\rm ev}A}$, and $S_{{\rm ev}C}$ of (2.13) and (2.14). For the reasons explained above, we assume that S_d is non-evanescent and $S_{\rm ev}$ is formally evanescent, so S does not contain any analytically evanescent terms. Moreover, the action (2.20) does not contain second metrics and is invariant under $SO(-\varepsilon)$ and the other global nonanomalous symmetries of the theory. We do not require that $S_{\rm ev}$ be invariant under rigid diffeomorphisms, but just that it be built with a unique metric tensor or vielbein. We denote the parameters contained in $S_{\rm ev}$ by ς_I and η_I, where ς_I multiply the dominant evanescent kinetic terms, and η_I multiply the other terms, as shown by formulas (2.14) and (2.15). For convenience, we assume that $S_{\rm ev}$ depends linearly on ς and η, and vanishes for $\varsigma=\eta=0$. We extend $S_{\rm ev}$ till it includes all the evanescent terms allowed by weighted power counting, constructed with the fields Φ, the sources K_ϕ and K_C, and their derivatives, multiplied by the maximum number of independent parameters ς and η. This will allow us to renormalize the divergent evanescences by means of ς and η redefinitions. It is consistent to choose $S_{\rm ev}$ independent of the sources $K_{\bar{C}}$ and K_B. Indeed, if we do so, the action S does not contain K_B and depends on $K_{\bar{C}}$ only through the last three terms of (2.4). Then, $K_{\bar{C}}$ and K_B cannot contribute to nontrivial diagrams, so the counterterms are also independent of them.

In total, we have physical parameters λ, contained in S_c, gauge-fixing parameters ξ, contained in Ψ, and regularizing parameters ς and η, contained in $S_{\rm ev}$. The action (2.20) is also written as $S(\Phi,K,\lambda,\xi,\varsigma,\eta)$.

Clearly, the CD regularized action $S=S_d+S_{\rm ev}$ satisfies the *deformed master equation*

$$(S,S)=\hat{\mathcal{O}}(\varepsilon), \tag{2.21}$$

where "$\hat{\mathcal{O}}(\varepsilon)$" denotes formally evanescent local terms. The right-hand side is the source of potential anomalies.

Given a regularized classical action $S(\Phi,K)$, the regularized generating functionals Z and W are given by

$$Z(J,K)=\int[\mathrm{d}\Phi]\exp\left(iS(\Phi,K)+i\int\Phi^\alpha J_\alpha\right)$$
$$=\exp iW(J,K). \tag{2.22}$$

The Legendre transform $\Gamma(\Phi,K)=W(J,K)-\int\Phi^\alpha J_\alpha$ of $W(J,K)$ with respect to J is the generating functional of one-particle irreducible diagrams. The anomaly functional is

$$\mathcal{A}=(\Gamma,\Gamma)=\langle(S,S)\rangle_S, \tag{2.23}$$

where $\langle\cdots\rangle_S$ denotes the average defined by the action S at arbitrary sources J and K. A quick way to prove the last equality of (2.23) is to make the change of field variables $\Phi^\alpha\to\Phi^\alpha+\bar{\theta}(S,\Phi^\alpha)$ inside $Z(J,K)$, where $\bar{\theta}$ is a constant anticommuting parameter. For details, see for example the appendixes of [4,24].

B. Truncation

When we quantize a nonrenormalizable theory, or study composite fields of high dimensions in any kind of theory, it may be convenient to truncate the tree-level action S_d in some way. For the arguments of this paper, the truncation is necessary to define a suitable higher-derivative regularization. Indeed, to make the HD theory superrenormalizable at fixed Λ, the higher-derivative regularizing terms must be placed well beyond the truncation.

Denote the gauge coupling of minimum dimension with κ. If there are more than one gauge coupling of minimum dimension, we call one of them κ and write any other as $r\kappa$, where the dimensionless ratio r is treated as a parameter of order one. The other gauge couplings g are written as $g=r_+\kappa$, where the ratios r_+ have positive dimensions and are also of order one. We parametrize the non-gauge-fixed solution $\bar{S}_d(\Phi,K,\kappa,\zeta)$ of the master equation as

$$\bar{S}_d(\Phi,K,\kappa,\zeta)=\frac{1}{\kappa^2}\bar{S}'_d(\kappa\Phi,\kappa K,\zeta),$$

where ζ are any other parameters besides κ, including r and r_+, and $\bar{S}'_d$ is analytic in ζ. We assume that each field Φ has a dominant kinetic term (2.16) normalized to one or multiplied by a dimensionless parameter of order one.

The gauge fixing must be parametrized similarly. We choose a gauge fermion Ψ of the form

$$\Psi(\Phi,\kappa,\xi)=\frac{1}{\kappa^2}\Psi'(\kappa\Phi,\xi),$$

where ξ are the gauge-fixing parameters and Ψ' depends analytically on ξ. We know that if the gauge algebra closes off shell, we can choose an $\bar{S}_d$ that is linear in K, as in formula (2.3). Then, the gauge-fixed solution $S_d=\bar{S}_d+(\bar{S}_d,\Psi)$ of the master equation has the structure

$$S_d(\Phi, K, \kappa, \zeta, \xi) = \frac{1}{\kappa^2} S'_d(\kappa\Phi, \kappa K, \zeta, \xi). \tag{2.24}$$

We parametrize the evanescent sector $S_{\rm ev}$ in the same way and define the parameters ς, η so that

$$S_{\rm ev}(\Phi, K, \kappa, \varsigma, \eta) = \frac{1}{\kappa^2} S'_{\rm ev}(\kappa\Phi, \kappa K, \varsigma, \eta). \tag{2.25}$$

In the end, the total action S, and all the tree-level functionals we work with, have the κ structure

$$X_{\rm tree}(\Phi, K, \kappa) = \frac{1}{\kappa^2} X'_{\rm tree}(\kappa\Phi, \kappa K). \tag{2.26}$$

Then, it is easy to prove that every loop carries an additional factor κ^2. Therefore, the renormalized action, the Γ functional, and the renormalized Γ functional have the κ structure

$$X(\Phi, K, \kappa) = \sum_{L\geq 0} \kappa^{2(L-1)} X'_L(\kappa\Phi, \kappa K), \tag{2.27}$$

where X_L collects the L-loop contributions.

The κ structures (2.26) and (2.27) are preserved by the antiparentheses: if two functionals $X(\Phi, K, \kappa)$ and $Y(\Phi, K, \kappa)$ satisfy (2.26), or (2.27), then the functional (X, Y) satisfies (2.26), or (2.27), respectively.

In perturbatively unitary theories, the propagating fields have standard dimensions in units of mass (because $N_\Phi = 2$ and $N_\Phi = 1$ for bosons and fermions, respectively). When the theory is not perturbatively unitary, such as higher-derivative quantum gravity [25], fields of negative or vanishing dimensions may be present. This is not a problem, as long as the tree-level action has the structure (2.24) and the other assumptions we make are fulfilled.

In the presence of gravity, the square root κ_N of Newton's constant is equal to κ times a ratio of non-negative dimension. The κ structure of the action becomes explicit when we expand around a background metric or vielbein. We also need to rescale the ghosts and the sources associated with diffeomorphisms and local Lorentz symmetry. For simplicity, we expand around flat space, although flat space may not be a solution of the classical field equations, because the renormalization of the theory and its anomalies do not depend on the background we choose. In that case, we can make the κ structures (2.24), (2.26), and (2.27) explicit by means of the canonical transformation

$$\begin{aligned} &e^{\bar a}_{\bar\mu} \to \delta^{\bar a}_{\bar\mu} + \kappa_N \phi^{\bar a}_{\bar\mu}, \qquad C^{\bar\rho} \to \kappa_N C^{\bar\rho}, \qquad C^{\bar a\,\bar b} \to \kappa_N C^{\bar a\,\bar b}, \\ &K^{\bar\mu}_{\bar a} \to \frac{1}{\kappa_N} K^{\bar\mu}_{\bar a}, \qquad K^C_{\bar\mu} \to \frac{1}{\kappa_N} K^C_{\bar\mu}, \qquad K^C_{\bar a\,\bar b} \to \frac{1}{\kappa_N} K^C_{\bar a\,\bar b}. \end{aligned} \tag{2.28}$$

Check this fact in formulas (2.4) and (2.19). Whenever we speak of κ structures we understand the replacements (2.28), although we do not make them explicit all the time.

Now we define the truncation. We organize the set of parameters $\zeta, \xi, \varsigma, \eta$ into two subsets $\bar s$ and s_-. The subset $\bar s$ contains the parameters of positive dimensions, as well as those of vanishing dimensions that are not treated perturbatively. Examples are the parameters that appear in the propagators. The parameters r and r_+ (but not κ) are also included in the set $\bar s$ because they are considered of order one. The set $\bar s$ also includes the parameters that cure infrared problems when superrenormalizable interactions are present. Examples are the masses, the cosmological constant $\Lambda_{\rm c}$ of formula (2.1), and the Chern-Simons coupling in three dimensions. If κ has a negative dimension (such as the square root of Newton's constant in Einstein gravity), the set $\bar s$ also includes the parameters ζ, ξ that multiply the power counting renormalizable vertices. An example is the constant $\lambda'_4 = \lambda_4/\kappa^2$ that appears when the four-scalar vertex $\lambda_4\varphi^4$ is written as $\lambda'_4(\kappa\varphi)^4/\kappa^2$ in the four-dimensional φ^4-theory coupled to Einstein gravity. If $[\kappa] = 0$, the parameters such as λ'_4 can be assumed to be of order one and also included in $\bar s$. We express each parameter contained in $\bar s$ as a dimensionless constant of order one times m^Δ, where Δ is a non-negative number and m is a generic mass scale.

The subset s_- contains the parameters $\zeta, \xi, \varsigma, \eta$ of negative dimensions. We write them as dimensionless constants of order one times $\Lambda_-^{-\Delta_-}$, where Λ_- is some energy scale and Δ_- is a positive number. The subset s_- includes the coefficients of the quadratic terms $\sim\Phi\partial^{N'_\Phi}\Phi$ with $N'_\Phi > N_\Phi$, which have to be treated perturbatively, since the dominant quadratic terms we perturb around are (2.16). Observe that κ is not included in the set s_-, even if it may have a negative dimension.

The Feynman diagrams are multiplied by various factors, but their core integrals depend only on the parameters of the subset $\bar s$ and the external momenta. Therefore, if we assume that m and the overall energy E are of the same order, each field Φ of dimension d_Φ contributes to the amplitudes as a power $\sim E^{d_\Phi} \sim m^{d_\Phi}$.

We assume that there exists a range of energies E such that

$$m \sim E \ll \Lambda_-, \tag{2.29}$$

and that κ is small enough; that is to say,

$$\kappa\Lambda_-^{-[\kappa]} \ll 1, \qquad \kappa E^{-[\kappa]} \ll 1. \tag{2.30}$$

If $[\kappa] < 0$, the first of these conditions, combined with (2.29), implies the second one. If $[\kappa] > 0$, the second condition implies the first one. If $[\kappa] = 0$, the two conditions obviously coincide.

It is easy to show that the conditions (2.29) and (2.30) are sufficient to have a well-defined perturbative expansion. Consider the contributions to the action S and the

logarithmic divergences. Factorizing the parameters in front of a generic local Lagrangian term $V(\partial, \Phi, K)$, we find the structure

$$\frac{\kappa^a m^c}{\Lambda_-^b}\left(1+\cdots+\frac{\kappa^{a'} m^{c'}}{\Lambda_-^{b'}}+\cdots\right)\int V(\partial,\Phi,K),$$

where the first factor is the tree-level coefficient and the ratio inside the parentheses is a generic contribution coming from the divergent parts of Feynman diagrams. We have $a \geq -1$,[1] $b \geq 0$, $c \geq 0$, $a[\kappa] + c = b$, $a' > 0$, and $a'[\kappa] + c' = b'$. The tree-level vertices have either $b = 0$ or $c = 0$. Then, $b' \geq 0$ or $c' \geq 0$, respectively, so we can write

$$\begin{aligned}\frac{\kappa^{a'} m^{c'}}{\Lambda_-^{b'}} &= (\kappa m^{-[\kappa]})^{a'}\left(\frac{m}{\Lambda_-}\right)^{b'} \ll 1 \quad \text{or}\\ \frac{\kappa^{a'} m^{c'}}{\Lambda_-^{b'}} &= (\kappa \Lambda_-^{-[\kappa]})^{a'}\left(\frac{m}{\Lambda_-}\right)^{c'} \ll 1,\end{aligned} \tag{2.31}$$

which shows that the expansion does work.

Next, consider the finite contributions to the Γ functional. They have the form

$$\sim \frac{\kappa^a E^{b-a[\kappa]}}{\Lambda_-^b} = (\kappa E^{-[\kappa]})^a\left(\frac{E}{\Lambda_-}\right)^b, \tag{2.32}$$

where $a \geq -1$ and $b \geq 0$. The power of E can be arbitrary and comes from the fields Φ, the sources K, the powers of $m \sim E$, and the evaluations of the core integrals of the Feynman diagrams. Clearly, formula (2.32) shows that the expansion works. It also ensures that a finite number of diagrams can contribute for each a and b. Indeed, by formula (2.27) a bounds the number of loops. Moreover, we can use only a finite number of vertices, because the power of κ bounds the numbers of Φ and K legs, while the power of $1/\Lambda_-$ bounds the number of derivatives.

It should be noticed that assumptions (2.29) and (2.30) are merely tools to organize the perturbative expansion and the proof of the Adler-Bardeen theorem. They ensure that we can reach all types of contributions (vertices, diagrams, counterterms, potential anomalies, etc.), working with finitely many of them at a time. They are not crucial for the validity of the proof itself. What we mean is that the proof of the theorem also holds when assumptions (2.29) and (2.30) are not valid, and the perturbative expansion is organized in a different way.

[1]According to the κ structure (2.26), the terms with $a = -1$ are linear in the fields Φ or the sources K. Such terms may be present when we expand around a configuration that is not a minimum of the action (for example when we expand the metric tensor around flat space in the presence of a cosmological term). All other terms have $a \geq 0$.

Now we define the truncation T of the theory. We divide it into two prescriptions, (T1) and (T2), which play different roles.

(T1) We switch off the $o(1/\Lambda_-^T)$ terms of the action $S = S_d + S_{\rm ev}$. All the terms of S_c and $S_{\rm ev}$ that are not $o(1/\Lambda_-^T)$ and satisfy the other assumptions of this paper are kept and multiplied by the maximum number of independent parameters.

In Sec. II D we explain that this prescription is also sufficient to truncate the action $S_\Lambda = S + S_{\rm HD}$ of the HD theory, because the higher-derivative terms $S_{\rm HD}$ can be chosen to be Λ_- independent. We can also take a Λ_--independent gauge fermion Ψ. The actions determined by the truncation T1 are denoted by S_{cT}, $\bar{S}_{dT}$, S_{dT}, S_T, $S_{\Lambda T}$, and so on.

Note that the prescription T1 just switches off portions of S, but leaves arbitrary powers of $1/\Lambda_-$ in the radiative corrections. This is sufficient to renormalize the HD theory, at Λ fixed, and prove that it satisfies the manifest Adler-Bardeen theorem.

(T2) For $[\kappa] < 0$, define $\sigma = -[\kappa]$ and

$$\bar{\ell} = \left[\frac{T}{2\sigma}\right]_{\rm int}, \tag{2.33}$$

$[\cdots]_{\rm int}$ denoting the integral part. For $[\kappa] \geq 0$, define $\sigma = 0$, $\bar{\ell} = \infty$. We define the truncation T2 as the truncation that keeps the ℓ-loop contributions up to $o(1/\Lambda_-^{T-2\ell\sigma})$, for $0 \leq \ell \leq \bar{\ell}$, and neglects the rest.

The truncation T2 is useful for the second part of the proof, when we study the limit $\Lambda \to \infty$ on the HD theory, renormalize the Λ divergences, and prove that the final theory satisfies the almost manifest Adler-Bardeen theorem. Indeed, these results are all proved within the truncation T2. This fact illustrates the meaning of the almost manifest Adler-Bardeen theorem, i.e. statement 3 of the Introduction.

Both prescriptions T1 and T2 are gauge invariant at $\varepsilon = 0$, since the gauge symmetries do not involve Λ_-. In power-counting renormalizable theories with $[\kappa] = 0$ we have $T = 0$.

If $[\kappa] < 0$, the quantity σ is strictly positive, so the prescription T2 reduces the powers of $1/\Lambda_-$ when the number of loops increases. The area that is covered by the truncation forms a triangle in the plane with axes T and L. In particular, the truncation only contains a finite number of loops, up to and including $\bar{\ell}$.

Note that we do not truncate the powers of κ. If we did, we would explicitly break the gauge invariant terms into gauge noninvariant pieces. For various arguments of the proof, it is convenient to define a truncation that is gauge invariant at $\varepsilon = 0$. Nevertheless, at the practical level, a sort of truncation on the powers of κ is implicitly contained in the conditions (2.30), because they imply that the contributions carrying sufficiently large powers of κ are smaller

than certain contributions neglected by the truncation. We keep the higher powers of κ anyway, because we want to concentrate on the potential anomalies that may break gauge invariance dynamically, so it is not wise to break gauge invariance artificially at the same time.

The reason why we adopt the prescription T2, when we renormalize the final theory, can be understood as follows. Consider an invariant $\mathcal{G}(\kappa\phi)$, equal to the integral of a local function of dimension $d_{\mathcal{G}}$. By power counting and formula (2.27), at L loops $\mathcal{G}$ may appear as a counterterm with the structure

$$\frac{(\kappa^2)^L m^p \Lambda^q}{\kappa^2 \Lambda_-^{\Delta+2L[\kappa]}} (\ln \Lambda)^{q'} \mathcal{G}(\kappa\phi) \tag{2.34}$$

times a product of dimensionless constants, where $\Delta = p + q + d_{\mathcal{G}} - d - 2[\kappa]$ and $q, q' \geq 0$. If the counterterm (2.34) is contained within the truncation, prescription T2 tells us that

$$\Delta + 2L[\kappa] \leq T - 2L\sigma. \tag{2.35}$$

Then we also have the inequality $\Delta \leq T$. This ensures that the truncated classical action S_{cT}, which obeys T1, also contains the invariant $\mathcal{G}$. There, it appears with one of the structures

$$\frac{\zeta}{\kappa^2 \Lambda_-^{\Delta-p-q}} \mathcal{G}(\kappa\phi), \qquad \frac{\zeta m^{p+q-\Delta}}{\kappa^2} \mathcal{G}(\kappa\phi), \tag{2.36}$$

depending on whether $\Delta > p + q$ or $\Delta \leq p + q$, where ζ is a dimensionless constant. In the end, a divergence of the form (2.34) can be subtracted by redefining ζ. If we replaced (2.35) by a different prescription, i.e. $\Delta + 2L[\kappa] \leq T$, we could be unable to subtract the counterterms (2.34) by redefining the parameters of S_{cT}, for $[\kappa] < 0$.

The same argument applies to the counterterms that depend on both $\kappa\Phi$ and κK and fall within the truncation. In particular, thanks to the prescriptions T1 and T2, the counterterms that are formally evanescent can be subtracted by redefining the parameters ς and η of S_{evT}. The counterterms that fall within the truncation but do not belong to either this class or the class (2.34) will be subtracted by means of canonical transformations.

For example, in pure quantum gravity ($[\kappa] = -1$) we have the counterterms

$$\int \sqrt{|g|} R^2, \qquad \int \sqrt{|g|} R_{\bar{\mu}\bar{\nu}} R^{\bar{\mu}\bar{\nu}}, \tag{2.37}$$

at one loop, which are Λ_- independent and have $\Delta = 2$. The minimal truncation containing them is the one that neglects $o(1/\Lambda_-^0)$ at one loop, which means $T + 2[\kappa] = 0$, i.e. $T = 2$. At the tree level, the same terms appear as

$$\frac{\zeta_1}{\kappa^2 \Lambda_-^2} \int \sqrt{|g|} R^2, \qquad \frac{\zeta_2}{\kappa^2 \Lambda_-^2} \int \sqrt{|g|} R_{\bar{\mu}\bar{\nu}} R^{\bar{\mu}\bar{\nu}}, \tag{2.38}$$

where $\zeta_{1,2}$ are dimensionless constants. Thus, if we truncated the powers of Λ_- by neglecting $o(1/\Lambda_-^0)$ at the tree level, the truncated classical action S_{cT} would not contain the terms (2.38), and we would not be able to subtract the divergences (2.37) by redefining appropriate parameters.

Now we discuss the truncated actions. We have $\bar{S}_{dT} = S_{cT} + S_K$, $S_{dT} = \bar{S}_{dT} + (S_K, \Psi)$, where, as anticipated before, we assume that Ψ is Λ_- independent. Since the truncation does not conflict with the gauge symmetries, S_{dT} and $\bar{S}_{dT}$ satisfy the master equations $(S_{dT}, S_{dT}) = (\bar{S}_{dT}, \bar{S}_{dT}) = 0$. Observe that, by prescription T1, S_{dT} does not contain any invariants $\mathcal{G}_i$ that fall beyond the truncation. We stress that, at the tree level, it is not enough to neglect those invariants: we must really switch them off. Indeed, if they were present, we would be unable to properly HD regularize the truncated theory. On the other hand, all the invariants $\mathcal{G}_i$ that are multiplied by powers $1/\Lambda_-^t$ with $t \leq T$ and satisfy the other assumptions of this paper [check, in particular, (II-i)–(II-iv) right below] must be contained in S_{dT}, multiplied by independent parameters, since we want to renormalize the divergences proportional to $\mathcal{G}_i$ that fall within the truncation by redefining those parameters. The evanescent part $S_{\rm ev}$ of the action S is truncated according to the same rules. In particular, the $o(1/\Lambda_-^T)$ monomials of S_{evT} must also be switched off and all the monomials of $S_{\rm ev}$ that are not $o(1/\Lambda_-^T)$ must be contained in S_{evT}, multiplied by independent parameters.

In the end, the truncated version of the action S is

$$S_T(\Phi, K) = S_{cT}(\phi) + (S_K, \Psi) + S_K + S_{evT} = S_{dT} + S_{evT} \tag{2.39}$$

and satisfies the master equation up to evanescent terms: $(S_T, S_T) = \hat{\mathcal{O}}(\varepsilon)$.

In general, the number of terms contained in the truncation may be infinite, because there can be fields Φ with $[\kappa\Phi] = 0$, or, as far as we know now, even fields with $[\kappa\Phi] < 0$. Now we make some assumptions that give us relative control on the power counting.

(II) We assume that
- (i) $[\kappa\Phi] \geq 0$ for every Φ;
- (ii) there exists at least one field with $N_\Phi \geq 1$;
- (iii) every field Φ with $[\kappa\Phi] = 0$ has $N_\Phi \geq 2$;
- (iv) the fields with $N_\Phi = 0$ are just the Lagrange multipliers B for the gauge fixing.

The integers N_Φ are those defined by formula (2.16).

Clearly, the standard model coupled to quantum gravity, as well as most of its extensions, satisfies these assumptions, with the gauge fermion (2.19). Assumption (II-i) excludes, for example, four-dimensional higher-derivative Yang-Mills theory coupled to Einstein gravity, because in that case

$[A] \leq 0$ and $[\kappa] = -1$. Assumption (II-ii) just excludes nonpropagating theories.

Assumptions (II-ii) and (II-iii) allow us to prove that the sources K_Φ satisfy $[\kappa K_\Phi] \geq N_\Phi/2$. Indeed, we know that

$$[\Phi] = \frac{d - N_\Phi}{2}, \qquad [K_\Phi] = \frac{d + N_\Phi}{2} - 1, \tag{2.40}$$

because $[R^\alpha] = [\Phi^\alpha] + 1$, while the form of S_K ensures that $[\Phi^\alpha] + [K^\alpha] = d - 1$. Now, if there exists a field $\bar{\Phi}$ with $[\kappa\bar{\Phi}] = 0$, we have $d = 2[\bar{\Phi}] + N_{\bar{\Phi}} = -2[\kappa] + N_{\bar{\Phi}} \geq 2 - 2[\kappa]$, which implies $[\kappa] \geq 1 - (d/2)$ and $[\kappa K_\Phi] \geq N_\Phi/2$ for every Φ. If all fields satisfy $[\kappa\Phi] > 0$, we have $d > N_\Phi - 2[\kappa]$, which implies $[\kappa] > (N_\Phi - d)/2$ for every Φ. Since there must be at least a Φ with $N_\Phi \geq 1$, we conclude that $[\kappa] > (1 - d)/2$ and $[\kappa K_\Phi] > (N_\Phi - 1)/2$ for every Φ. If g denotes the gauge coupling associated with the gauge field ϕ_g [which is the fluctuation $\phi^{\bar{a}}_{\bar{\mu}}$ of formula (2.28) in the case of gravity], and s_g denotes the spin of ϕ_g, we have $[g\phi_g] = 2 - s_g$, which is integer or semi-integer. Since $[\Phi]$ and $[K_\Phi]$ are also integer or semi-integer, so is $[g]$, as well as $[\kappa]$, $[\kappa\Phi]$, and $[\kappa K_\Phi]$. Then, the inequality $[\kappa K_\Phi] > (N_\Phi - 1)/2$ gives $[\kappa K_\Phi] \geq N_\Phi/2$.

We have already remarked that the sources K_B and $K_{\bar{C}}$ do not contribute to nontrivial one-particle irreducible diagrams. Thus, assumption (II-iv) ensures that all sources that contribute to nontrivial diagrams satisfy the stronger inequality $[\kappa K_\Phi] \geq 1/2$.

It is easy to check that the relations $[\kappa_N \phi^{\bar{a}}_{\bar{\mu}}] = 0$, $[gA_{\bar{\mu}}] = 1$, $[\kappa\phi^{\bar{a}}_{\bar{\mu}}] \geq 0$, $[\kappa A_{\bar{\mu}}] \geq 0$, and formula (2.40) imply $[g] \geq [\kappa_N]$ and $N_A \leq N_\phi \leq N_A + 2$, where N_ϕ and N_A are the numbers of $\bar{\partial}$ derivatives of the dominant kinetic terms (2.16) of the graviton field $\phi^{\bar{a}}_{\bar{\mu}}$ and the Yang-Mills gauge fields $A_{\bar{\mu}}$, respectively. Thus, in the presence of gravity the square root κ_N of Newton's constant is always a gauge coupling of minimum dimension, and we can take $\kappa = \kappa_N$.

Note that the remarks made after formula (2.40) ensure that the powers of $1/\Lambda_-$ appearing in the action are also integer or semi-integer.

C. Key assumptions

Now we formulate the key assumptions that allow us to characterize the counterterms and ensure the triviality of the one-loop gauge anomalies. The action obtained from S_d by switching off all parameters ζ that belong to the subset s_- is called *basic action* and is denoted by $S_{d\mathrm{b}}$. The basic action can also be formally obtained from S_{dT} by taking the limit $\Lambda_- \to \infty$.

For example, in the case of the standard model coupled to quantum gravity, the basic action $S_{d\mathrm{b}}$ is equal to $S_{c\mathrm{SMG}} + (S_K, \Psi) + S_K$, where $S_{c\mathrm{SMG}}$ is the low-energy classical action of formula (2.1), if $\mathcal{L}_m$ is extended appropriately. Note that the matter Lagrangian $\mathcal{L}_m$ of $S_{c\mathrm{SMG}}$ is at most linear in $D_{\bar{\mu}}\psi$, and at most quadratic in $D_{\bar{\mu}}H$, where ψ are the fermions and H is the Higgs field. The scalar mass terms, the Yukawa couplings, and the vertices $(H^\dagger H)^2$ and $R(H^\dagger H)$ have the structures

$$\frac{m^2}{\kappa^2}\int \sqrt{|g|}(\kappa\varphi)^2, \qquad \frac{m}{\kappa^2}\int \sqrt{|g|}(\kappa\varphi)(\kappa\bar{\psi})(\kappa\psi),$$
$$\frac{m^2}{\kappa^2}\int \sqrt{|g|}(\kappa\varphi)^4, \qquad \frac{\zeta}{\kappa^2}\int \sqrt{|g|}R(\kappa\varphi^\dagger)(\kappa\varphi), \tag{2.41}$$

where ζ is dimensionless. Therefore, they survive the limit $\Lambda_- \to \infty$ and are contained in $S_{d\mathrm{b}}$. For the same reason, arbitrary powers of $\kappa\varphi$ are contained in $\mathcal{L}_m$. The basic action $\check{S}_{d\mathrm{b}}$ associated with the extended theory $\check{S}_{dT}$ contains the vertices $(LH)^2$ and the four fermion vertices that break baryon number conservation. Indeed, although those vertices are power counting nonrenormalizable, they also survive the limit $\Lambda_- \to \infty$, because their structures are

$$\frac{m}{\kappa^2}\int \sqrt{|g|}(\kappa\varphi)^2(\kappa\bar{\psi})(\kappa\psi), \qquad \frac{\lambda}{\kappa^2}\int \sqrt{|g|}(\kappa\bar{\psi})^2(\kappa\psi)^2, \tag{2.42}$$

where λ is dimensionless.

If the nonanomalous accidental symmetries are unbroken, the standard model coupled to quantum gravity does not satisfy the Kluberg-Stern–Zuber assumption (2.11). Nevertheless, we can formulate a less restrictive assumption that is sufficient to give us control over the counterterms. Precisely, we assume that

(III) the basic action $S_{d\mathrm{b}}$ is cohomologically complete [that is to say, $\check{S}_{d\mathrm{b}}$ satisfies the extended Kluberg-Stern–Zuber assumption (2.12)] and the group G_{nas} is compact.

Moreover, we assume that

(IV) the basic action $S_{d\mathrm{b}}$ has trivial one-loop gauge anomalies $\mathcal{A}^{(1)}_{\mathrm{b}}$; i.e. there exists a local functional $\mathcal{X}(\Phi, K)$ such that $\mathcal{A}^{(1)}_{\mathrm{b}} = (S_{d\mathrm{b}}, \mathcal{X})$.

To subtract the potential anomalies of the higher-derivative theory, which is defined at Λ fixed, in a way that preserves its structure and nice properties, we actually need a stronger assumption, that is to say,

(V) a local functional $\mathcal{F}(\Phi)$ of ghost number one that is trivial in the $S_{d\mathrm{b}}$ cohomology is also trivial in the S_K cohomology; i.e. if there exists a local functional $\mathcal{X}(\Phi, K)$ such that $\mathcal{F} = (S_{d\mathrm{b}}, \mathcal{X})$, then there also exists a local functional $\chi(\Phi)$ such that $\mathcal{F} = (S_K, \chi)$.

In Sec. VIII we show that the standard model coupled to quantum gravity satisfies all the assumptions of our proof, so it is free of gauge anomalies to all orders.

When assumptions (IV) and (V) do not hold, or only one of them holds, we may replace them with the assumption that

(IV′) the one-loop anomalies of the higher-derivative theory defined in Sec. II D are trivial in the S_K cohomology;

i.e., there exists a local functional $\chi(\Phi)$ such that they can be written as (S_K,χ).

Indeed, assumptions (IV) and (V) are just needed to prove (IV′) [see the arguments of Sec. V from formula (4.3) to formula (5.14)]. In some practical situations it may be easier to prove (IV′) rather than (III) and (IV).

D. CDHD regularization

To find the subtraction scheme where the Adler-Bardeen theorem is almost manifest, we must merge the CD regularization with a suitable gauge invariant higher-derivative regularization. The resulting technique is called chiral-dimensional/higher-derivative regularization. It resembles the dimensional/higher-derivative (DHD) regularization of Ref. [4] in various respects, but there are a few crucial differences. First, the usual dimensional regularization is replaced by the CD regularization to overcome the difficulties mentioned in Sec. II A. Second, the DHD regularization is good for renormalizable theories, while we also want to apply the CDHD technique to nonrenormalizable theories. To this purpose, the HD regularizing terms must be adapted to the truncation. For several arguments of our derivations, we actually need to place them well beyond the truncation, and we must show that it is always possible to arrange them to meet our needs. As in Ref. [4], the HD regularization must preserve gauge invariance in d dimensions, to ensure that it is as transparent as possible to potential anomalies.

In this section we build the HD and CDHD regularizations. In general terms, they can be defined independently of the truncation, so we first work with the untruncated theory. Nevertheless, we cannot satisfy all the requirements we need in this paper, until we introduce the truncation. We do that at a second stage and emphasize why the truncation is so crucial for our purposes.

We introduce higher-derivative local functionals $S^I_{\rm HD}$, where I is an index labeling them, a higher-derivative gauge fermion $\Psi_{\rm HD}$, and higher-derivative formally evanescent terms $S_{{\rm ev}\Lambda}$. We use them to define a regularized action S_Λ whose propagators fall off as rapidly as we want, when the momenta p become large.

We take the functionals $S^I_{\rm HD}$ to be gauge invariant in d dimensions, i.e. satisfy $(S_K, S^I_{\rm HD}) = 0$ and are of the form $S^I_{\rm HD}(\kappa\phi, r, r_+)$. In particular, they just depend on the physical fields ϕ. We normalize each $S^I_{\rm HD}$ so that its quadratic terms (if any) have the form $\sim\kappa^2\phi\partial^{\bar N_I+N_\phi}\phi$, where $\bar N_I$ are non-negative integers and N_ϕ are the integers of formula (2.16). The invariants $S^I_{\rm HD}$ are extended from d to D dimensions by preserving the identity $(S_K, S^I_{\rm HD}) = 0$, according to the rules of the CD regularization [6].

Specifically, for the standard model coupled to quantum gravity, examples of the functionals $S^I_{\rm HD}$ are the integrals of $\sqrt{|g|}$ times

$$g^{\bar\mu\bar\nu}(\kappa\mathcal{D}_{\bar\mu}\bar\varphi)(\mathcal{D}^2)^{\bar N_\varphi/2}(\kappa\mathcal{D}_{\bar\nu}\varphi), \qquad (\kappa\bar\psi)(\gamma^{\bar\mu}\mathcal{D}_{\bar\mu})^{\bar N_\psi+1}(\kappa\psi),$$
$$(\kappa F_{\mu\nu})(\mathcal{D}^2)^{\bar N_A/2}(\kappa F^{\mu\nu}), \qquad R_{\mu\nu}(\mathcal{D}^2)^{(\bar N_G-2)/2}R^{\mu\nu},$$
$$R(\mathcal{D}^2)^{(\bar N_G-2)/2}R, \tag{2.43}$$

where $\mathcal{D}_{\bar\mu}$ denotes the covariant derivative, $\mathcal{D}^2 = g^{\bar\mu\bar\nu}\mathcal{D}_{\bar\mu}\mathcal{D}_{\bar\nu}$, and the integers $\bar N_\varphi$, $\bar N_\psi$, $\bar N_A$, $\bar N_G$ are large enough (see below). The same invariants work for any Einstein–Yang-Mills theory, as well as any higher-derivative theories of quantum gravity, Yang-Mills gauge fields, scalars, and fermions.

The classical action $S_c(\phi)$ is extended to

$$S_{c\Lambda}(\phi) = S_c(\phi) + \frac{1}{\kappa^2}\sum_I \frac{1}{\Lambda^{2\bar N_I}} S^I_{\rm HD}(\kappa\phi, r, r_+), \tag{2.44}$$

where Λ is the energy scale associated with the HD regularization. The new non-gauge-fixed action then reads

$$\bar S_{d\Lambda}(\Phi, K) = S_{c\Lambda}(\phi) + S_K = S_{c\Lambda}(\phi) - \int R^\alpha(\Phi)K_\alpha \tag{2.45}$$

and solves $(\bar S_{d\Lambda}, \bar S_{d\Lambda}) = 0$ in arbitrary D dimensions.

Divide the set ϕ of the physical fields into two subsets, called ϕ'_g and ϕ_m. The set ϕ_m contains the matter fields ϕ that have $[\kappa\phi] > 0$. The set ϕ'_g contains the gauge fields ϕ_g plus the matter fields ϕ that have $[\kappa\phi] = 0$. We decompose Φ as $\{\Phi'_g, \phi_m\}$, where Φ'_g contains the fields ϕ'_g, the ghosts C, the antighosts $\bar C$ and the Lagrange multipliers B. Similarly, we decompose the sources K as $\{K'_g, K_m\}$. The transformations $R_g(\Phi)$ of the fields Φ'_g are independent of ϕ_m, and the transformations $R_m(\Phi)$ of the fields ϕ_m are linear in the fields ϕ_m themselves and vanish at $\phi_m = 0$.

In the case of the standard model coupled to quantum gravity, the set ϕ'_g contains the bosons, while the set ϕ_m contains the fermions.

If we organize the HD regularization properly, we can show that the counterterms and the local contributions to potential anomalies at finite Λ are independent of the matter fields ϕ_m. The transformations $R^\alpha(\Phi, g)$ do not depend on other parameters besides the gauge couplings g, so, after the replacements (2.28), we can write

$$S_K(\Phi, K, \kappa) = -\int R^\alpha(\Phi, g)K_\alpha = -\frac{1}{\kappa^2}\int R'^\alpha(\kappa\Phi, r, r_+)(\kappa K_\alpha). \tag{2.46}$$

We organize the invariants $S^I_{\rm HD}$ into invariants $S^I_{g\rm HD}$ that are ϕ_m-independent and invariants $S^I_{m\rm HD}$ that are quadratic in the fields ϕ_m. We ignore any ϕ_m-dependent invariants $S^I_{\rm HD}$ that are not quadratic in ϕ_m because they are not

necessary for our purposes. The examples (2.43) fulfill this requirement.

We require that the modified gauge fermion $\Psi_{\rm HD}$ be invariant under rigid diffeomorphisms and independent of the matter fields. Moreover, we organize it so that each term contains an even power $2k$ of $1/\Lambda$, and at least k derivatives $\bar\partial$ act on the antighosts $\bar C$ and k derivatives $\bar\partial$ act on the Lagrange multipliers B, whenever $\bar C$ and/or B are present. The prototype of this kind of gauge fermion is

$$\Psi_{\rm HD}(\Phi) = \sum_i \int \sqrt{|g|}\bar C_i \times \left(Q_i(\Box)G_i(\phi,\xi) + \frac{1}{2}Q_i'(\Box)B_i\right), \quad (2.47)$$

where i is a generic label to distinguish different types of contributions, and Q_i and Q_i' are operators acting as follows:

$$\int \sqrt{|g|}\bar C_i Q_i(\Box)G_i(\phi) \equiv \int \sqrt{|g|}\sum_{k=0}^{N_i}\frac{c_{ik}}{\Lambda^{2k}}(\partial_{\bar\rho_1}\ldots\partial_{\bar\rho_{2k}}\bar C_I)g^{\bar\rho_1\bar\rho_2}\ldots g^{\bar\rho_{2k-1}\bar\rho_{2k}}G^I(\phi,\xi),$$
$$\int \sqrt{|g|}\bar C_i Q_i'(\Box)B_i \equiv \int \sqrt{|g|}\sum_{k=0}^{N_i'}\frac{c_{ik}'}{\Lambda^{2k}}(\partial_{\bar\rho_1}\ldots\partial_{\bar\rho_k}\bar C_I)g^{\bar\rho_1\bar\sigma_1}\ldots g^{\bar\rho_k\bar\sigma_k}g^{IJ}P(\phi,\xi',\partial)(\partial_{\bar\sigma_1}\ldots\partial_{\bar\sigma_k}B_J). \quad (2.48)$$

The functions $G^I(\phi,\xi)$ and the operators $P(\phi,\xi',\partial)$ can be read by comparing $\Psi_{\rm HD}$ with the gauge fermion Ψ of S_d in the limit $\Lambda=\infty$, while N_i, N_i' are integer numbers and c_{ik}, c_{ik}' are constants. In the case of diffeomorphisms, $\bar C_I = \bar C_{\bar\mu}$, $G^I = G^{\bar\mu}$, $B_J = B_{\bar\nu}$, and $g^{IJ} = g^{\bar\mu\bar\nu}$. In the case of Yang-Mills symmetries, $\bar C_I = \bar C^a$, $G^I = G^a$, $B_J = B^b$, and $g^{IJ} = \delta^{ab}$. In the case of local Lorentz symmetry, $\bar C_I = \bar C_{\bar a\bar b}$, $G^I = G^{\bar a\bar b}$, $B_J = B_{\bar c\bar d}$, and $g^{IJ} = (\delta^{\bar a\bar c}\delta^{\bar b\bar d} - \delta^{\bar a\bar d}\delta^{\bar b\bar c})/2$. Thanks to the structure (2.47), we will be able to prove that the antighosts and the Lagrange multipliers cannot contribute to the counterterms and the potential anomalies at finite Λ.

Specifically, in the case of perturbatively unitary theories, such as the standard model coupled to quantum gravity, we extend (2.19) to

$$\Psi_{\rm HD}(\Phi) = \int \sqrt{|g|}\bar C^a\left(Q_1(\Box)g^{\bar\mu\bar\nu}\partial_{\bar\mu}A^a_{\bar\nu} + \frac{1}{2}Q_1'(\Box)B^a\right) + \int \sqrt{|g|}\bar C_{\bar a\bar b}\left(\frac{1}{\kappa}Q_2(\Box)e^{\bar\rho\,\bar a}g^{\bar\mu\bar\nu}\partial_{\bar\mu}\partial_{\bar\nu}e^{\bar b}_{\bar\rho} + \frac{1}{2}Q_2'(\Box)B^{\bar a\bar b}\right)$$
$$- \int \sqrt{|g|}\bar C_{\bar\mu}\left(\frac{1}{\kappa}Q_3(\Box)\partial_{\bar\nu}g^{\bar\mu\bar\nu} + \frac{1}{\kappa}Q_4(\Box)g^{\bar\mu\bar\nu}g_{\bar\rho\bar\sigma}\partial_{\bar\nu}g^{\bar\rho\bar\sigma} - \frac{Q_3'(\Box)}{2}g^{\bar\mu\bar\nu}B_{\bar\nu}\right). \quad (2.49)$$

The gauge-fixed action is then

$$S_{d\Lambda}(\Phi,K) = \bar S_{d\Lambda} + (S_K,\Psi_{\rm HD}) \quad (2.50)$$

and satisfies $(S_{d\Lambda},S_{d\Lambda}) = 0$ in arbitrary D. It is obvious that the higher-derivative terms can make the propagators of all fields fall off as rapidly as we want, when the physical components $\bar p$ of the momenta p become large.

Finally, the HD regularized action

$$S_\Lambda = S_{d\Lambda} + S_{{\rm ev}\Lambda} = S_c(\phi) + \frac{1}{\kappa^2}\sum_I \frac{1}{\Lambda^{2\bar N_I}}S^I_{\rm HD}(\kappa\phi,r,r_+) + (S_K,\Psi_{\rm HD}) + S_K + S_{{\rm ev}\Lambda} \quad (2.51)$$

is obtained by adding suitable formally evanescent terms $S_{{\rm ev}\Lambda}$ compatible with weighted power counting and the nonanomalous global symmetries of the theory. We also require that $S_{{\rm ev}\Lambda}$ be built with a unique metric tensor or vielbein. The scale Λ has weight 1, equal to its dimension. The important terms of $S_{{\rm ev}\Lambda} - S_{\rm ev}$ are the kinetic ones, which must complete the regularized propagators, according to weighted power counting (more details on this are given in the next subsection). We can choose the other contributions to $S_{{\rm ev}\Lambda} - S_{\rm ev}$ at our discretion or suppress them. The kinetic terms of $S_{{\rm ev}\Lambda}$ can be constructed, for example, by inserting higher derivatives $\bar\partial/\Lambda$ and $\hat\partial^2/(M\Lambda)$ into the evanescent terms of $S_{\rm ev}$, such as (2.13) and (2.14). We assume that the difference $S_{{\rm ev}\Lambda} - S_{\rm ev}$ is K independent, since K-dependent higher-derivative terms are unnecessary for our purposes. We also assume that $S_{{\rm ev}\Lambda} - S_{\rm ev}$ is a sum of terms that are either independent of the fields ϕ_m or quadratic in ϕ_m, and that the ϕ_m-dependent terms are independent of $\bar C$ and B. Finally, we assume that each term of $S_{{\rm ev}\Lambda} - S_{\rm ev}$ contains an even power $2k$ of $1/\Lambda$, and at least k derivative operators $\bar\partial \sim \hat\partial^2/M$ act

on the antighosts $\bar{C}$ and k derivative operators $\bar{\partial} \sim \hat{\partial}^2/M$ act on the Lagrange multipliers B, whenever $\bar{C}$ and/or B are present.

The action (2.51) clearly satisfies

$$(S_\Lambda, S_\Lambda) = \hat{\mathcal{O}}(\varepsilon). \tag{2.52}$$

The HD sector $S_{\rm HD} \equiv S_\Lambda - S$ is also K independent. It must have the κ structure (2.26) and be organized so that all the propagators have the structure (2.17). The parameters on which $S_{\rm HD}$ depends, besides κ, r, r_+, and Λ, must have non-negative dimensions. We include them in a set λ_+, together with r, r_+, and write

$$S_{\rm HD} = S_{\rm HD}(\Phi, \kappa, \Lambda, \lambda_+) = \frac{1}{\kappa^2} S'_{\rm HD}(\kappa\Phi, \Lambda, \lambda_+). \tag{2.53}$$

Note that each contribution to $S_{\rm HD}$ is either independent of the fields ϕ_m or quadratic in them. Formula (2.53) is also implicitly assuming that $S_{\rm HD}$ is Λ_- independent. Then, it coincides with its own truncation. More conditions on the higher-derivative sector $S_{\rm HD}$ are given in the next section.

Now we come to the truncation. The prescription T1 of Sec. II B tells us that the truncated action $S_{\Lambda T}$ is obtained by switching off the $o(1/\Lambda_-^T)$ terms of S_Λ. Since $S_{\rm HD}$ is Λ_- independent, we just get the sum of S_T and $S_{\rm HD}$:

$$\begin{aligned} S_{\Lambda T} &= S_T + S_{\rm HD} \\ &= S_{cT}(\phi) + \frac{1}{\kappa^2} \sum_I \frac{1}{\Lambda^{2\bar{N}_I}} S^I_{\rm HD}(\kappa\phi, r, r_+) \\ &\quad + (S_K, \Psi_{\rm HD}) + S_K + S_{{\rm ev}\Lambda T}. \end{aligned} \tag{2.54}$$

Again, the action $S_{\Lambda T}$ satisfies the master equation up to formally evanescent terms, which means

$$(S_{\Lambda T}, S_{\Lambda T}) = \hat{\mathcal{O}}(\varepsilon). \tag{2.55}$$

At finite Λ, the theory defined by the action $S_{\Lambda T}$, regularized and renormalized by means the CD technique, is called (truncated) "higher-derivative theory," or HD theory. The theory defined by the same action $S_{\Lambda T}$, but regularized and renormalized by means of the CDHD technique, is called (truncated) *final* theory. The HD theory is renormalized by studying the limit $\varepsilon \to 0$ and removing the divergences and potential anomalies at Λ fixed. Once that is done, the final theory is reached by studying the limit $\Lambda \to \infty$ on the HD theory, removing the Λ divergences and proving that the cancellation of anomalies survives these operations.

At this point, we have two regulators and two types of divergences: the poles in ε and the Λ divergences. The latter are products $\Lambda^k \ln^{k'} \Lambda$, with $k, k' \geq 0$, $k + k' > 0$, times local monomials of the fields, the sources, and their derivatives. From the point of view of the CD regularization, those monomials may be nonevanescent or formally evanescent, and their coefficients must be evaluated in the analytic limit $\varepsilon \to 0$. To complete the CDHD regularization, we must specify how the regularization parameters ε and Λ are removed. If the HD sector of the regularization is organized in a suitable way, which we specify in the next section, the HD theory is superrenormalizable and only a few one-loop diagrams diverge. After studying the poles in ε and the one-loop potential anomalies, at Λ fixed, we prove that it is possible to remove both. We also show that these operations are sufficient to remove both divergences and anomalies to all orders, in the HD theory. Then we study the limit $\Lambda \to \infty$ and show that we can remove the divergences and potential anomalies appearing in that limit, preserving gauge invariance. We call the set of such operations the *CDHD limit*.

For more clarity, we describe how the CDHD limit works with the help of a set of symbolic expressions. When we study the HD theory, we expand around $\varepsilon = 0$ at Λ fixed. Then we find poles, finite terms, and evanescent terms of the form

$$\frac{1}{\varepsilon}, \qquad \frac{\hat{\delta}}{\varepsilon}, \qquad \varepsilon^0, \qquad \hat{\delta}\varepsilon^0, \qquad \varepsilon, \qquad \hat{\delta}\varepsilon,$$

where $1/\varepsilon$ denotes any divergent expression, $\hat{\delta}$ is any formally evanescent expression, ε^0 is any expression that is convergent and nonevanescent in the analytic limit $\varepsilon \to 0$, and ε denotes any analytic evanescence. Next, we subtract the divergent parts, that is to say, the first two terms of the list. The coefficients of the surviving terms, which are

$$\varepsilon^0, \qquad \hat{\delta}\varepsilon^0, \qquad \varepsilon, \qquad \hat{\delta}\varepsilon, \tag{2.56}$$

are then expanded around $\Lambda = \infty$, which gives the structures

$$\begin{aligned} &\varepsilon^0\Lambda, \quad \hat{\delta}\varepsilon^0\Lambda, \quad \varepsilon^0\Lambda^0, \quad \hat{\delta}\varepsilon^0\Lambda^0, \quad \frac{\varepsilon^0}{\Lambda}, \quad \frac{\hat{\delta}\varepsilon^0}{\Lambda}, \\ &\varepsilon\Lambda, \quad \hat{\delta}\varepsilon\Lambda, \quad \varepsilon\Lambda^0, \quad \hat{\delta}\varepsilon\Lambda^0, \quad \frac{\varepsilon}{\Lambda}, \quad \frac{\hat{\delta}\varepsilon}{\Lambda}, \end{aligned} \tag{2.57}$$

where Λ denotes any expression that diverges when $\Lambda \to \infty$ (i.e. it is multiplied by a coefficient that behaves like $\Lambda^k \ln^{k'} \Lambda$, with $k, k' \geq 0$, $k + k' > 0$), Λ^0 is any expression that is convergent, but not evanescent, in the same limit, while $1/\Lambda$ is any expression that vanishes in the limit. The first two terms of the list (2.57) are the Λ divergences of the CDHD limit and must be subtracted. For convenience, we include the terms $\hat{\delta}\varepsilon^0\Lambda$ (which are local) in this subtraction, although they are going to be dropped at a later stage. We cannot include the terms $\varepsilon\Lambda$, instead, because they are not local. After these new subtractions, we remain with

$$\varepsilon^0\Lambda^0, \qquad \hat{\delta}\varepsilon^0\Lambda^0, \qquad \frac{\varepsilon^0}{\Lambda}, \qquad \frac{\hat{\delta}\varepsilon^0}{\Lambda}, \qquad \varepsilon\Lambda,$$
$$\hat{\delta}\varepsilon\Lambda, \qquad \varepsilon\Lambda^0, \qquad \hat{\delta}\varepsilon\Lambda^0, \qquad \frac{\varepsilon}{\Lambda}, \qquad \frac{\hat{\delta}\varepsilon}{\Lambda}. \tag{2.58}$$

Finally, the CDHD limit is taken by dropping all the contributions of this list but the $\varepsilon^0\Lambda^0$ terms. Note that the terms proportional to ε vanish in the CDHD limit, even if they are divergent in Λ, because the limit $\varepsilon \to 0$ is taken before the limit $\Lambda \to \infty$.

III. PROPERTIES OF THE HD THEORY

In this section we organize the higher-derivative regularization and study its properties. We want to show that for every truncation T1 of Sec. II B we can arrange the higher-derivative sector $S_{\rm HD} = S_{\Lambda T} - S_T$ so that it satisfies a number of conditions that will be useful to prove the Adler-Bardeen theorem. So far, for example, we have not specified the numbers of higher derivatives that we need. We anticipate that, besides being sufficiently many, they should not conflict with the truncated action S_T, that is to say, they should all be placed well beyond the truncation. The tree-level truncation T1 will be enough to give us complete control on the radiative corrections of the HD theory, to all orders in $\hbar$ and for arbitrarily large powers of $1/\Lambda_-$. We do not apply the truncation T2 till Sec. VII, where we study the limit $\Lambda \to \infty$ and the final theory.

The numbers of higher derivatives are governed by the Λ exponents $\bar{N}_I$ appearing in formula (2.44), analogous exponents $\hat{N}_I$ appearing inside $S_{{\rm ev}\Lambda}$, and the exponents N_i, N'_i of $\Psi_{\rm HD}$, appearing in (2.48). The Φ kinetic terms of $S_{\rm HD}$ that are dominant in the large momentum limits $\bar{p} \to \infty$ and $\hat{p} \to \infty$ have the form

$$\bar{c}_\Phi \int \Phi \left(\frac{\bar{\partial}^2}{\Lambda^2}\right)^{\bar{N}_\Phi} \bar{\partial}^{N_\Phi}\Phi + \hat{c}_\Phi \int \Phi \left(\frac{\hat{\partial}^2}{M\Lambda}\right)^{2\hat{N}_\Phi} \left(\frac{\hat{\partial}^2}{M}\right)^{N_\Phi}\Phi, \tag{3.1}$$

where $\bar{c}_\Phi$ and $\hat{c}_\Phi$ are weightless constants, $2\bar{N}_\Phi$ is the maximum number of higher derivatives $\bar{\partial}$ and $4\hat{N}_\Phi$ the maximum number of higher derivatives $\hat{\partial}$. Weighted power counting requires $\bar{N}_\Phi = \hat{N}_\Phi$. For reasons that will be clear below, we need to take the same $\bar{N}_{\phi'_g} = \hat{N}_{\phi'_g} \equiv N_+$ for all fields ϕ'_g, and the same $\bar{N}_{\phi_m} = \hat{N}_{\phi_m} \equiv N_-$ for all fields ϕ_m. Then we set $N_i = N'_i = N_+$ in (2.48). We switch off all terms of $S_{\rm HD}$ that are multiplied by more than $2N_+$ powers of $1/\Lambda$, and all ϕ_m-dependent $S_{\rm HD}$ terms that are multiplied by more than $2N_-$ powers of $1/\Lambda$. We also need to take N_+, N_-, and $N_+ - N_- > 0$ sufficiently large. The first task of this section is to determine the bounds on these numbers and show that it is always possible to choose them so that they satisfy the requirements we need.

Define tilde fields and sources as

$$\tilde{\Phi}'_g = \frac{\Phi'_g}{\Lambda^{N_+}}, \qquad \tilde{\phi}_m = \frac{\phi_m}{\Lambda^{N_-}},$$
$$\tilde{K}'_g = \Lambda^{N_+}K'_g, \qquad \tilde{K}_m = \Lambda^{N_-}K_m, \tag{3.2}$$

and tilde parameters $\tilde{\kappa} = \kappa\Lambda^{N_+}$, $\tilde{r} = r$, and $\tilde{r}_+ = r_+$. We have

$$\tilde{\kappa}\tilde{\Phi}'_g = \kappa\Phi'_g, \qquad \tilde{\kappa}\tilde{K}'_g = \Lambda^{2N_+}\kappa K'_g,$$
$$\tilde{\kappa}\tilde{\phi}_m = \kappa\phi_m\Lambda^{N_+-N_-}, \qquad \tilde{\kappa}\tilde{K}_m = \Lambda^{N_++N_-}\kappa K_m. \tag{3.3}$$

Observe that (3.2) is a canonical transformation. After the redefinitions, the dominant kinetic terms (3.1) of $S_{\rm HD}$ are Λ independent. Those of the fields ϕ'_g and ϕ_m are

$$\int \tilde{\phi}'_g \left[\bar{c}_g\bar{\partial}^{2N_+ + N_{\phi'_g}} + \hat{c}_g\left(\frac{\hat{\partial}^2}{M}\right)^{2N_+ + N_{\phi'_g}}\right]\tilde{\phi}'_g$$
$$+ \int \tilde{\phi}_m \left[\bar{c}_m\bar{\partial}^{2N_- + N_{\phi_m}} + \hat{c}_m\left(\frac{\hat{\partial}^2}{M}\right)^{2N_- + N_{\phi_m}}\right]\tilde{\phi}_m.$$

Those of the ghosts C, the antighosts $\bar{C}$, and the Lagrange multipliers B follow from the choices of $G(\phi, \xi)$ and $P(\phi, \xi', \partial)$ in (2.5).

Recall that $S_{\rm HD}$ has the structure (2.53), $\Psi_{\rm HD}$ is independent of the matter fields, and each contribution to $S_{\rm HD}$ is either quadratic in the matter fields ϕ_m or independent of them. Then, we can write

$$S_{\rm HD} = \frac{1}{\tilde{\kappa}^2} S''_{\rm HD}(\tilde{\kappa}\tilde{\Phi}'_g, \tilde{\kappa}\tilde{\phi}_m, \tilde{\lambda}_+), \tag{3.4}$$

where $S''_{\rm HD}$ is Λ independent in the tilde parametrization and $\tilde{\lambda}_+$ are parameters of non-negative dimensions, equal to products $\lambda_+\Lambda^k$, with $k \geq 0$. To simplify some arguments, we switch off all the parameters λ_+ such that $\tilde{\lambda}_+ = \lambda_+\Lambda^k$ with $k > 0$, because they are not necessary to make the higher-derivative regularization work. Thus, from now on we assume that the parameters λ_+ have non-negative weights and satisfy $\lambda_+ = \tilde{\lambda}_+$. Examples are the ratios $r = \tilde{r}$, $r_+ = \tilde{r}_+$ between the gauge couplings g and κ.

As far as the truncated action $S_{\Lambda T}$ is concerned, we have

$$S_{\Lambda T}(\Phi, K)$$
$$= \frac{\Lambda^{2N_+}}{\tilde{\kappa}^2} S'_T(\tilde{\kappa}\tilde{\Phi}'_g, \Lambda^{N_- - N_+}\tilde{\kappa}\tilde{\phi}_m, \Lambda^{-2N_+}\tilde{\kappa}\tilde{K}'_g, \Lambda^{-N_+ - N_-}\tilde{\kappa}\tilde{K}_m)$$
$$+ \frac{1}{\tilde{\kappa}^2} S''_{\rm HD}(\tilde{\kappa}\tilde{\Phi}'_g, \tilde{\kappa}\tilde{\phi}_m, \tilde{\lambda}_+), \tag{3.5}$$

where $S'_T = S'_{dT} + S'_{{\rm ev}T}$ and S'_{dT} and $S'_{{\rm ev}T}$ are defined by applying the truncation T1 to formulas (2.24) and (2.25).

If N_+ is large enough, the dimension $[\tilde{\kappa}]$ of $\tilde{\kappa}$ is strictly positive, which is a necessary condition to have superrenormalizability. Actually, for later use we assume that $[\tilde{\kappa}]$ is greater than some given $t > 0$, that is to say,

$$N_+ > t - [\kappa]. \tag{3.6}$$

The right-hand side of (3.4) contains only parameters of non-negative dimensions in units of mass, apart from the overall factor $1/\tilde{\kappa}^2$. Instead, S'_T, written in the tilde parametrization, contains parameters that can have positive, vanishing, or negative dimensions, as well as factors $\Lambda^{N_- - N_+}$ and $\Lambda^{-N_- - N_+}$. However, we can show that the overall factor Λ^{2N_+} that multiplies $S'_T/\tilde{\kappa}^2$ in formula (3.5) allows us to turn $\Lambda^{2N_+} S'_T$ into a functional that contains only parameters of positive (and arbitrarily large) dimensions, at least within the truncation T1.

We begin with the functional S_K. By formula (2.46) and the properties recalled right below formula (2.45), we have, in the tilde parametrization

$$S_K = -\frac{1}{\tilde{\kappa}^2} \sum_g \int R'^{\alpha}(\tilde{\kappa}\,\tilde{\Phi}, \tilde{r}, \tilde{r}_+)(\tilde{\kappa}\tilde{K}_\alpha), \tag{3.7}$$

which are of the form we want, that is to say, the tilde version of (2.27).

Next, consider the K-independent contributions to $\Lambda^{2N_+} S'_T/\tilde{\kappa}^2$ in formula (3.5). They have the form

$$\lambda \frac{\Lambda^{2N_+}}{\Lambda_-^u \tilde{\kappa}^2} \partial^p \prod_g (\tilde{\kappa}\tilde{\Phi}'_g)^{q_g} \prod_m (\Lambda^{N_- - N_+} \tilde{\kappa}\tilde{\phi}_m)^{q_m}, \tag{3.8}$$

where u, p, q_g, q_m are non-negative integers and λ is a Λ-independent product of parameters of non-negative dimensions. The truncated action $S_T = \Lambda^{2N_+} S'_T/\tilde{\kappa}^2$ contains a finite number of matter fields ϕ_m, because $[\Lambda^{N_- - N_+}\tilde{\kappa}\tilde{\phi}_m] = [\kappa\phi_m] > 0$, $[\tilde{\kappa}\tilde{\Phi}'_g] = [\kappa\Phi'_g] \geq 0$, by assumption (II-i) of Sec. II B, and $u \leq T$, by prescription T1. Thus, there exists a $q_{\max}$ such that $\sum_m q_m \leq q_{\max}$. Then, if we choose N_+ and N_- such that the condition

$$2N_+ > q_{\max}(N_+ - N_-) + T + 2t + 2|[\kappa]| \tag{3.9}$$

holds, besides (3.6), the structure (3.8) becomes

$$\frac{\tilde{\lambda}}{\tilde{\kappa}^2} \partial^p \prod_g (\tilde{\kappa}\tilde{\Phi}'_g)^{q_g} \prod_m (\tilde{\kappa}\tilde{\phi}_m)^{q_m}, \tag{3.10}$$

where the constants $\tilde{\lambda} = \lambda \Lambda^{d_\lambda}/\Lambda_-^u$, with $d_\lambda = 2N_+ - (N_+ - N_-)\sum_m q_m$, have dimensions greater than $2t + 2|[\kappa]|$.

For future use, we observe that if ω denotes ζ, ς, or η, all terms of $S_c(\phi)$ and $S_{\rm ev}$ that just depend on ϕ'_g have the κ structure

$$\omega F(\phi'_g, \kappa, r, r_+) = \frac{\omega}{\kappa^2} F'(\kappa\phi'_g, r, r_+) = \frac{\tilde{\omega}}{\tilde{\kappa}^2} F'(\tilde{\kappa}\tilde{\phi}'_g, \tilde{r}, \tilde{r}_+), \tag{3.11}$$

where $\tilde{\omega} = \omega\Lambda^{2N_+}$.

Collecting (3.7) and (3.10), we can define a truncated functional $S''_{d\Lambda T}$ that depends analytically on $\tilde{\lambda}$, such that

$$S_{d\Lambda T}(\Phi, K, \kappa) = \frac{1}{\tilde{\kappa}^2} S''_{d\Lambda T}(\tilde{\kappa}\tilde{\Phi}'_g, \tilde{\kappa}\tilde{\phi}_m, \tilde{\kappa}\tilde{K}'_g, \tilde{\kappa}\tilde{K}_m, \tilde{\lambda}). \tag{3.12}$$

It remains to study the K-dependent contributions to the first term on the right-hand side of (3.5). Actually, we have already studied those contained in S_K, which are rearranged in formula (3.7). The remaining ones are contained in $S_{{\rm ev}T}$. Write

$$S_{{\rm ev}T}(\Phi, K, \kappa) = \frac{\Lambda^{2N_+}}{\tilde{\kappa}^2} S'_{{\rm ev}T}(\tilde{\kappa}\tilde{\Phi}'_g, \Lambda^{N_- - N_+}\tilde{\kappa}\tilde{\phi}_m, \kappa K'_g, \kappa K_m). \tag{3.13}$$

Using $[\kappa K_\Phi] \geq 1/2$, which was proved in Sec. II B, a condition like (3.9), with a possibly different $q_{\max}$, is also sufficient to rewrite each contribution to $S_{{\rm ev}T}$ in the form

$$\frac{\tilde{\varsigma}}{\tilde{\kappa}^2} \partial^p \prod_g (\tilde{\kappa}\tilde{\Phi}'_g)^{q_g} \prod_m (\tilde{\kappa}\tilde{\phi}_m)^{q_m} \prod_K (\kappa K)^{q_K}, \tag{3.14}$$

where $\tilde{\varsigma}$ are new parameters of dimensions greater than $2t + 2|[\kappa]|$, which include the tilde versions of both ς and η. Finally, we can write

$$S_{{\rm ev}T}(\Phi, K, \kappa) = \frac{1}{\tilde{\kappa}^2} S''_{{\rm ev}T}(\tilde{\kappa}\tilde{\Phi}'_g, \tilde{\kappa}\tilde{\phi}_m, \tilde{\varsigma}\kappa^p K^p, \tilde{\varsigma}), \tag{3.15}$$

with $S_{{\rm ev}T} = 0$ at $\tilde{\varsigma} = 0$. The argument $\tilde{\varsigma}\kappa^p K^p$ of $S''_{{\rm ev}T}$ is there to remind us that all nontilde products of κK must be multiplied by parameters $\tilde{\varsigma}$. From now on we assume that the $q_{\max}$ of condition (3.9) is raised to a value that is good for both (3.10) and (3.14).

The T1 truncated HD theory has the basic features of a superrenormalizable theory, since its parameters have non-negative dimensions in units of mass, and $\tilde{\kappa}$ has a strictly positive dimension. The proof of superrenormalizability is completed in the next sections, where we show that the divergences can be renormalized by redefining a few parameters. In the tilde parametrization, the action $S_{\Lambda T}$ becomes

$$\tilde{S}_{\Lambda T} = \frac{1}{\tilde{\kappa}^2} S''_{d\Lambda T}(\tilde{\kappa}\,\tilde{\Phi}, \tilde{\kappa}\,\tilde{K}, \tilde{\lambda}) + \frac{1}{\tilde{\kappa}^2} S''_{{\rm ev}T}(\tilde{\kappa}\,\tilde{\Phi}, \tilde{\varsigma}\kappa^p K^p, \tilde{\varsigma}) + \frac{1}{\tilde{\kappa}^2} S''_{\rm HD}(\tilde{\kappa}\,\tilde{\Phi}, \tilde{\lambda}_+), \tag{3.16}$$

and $\tilde{\kappa}, \tilde{\lambda}_+$ are the only tilde parameters that may have (non-negative) dimensions smaller than or equal to $2t + 2|[\kappa]|$.

Only the first and third functionals on the right-hand side of (3.16) have the expected form, which is the tilde version of (2.27). The second functional cannot be written like the rest. This will force us to do some extra effort. However, since the terms of $S''_{\mathrm{ev}T}$ are multiplied by parameters $\tilde{\varsigma}$, which have sufficiently large dimensions, we will still be able to prove the properties we need.

Finally, it is possible to choose N_+ and N_- so that the HD theory satisfies other properties that will be important for the arguments of the next subsections. For example, it is sufficient to require

$$N_+ + N_- > 2t - \min_K[\kappa K],$$
$$N_+ - N_- > 2t - \min_m[\kappa\phi_m] \tag{3.17}$$

to make all products $\tilde{\kappa}\tilde{K}$ and $\tilde{\kappa}\tilde{\phi}_m$ have dimensions (equal to their weights) greater than $2t$.

Another condition allows us to have control on the dependences on the antighosts $\bar{C}$ and the Lagrange multipliers B. Checking the action (2.54), we see that $\bar{C}$ and B appear inside the term $-\int BK_{\bar{C}}$ of S_K (which cannot contribute to nontrivial diagrams), as well as $(S_K, \Psi_{\mathrm{HD}})$ and $S_{\mathrm{ev}\Lambda T}$. The gauge fermion Ψ_{HD} contains $\bar{C}$ and B according to the structure (2.48), where now the integers N_i and N'_i are replaced by N_+. Since we have suppressed the parameters λ_+ of S_{HD} that have $[\tilde{\lambda}_+] > [\lambda_+]$, the terms of Ψ_{HD} with $0 < k < N_+$ are absent. Working out $(S_K, \Psi_{\mathrm{HD}} - \Psi)$ explicitly, it is easy to prove that at least N_+ derivatives $\bar{\partial}$ act on the antighosts $\bar{C}$ and N_+ derivatives $\bar{\partial}$ act on the Lagrange multipliers B. By construction, the formally evanescent higher-derivative terms $S_{\mathrm{ev}\Lambda T} - S_{\mathrm{ev}T}$ depend on $\bar{C}$ and B in the same way, with derivatives $\bar{\partial}$ possibly replaced by $\hat{\partial}^2/M$. In the end, the dependence on $\bar{C}$ and B of the full higher-derivative sector S_{HD} of the action $S_{\Lambda T}$ has this structure.

When we switch to the tilde parametrization, the powers of Λ disappear from the denominators. With the sole exception of $-\int BK_{\bar{C}}$, every term of $\tilde{S}_{\Lambda T}$ that depends on $\tilde{\kappa}\tilde{\bar{C}}$ and/or $\tilde{\kappa}\tilde{B}$ is multiplied by a parameter $\tilde{\lambda}$ or $\tilde{\varsigma}$, or has at least N_+ derivatives $\bar{\partial} \sim \hat{\partial}^2/M$ acting on each leg $\tilde{\kappa}\tilde{\bar{C}}$ and $\tilde{\kappa}\tilde{B}$. It is easy to check that $(\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T})$ has the same structure. These observations will be useful later on, because the parameters $\tilde{\lambda}$ or $\tilde{\varsigma}$, as well as the derivatives $\bar{\partial} \sim \hat{\partial}^2/M$ acting on the external legs $\tilde{\kappa}\tilde{\bar{C}}$ and $\tilde{\kappa}\tilde{B}$, lower the degrees of divergence of the diagrams, and allow us to prove that certain types of counterterms and local contributions to anomalies are absent.

For our purposes, it is sufficient to require that the N_+ derivatives $\bar{\partial} \sim \hat{\partial}^2/M$ that act on $\tilde{\kappa}\tilde{\bar{C}}$ and $\tilde{\kappa}\tilde{B}$ inside S_{HD} have weights greater than $2t$, which means

$$N_+ > 2t. \tag{3.18}$$

There is no difficulty to choose N_+ and N_- such that requirements (3.6), (3.9), (3.17), and (3.18) are fulfilled at the same time, no matter how large we want t to be. In the next subsections we show that, if we choose t in a clever way, we can ensure that the higher-derivative theory has no divergences and no local contributions to anomalies beyond one loop, and that the one-loop divergences, as well as the one-loop potential anomalies, are independent of the sources, the matter fields ϕ_m, the antighosts, and the Lagrange multipliers. We begin by studying the structure of the counterterms.

A. HD theory: Structure of counterterms

Ignoring the factors $\tilde{\kappa}$ and κ attached to the sources $\tilde{K}$ and K, which are external to the diagrams, each vertex of the action (3.16) is multiplied by a power of $\tilde{\kappa}$ that is equal to the number of its Φ legs minus 2. Then each loop carries an extra factor $\tilde{\kappa}^2$, and the counterterms have the form

$$(\tilde{\kappa}^2)^{L-1}\tilde{\lambda}_+^u\tilde{\lambda}^r\tilde{\varsigma}^s\partial^p\prod_g(\tilde{\kappa}\tilde{\Phi}'_g)^{q_g}\prod_m(\tilde{\kappa}\tilde{\phi}_m)^{q_m} \times\prod_K(\tilde{\kappa}\,\tilde{K})^{q_K}\prod_K(\kappa K)^{q'_K}, \tag{3.19}$$

where u, r, s, p, q_g, q_m, q_K, and q'_K are non-negative integers. Every factor has a non-negative dimension for $L \geq 1$, since $[\tilde{\kappa}\tilde{\Phi}] \geq [\kappa\Phi] \geq 0$ and $[\tilde{\kappa}\tilde{K}] > [\kappa K] \geq 1/2$. Recalling that $[\tilde{\kappa}^2] > 2t$, we see that, if we choose $t > d/2$, the expressions (3.19) have dimensions greater than d for every $L \geq 2$. Thus, no divergences may be present beyond one loop. Moreover, at $L = 1$ we must have $r = s = 0$, because the dimensions of $\tilde{\lambda}$ and $\tilde{\varsigma}$ are also greater than d. Then, we also have $q'_K = 0$, because the last product of (3.19) is always accompanied by some parameters $\tilde{\varsigma}$. Finally, since by (3.17) the dimensions of $\tilde{\kappa}\tilde{\phi}_m$ and $\tilde{\kappa}\,\tilde{K}$ are greater than d, the divergences of the higher-derivative theory are just one loop and have the form

$$\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}(\tilde{\kappa}\tilde{\Phi}'_g, \tilde{\lambda}_+) = \Gamma^{(1)}_{\Lambda T\mathrm{div}}(\kappa\Phi'_g, \lambda_+). \tag{3.20}$$

To write the last equality we have used the fact that the parameters λ_+ with $[\tilde{\lambda}_+] > [\lambda_+]$ have been switched off.

We can also show that $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$ cannot depend on the antighosts and the Lagrange multipliers, since, by the observations of the previous subsection and condition (3.18), a nontrivial Feynman diagram that has $\tilde{\kappa}\tilde{\bar{C}}$ and/or $\tilde{\kappa}\tilde{B}$ among its external legs either is multiplied by parameters $\tilde{\lambda}$ and $\tilde{\varsigma}$ or has derivative operators of weights greater than d acting on all external legs $\tilde{\kappa}\tilde{\bar{C}}$ and $\tilde{\kappa}\tilde{B}$. Finally, since $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$ has ghost number zero, it cannot even depend on the ghosts, because we have already excluded all fields and sources that have negative ghost numbers. In the end, we have

$$\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}} = \tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}(\tilde{\kappa}\tilde{\phi}'_g, \tilde{\lambda}_+) = \Gamma^{(1)}_{\Lambda T\mathrm{div}}(\kappa\phi'_g, \lambda_+). \tag{3.21}$$

We stress that $\Gamma^{(1)}_{\Lambda T\mathrm{div}}$ is independent of Λ. Moreover, it is independent of Λ_-, which implies that it is fully contained in every truncation T2 such that $T \geq 2\sigma$. From now on we assume that T is larger than 2σ.

B. HD theory: Structure of anomalies

We call "local contributions to (potential) anomalies" the local terms originated by the simplification between overall divergences and evanescences in Feynman diagrams (see Sec. IV for details). The local contributions to anomalies may still be divergent, or nonevanescent, or even evanescent. What is important for us is that they inherit the basic properties of divergences. Besides being local, they are polynomial in the parameters that have positive dimensions. If the gauge anomalies do not vanish at one loop, the anomaly functional $\mathcal{A}$ receives in general nonlocal contributions at higher orders. If the gauge anomalies vanish up to and including n loops, $\mathcal{A}$ receives only local contributions at $n+1$ loops, up to evanescent corrections. In view of the applications of the next sections, now we investigate the structure of the local contributions to the gauge anomalies of the HD theory.

We must concentrate on $(\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T})$ and the average $\langle(\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T})\rangle_{\tilde{S}_{\Lambda T}}$. Using (3.16) we find

$$(\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T}) = \frac{1}{\tilde{\kappa}^2} U(\tilde{\kappa}\tilde{\Phi}, \tilde{\kappa}\tilde{K}, \tilde{\lambda}, \tilde{\lambda}_+) + \frac{\Lambda^{-2N_+}}{\tilde{\kappa}^2} V(\tilde{\kappa}\tilde{\Phi}, \tilde{\kappa}\tilde{K}, \tilde{\varsigma}\kappa^p K^p, \tilde{\lambda}, \tilde{\lambda}_+, \tilde{\varsigma}, \Lambda), \tag{3.22}$$

where U and V are formally evanescent functionals, and $V = 0$ at $\tilde{\varsigma} = 0$. We have added the argument Λ to V, to emphasize that V can contain positive powers of Λ, which are generated, together with the overall factor Λ^{-2N_+}, by the presence of nontilde products κK inside $S''_{\mathrm{ev}T}$. The factor Λ^{-2N_+} in front of V deserves some attention, because it can be a source of trouble, from the point of view of power counting. We can bypass this difficulty as follows. Denoting the Γ functional associated with the action $\tilde{S}_{\Lambda T}$ by $\tilde{\Gamma}_{\Lambda T}$, the anomaly functional is

$$\tilde{\mathcal{A}}_{\Lambda T} = (\tilde{\Gamma}_{\Lambda T}, \tilde{\Gamma}_{\Lambda T}) = \langle(\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T})\rangle_{\tilde{S}_{\Lambda T}} = \frac{1}{\tilde{\kappa}^2}\langle U\rangle_{\tilde{S}_{\Lambda T}} + \frac{\Lambda^{-2N_+}}{\tilde{\kappa}^2}\langle V\rangle_{\tilde{S}_{\Lambda T}}. \tag{3.23}$$

It is easy to see that the averages have the following structures:

$$\frac{1}{\tilde{\kappa}^2}\langle U\rangle_{\tilde{S}_{\Lambda T}} = \sum_{L=0}^{\infty}(\tilde{\kappa}^2)^{L-1}\mathcal{U}_L(\tilde{\kappa}\tilde{\Phi}, \tilde{\kappa}\tilde{K}, \tilde{\varsigma}\kappa^p K^p, \tilde{\lambda}, \tilde{\lambda}_+, \tilde{\varsigma}), \tag{3.24}$$

$$\frac{\Lambda^{-2N_+}}{\tilde{\kappa}^2}\langle V\rangle_{\tilde{S}_{\Lambda T}} = \sum_{L=0}^{\infty}\kappa^2(\tilde{\kappa}^2)^{L-2} \times \mathcal{V}_L(\tilde{\kappa}\tilde{\Phi}, \tilde{\kappa}\tilde{K}, \tilde{\varsigma}\kappa^p K^p, \tilde{\lambda}, \tilde{\lambda}_+, \tilde{\varsigma}, \Lambda), \tag{3.25}$$

where $\mathcal{V}_L = 0$ at $\tilde{\varsigma} = 0$. Recall that $[\tilde{\kappa}^2] > 2t$ and $[\kappa^2\tilde{\varsigma}] > 2t$. If we choose a t such that $2t > d+1$ (instead of $2t > d$, which was the condition of the previous subsection), then all local contributions to anomalies (which must be integrals of local functions of weight $d+1$) vanish by weighted power counting for $L \geq 2$. Indeed, the right-hand side of (3.24) contains at least one factor $\tilde{\kappa}^2$ times objects of non-negative weights, while the right-hand side of (3.25) contains one factor $\kappa^2\tilde{\varsigma}$ times objects of non-negative weights.

Now we study the functionals $\mathcal{U}_1$ and $\mathcal{V}_1$. Since they collect one-loop diagrams that contain insertions of formally evanescent vertices, they are sums of local divergent evanescences, plus local nonevanescent terms (which arise from simplified divergences), plus possibly nonlocal evanescent terms. We concentrate our attention on the nonevanescent contributions $\mathcal{U}_{1\mathrm{nonev}}$ and $\mathcal{V}_{1\mathrm{nonev}}$ to $\mathcal{U}_1$ and $\mathcal{V}_1$.

The nonevanescent part $\mathcal{U}_{1\mathrm{nonev}}$ of $\mathcal{U}_1$ is independent of $\tilde{\lambda}$, $\tilde{\varsigma}$, $\tilde{\kappa}\tilde{K}$, and $\tilde{\kappa}\tilde{\phi}_m$, because such objects have weights greater than $d+1$. Moreover, $\mathcal{U}_{1\mathrm{nonev}}$ is independent of the antighosts and the Lagrange multipliers, because the choice $2t > d+1$ and the condition (3.18) ensure that every Feynman diagram that contributes to $\tilde{\mathcal{A}}_{\Lambda T}$ and has external legs $\tilde{\kappa}\tilde{\bar{C}}$ and/or $\tilde{\kappa}\tilde{B}$ is either multiplied by parameters $\tilde{\lambda}$ and $\tilde{\varsigma}$ or has derivative operators of weights greater than $d+1$ acting on each external leg $\tilde{\kappa}\tilde{\bar{C}}$ and $\tilde{\kappa}\tilde{B}$. In this respect, it is important to recall that not only $\tilde{S}_{\Lambda T}$ but also $(\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T})$ has the structure explained before formula (3.18). Since $\mathcal{U}_{1\mathrm{nonev}}$ has ghost number one, and cannot contain any fields or sources of negative ghost numbers, it must be proportional to the ghosts. Precisely,

$$\mathcal{U}_{1\mathrm{nonev}} = \int(\tilde{\kappa}\,\tilde{C})^I\mathcal{A}_I(\tilde{\kappa}\tilde{\phi}'_g, \tilde{\lambda}_+) = \int(\kappa C)^I\mathcal{A}_I(\kappa\phi'_g, \lambda_+), \tag{3.26}$$

where $\mathcal{A}_I$ are local functions of the fields ϕ'_g.

The nonevanescent part $\mathcal{V}_{1\mathrm{nonev}}$ of $\mathcal{V}_1$ actually vanishes. We know that it must be polynomial in $\tilde{\varsigma}$ and vanish for $\tilde{\varsigma} = 0$. If we differentiate the one-loop contributions to (3.23) with respect to $\tilde{\varsigma}$, and take their nonevanescent parts, we find

$$\frac{\Lambda^{-2N_+}}{2}\tilde{\varsigma}\frac{\partial\mathcal{V}_{1\text{nonev}}}{\partial\tilde{\varsigma}}=\left(\tilde{S}_{\Lambda T},\tilde{\varsigma}\frac{\partial\tilde{\Gamma}^{(1)}_{\Lambda T}}{\partial\tilde{\varsigma}}\right)\Big|_{\text{nonev}}+\left(\tilde{\Gamma}^{(1)}_{\Lambda T},\tilde{\varsigma}\frac{\partial\tilde{S}_{\Lambda T}}{\partial\tilde{\varsigma}}\right)\Big|_{\text{nonev}},\qquad(3.27)$$

where $\tilde{\Gamma}^{(1)}_{\Lambda T}$ is the one-loop contribution to the Γ functional $\tilde{\Gamma}_{\Lambda T}$. We have used the fact that $\mathcal{U}_{1\text{nonev}}$ is independent of $\tilde{\varsigma}$. Now, $\tilde{\varsigma}\partial\tilde{S}_{\Lambda T}/\partial\tilde{\varsigma}$ is formally evanescent, so the last term of (3.27) vanishes. On the other hand, we have

$$\tilde{\varsigma}\frac{\partial\tilde{\Gamma}_{\Lambda T}}{\partial\tilde{\varsigma}}\Big|^{\text{one-loop}}_{\text{nonev}}=\left\langle\tilde{\varsigma}\frac{\partial\tilde{S}_{\Lambda T}}{\partial\tilde{\varsigma}}\right\rangle^{\text{one-loop}}_{\text{nonev}}.\qquad(3.28)$$

The average appearing on the right-hand side of this formula collects the diagrams that contain one insertion of $\tilde{\varsigma}\partial\tilde{S}_{\Lambda T}/\partial\tilde{\varsigma}$. At one loop, the formally evanescent vertices provided by this functional can give a nonevanescent result only by simplifying some divergences. Therefore, expression (3.28) is a local functional. It is equal to the integral of a local function of dimension d that has the structure (3.19), with $L=1$ and $s>0$. This means that it vanishes, since $[\tilde{\varsigma}]>d$. Consequently, (3.27) also vanishes, and so does $\mathcal{V}_{1\text{nonev}}$.

In the end, we take

$$t>\frac{d+1}{2},\qquad(3.29)$$

because with this choice (a) the truncated HD theory is superrenormalizable, (b) there are no divergences and no local contributions to anomalies beyond one loop, (c) the one-loop divergences have the form (3.21), and (d) the one-loop nonevanescent contributions to anomalies have the form (3.26).

We have not discussed the divergent evanescences contained in $\mathcal{U}_1$ and $\mathcal{V}_1$. The reason is that we do not need to, because as soon as we renormalize the one-loop divergences of the Γ functional $\tilde{\Gamma}_{\Lambda T}$, the anomaly functional $\tilde{\mathcal{A}}_{\Lambda T}=(\tilde{\Gamma}_{\Lambda T},\tilde{\Gamma}_{\Lambda T})$ is automatically one-loop convergent.

C. The CDHD limit

In the CDHD limit, it is important to avoid conflicts between the higher-derivative terms contained in the action $S_{\Lambda T}$ and the powerlike divergences. In particular, if Γ_{nRT} denotes the Γ functional CDHD renormalized up to and including n loops, when we take the $(n+1)$-loop Λ-divergent part of expressions such as $(\Gamma_{nRT},\Gamma_{nRT})$, we have to be sure that $(S_{\text{HD}},\Gamma^{(n+1)}_{nRT\text{div}})$ vanishes for $\Lambda\to\infty$, where $\Gamma^{(n+1)}_{nRT\text{div}}$ denotes the $(n+1)$-loop divergent part of Γ_{nRT}. It is impossible to satisfy this requirement without a truncation, because the powerlike divergences $\sim\Lambda^k$ of $\Gamma^{(n+1)}_{nR\text{div}}$ can have k arbitrarily large and beat the powers Λ^{-2N+} and Λ^{-2N-} that appear in S_Λ. This is the main reason why we cannot provide a subtraction scheme where the Adler-Bardeen theorem is manifest to all orders.

Given a truncation, on the other hand, it is possible to fulfill a satisfactory requirement by choosing higher-derivative regularizing terms S_{HD} that lie *well beyond the truncation* and subtracting just the contributions to $\Gamma^{(n+1)}_{nRT\text{div}}$ that lie within the truncation. We recall that the truncation T2 of Sec. II B prescribes that we ignore the L-loop contributions that are $o(1/\Lambda_-^{T-2L\sigma})$. We anticipate that, to provide a scheme where the Adler-Bardeen theorem is almost manifest within the truncation, we need to satisfy

$$\lim_{\Lambda\to\infty}(S_{\text{HD}},\Gamma^{(n+1)}_{nRT\text{div}})=o(1/\Lambda_-^{T-2(n+1)\sigma}).\qquad(3.30)$$

By this formula we mean that the limit exists and vanishes up to corrections $o(1/\Lambda_-^{T-2(n+1)\sigma})$ (but such corrections may not have a regular limit for $\Lambda\to\infty$).

To find a condition that ensures (3.30), we first observe that the powerlike divergences of $\Gamma^{(n+1)}_{nRT\text{div}}$ have the form

$$\ln^{q'}\Lambda\frac{\Lambda^q}{\Lambda_-^{q_-}}\delta_+(\kappa^2)^n\partial^p(\kappa\Phi)^{n_\Phi}(\kappa K)^{n_K},\qquad(3.31)$$

where $q>0$, $q',q_-\geq 0$, and δ_+ is a product of parameters of non-negative dimensions. We can concentrate on the contributions (3.31) that have $q_-\leq T-2(n+1)\sigma$, because the ones with $q_->T-2(n+1)\sigma$ satisfy (3.30) in an obvious way. We know that $[\kappa\Phi]\geq 0$ and $[\kappa K]\geq 1/2$. Then, distinguishing the cases $[\kappa]\geq 0$ and $[\kappa]<0$, we can easily check that

$$q\leq T+d-2\sigma.\qquad(3.32)$$

In perturbatively unitary, power-counting renormalizable theories with $T=0$ we obviously have $q\leq d$.

To ensure that $(S_{\text{HD}},\Gamma^{(n+1}_{nRT\text{div}})$ vanishes for $\Lambda\to\infty$ within the truncation, it is sufficient to require $S_{\text{HD}}=\mathcal{O}(1/\Lambda^{T+d-2\sigma+1})$. In particular, we must have

$$2N_+>2N_->T+d-2\sigma.\qquad(3.33)$$

Moreover, the HD regularized theory cannot contain higher-derivative terms of orders $\mathcal{O}(1/\Lambda^k)$ with $k\leq T+d-2\sigma$. However, this is an automatic consequence of another choice we have already made, when we switched off the parameters λ_+ of S_{HD} such that $[\tilde{\lambda}_+]>[\lambda_+]$. Thus, in our framework condition (3.33) is sufficient to ensure (3.30).

Given any truncation T, it is always possible to satisfy all the conditions on N_+ and N_- mentioned so far, at the same time. They are (3.6), (3.9), (3.17), (3.18), (3.29), and (3.33).

IV. RENORMALIZATION OF THE HD THEORY

In this section and the next two, we study the truncated higher-derivative theory with action $\tilde{S}_{\Lambda T}$, which is defined by keeping Λ fixed and regularized by means of the CD technique. We mostly use the tilde parametrization, but sometimes need to switch to the nontilde one. The first task is to work out the renormalization of this theory. Then we must study its one-loop anomalies, and finally prove that it satisfies the manifest Adler-Bardeen theorem.

The anomaly functional (2.23) of the higher-derivative theory is (3.23), in the tilde parametrization. Its one-loop contribution $\tilde{\mathcal{A}}^{(1)}_{\Lambda T}$ is

$$\tilde{\mathcal{A}}^{(1)}_{\Lambda T} = 2(\tilde{S}_{\Lambda T}, \tilde{\Gamma}^{(1)}_{\Lambda T}) = \langle (\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T}) \rangle_{\tilde{S}_{\Lambda T}} \Big|_{\text{one-loop}}. \tag{4.1}$$

We know that $(\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T}) = \hat{\mathcal{O}}(\varepsilon)$. The right-hand side of (4.1) collects one-loop Feynman diagrams that contain insertions of formally evanescent vertices. The formal evanescences can either remain as such or generate factors of ε. In the former case, they give local divergent evanescences, plus evanescences. In the latter case, a factor ε can simplify a local divergent part and give local nonevanescent contributions, in addition to evanescences. Therefore, we can write

$$\tilde{\mathcal{A}}^{(1)}_{\Lambda T} = \tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{nev}} + \tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{divev}} + \tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{ev}}, \tag{4.2}$$

where $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{nev}}$ is local, convergent, and nonevanescent, $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{divev}}$ is local and divergent evanescent, and $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{ev}}$ is evanescent and possibly nonlocal. The analysis of Sec. III B and formula (3.26) tell us that

$$\tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{nev}} = \int (\tilde{\kappa}\,\tilde{C})^I \mathcal{A}_I(\tilde{\kappa}\tilde{\phi}'_g, \tilde{\lambda}_+) = \int (\kappa C)^I \mathcal{A}_I(\kappa\phi'_g, \lambda_+). \tag{4.3}$$

Clearly, $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{nev}}$ is independent of Λ_- and Λ. In particular, it is fully contained in any truncation that has $T \geq 2\sigma$.

Taking the divergent part of Eq. (4.1), we find

$$(\tilde{S}_{\Lambda T}, \tilde{\Gamma}^{(1)}_{\Lambda T\text{div}}) = \frac{1}{2}\tilde{\mathcal{A}}^{(1)}_{\Lambda T\text{divev}}. \tag{4.4}$$

Formula (3.21) tells us that $\tilde{\Gamma}^{(1)}_{\Lambda T\text{div}}$ is just a functional of $\tilde{\kappa}\tilde{\phi}'_g$, fully contained within any truncation T2 with $T \geq 2\sigma$. In particular, its antiparentheses with $\tilde{S}_{\Lambda T}$ are only sensitive to $\tilde{S}_K$ and the K-dependent contributions to $\tilde{S}_{\text{ev}T}$. Moreover, we can further decompose $\tilde{\Gamma}^{(1)}_{\Lambda T\text{div}}$ as the sum of a nonevanescent divergent part $\tilde{\Gamma}^{(1)}_{\Lambda T\text{nev div}}$ and a divergent evanescence $\tilde{\Gamma}^{(1)}_{\Lambda T\text{divev}}$. Taking the nonevanescent divergent part of (4.4), we obtain

$$(\tilde{S}_K, \tilde{\Gamma}^{(1)}_{\Lambda T\text{nev div}}) = 0, \tag{4.5}$$

which just states that $\tilde{\Gamma}^{(1)}_{\Lambda T\text{nev div}}$ is gauge invariant.

Since $\tilde{\Gamma}^{(1)}_{\Lambda T\text{div}}$ is Λ_- independent, the arguments that lead to formula (2.36) ensure that $\tilde{\Gamma}^{(1)}_{\Lambda T\text{nev div}}$ is a linear combination of the invariants $\mathcal{G}_i$ contained in the T1 truncated classical action $S_{cT}(\phi)$ with $T = 2\sigma$ [check formula (2.10)]. Since we are assuming $T \geq 2\sigma$, we can remove $\tilde{\Gamma}^{(1)}_{\Lambda T\text{nev div}}$ by redefining a few parameters λ_i of S_{cT}. The rest, which is $\tilde{\Gamma}^{(1)}_{\Lambda T\text{divev}}$, can be subtracted by redefining the parameters ς and η of $S_{\text{ev}T}$.

In the case of the standard model coupled to quantum gravity, $\Gamma^{(1)}_{\Lambda T\text{nev div}}$ is a linear combination of terms of dimensions smaller than or equal to four, such as

$$\begin{aligned}\Gamma^{(1)}_{\Lambda T\text{nev div}} &= \int \sqrt{|g|}\Big(c_1 + c_2 R + c_3 R^2 + c_4 R_{\bar\mu\bar\nu}R^{\bar\mu\bar\nu} \\ &\quad + c_5\kappa^2 F^a_{\bar\mu\bar\nu}F^{a\bar\mu\bar\nu} + c_6\kappa^2 F^a_{\bar\mu\bar\nu}\mathcal{D}^2 F^{a\bar\mu\bar\nu} \\ &\quad + c_7\kappa^2 R F^a_{\bar\mu\bar\nu}F^{a\bar\mu\bar\nu} + c_8\kappa^4 F^a_{\bar\mu\bar\nu}F^{a\bar\mu\bar\nu}F^b_{\bar\rho\bar\sigma}F^{b\bar\rho\bar\sigma} + \cdots\Big),\end{aligned}$$

where the coefficients c_i are products of parameters of non-negative dimensions. This list also contains invariants that in principle can be subtracted by means of field redefinitions, rather than redefinitions of parameters. Among those invariants, we mention $\int\sqrt{|g|}R^2$ and $\int\sqrt{|g|}R_{\bar\mu\bar\nu}R^{\bar\mu\bar\nu}$. However, if we use the Einstein equations, which read

$$R_{\bar\mu\bar\nu} - \frac{1}{2}R g_{\bar\mu\bar\nu} - \Lambda_c g_{\bar\mu\bar\nu} = \kappa^2 T_{\bar\mu\bar\nu},$$

where the energy-momentum tensor $T_{\bar\mu\bar\nu}$ can contain purely gravitational contributions due to the higher-derivative corrections, we do not really remove the invariants in question, but rather convert them into other invariants, such as $\int\sqrt{|g|}\kappa^4 T_{\bar\mu\bar\nu}T^{\bar\mu\bar\nu}$, which may depend on the matter fields ϕ_m and spoil the nice structure of the HD theory. For this reason, it is not convenient to use canonical transformations to remove $\Gamma^{(1)}_{\Lambda T\text{nev div}}$, or parts of it. As anticipated in Sec. II, all the invariants of $\Gamma^{(1)}_{\Lambda T\text{nev div}}$ are included in the basis $\{\mathcal{G}_i(\phi)\}$, so we can completely remove $\Gamma^{(1)}_{\Lambda T\text{nev div}}$ by redefining the parameters λ_i. We recall that it is possible to get rid of the redundant invariants at the very end (after subtracting the Λ divergences and proving the almost manifest Adler-bardeen theorem), by means of a procedure like the one described in Ref. [17], which consists of making a canonical transformation, re-renormalize the theory, and re-fine-tune the finite local counterterms to recover the cancellation of gauge anomalies.

In the end, to renormalize the HD theory we just need to redefine some parameters λ_i, ς, and η of S_{cT} and S_{evT}, which multiply terms of the form (3.11). The renormalized action, which we denote by $\hat{S}_{\Lambda T}$, is obtained by making the replacements

$$\tilde{\lambda}_i \to \tilde{\lambda}_i + \frac{f_i}{\varepsilon}\tilde{\kappa}^2, \qquad \tilde{\varsigma} \to \tilde{\varsigma} + \frac{f_\varsigma}{\varepsilon}\tilde{\kappa}^2, \qquad \tilde{\eta} \to \tilde{\eta} + \frac{f_\eta}{\varepsilon}\tilde{\kappa}^2, \tag{4.6}$$

inside $S_{\Lambda T}$, where f_i, f_ς, and f_η are calculable factors that may depend on the parameters $\tilde{\lambda}_+$ appearing in (3.21). Switching to the nontilde parametrization, the redefinitions (4.6) are equivalent to

$$\lambda_i \to \lambda_i + \frac{f_i}{\varepsilon}\kappa^2, \qquad \varsigma \to \varsigma + \frac{f_\varsigma}{\varepsilon}\kappa^2, \qquad \eta \to \eta + \frac{f_\eta}{\varepsilon}\kappa^2. \tag{4.7}$$

Since $S_{\Lambda T}$ is linear in λ_i, ς, and η, we have

$$\hat{S}_{\Lambda T} = \tilde{S}_{\Lambda T} - \tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}. \tag{4.8}$$

Using (4.4) and $(\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}, \tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}) = 0$ (which holds because $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$ is K independent), we find

$$(\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T}) = (\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T}) - \tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{divev}}. \tag{4.9}$$

The generating functional $\hat{\Gamma}_{\Lambda T}$ defined by $\hat{S}_{\Lambda T}$ is convergent to all orders within the truncation, because it is convergent at one loop and the tilde structure of $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$ has the expected form, that is to say, the tilde version of (2.27). Then, the counterterms keep the form (3.19), which forbids divergences beyond one loop. Finally, $\hat{\Gamma}_{\Lambda T}$ and the anomaly functional $\hat{\mathcal{A}}_{\Lambda T} = (\hat{\Gamma}_{\Lambda T}, \hat{\Gamma}_{\Lambda T})$ are obtained by making the replacements (4.6) inside $\tilde{\Gamma}_{\Lambda T}$ and $\tilde{\mathcal{A}}_{\Lambda T} = (\tilde{\Gamma}_{\Lambda T}, \tilde{\Gamma}_{\Lambda T})$, respectively. Clearly, $\hat{\mathcal{A}}_{\Lambda T}$ is convergent, because $\hat{\Gamma}_{\Lambda T}$ is convergent.

V. ONE-LOOP ANOMALIES

In this section we study the one-loop anomalies and relate those of the basic theory, which are trivial by assumption (IV) of Sec. II C, to those of the HD theory, which turn out to also be trivial.

We begin with the relation between the one-loop contributions $\hat{\mathcal{A}}^{(1)}_{\Lambda T}$ and $\tilde{\mathcal{A}}^{(1)}_{\Lambda T}$ to $\hat{\mathcal{A}}_{\Lambda T}$ and $\tilde{\mathcal{A}}_{\Lambda T}$. Observe that

$$\begin{aligned}\hat{\mathcal{A}}_{\Lambda T} &= (\hat{\Gamma}_{\Lambda T}, \hat{\Gamma}_{\Lambda T}) = \langle (\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T})\rangle_{\hat{S}_{\Lambda T}} \\ &= \langle (\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T})\rangle_{\tilde{S}_{\Lambda T} - \tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}} = \langle (\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T})\rangle_{\tilde{S}_{\Lambda T}} + \mathcal{O}(\hbar^2).\end{aligned}$$

The last equality is proved as follows. The functional $\hat{\mathcal{A}}_{\Lambda T}$ collects the one-particle irreducible diagrams that contain one insertion of a vertex coming from $(\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T})$. If we also use $\mathcal{O}(\hbar)$ vertices provided by $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$, we must close at least one loop, to connect them with the vertex of $(\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T})$. This can only give $\mathcal{O}(\hbar^2)$ corrections.

Using (4.9), we have

$$\begin{aligned}\hat{\mathcal{A}}_{\Lambda T} &= \langle (\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T})\rangle_{\tilde{S}_{\Lambda T}} - \tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{divev}} + \mathcal{O}(\hbar^2) \\ &= \tilde{\mathcal{A}}_{\Lambda T} - \tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{divev}} + \mathcal{O}(\hbar^2),\end{aligned}$$

and thus (4.2) gives

$$\hat{\mathcal{A}}^{(1)}_{\Lambda T} = \tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}} + \tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{ev}}. \tag{5.1}$$

The divergent evanescences $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{divev}}$ had to disappear from $\hat{\mathcal{A}}^{(1)}_{\Lambda T}$, because $\hat{\mathcal{A}}_{\Lambda T}$ is convergent.

Since the structure of $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$ is the one of formula (3.21), we can straightforwardly extend the analysis of Sec. III B to the renormalized action $\hat{S}_{\Lambda T}$. The anomaly functional is still the sum of contributions of the forms (3.24) and (3.25). Therefore, all local contributions to anomalies vanish from two loops onwards.

Anomalies satisfy the Wess-Zumino consistency conditions [14], which, in the Batalin-Vilkovisky formalism, are consequences of a well-known property of the antiparentheses, stating that every functional X satisfies the identity $(X, (X, X)) = 0$. Taking $X = \hat{\Gamma}_{\Lambda T}$, we obtain

$$(\hat{\Gamma}_{\Lambda T}, \hat{\mathcal{A}}_{\Lambda T}) = 0. \tag{5.2}$$

At one loop we have

$$(\tilde{S}_{\Lambda T}, \hat{\mathcal{A}}^{(1)}_{\Lambda T}) = -(\hat{\Gamma}^{(1)}_{\Lambda T}, (\tilde{S}_{\Lambda T}, \tilde{S}_{\Lambda T})). \tag{5.3}$$

Since the antiparentheses of an evanescent functional, such as $(\tilde{S}_\Lambda, \tilde{S}_\Lambda)$, with a convergent functional, such as $\hat{\Gamma}^{(1)}_{\Lambda T}$, are evanescent, we have

$$(\tilde{S}_{\Lambda T}, \hat{\mathcal{A}}^{(1)}_{\Lambda T}) = \mathcal{O}(\varepsilon).$$

Using (5.1) we also find

$$(\tilde{S}_{\Lambda T}, \tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}}) = \mathcal{O}(\varepsilon). \tag{5.4}$$

By formula (4.3), $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}}$ is independent of the sources K. Then, only the K-dependent terms of $\tilde{S}_{\Lambda T}$, which are contained in $\tilde{S}_K$ and $\tilde{S}_{evT}$, can contribute to the left-hand side of (5.4). Taking the nonevanescent part of both sides, we find

$$(\tilde{S}_K, \tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}}) = 0. \tag{5.5}$$

A. Relation between the anomalies of the HD theory and those of the basic theory

Now we relate the potential one-loop anomalies $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}}$ of the HD theory to the potential one-loop anomalies $\mathcal{A}^{(1)}_{\mathrm{b}}$ of the basic theory, which are trivial by assumption (IV) of

Sec. II C. We recall that the action $S_{d\mathrm{b}}$ of the basic theory can be retrieved by taking the formal limit $\Lambda_- \to \infty$ of S_{dT}. In the same limit, the CD regularized action S_T is equal to the basic action $S_{d\mathrm{b}}$ plus the evanescent terms $S_{\mathrm{ev}T}$ (calculated at $\Lambda_- = \infty$). The CDHD regularized action is still obtained by adding S_{HD} (which is Λ_- independent), or by taking the formal limit $\Lambda_- \to \infty$ of $S_{\Lambda T}$.

Once the formal limit $\Lambda_- \to \infty$ is taken, the one-loop CDHD divergences must be subtracted just as they come, rather than by redefining parameters (since the basic action misses the parameters of the subset s_-). For example, the one-loop divergences $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$ of the HD theory can still be subtracted by formula (4.8), which, however, cannot be seen as implied by the redefinitions (4.6) or (4.7). In this section we understand that $\Lambda_- = \infty$ everywhere, so the final theory is the one associated with the basic action. Since $\tilde{\Gamma}^{(1)}_{\Lambda T\mathrm{div}}$ and $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}}$ do not depend on Λ_-, we do not lose any relevant information.

The last expression of formula (4.3) tells us that $\tilde{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}}$ is Λ independent in the nontilde parametrization, where we denote it by $\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}$. Now we show that actually $\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}$ is equivalent to the one-loop anomaly $\mathcal{A}^{(1)}_{\mathrm{b}}$ of the basic theory.

To prove this fact, we need to study the Λ-divergent parts and take the CDHD limit at one loop. In this subsection we denote the terms that are Λ divergent in the CDHD limit as "Ddiv," to distinguish them from the poles in ε. Recall that the Λ divergences can be nonevanescent or formally evanescent, from the point of view of the dimensional regularization, but not analytically evanescent. They are the terms $\varepsilon^0\Lambda$ and $\hat{\delta}\varepsilon^0\Lambda$ of the list (2.57).

Consider $\hat{\mathcal{A}}_{\Lambda T} = (\hat{\Gamma}_{\Lambda T}, \hat{\Gamma}_{\Lambda T})$ and take the one-loop CDHD-divergent part of this equation. Using (5.1) and recalling that $\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}$ is Λ independent, we get

$$\begin{aligned}\frac{1}{2}\mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}}|_{\mathrm{Ddiv}} &= (S_{\Lambda T}, \hat{\Gamma}^{(1)}_{\Lambda T})|_{\mathrm{Ddiv}} = (S_{\Lambda T}, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}})|_{\mathrm{Ddiv}} \\ &= (S_T, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}}) + (S_{\mathrm{HD}}, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}})|_{\mathrm{Ddiv}}, \qquad (5.6)\end{aligned}$$

where $\hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}}$ is the one-loop CDHD-divergent part of $\hat{\Gamma}_{\Lambda T}$. Note that although $\mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}}|_{\mathrm{Ddiv}}$ is evanescent from the point of view of the CD regularization, it can be nontrivial, because it can contain the terms $\hat{\delta}\varepsilon^0\Lambda$ of the list (2.57).

The one-loop powerlike divergences at $\Lambda_- = \infty$ have the form

$$\Lambda^q \delta_+ \partial^p (\kappa\Phi)^{n_\Phi} (\kappa K)^{n_K},$$

where $q > 0$, and δ_+ is a product of parameters of nonnegative dimensions. Recalling that $[\kappa\Phi] \geq 0$ and $[\kappa K] \geq 1/2$, the exponent q is smaller than or equal to d. Since $T \geq 2\sigma$ and $S_{\mathrm{HD}} = \mathcal{O}(1/\Lambda^{T+d-2\sigma+1})$, by inequality (3.33), the antiparentheses $(S_{\mathrm{HD}}, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}})$, specialized to the basic theory, tend to zero in the CDHD limit. Thus, (5.6) gives

$$\frac{1}{2}\mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}}|_{\mathrm{Ddiv}} = (S_T, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}}). \qquad (5.7)$$

The one-loop CDHD-renormalized action $\hat{S}_{fT}$ of the final theory associated with the basic action reads

$$\hat{S}_{fT} = \hat{S}_{\Lambda T} - \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}} - \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}} + \mathcal{O}(\hbar^2), \qquad (5.8)$$

where $\hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}}$ denote arbitrary local counterterms that are finite and nonevanescent in the CDHD limit [i.e. terms of the type $\varepsilon^0\Lambda^0$ of the list (2.58)]. For the purposes of this section, the generic subtraction (5.8) is enough. In Sec. VII we will be more precise about the removal of divergences (at $\Lambda_- < \infty$), as well as the finite local counterterms $\hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}}$ and the higher-order corrections $\mathcal{O}(\hbar^2)$. The anomaly is then

$$\mathcal{A}_{fT} = \langle (\hat{S}_{fT}, \hat{S}_{fT}) \rangle_{\hat{S}_{fT}},$$

and its one-loop nonevanescent part $\mathcal{A}^{(1)}_{\mathrm{b}}$ is the quantity we want. Denoting the sum $\hat{\Gamma}^{(1)}_{\Lambda T\mathrm{Ddiv}} + \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}}$ by $\Delta\hat{\Gamma}^{(1)}_{\Lambda T}$ and using (5.1), we find

$$\begin{aligned}\mathcal{A}_{fT} &= \langle (\hat{S}_{\Lambda T} - \Delta\hat{\Gamma}^{(1)}_{\Lambda T}, \hat{S}_{\Lambda T} - \Delta\hat{\Gamma}^{(1)}_{\Lambda T}) \rangle_{\hat{S}_{\Lambda T} - \Delta\hat{\Gamma}^{(1)}_{\Lambda T}} + \mathcal{O}(\hbar^2) \\ &= \hat{\mathcal{A}}_{\Lambda T} - 2(S_{\Lambda T}, \Delta\hat{\Gamma}^{(1)}_{\Lambda T}) + \mathcal{O}(\hbar^2) \\ &= (S_{\Lambda T}, S_{\Lambda T}) + \mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}} + \mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}} - 2(S_T, \Delta\hat{\Gamma}^{(1)}_{\Lambda T}) \\ &\quad - 2(S_{\mathrm{HD}}, \Delta\hat{\Gamma}^{(1)}_{\Lambda T}) + \mathcal{O}(\hbar^2). \qquad (5.9)\end{aligned}$$

In these manipulations we have used the formula

$$\hat{\mathcal{A}}_{\Lambda T} = \langle (\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T}) \rangle_{\hat{S}_{\Lambda T}} = \langle (\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T}) \rangle_{\hat{S}_{\Lambda T} - \Delta\hat{\Gamma}^{(1)}_{\Lambda T}} + \mathcal{O}(\hbar^2),$$

which holds because at one loop the vertices of $\Delta\hat{\Gamma}^{(1)}_{\Lambda T}$, which are already $\mathcal{O}(\hbar)$, cannot contribute to one-particle irreducible diagrams that contain one insertion of $(\hat{S}_{\Lambda T}, \hat{S}_{\Lambda T})$.

At one loop, using (5.7), we obtain

$$\begin{aligned}\mathcal{A}^{(1)}_{fT} &= \mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}} + \mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}} - \mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}}|_{\mathrm{Ddiv}} - 2(S_T, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}}) \\ &\quad - 2(S_{\mathrm{HD}}, \Delta\hat{\Gamma}^{(1)}_{\Lambda T}). \qquad (5.10)\end{aligned}$$

Now we take the CDHD limit. Since $\Delta\hat{\Gamma}^{(1)}_{\Lambda T}$ is Λ independent, the antiparentheses $(S_{\mathrm{HD}}, \Delta\hat{\Gamma}^{(1)}_{\Lambda T})$ vanish when $\Lambda \to \infty$. Moreover, $\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}$ is independent of Λ. On the other hand, $\mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}} - \mathcal{A}^{(1)}_{\Lambda T\mathrm{ev}}|_{\mathrm{Ddiv}}$ vanishes in the CDHD

limit, because the terms $\hat{\delta}\varepsilon^0\Lambda$ are subtracted away in the difference. Since $S_T - S_{d\mathrm{b}} = \mathcal{O}(\varepsilon)$ at $\Lambda_- = \infty$, we can replace $(S_T, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}})$ by $(S_{d\mathrm{b}}, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}})$. Finally, using formula (4.3) we get

$$\mathcal{A}^{(1)}_{\mathrm{b}} = \mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}} - 2(S_{d\mathrm{b}}, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}}) = \int (\kappa C)^I \mathcal{A}_I(\kappa\phi'_g, \lambda_+) - 2(S_{d\mathrm{b}}, \hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}}). \quad (5.11)$$

In particular, by formula (5.5) and $(S_{d\mathrm{b}}, S_{d\mathrm{b}}) = 0$, the one-loop anomaly functional $\mathcal{A}^{(1)}_{\mathrm{b}}$ of the basic theory solves the condition

$$(S_{d\mathrm{b}}, \mathcal{A}^{(1)}_{\mathrm{b}}) = 0. \quad (5.12)$$

At this point, we are ready to use assumption (IV) of Sec. II C, which tells us that there exists a local functional $\mathcal{X}(\Phi, K)$ such that $\mathcal{A}^{(1)}_{\mathrm{b}} = (S_{d\mathrm{b}}, \mathcal{X})$. Using this piece of information and (5.11), we obtain

$$\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}} = \int (\kappa C)^I \mathcal{A}_I(\kappa\phi'_g, \lambda_+) = (S_{d\mathrm{b}}, \mathcal{X}') \quad (5.13)$$

for $\mathcal{X}' = \mathcal{X} + 2\hat{\Gamma}^{(1)}_{\Lambda T\mathrm{fin}}$.

We know that the functional $\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}$ satisfies both (5.5) and (5.13). To subtract it in a way that preserves the structure of the HD theory, we need to know that, in addition, we can find a K-independent $\mathcal{X}'$. This is ensured by assumption (V) of Sec. II C, which tells us that there exists a local functional of vanishing ghost number $\chi(\kappa\Phi, \lambda_+)$, equal to the integral of a local function of dimension d, such that

$$\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}} = (S_K, \chi). \quad (5.14)$$

Since $\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}$ is ϕ_m independent, we can assume that χ is also ϕ_m independent. Indeed, recall that the transformations $R_g(\Phi)$ of the fields Φ'_g are independent of ϕ_m and the transformations $R_m(\Phi)$ of the fields ϕ_m are proportional to ϕ_m. Write $\chi(\kappa\Phi) = \chi_0(\kappa\Phi'_g) + \chi_m$, where $\chi_m = 0$ at $\phi_m = 0$. Then, $(S_K, \chi) = (S_K, \chi_0)$, as we can see by calculating these expressions at $\phi_m = 0$. From now on we drop χ_m and just write $\chi = \chi(\kappa\Phi'_g, \lambda_+)$.

Clearly, assumption (IV′) of Sec. II C is sufficient to justify (5.14), with $\chi = \chi(\kappa\Phi'_g, \lambda_+)$, in alternative to assumptions (IV) and (V).

Since χ is one loop, its κ structure agrees with the $L = 1$ sector of formula (2.27).

B. Cancellation of anomalies in the HD theory

Now we go back to the HD theory. We can cancel its potential anomalies by redefining the action. Indeed, if we take

$$\check{S}_{\Lambda T} = \hat{S}_{\Lambda T} - \frac{1}{2}\chi = S_{\Lambda T} - \Gamma^{(1)}_{\Lambda T\mathrm{div}} - \frac{1}{2}\chi \quad (5.15)$$

as the new action, we find

$$\check{\mathcal{A}}_{\Lambda T} = \langle(\check{S}_{\Lambda T}, \check{S}_{\Lambda T})\rangle_{\check{S}_{\Lambda T}} = \hat{\mathcal{A}}_{\Lambda T} - (S_{\Lambda T}, \chi) + \mathcal{O}(\hbar^2). \quad (5.16)$$

Since χ is K independent, only the K-dependent sector of $S_{\Lambda T}$, which is made of S_K and $S_{\mathrm{ev}T}$, can contribute to $(S_{\Lambda T}, \chi)$. Taking the one-loop nonevanescent part of (5.16), and using (5.1) and (5.14), we get

$$\check{\mathcal{A}}^{(1)}_{\Lambda T\mathrm{nev}} = \mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}} - (S_K, \chi) = 0. \quad (5.17)$$

The new Γ functional $\check{\Gamma}_{\Lambda T}$ defined by the action $\check{S}_{\Lambda T}$ of formula (5.15) is still convergent to all orders. Indeed, it is convergent at one loop and, once we switch to the tilde parametrization, the functional χ is written as a functional $\tilde{\chi}(\tilde{\kappa}\tilde{\Phi}'_g, \tilde{\lambda}_+)$. This fact, together with formulas (3.16) and (3.21), ensures that the counterterms keep the form (3.19), which forbids divergences beyond one loop. The anomaly functional $\check{\mathcal{A}}_{\Lambda T} = (\check{\Gamma}_{\Lambda T}, \check{\Gamma}_{\Lambda T})$ is also convergent to all orders. Since its one-loop contribution $\check{\mathcal{A}}^{(1)}_{\Lambda T}$ has no divergent part and, by formula (5.17), no nonevanescent part, it is just evanescent: $\check{\mathcal{A}}^{(1)}_{\Lambda T} = \mathcal{O}(\varepsilon)$. Including the tree-level contribution $(S_{\Lambda T}, S_{\Lambda T})$, which is also $\mathcal{O}(\varepsilon)$, we can write

$$\check{\mathcal{A}}_{\Lambda T} = \mathcal{O}(\varepsilon) + \mathcal{O}(\hbar^2). \quad (5.18)$$

The next step is to prove the anomaly cancellation to all orders in the higher-derivative theory, which we do in the next section. After that, we complete the CDHD limit by renormalizing the Λ divergences.

VI. MANIFEST ADLER-BARDEEN THEOREM IN THE HD THEORY

In this section we prove that, from two loops onwards, the gauge anomalies manifestly vanish in the HD theory. We have to study the diagrams with two or more loops, with one insertion of

$$\mathcal{E}_T \equiv (\check{S}_{\Lambda T}, \check{S}_{\Lambda T}) = (S_{\Lambda T}, S_{\Lambda T}) - \mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}} - \mathcal{A}^{(1)}_{\Lambda T\mathrm{divev}} - (S_{\mathrm{ev}T}, \chi), \quad (6.1)$$

calculated with the action (5.15). To derive the right-hand side of (6.1), we have used the fact that $\Gamma^{(1)}_{\Lambda T\mathrm{div}}$ and χ are K independent, then applied formula (4.4), and replaced (S_K, χ) with $\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}$. The action $\check{S}_{\Lambda T}$ has the structure (3.16) plus one-loop corrections of the form $F(\tilde{\kappa}\tilde{\Phi}'_g, \tilde{\lambda}_+)$. Therefore, its counterterms have the structure (3.19). On the other hand, $\mathcal{E}_T$ has the structure (3.22) plus (possibly

nonevanescent and divergent-evanescent) one-loop corrections that have the same form times $\tilde{\kappa}^2$, such that V still vanishes at $\tilde{\varsigma}=0$. This fact implies that $\breve{\mathcal{A}}_{\Lambda T}=\langle\mathcal{E}_T\rangle$ is still the sum of contributions that have the structures (3.24) and (3.25), with $\mathcal{V}_L=0$ at $\tilde{\varsigma}=0$.

The functional $\mathcal{E}_T$ is made of the tree-level local evanescent functional $(S_{\Lambda T},S_{\Lambda T})$, plus one-loop local corrections. Formula (5.18) tells us that such corrections make the average $\langle\mathcal{E}_T\rangle$ evanescent at one loop. Then, the theory of evanescent operators [4,10] tells us that the two-loop nonevanescent part of $\langle\mathcal{E}_T\rangle$ is local. Briefly, the reason is as follows. Writing $\hat{\partial}^\mu=\hat{\eta}^{\mu\nu}\partial_\nu$ and $\hat{p}^\mu=\hat{\eta}^{\mu\nu}p_\nu$ everywhere inside $(S_{\Lambda T},S_{\Lambda T})$, we can express each vertex of $(S_{\Lambda T},S_{\Lambda T})$ in a factorized form $\mathcal{T}_k\hat{\delta}_k$, where $\hat{\delta}_k$ denotes a formally evanescent part, made of tensors $\eta^{\hat{\mu}\hat{\nu}}$ and other structures that stay outside of the diagrams, while $\mathcal{T}_k$ is a nonevanescent local functional and collects the momenta. The average $\langle\mathcal{T}_k\hat{\delta}_k\rangle$ is the sum of the one-particle irreducible diagrams G that contain one insertion of $\mathcal{T}_k\hat{\delta}_k$. Leaving $\hat{\delta}_k$ outside the diagrams, consider the average $\langle\mathcal{T}_k\rangle$, and let $\mathcal{T}^{(1)}_{k\mathrm{div}}$ denote its one-loop divergent part. Using (4.1) and (4.2), we find

$$\sum_k\mathcal{T}^{(1)}_{k\mathrm{div}}\hat{\delta}_k=\mathcal{A}^{(1)}_{\Lambda T\mathrm{nev}}+\mathcal{A}^{(1)}_{\Lambda T\mathrm{divev}}+L^{(1)}_{\mathrm{ev}},$$

where $L^{(1)}_{\mathrm{ev}}$ are unspecified local evanescences. The theorem on the locality of counterterms ensures that the divergent part of $\langle\mathcal{T}_k-\mathcal{T}^{(1)}_{k\mathrm{div}}\rangle$ is local at two loops. Accordingly, the nonevanescent and divergent parts of

$$\langle\mathcal{E}_T\rangle=\left\langle\sum_k(\mathcal{T}_k-\mathcal{T}^{(1)}_{k\mathrm{div}})\hat{\delta}_k+L^{(1)}_{\mathrm{ev}}-(S_{\mathrm{ev}T},\chi)\right\rangle$$

are also local at two loops. In Sec. III B we proved that the local functionals that have the structures (3.24) and (3.25) vanish from two loops onwards, by simple power counting. Therefore, $\langle\mathcal{E}_T\rangle$ is evanescent at two loops, which means that formula (5.18) can be improved by one order and turned into

$$\breve{\mathcal{A}}_{\Lambda T}=\mathcal{O}(\varepsilon)+\mathcal{O}(\hbar^3).$$

The argument can be iterated to all orders, because if an evanescent operator E is renormalized, and equipped with finite local subtractions such that its average $\langle\mathsf{E}\rangle$ is evanescent up to and including ℓ loops, then the $\mathcal{O}(\hbar^{\ell+1})$ nonevanescent and divergent parts $\langle\mathsf{E}\rangle^{(\ell+1)}_{\mathrm{nonev}}$ and $\langle\mathsf{E}\rangle^{(\ell+1)}_{\mathrm{div}}$ of $\langle\mathsf{E}\rangle$ must be local. In the case we are considering here, which is $\mathsf{E}=\mathcal{E}_T$, $\langle\mathsf{E}\rangle^{(\ell+1)}_{\mathrm{nonev}}$ and $\langle\mathsf{E}\rangle^{(\ell+1)}_{\mathrm{div}}$ must also have the structures (3.24) and (3.25), but then they vanish.

We infer that the anomaly functional $\breve{\mathcal{A}}_{\Lambda T}$ is evanescent to all orders, that is to say,

$$\breve{\mathcal{A}}_{\Lambda T}=(\breve{\Gamma}_{\Lambda T},\breve{\Gamma}_{\Lambda T})=\mathcal{O}(\varepsilon),\tag{6.2}$$

which proves the manifest Adler-Bardeen theorem for the HD theory $S_{\Lambda T}$. Therefore, the HD theory is free of gauge anomalies to all orders in the limit $D\to d$.

This concludes the proof that the HD theory is superrenormalizable and anomaly free to all orders. We stress again that only the truncation T1 of the action S_Λ is necessary, and the result (6.2) holds to all orders in $\hbar$ and for arbitrarily large powers of $1/\Lambda_-$. The truncation T2 of Sec. II B is important for the second part of the proof, which is worked out in the next section.

VII. ALMOST MANIFEST ADLER-BARDEEN THEOREM IN THE FINAL THEORY

We are finally ready to prove the cancellation of gauge anomalies to all orders in the final theory. The task consists of studying the Λ dependence of the HD theory, for Λ large, subtract the Λ divergences, and complete the CDHD limit, according to the rules explained in Sec. II D. The subtraction of the Λ divergences is done inductively and preserves the master equation up to $\mathcal{O}(\varepsilon)$ terms that vanish in the CDHD limit.

Before beginning the proof, let us recall that our approach uses two regularizations, the chiral dimensional one, with regularizing parameter ε, and the higher-derivative one, with energy scale Λ. So far, we have taken care of the renormalization and the cancellation of anomalies to all orders at the CD level. Now we consider the Λ divergences. As far as those are concerned, once we have adjusted the orders $\hbar^n$, $k\le n$, we can concentrate on the order $\hbar^{n+1}$ and neglect higher-order corrections, as is done in most common renormalization procedures. However, at each step of the subtraction of the Λ divergences, we must preserve the properties gained so far with respect to the CD renormalization, and those must hold to all orders in $\hbar$, like Eq. (6.2).

Because of the truncation T2, we say that an action S_k is CDHD renormalized up to and including k loops, when the ℓ-loop contributions to its Γ functional Γ_k are CDHD convergent up to $o(1/\Lambda_-^{T-2\ell\sigma})$, for $0\le\ell\le k$.

We work inductively in the number n of loops. We assume that for every $k\le n<\bar{\ell}$, where $\bar{\ell}$ is given by (2.33), there exists an action $S_{kT}=S_{\Lambda T}+\mathcal{O}(\hbar)$, obtained from $S_{\Lambda T}$ by means of ε-convergent, possibly Λ-divergent canonical transformations and redefinitions of parameters, with the following properties: we can ε-renormalize S_{kT} at Λ fixed, to all orders in $\hbar$, and fine-tune its finite local counterterms, so that the so-renormalized action S_{kRT} is also CDHD renormalized up to and including k loops, and the renormalized Γ functional Γ_{kRT} associated with S_{kRT} is free of gauge anomalies to all orders in $\hbar$ at Λ fixed, i.e.

$$(\Gamma_{kRT},\Gamma_{kRT})=\langle(S_{kRT},S_{kRT})\rangle_{S_{kRT}}=\mathcal{O}(\varepsilon),\qquad k\le n.\tag{7.1}$$

At $n=0$ we take $S_{0T}=S_{\Lambda T}$, so $S_{0RT}=\check{S}_{\Lambda T}$. Clearly, Γ_{0RT} coincides with $\check{\Gamma}_{\Lambda T}$ and satisfies (6.2).

Note that, by assumption, Γ_{kRT} has a regular limit for $\varepsilon\to 0$ at Λ fixed, and not just within the truncation T2, but also beyond. More precisely, Γ_{kRT} is a sum of ℓ-loop contributions of the form (2.58) up to $o(1/\Lambda_-^{T-2\ell\sigma})$ for $0\leq\ell\leq k$ (because it is CDHD convergent in that sector), and a sum of terms (2.56) everywhere else. Instead, $(\Gamma_{kRT},\Gamma_{kRT})$ is a sum of ℓ-loop contributions (2.58) except $\varepsilon^0\Lambda^0$ and ε^0/Λ up to $o(1/\Lambda_-^{T-2\ell\sigma})$ for $0\leq\ell\leq k$, plus terms (2.56) except ε^0 everywhere else. Note that assumption (7.1) also holds beyond the truncation T2 [where the "$\mathcal{O}(\varepsilon)$" may contain terms $\hat{\delta}\varepsilon^0\Lambda$].

The theorem on the locality of counterterms ensures that the $(n+1)$-loop CDHD divergent part $\Gamma^{(n+1)}_{nRT\text{div}}$ of Γ_{nRT} is a local functional, up to $o(1/\Lambda_-^{T-2n\sigma})$. Since Γ_{nRT} has a regular limit for $\varepsilon\to 0$ at Λ fixed, $\Gamma^{(n+1)}_{nRT\text{div}}$ contains only divergences in Λ, but not in ε. Precisely, we can write

$$\Gamma^{(n+1)}_{nRT\text{div}}=\Gamma^{(n+1)}_{nRT\text{div nev}}+\Gamma^{(n+1)}_{nRT\text{div fev}}+o(1/\Lambda_-^{T-2(n+1)\sigma}), \tag{7.2}$$

where $\Gamma^{(n+1)}_{nRT\text{div nev}}$ and $\Gamma^{(n+1)}_{nRT\text{div fev}}$ collect the terms $\varepsilon^0\Lambda$ and $\hat{\delta}\varepsilon^0\Lambda$ of the list (2.57), respectively.

Now, we take the $(n+1)$-loop CDHD-divergent non-ε-evanescent part of Eq. (7.1) for $k=n$, within the truncation, which means the terms of types $\varepsilon^0\Lambda$ of the list (2.57), up to $o(1/\Lambda_-^{T-2(n+1)\sigma})$. Expand Γ_{nRT} in powers of $\hbar$, by writing it as $\sum_{k=0}^{\infty}\hbar^k\Gamma^{(k)}_{nRT}$. Observe that the contributions $(\Gamma^{(k)}_{nRT},\Gamma^{(n+1-k)}_{nRT})$ with $0<k<n+1$ can be dropped, because they are convergent in the CDHD limit, up to $o(1/\Lambda_-^{T-2(n+1)\sigma})$. We remain with $2(\Gamma^{(0)}_{nRT},\Gamma^{(n+1)}_{nRT})=2(S_{\Lambda T},\Gamma^{(n+1)}_{nRT})$. Taking the Λ divergent part of this expression, and recalling that, by formula (3.30), $(S_{\text{HD}},\Gamma^{(n+1)}_{nRT\text{div}})$ tends to zero for $\Lambda\to\infty$ within the truncation, we get $2(S_T,\Gamma^{(n+1)}_{nRT\text{div}})+o(1/\Lambda_-^{T-2(n+1)\sigma})$. Taking the non-$\varepsilon$-evanescent part and recalling that S_T is equal to $S_{dT}+S_{\text{ev}T}$, where S_{dT} is non-ε-evanescent, the left-hand side of (7.1) at $k=n$ gives $2(S_{dT},\Gamma^{(n+1)}_{nRT\text{div nev}})+o(1/\Lambda_-^{T-2(n+1)\sigma})$. Noting that the CDHD-divergent part of the right-hand side is just made of terms $\hat{\delta}\varepsilon^0\Lambda$, within the truncation, we obtain

$$(S_{dT},\Gamma^{(n+1)}_{nRT\text{div nev}})=o(1/\Lambda_-^{T-2(n+1)\sigma}). \tag{7.3}$$

A. Solution of the cohomological problem

We work out the solution of the cohomological problem (7.3) by applying the assumption (III) of Sec. II C. Let us imagine that, instead of working with the classical action S_c, we work with its extension $\check{S}_c$, which includes the invariants $\check{\mathcal{G}}_i$ that break the nonanomalous accidental symmetries belonging to the group G_{nas}. Similarly, we extend S_d to $\check{S}_d$, S_{ev} to $\check{S}_{\text{ev}}$, and $S=S_d+S_{\text{ev}}$ to $\check{S}=\check{S}_d+\check{S}_{\text{ev}}$. Every extended functional reduces to the nonextended one when we set $\check{\lambda}=\check{\eta}=0$, where $\check{\lambda}$ and $\check{\eta}$ are the extra parameters contained in $\check{S}_c$ and $\check{S}_{\text{ev}}$, respectively. There is no need to extend the higher-derivative sector S_{HD}.

If we repeat the operations that lead to (7.3), we obtain an extended, nonevanescent local functional $\check{\Gamma}^{(n+1)}_{nRT\text{div nev}}$ that satisfies $(\check{S}_{dT},\check{\Gamma}^{(n+1)}_{nRT\text{div nev}})=o(1/\Lambda_-^{T-2(n+1)\sigma})$. Taking the limit $\Lambda_-\to\infty$ of this equation and recalling that $T\geq 2(n+1)\sigma$ (because $n<\bar{\ell}$), we get

$$(\check{S}_{d\text{b}},\check{V}_0)=0,$$

where $\check{V}_0$ denotes the $\Lambda_-\to\infty$ limit of $\check{\Gamma}^{(n+1)}_{nRT\text{div nev}}$. Assumption (III) tells us that the action $\check{S}_{d\text{b}}$ satisfies the extended Kluberg-Stern–Zuber assumption, and the group G_{nas} is compact. Thus, there exist constants a_{i0} and b_{i0}, which depend on the parameters of $\check{V}_0$, and a local functional $\check{Y}_0$ such that

$$\check{V}_0=\sum_i a_{i0}\mathcal{G}_i+\sum_i b_{i0}\check{\mathcal{G}}_i+(\check{S}_{d\text{b}},\check{Y}_0).$$

Recall that in Sec. II B we showed that only integer and semi-integer powers of $1/\Lambda_-$ can appear. Define

$$\check{X}_1=\Lambda_-^{1/2}\left[\check{\Gamma}^{(n+1)}_{nRT\text{div nev}}-\sum_i a_{i0}\mathcal{G}_i-\sum_i b_{i0}\check{\mathcal{G}}_i-(\check{S}_{dT},\check{Y}_0)\right].$$

The local functional $\check{X}_1$ is analytic in $1/\Lambda_-^{1/2}$, because $\check{\Gamma}^{(n+1)}_{nRT\text{div nev}}=\check{V}_0+\mathcal{O}(1/\Lambda_-^{1/2})$ and $\check{S}_{dT}=\check{S}_{d\text{b}}+\mathcal{O}(1/\Lambda_-^{1/2})$. Moreover, since $(\check{S}_{dT},\mathcal{G}_i)=(\check{S}_{dT},\check{\mathcal{G}}_i)=(\check{S}_{dT},\check{S}_{dT})=0$, $\check{X}_1$ satisfies $(\check{S}_{dT},\check{X}_1)=o(1/\Lambda_-^{T-2(n+1)\sigma-1/2})$. Then we repeat the argument just given with $\check{\Gamma}^{(n+1)}_{nRT\text{div nev}}$ replaced by $\check{X}_1$, and continue like this till we can. For $0\leq m\leq 2T-4(n+1)\sigma+1$, we find constants a_{im-1} and b_{im-1}, depending on the parameters, and local functionals $\check{Y}_{m-1}$ such that the combinations

$$\check{X}_m=\Lambda_-^{1/2}\left[\check{X}_{m-1}-\sum_i a_{im-1}\mathcal{G}_i-\sum_i b_{im-1}\check{\mathcal{G}}_i-(\check{S}_{dT},\check{Y}_{m-1})\right]$$

are analytic in $1/\Lambda_-^{1/2}$ and satisfy $(\check{S}_{dT},\check{X}_m)=o(1/\Lambda_-^{T-2(n+1)\sigma-m/2})$, with $\check{X}_0=\check{\Gamma}^{(n+1)}_{nRT\text{div nev}}$. In the end, there exist constants $\Delta\lambda'_{ni}$, $\Delta\check{\lambda}'_{ni}$ depending on the parameters, and local functionals $\check{\chi}_{nT}$,

$$\Delta\lambda'_{ni} = \sum_{m=0}^{2T-4(n+1)\sigma} \frac{a_{im}}{\Lambda_-^{m/2}}, \qquad \Delta\check{\lambda}'_{ni} = \sum_{m=0}^{2T-4(n+1)\sigma} \frac{b_{im}}{\Lambda_-^{m/2}},$$
$$\check{\chi}_{nT} = \sum_{m=0}^{2T-4(n+1)\sigma} \frac{\check{Y}_m}{\Lambda_-^{m/2}}, \tag{7.4}$$

such that

$$\check{\Gamma}^{(n+1)}_{nRT\text{div nev}} = \sum_i \Delta\lambda'_{ni}\mathcal{G}_i + \sum_i \Delta\check{\lambda}'_{ni}\check{\mathcal{G}}_i + (\check{S}_{dT}, \check{\chi}_{nT}) + o(1/\Lambda_-^{T-2(n+1)\sigma}). \tag{7.5}$$

Clearly, $\Delta\lambda'_{ni}$, $\Delta\check{\lambda}'_{ni}$, and $\check{\chi}_{nT}$ are of order $\hbar^{n+1}$. If we set $\check{\lambda} = \check{\eta} = 0$ in Eq. (7.5), we obtain

$$\Gamma^{(n+1)}_{nRT\text{div nev}} = \sum_i \bar{\Delta}\,\lambda_{ni}\mathcal{G}_i + \sum_i \Delta\check{\lambda}_{ni}\check{\mathcal{G}}_i + (S_{dT}, \bar{\chi}_{nT}) + o(1/\Lambda_-^{T-2(n+1)\sigma}), \tag{7.6}$$

where $\bar{\Delta}\lambda_{ni}$, $\Delta\check{\lambda}_{ni}$, and $\bar{\chi}_{nT}$ are equal to $\Delta\lambda'_{ni}$, $\Delta\check{\lambda}'_{ni}$, and $\check{\chi}_{nT}$ at $\check{\lambda} = \check{\eta} = 0$. However, $\Gamma^{(n+1)}_{nRT\text{div nev}}$ is invariant under the nonanomalous accidental symmetries that belong to the group G_{nas}, while the functionals $\check{\mathcal{G}}_i$ are not. Since G_{nas} is assumed to be compact, we can average on it. When we do that, the invariants $\check{\mathcal{G}}_i$ disappear (or turn into linear combinations of $\mathcal{G}_i$) and $\bar{\chi}_{nT}$ turns into some χ_{nT}. We finally obtain

$$\Gamma^{(n+1)}_{nRT\text{div nev}} = \sum_i \Delta\lambda_{ni}\mathcal{G}_i + (S_{dT}, \chi_{nT}) + o(1/\Lambda_-^{T-2(n+1)\sigma}), \tag{7.7}$$

for possibly new constants $\Delta\lambda_{ni}$ of order $\hbar^{n+1}$ that depend on the parameters.

The arguments of this subsection, which lead from formula (7.3) to formula (7.7), are purely algebraic and can be applied in more general contexts. For example, taking $T \to \infty$, formula (7.5) proves that the action S_d is also cohomologically complete. Instead, formula (7.7) at $T = \infty$ proves that S_d satisfies what we can call the *physical* Kluberg-Stern–Zuber assumption, which states that if a nonevanescent local functional Γ_{div} solves $(S_d, \Gamma_{\text{div}}) = 0$ and is generated by renormalization as a local divergent part of the Γ functional, then there exists constants a_i and a local functional Y of ghost number -1 such that

$$\Gamma_{\text{div}} = \sum_i a_i\mathcal{G}_i + (S_d, Y). \tag{7.8}$$

Indeed, we can always lift the discussion to the extended theory $\check{S}_d$, which gives an extended functional $\check{\Gamma}_{\text{div}}$ that solves $(\check{S}_d, \check{\Gamma}_{\text{div}}) = 0$. Then $\check{\Gamma}_{\text{div}}$ can be expanded like the right-hand side of (7.5) at $T = \infty$. When we go back down to S_d, we find (7.8).

B. Subtraction of divergences

Now we work out the operations that subtract the divergences $\Gamma^{(n+1)}_{nRT\text{div}}$ within the truncation. We recall from Sec. II B that the truncated classical action S_{cT} contains enough independent parameters λ_i to subtract the divergences proportional to $\mathcal{G}_i$ of (7.7) by means of λ_i redefinitions, within the truncation T2. If we make the canonical transformation generated by

$$F_n(\Phi, K') = \int \Phi^\alpha K'_\alpha - \chi_{nT}(\Phi, K') \tag{7.9}$$

and the redefinitions

$$\lambda_i \to \lambda_i - \Delta\lambda_{ni} \tag{7.10}$$

on S_{dT}, we get

$$S_{dT} \to S_{dT} - \sum_i \Delta\lambda_{ni}\mathcal{G}_i - (S_{dT}, \chi_{nT}) + \mathcal{O}(\hbar^{n+2}). \tag{7.11}$$

Observe that the operations (7.9) and (7.10) are independent of ε and divergent in Λ, because so is $\Gamma^{(n+1)}_{nRT\text{div nev}}$.

Formula (7.11) is equivalent to

$$S_{dT} \to S_{dT} - \Gamma^{(n+1)}_{nRT\text{div nev}} + \mathcal{O}(\hbar^{n+2}) + \mathcal{O}(\hbar^{n+1})o(1/\Lambda_-^{T-2(n+1)\sigma}),$$

which shows that we can fully subtract the ε-nonevanescent Λ divergences $\Gamma^{(n+1)}_{nRT\text{div nev}}$, by making the operations (7.9) and (7.10) on S_{dT}, up to $\mathcal{O}(\hbar^{n+1})o(1/\Lambda_-^{T-2(n+1)\sigma})$.

However, the truncated classical action we have been using is not S_{dT}, nor $S_T = S_{dT} + S_{\text{ev}T}$, but S_{nT}, whose classical limit is $S_{\Lambda T}$, so we must inquire what happens when we make the operations (7.9) and (7.10) on $S_{\Lambda T}$.

Let us begin from $S_T = S_{dT} + S_{\text{ev}T}$. Since the operations (7.9) and (7.10) are independent of ε and divergent in Λ, when we apply them to $S_{\text{ev}T}$ we generate new formally ε-evanescent, Λ-divergent terms of order $\hbar^{n+1}$, which change $\Gamma^{(n+1)}_{nRT\text{div fev}}$ [check formula (7.2)] into some new $\Gamma'^{(n+1)}_{nRT\text{div fev}}$, plus $\mathcal{O}(\hbar^{n+2})$. The divergences $\Gamma'^{(n+1)}_{nRT\text{div fev}}$ are not constrained by gauge invariance, but just locality, weighted power counting, and the nonanomalous global symmetries of the theory. In Sec. II B we remarked that, within the truncation T2, that is to say, up to $o(1/\Lambda_-^{T-2(n+1)\sigma})$, they can be subtracted by redefining the parameters ς and η of $S_{\text{ev}T}$.

Let $\mathcal{R}_n$ denote the set of operations made by the canonical transformation (7.9), the λ redefinitions (7.10), and the ς and η redefinitions that subtract $\Gamma'^{(n+1)}_{nRT\text{div fev}}$. We have

$$\mathcal{R}_n S_T = S_T - \Gamma^{(n+1)}_{nRT\text{div}} + \mathcal{O}(\hbar^{n+2}) + \mathcal{O}(\hbar^{n+1}) o(1/\Lambda_-^{T-2(n+1)\sigma}). \quad (7.12)$$

It remains to check what happens when the operations $\mathcal{R}_n$ act on $S_{\text{HD}} = S_{\Lambda T} - S_T$. Note that $\mathcal{R}_n$ are equal to the identity plus $\mathcal{O}(\hbar^{n+1})$, and they are independent of ε and divergent in Λ. Moreover, by formula (7.4) and the arguments of Sec. III C, they do not involve powers of Λ greater than $T + d - 2\sigma$, at the order $\mathcal{O}(\hbar^{n+1})$. Recalling that the difference S_{HD} is $\mathcal{O}(1/\Lambda^{T+d-2\sigma+1})$, we have that $(\mathcal{R}_n - 1)S_{\text{HD}}$ vanishes in the CDHD limit to the order $\mathcal{O}(\hbar^{n+1})$.

Define

$$S_{n+1T} = \mathcal{R}_n S_{nT} = \mathcal{R}_n \circ \cdots \circ \mathcal{R}_0 S_{\Lambda T} \equiv \mathcal{U}_n S_{\Lambda T}. \quad (7.13)$$

Using (7.12), we find

$$\begin{aligned} S_{n+1T} &= S_{nT} + (\mathcal{R}_n - 1)S_{\Lambda T} + \mathcal{O}(\hbar^{n+2}) \\ &= S_{nT} + (\mathcal{R}_n - 1)S_T + (\mathcal{R}_n - 1)S_{\text{HD}} + \mathcal{O}(\hbar^{n+2}) \\ &= S_{nT} - \Gamma^{(n+1)}_{nRT\text{div}} + (\mathcal{R}_n - 1)S_{\text{HD}} + \mathcal{O}(\hbar^{n+2}) \\ &\quad + \mathcal{O}(\hbar^{n+1}) o(1/\Lambda_-^{T-2(n+1)\sigma}). \end{aligned} \quad (7.14)$$

Thus, the operations $\mathcal{R}_n$ do renormalize the Λ divergences to the order $n + 1$, as we want.

The operations $\mathcal{U}_n = \mathcal{R}_n \circ \cdots \circ \mathcal{R}_0$ are combinations of local canonical transformations and redefinitions of parameters. They act on the action $S_{\Lambda T}$, and, from the point of view of the HD theory, where Λ is fixed, they are convergent. In general, a canonical transformation may destroy the nice properties of the HD theory, such as its manifest superrenormalizability, its structure in the tilde parametrization, and the manifest cancellation of its gauge anomalies. To overcome these problems, we must re-renormalize the ε divergences and recancel the gauge anomalies after making the operations $\mathcal{U}_n$. We can achieve these goals with the help of the theorem proved in Ref. [17].

C. Renormalization and almost manifest Adler-Bardeen theorem

Now we must renormalize S_{n+1T} at Λ fixed. We use the theorem proved in Ref. [17], which ensures that if we make a convergent local canonical transformation [equal to the identity transformation plus $\mathcal{O}(\theta)$, where θ is some expansion parameter] on the action S of a theory that is free of gauge anomalies, it is possible to re-renormalize the divergences of the transformed theory and re-fine-tune its finite local counterterms, continuously in θ, so as to preserve the cancellation of gauge anomalies to all orders. Clearly, we can achieve the same goal if we combine canonical transformations and redefinitions of parameters, as long as they are both convergent.

Before proceeding, let us recapitulate the situation. The HD theory has the action $S_{\Lambda T}$, which is superrenormalizable and has a particularly nice structure, once we use the tilde parametrization. Its renormalized action is the action $\breve{S}_{\Lambda T}$ of formula (5.15), which contains both the counterterms $\Gamma^{(1)}_{\Lambda T\text{div}}$ that subtract the ε-divergences at Λ fixed, and the finite local counterterms $-\chi/2$ that subtract the trivial anomalous terms. Formula (6.2) ensures that $\breve{\Gamma}_{\Lambda T}$ is free of gauge anomalies to all orders.

Now we need to make the operations $\mathcal{U}_n$ on the action $S_{\Lambda T}$. From the point of view of the HD theory, where Λ is fixed, those operations are completely convergent, because they are convergent in ε (although possibly divergent in Λ). However, the canonical transformations can ruin the manifest superrenormalizability of $S_{\Lambda T}$, as well as the nice structure exhibited by $S_{\Lambda T}$ in the tilde parametrization. Because of this, the arguments that allowed us to prove the cancellation of gauge anomalies in the HD theory cannot be used after the transformations. Nevertheless, we expect that the superrenormalizability of $S_{\Lambda T}$ and the cancellation of its gauge anomalies survive in some nonmanifest form.

What happens is that, after the operations $\mathcal{U}_n$, the (nonlinear part of the) canonical transformation generates new poles in ε, and not just at one loop, but at each order of the perturbative expansion. Then, the first thing to do is re-renormalize the transformed HD theory at Λ fixed, to remove the new divergences. Moreover, the cancellation of gauge anomalies, which is in general ruined by the operations $\mathcal{U}_n$, can be enforced again by re-fine-tuning all sorts of finite local counterterms. The theorem proved in Ref. [17] ensures that this goal can indeed be achieved, to all orders in $\hbar$ and $1/\Lambda_-$. In these arguments, the truncation T2 plays no role.

We know that each $\mathcal{R}_n$ is equal to the identity plus $\mathcal{O}(\hbar^{n+1})$, and so is the canonical transformation (7.9). If we replace the factor $\hbar^{n+1}$ by a parameter θ_n, we can define operations $\mathcal{R}_n(\theta_n)$ that are equal to the identity plus $\mathcal{O}(\theta_n)$. Then we also have operations $\mathcal{U}_n(\theta_1, \ldots, \theta_n)$, which we sometimes denote for brevity by $\mathcal{U}_n(\theta)$. Clearly, $\mathcal{U}_{n-1}(\theta_1, \ldots, \theta_{n-1}) = \mathcal{U}_n(\theta_1, \ldots, \theta_{n-1}, 0)$. For a while, we work on the actions $\bar{S}_{k+1T} \equiv \mathcal{U}_k(\theta) S_{\Lambda T}$ at Λ fixed, for $0 \le k \le n$. Applying the results of Ref. [17] to the operations $\mathcal{U}_k(\theta)$, we know that we can ε-renormalize the actions $\bar{S}_{k+1T}$ at Λ fixed and fine-tune the finite local counterterms, continuously in θ, so as to preserve the cancellation of gauge anomalies for arbitrary values of each θ. Call the so-renormalized actions $\bar{S}_{k+1RT}$ and their Γ functionals $\bar{\Gamma}_{k+1RT}$. We have

$$(\bar{\Gamma}_{k+1RT}, \bar{\Gamma}_{k+1RT}) = \mathcal{O}(\varepsilon), \qquad k \le n. \quad (7.15)$$

Observe that

$$\bar{S}_{k+1RT} = \bar{S}_{k+1T} + \breve{S}_{\Lambda T} - S_{\Lambda T} + \mathcal{O}(\hbar)\mathcal{O}(\theta), \qquad k \le n.$$

Indeed, $\breve{S}_{\Lambda T} - S_{\Lambda T}$ are the counterterms that ε-renormalize the theory and cancel the gauge anomalies at $\theta = 0$. Every other counterterm must be both $\mathcal{O}(\hbar)$ and $\mathcal{O}(\theta)$. Thus,

$$\bar{S}_{k+1RT} - \bar{S}_{kRT} = \bar{S}_{k+1T} - \bar{S}_{kT} + \mathcal{O}(\hbar)\mathcal{O}(\theta_k), \qquad k \leq n. \tag{7.16}$$

We have replaced $\mathcal{O}(\hbar)\mathcal{O}(\theta)$ with $\mathcal{O}(\hbar)\mathcal{O}(\theta_k)$ in this formula, because at $\theta_k = 0$ we have $\bar{S}_{k+1RT} = \bar{S}_{kRT}$ and $\bar{S}_{k+1T} = \bar{S}_{kT}$.

By formula (7.13), when we replace θ_i with $\hbar^{i+1}$, $i = 1, \ldots, k$, inside $\bar{S}_{k+1T}$, we obtain the actions S_{k+1T}, $k \leq n$. When we replace θ_i with $\hbar^{i+1}$ inside $\bar{S}_{k+1RT}$, we obtain the renormalized actions S_{k+1RT}. The actions S_{k+1T} and S_{k+1RT} with $k < n$ are those that are assumed to satisfy the inductive hypotheses mentioned at the beginning of this section. We must show that the actions

$$S_{n+1T} = \bar{S}_{n+1T}|_{\theta_i = \hbar^{i+1}}, \qquad S_{n+1RT} = \bar{S}_{n+1RT}|_{\theta_i = \hbar^{i+1}}, \tag{7.17}$$

satisfy analogous properties, that is to say, (a) S_{n+1RT} is ε-renormalized to all orders in $\hbar$ at Λ fixed; (b) it is CDHD renormalized up to and including $n+1$ loops; and (c) the Γ functional Γ_{n+1RT} associated with S_{n+1RT} is free of gauge anomalies to all orders in $\hbar$ at Λ fixed.

The action S_{n+1RT} defined by formula (7.17) is ε-renormalized to all orders at Λ fixed, because so is the action $\bar{S}_{n+1RT}$, by construction. To show that S_{n+1RT} is properly CDHD renormalized, we use, in the order, (7.17), (7.16), and (7.14). We obtain

$$\begin{aligned} S_{n+1RT} - S_{nRT} &= \bar{S}_{n+1RT}|_{\theta_i = \hbar^{i+1}} - \bar{S}_{nRT}|_{\theta_i = \hbar^{i+1}} \\ &= \bar{S}_{n+1T}|_{\theta_i = \hbar^{i+1}} - \bar{S}_{nT}|_{\theta_i = \hbar^{i+1}} + \mathcal{O}(\hbar^{n+2}) \\ &= S_{n+1T} - S_{nT} + \mathcal{O}(\hbar^{n+2}) \\ &= -\Gamma^{(n+1)}_{nRT\text{div}} + (\mathcal{R}_n - 1)S_{\text{HD}} \\ &\quad + \mathcal{O}(\hbar^{n+2}) + \mathcal{O}(\hbar^{n+1})o(1/\Lambda_-^{T-2(n+1)\sigma}). \end{aligned} \tag{7.18}$$

By the inductive assumption, the action S_{nRT} is CDHD renormalized up to and including n loops, which means that the ℓ-loop contributions to Γ_{nRT} are CDHD convergent up to $o(1/\Lambda_-^{T-2\ell\sigma})$, for $0 \leq \ell \leq n$. Moreover, Γ_{n+1RT} and Γ_{nRT} coincide up to $\mathcal{O}(\hbar^{n+1})$, as well as S_{n+1RT} and S_{nRT}. Now, $\Gamma_{n+1RT} = \Gamma_{nRT} + S_{n+1RT} - S_{nRT} + \mathcal{O}(\hbar^{n+2})$, and $(\mathcal{R}_n - 1)S_{\text{HD}}$ vanishes in the CDHD limit, up to $\mathcal{O}(\hbar^{n+2})$. Thus, formula (7.18) proves that the ℓ-loop contributions to Γ_{n+1RT} are CDHD convergent up to $o(1/\Lambda_-^{T-2\ell\sigma})$, for $0 \leq \ell \leq n+1$, which means that Γ_{n+1RT} is CDHD renormalized up to and including $n+1$ loops.

The last thing to do is show that Γ_{n+1RT} is free of gauge anomalies. This result follows from formula (7.15) for $k = n$. Indeed, by (7.17), when we replace θ_i with $\hbar^{i+1}$, $i = 1, \ldots, n$, the functional $\bar{\Gamma}_{n+1RT}$ turns into Γ_{n+1RT}. We finally obtain

$$(\Gamma_{n+1RT}, \Gamma_{n+1RT}) = \mathcal{O}(\varepsilon), \tag{7.19}$$

which means that we have successfully promoted the inductive hypotheses to $n+1$ loops.

Iterating the argument, we can make it work till it makes sense, which means for $n = 0, \ldots, \bar{\ell} - 1$, where $\bar{\ell}$ is given by formula (2.33) for $[\kappa] < 0$ and ∞ for $[\kappa] \geq 0$. Finally, we obtain

$$\mathcal{A}_{RT} \equiv (\Gamma_{RT}, \Gamma_{RT}) = \mathcal{O}(\varepsilon), \tag{7.20}$$

where $\Gamma_{RT} = \Gamma_{\bar{\ell}RT}$. Observe that the right-hand side of (7.20) tends to zero everywhere at Λ fixed. However, only within the truncation T2 is $\Gamma_{\bar{\ell}RT}$ convergent in the CDHD limit. Thus, the ℓ-loop contributions to the right-hand side vanish in the CDHD limit up to $o(1/\Lambda_-^{T-2\ell\sigma})$, for $0 \leq \ell \leq \bar{\ell}$. In other words, Γ_{RT} is free of gauge anomalies within the truncation T2. This proves the almost manifest Adler-Bardeen theorem.

D. Adler-Bardeen theorem

The result just achieved is also sufficient to prove the Adler-Bardeen theorem, i.e. statement 1 of the Introduction. So far, we have suppressed the $o(1/\Lambda_-^T)$ terms of the action S and its HD regularized extension S_Λ, according to the prescription T1 of Sec. II B. Now we restore those terms, all of which fall outside the truncation T2. Clearly, the results we have obtained still hold within the truncation T2. The CD, HD, and CDHD regularizations are still well defined, because the divergences not cured by the HD technique are cured by the dimensional one. Note that, however, the HD theory S_Λ is not superrenormalizable, but nonrenormalizable.

Consider the contributions to the gauge anomalies that lie outside the truncation T, and classify them according to the number of loops and the power of $1/\Lambda_-$. Let $\mathcal{A}_{>T}$ denote any finite class of them. Clearly, the terms of $\mathcal{A}_{>T}$ lie inside some other truncation $T' > T$, as long as T' is sufficiently large. Now, different truncations just define different subtraction schemes (by means of different higher-derivative theories and different CDHD regularizations), and different subtraction schemes differ by finite local counterterms. Let s_T and s'_T denote the schemes defined by the truncations T and T', respectively. We can assume that they give exactly the same results (which means that Γ_{RT} and $\Gamma_{RT'}$ coincide) within the truncation T, up to corrections $\mathcal{E}_{\text{CDHD}}$ that vanish in the CDHD limit. We prove this fact by proceeding inductively. Assume that

$$\Gamma_{RT'} = \Gamma_{RT} + \mathcal{O}(\hbar^{n+1}) + \sum_{k=0}^{n} \mathcal{O}(\hbar^k)o(1/\Lambda_-^{T-2k\sigma}) + \mathcal{E}_{\text{CDHD}} \tag{7.21}$$

till some order $n < \bar{\ell}$. The assumption is certainly true for $n = 0$. Then, the CDHD nonevanescent $(n+1)$-loop

contributions to Γ_{RT} and $\Gamma_{RT'}$ differ by finite local terms ΔS_{n+1}, up to $o(1/\Lambda_-^{T-2(n+1)\sigma})$, which means

$$\Gamma_{RT'} = \Gamma_{RT} + \Delta S_{n+1} + \mathcal{O}(\hbar^{n+2}) + \sum_{k=0}^{n+1} \mathcal{O}(\hbar^k) o(1/\Lambda_-^{T-2k\sigma}) + \mathcal{E}_{\rm CDHD}. \quad (7.22)$$

Both Γ_{RT} and $\Gamma_{RT'}$ satisfy the almost manifest Adler-Bardeen theorem, that is to say, formula (7.20) and its T' version. The right-hand sides of (7.20) and its T' version vanish in the CDHD limit, within the respective truncations, because Γ_{RT} and $\Gamma_{RT'}$ are convergent there. Thus,

$$\begin{aligned}\mathcal{A}_{RT} &= (\Gamma_{RT}, \Gamma_{RT}) = \mathcal{E}_{\rm CDHD} + \mathcal{O}(\hbar^{\bar\ell+1}) + \sum_{k=0}^{\bar\ell} \mathcal{O}(\hbar^k) o(1/\Lambda_-^{T-2k\sigma}),\\ \mathcal{A}_{RT'} &= (\Gamma_{RT'}, \Gamma_{RT'}) = \mathcal{E}_{\rm CDHD} + \mathcal{O}(\hbar^{\bar\ell'+1}) + \sum_{k=0}^{\bar\ell'} \mathcal{O}(\hbar^k) o(1/\Lambda_-^{T'-2k\sigma}). \end{aligned} \quad (7.23)$$

Using (7.22) inside these equations, and taking the CDHD convergent $(n+1)$-loop contributions to the difference, we obtain

$$(S_{dT}, \Delta S_{n+1}) = o(1/\Lambda_-^{T-2(n+1)\sigma}),$$

which is a cohomological problem analogous to (7.3). It can be solved in the same way, and the solution is the analogue of (7.7), i.e.

$$\Delta S_{n+1} = \sum_i \Delta\tilde\lambda_{ni}\mathcal{G}_i + (S_{dT}, \Delta\tilde\chi_{nT}) + o(1/\Lambda_-^{T-2(n+1)\sigma}),$$

where $\Delta\tilde\lambda_{ni}$ are convergent constants and $\Delta\tilde\chi_{nT}$ is a convergent local functional. At this point, we can attach $\Delta\tilde\lambda_{ni}$ and $\Delta\tilde\chi_{nT}$ to the constants $\Delta\lambda_{niT'}$ and the functional $\chi_{nT'}$ that subtract the $(n+1)$-loop divergences belonging to the truncation T', given by the T' version of formula (7.7). After that, we can go through the T' versions of the arguments that lead from formula (7.7) to formula (7.19) with no difficulty. So doing, we promote assumption (7.21) to the order $n+1$ and iterate the procedure till we get

$$\Gamma_{RT'} = \Gamma_{RT} + \mathcal{O}(\hbar^{\bar\ell+1}) + \sum_{k=0}^{\bar\ell} \mathcal{O}(\hbar^k) o(1/\Lambda_-^{T-2k\sigma}) + \mathcal{E}_{\rm CDHD}.$$

Once this is done, the subtraction schemes s_T and s'_T give the same results within the truncation T, up to $\mathcal{E}_{\rm CDHD}$.

Now we compare s_T and s'_T in between the truncations T and T'. First, we extend the subtraction scheme s_T in a generic way beyond the truncation T and within the truncation T', and renormalize the action $S_{\Lambda T}$ accordingly. Then, we adapt the extended scheme order by order to make it give the same results as the scheme $s_{T'}$ within the truncation T', up to $\mathcal{E}_{\rm CDHD}$. Let $s_{n,TT'}$ denote the extended scheme adapted up to and including $n < \bar\ell'$ loops. Precisely, we assume that $s_{n,TT'}$ gives

$$\Gamma_{RT'} = \Gamma_{RT} + \mathcal{O}_{n+1} + \sum_{k=0}^{n} \mathcal{O}(\hbar^k) o(1/\Lambda_-^{T'-2k\sigma}) + \mathcal{E}_{\rm CDHD}, \quad (7.24)$$

where

$$\mathcal{O}_{n+1} = \mathcal{O}(\hbar^{n+1}) \quad \text{for } n \geq \bar\ell,$$

$$\mathcal{O}_{n+1} = \mathcal{O}(\hbar^{\bar\ell+1}) + \sum_{k=n+1}^{\bar\ell} \mathcal{O}(\hbar^k) o(1/\Lambda_-^{T-2k\sigma}) \quad \text{for } n < \bar\ell.$$

Again, this assumption is satisfied at $n = 0$. Then, within the truncation T' the $(n+1)$-loop contributions to $\Gamma_{RT'}$ and Γ_{RT} differ by finite local terms, which we call $\Delta S_{n+1,T'}$, up to $\mathcal{E}_{\rm CDHD}$:

$$\Gamma_{RT'} = \Gamma_{RT} + \Delta S_{n+1,T'} + \mathcal{O}_{n+2} + \sum_{k=0}^{n+1} \mathcal{O}(\hbar^k) o(1/\Lambda_-^{T'-2k\sigma}) + \mathcal{E}_{\rm CDHD}.$$

Note that for $n < \bar\ell$, $\Delta S_{n+1,T'} = \mathcal{O}(\hbar^{n+1}) o(1/\Lambda_-^{T-2(n+1)\sigma})$. Now, replacing the renormalized action S_{RT} that defines Γ_{RT} with $S_{RT} - \Delta S_{n+1,T'}$, we cancel out $\Delta S_{n+1,T'}$ and promote the inductive assumption (7.24) from order n to order $n+1$. Iterating the procedure, we arrive at formula (7.24) with $n = \bar\ell'$. In the end, $\Gamma_{RT'}$ coincides with Γ_{RT} within the truncation T', up to $\mathcal{E}_{\rm CDHD}$. Finally, formula (7.23) ensures that Γ_{RT} is free of gauge anomalies within the truncation T'.

In other words, it is possible to modify the scheme s_T by fine-tuning the finite local counterterms so as to cancel the potentially anomalous contributions that belong to the class $\mathcal{A}_{>T}$. Since this conclusion applies to every class $\mathcal{A}_{>T}$, Theorem 1 follows.

VIII. STANDARD MODEL COUPLED TO QUANTUM GRAVITY

In this section we prove that the standard model coupled to quantum gravity satisfies the assumptions of the proof. In particular, although it does not satisfy the Kluberg-Stern–Zuber assumption (2.11), it satisfies assumption (III) of Sec. II C, since its basic action $S_{d\rm b}$ is cohomologically complete, and the group $G_{\rm nas}$ is compact. We also comment on the physical meaning of that assumption. We also show that the standard model coupled to quantum gravity

satisfies assumptions (IV) and (V) of Sec. II C, which concern the one-loop gauge anomalies.

We start by considering the class of four-dimensional Einstein–Yang-Mills theories that have classical actions of the form

$$S_{c\mathrm{EYM}} = \int \sqrt{|g|}\Big[-\frac{1}{2\kappa^2}(R+2\Lambda_{\mathrm{c}}) - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \mathcal{L}_\varphi(\varphi, D\varphi) + \mathcal{L}_\psi(\psi, D\psi) + \mathcal{L}_{\varphi\psi}(\varphi,\psi)\Big], \tag{8.1}$$

where $F^a_{\mu\nu}$ are the field strengths of the Abelian and non-Abelian Yang-Mills gauge fields, while $\mathcal{L}_\varphi$, $\mathcal{L}_\psi$, and $\mathcal{L}_{\varphi\psi}$ are the matter Lagrangians, which depend on the scalar fields φ, the fermions ψ, and their covariant derivatives $D\varphi$, $D\psi$, as specified by their arguments. Moreover, $\mathcal{L}_\varphi$ is at most quadratic in $D\varphi$, and $\mathcal{L}_\psi$ is at most linear in $D\psi$. The actions $\bar{S}_{d\mathrm{EYM}}$ and $S_{d\mathrm{EYM}}$ of formulas (2.3) and (2.8), built by taking $S_{c\mathrm{EYM}}$ as the classical action S_c, are known to satisfy the Kluberg-Stern–Zuber assumption (2.11) in two cases: when the Yang-Mills gauge group is semisimple and when there are no accidental symmetries [18]. When the Yang-Mills gauge group contains $U(1)$ factors and $S_{c\mathrm{EYM}}$ is invariant under accidental symmetries, there exist extra local solutions X of $(S_{d\mathrm{EYM}}, X) = 0$ that cannot be written in the form $(S_{d\mathrm{EYM}}, Y)$ with Y a local functional [18]. We denote them by $\mathcal{G}^{\mathrm{new}}_I$. They depend on the sources K, the $U(1)$ gauge fields, and the Noether currents associated with the accidental symmetries.

Consider first the standard model in flat space. We denote its basic action $S_{d\mathrm{b}}$ by $S_{d\mathrm{SM}}$. Clearly, $S_{d\mathrm{SM}}$ has the form (8.1) (with gravity switched off), but does not satisfy the Kluberg-Stern–Zuber assumption (2.11), because the Yang-Mills gauge group $SU(3)\times SU(2)\times U(1)_Y$ is not semisimple and $S_{d\mathrm{SM}}$ has accidental symmetries. One accidental symmetry is the conservation of the baryon number B. If the right-handed neutrinos are present and have Majorana masses, there are no other accidental symmetries. If the right-handed neutrinos are present, but do not have Majorana masses, there is an additional accidental symmetry, which is the conservation of the lepton number L. If the right-handed neutrinos are absent, the lepton numbers L_e, L_μ, and L_τ of each family are also conserved. The group of accidental symmetries is $U(1)^{I_{\max}}$, where $I_{\max} = 1$, 2, or 4, depending on the case.

The extra solutions X to the condition $(S_{d\mathrm{SM}}, X) = 0$ can be built as follows. It is well-known that the hypercharges of the matter fields are not uniquely fixed by the symmetries of the standard model Lagrangian. If we deform the standard model action $S_{d\mathrm{SM}}$ by giving arbitrary hypercharges to the matter fields, and later impose $U(1)_Y$ invariance, then one, two, or four arbitrary charges q_I ($I = 1, \ldots, I_{\max}$) survive (depending on the group of accidental symmetries), besides the overall $U(1)_Y$ charge. Call the deformed action $S_{d\mathrm{SM}q}(\Phi, K, q_I)$. Clearly, $S_{d\mathrm{SM}q}$ satisfies the master equation

$$(S_{d\mathrm{SM}q}, S_{d\mathrm{SM}q}) = 0 \tag{8.2}$$

in arbitrary D dimensions and for arbitrary values of the charges q_I. If we differentiate (8.2) with respect to each q_I, and then set the q_I to zero, we get

$$(S_{d\mathrm{SM}}, \mathcal{G}^{\mathrm{new}}_{I\mathrm{SM}}) = 0, \qquad \mathcal{G}^{\mathrm{new}}_{I\mathrm{SM}} \equiv \frac{\partial S_{d\mathrm{SM}q}}{\partial q_I}\bigg|_{q=0}.$$

The local functionals $\mathcal{G}^{\mathrm{new}}_{I\mathrm{SM}}(\Phi, K)$ depend explicitly on the sources K, because the charges q_I appear in the functional S_K of formula (2.4). It can be shown [18] that $\mathcal{G}^{\mathrm{new}}_{I\mathrm{SM}}$ cannot be written in the form $(S_{d\mathrm{SM}}, Y)$ for a local Y. This is why the Kluberg-Stern–Zuber requirement is not satisfied by the standard model.

The argument just given in flat space can be repeated for the standard model coupled to quantum gravity, with obvious modifications. Let us denote its basic action $S_{d\mathrm{b}}$ by $S_{d\mathrm{SMG}}$. It is built on the classical action $S_{c\mathrm{SMG}}$ of formula (2.1), which has the form (8.1). If we deform it into $S_{d\mathrm{SMG}q}(\Phi, K, q_I)$ and differentiate with respect to q_I, we find extra solutions X of $(S_{d\mathrm{SMG}}, X) = 0$ that cannot be written in the form $(S_{d\mathrm{SMG}}, Y)$ for a local Y. We denote them by $\mathcal{G}^{\mathrm{new}}_{I\mathrm{SMG}}(\Phi, K)$.

In principle, the invariants $\mathcal{G}^{\mathrm{new}}_{I\mathrm{SM}}$, or $\mathcal{G}^{\mathrm{new}}_{I\mathrm{SMG}}$, could be generated as counterterms by renormalization, because they satisfy $(S_{d\mathrm{SM}}, \mathcal{G}^{\mathrm{new}}_{I\mathrm{SM}}) = 0$, or $(S_{d\mathrm{SMG}}, \mathcal{G}^{\mathrm{new}}_{I\mathrm{SMG}}) = 0$. If this happened, however, we would have a big problem: some hypercharges would be allowed to run independently from one another and violate the conditions for the cancellation of gauge anomalies at one loop, required by assumption (IV). Indeed, it is easy to check that, in general, the deformation $S_{d\mathrm{SM}q}$ (and therefore also $S_{d\mathrm{SMG}q}$) is not compatible with the one-loop cancellation of the gauge anomalies [26].

In fact, in Sec. VII A it was shown that, if assumption (III) of Sec. II C holds, the extra invariants $\mathcal{G}^{\mathrm{new}}_I$, such as $\mathcal{G}^{\mathrm{new}}_{I\mathrm{SM}}$ or $\mathcal{G}^{\mathrm{new}}_{I\mathrm{SMG}}$, *are not* generated by renormalization. Indeed, they do not appear on the right-hand side of formula (7.7), which just contains the invariants $\mathcal{G}_i(\phi)$. Thus, the meaning of cohomological completeness is to ensure that renormalization has this key property.

To show that the standard model coupled to quantum gravity satisfies assumption (III), we lift the discussion to the extended theory $\check{S}_d$ of Sec. II and denote its basic action by $\check{S}_{d\mathrm{SMG}}$. It is easy to show that $\check{S}_{d\mathrm{SMG}}$ has no accidental symmetries, because it contains both the four fermion vertices and the vertex $(LH)^2$ that break B, L_e, L_μ, and L_τ. Indeed, in the parametrization (2.24) such vertices are not multiplied by parameters ζ belonging to the subsets s_-: the

coefficients ζ of the four fermion vertices are dimensionless, while the coefficient ζ of $(LH)^2$ has dimension one, as shown by formula (2.42). The functionals $\mathcal{G}_I^{\rm new}$ do not satisfy $(\check{S}_{d\rm SMG}, \mathcal{G}_I^{\rm new}) = 0$, and the theory with action $\check{S}_{d\rm SMG}$ cannot generate them as counterterms. By the results of Ref. [18], the action $\check{S}_{d\rm SMG}$, which has the form (8.1), satisfies the extended Kluberg-Stern–Zuber assumption (2.12); i.e. $S_{d\rm SMG}$ is cohomologically complete. The group $G_{\rm nas}$ of nonanomalous accidental symmetries of the action $S_{d\rm SMG}$ is certainly compact, so assumption (III) holds.

Let us now move to assumption (IV). Formula (5.12) tells us that the one-loop anomaly functional $\mathcal{A}_{\rm b}^{(1)}$ associated with the basic action $S_{d\rm SMG}$ of the standard model coupled to quantum gravity solves the equation $(S_{d\rm SMG}, \mathcal{A}_{\rm b}^{(1)}) = 0$. The most general solution to this condition reads

$$\mathcal{A}_{\rm b}^{(1)} = \mathcal{A}_{\rm nt} + (S_{d\rm SMG}, \mathcal{X}) \tag{8.3}$$

and is the sum of nontrivial terms $\mathcal{A}_{\rm nt}$ plus trivial terms $(S_{d\rm SMG}, \mathcal{X})$, where $\mathcal{X}$ is a local functional of ghost number zero. The nontrivial terms have been classified in Ref. [18]. They are (i) Bardeen terms [27]

$$\int \mathrm{d}^D x \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left[\partial_\mu C\left(A_\nu \partial_\rho A_\sigma + \frac{g}{2} A_\nu A_\rho A_\sigma\right)\right]$$

for non-Abelian Yang-Mills symmetries, where $C = C^{\dot a} T^{\dot a}$, $A_\mu = A_\mu^{\dot a} T^{\dot a}$, while $C^{\dot a}$, $A_\mu^{\dot a}$ are the non-Abelian Yang-Mills ghosts and gauge fields, respectively, and the index $\dot a$ runs on each simple subalgebra of the Yang-Mills Lie algebra; (ii) terms of the Bardeen type

$$\int \mathrm{d}^D x \varepsilon^{\mu\nu\rho\sigma} C_V (\partial_\mu V_\nu)(\partial_\rho V_\sigma),$$

$$\int \mathrm{d}^D x \varepsilon^{\mu\nu\rho\sigma} C^{\dot a} (\partial_\mu V_\nu)(\partial_\rho A_\sigma^{\dot a}),$$

$$\int \mathrm{d}^D x \varepsilon^{\mu\nu\rho\sigma} C_V F_{\mu\nu}^{\dot a} F_{\rho\sigma}^{\dot a},$$

$$\int \mathrm{d}^D x \varepsilon^{\mu\nu\rho\sigma} C_V R_{\mu\nu}^{\bar a \bar b} R_{\rho\sigma}^{\bar a \bar b},$$

$$\int \mathrm{d}^D x \varepsilon^{\mu\nu\rho\sigma} C_V R_{\mu\nu}^{\bar a \bar b} R_{\rho\sigma}^{\bar c \bar d} \varepsilon_{\bar a \bar b \bar c \bar d},$$

involving $U(1)$ gauge fields V_μ and/or $U(1)$ ghosts C_V; (iii) terms of the form $\int C_V \mathcal{L}$, where $\mathcal{L}$ is a Lagrangian density that depends only on the fields, is not a total derivative, and satisfies $(S_K, \int \mathcal{L}) = 0$; (iv) K-dependent extra terms $\mathcal{A}_{I\rm SMG}^{\rm new}$ of ghost number one, analogous to the extra terms $\mathcal{G}_{I\rm SMG}^{\rm new}$ of ghost number zero discussed above. The terms of class (iv) are absent unless the gauge group contains $U(1)$ factors and the theory has accidental symmetries. We recall that there are no Lorentz anomalies in four dimensions.

To study the anomalies $\mathcal{A}_{\rm nt}$ of Eq. (8.3) we can switch to the framework we prefer. A change of framework affects the finite local counterterms contained in the functional $\hat\Gamma_{\Lambda T\rm fin}^{(1)}$ of formula (5.8). As far as $\mathcal{A}_{\rm b}^{(1)}$ is concerned, formula (5.11) ensures that it only affects the functional $\mathcal{X}$ of (8.3).

Consider first the terms $\mathcal{A}_{\rm nt}$ that belong to the classes (i) and (ii). The most economic framework to study them is the standard dimensional regularization. For definiteness, we use a basis where all the fermionic fields are left handed, and we denote them by ψ_L. Associate a right-handed partner ψ_R with each ψ_L and extend the action $S_{d\rm SMG}$ by adding the correction

$$S_{LR}(\Phi) = \int \bar\psi_R i\tilde\gamma^\mu \partial_\mu \psi_L + \int \bar\psi_L i\tilde\gamma^\mu \partial_\mu \psi_R + \int \bar\psi_R i\tilde\gamma^\mu \partial_\mu \psi_R$$

to it, where the flat-space vielbein is used and $\tilde\gamma^\mu$ denote the standard γ matrices in D dimensions, which satisfy $\{\tilde\gamma^\mu, \tilde\gamma^\nu\} = 2\eta^{\mu\nu}$. Let $S_{\rm ext}(\Phi, K) = S_{d\rm SMG}(\Phi, K) + S_{LR}(\Phi)$ denote the extended action. Expanding around flat space as usual, the total kinetic terms of ψ_L and ψ_R are $\int i\bar\psi\tilde\gamma^\mu\partial_\mu\psi$, where $\psi = \psi_L + \psi_R$. Since ψ_R appears just in S_{LR}, no nontrivial one-particle irreducible diagrams with ψ_R external legs can be built, so the partners ψ_R decouple at $\varepsilon = 0$. Moreover, $S_{d\rm SMG}$ is gauge invariant, while S_{LR} is not, which means that $(S_{\rm ext}, S_{\rm ext})$ is cubic in the fields Φ. More precisely, $(S_{\rm ext}, S_{\rm ext})$ is bilinear in the fermions and linear in the ghosts. The anomaly functional is $\mathcal{A} = \langle (S_{\rm ext}, S_{\rm ext}) \rangle$. The nontrivial terms $\mathcal{A}_{\rm nt}$ of classes (i) and (ii) do not contain fermions, so they can only arise from the one-loop polygon diagrams that have $(S_{\rm ext}, S_{\rm ext})$ and gauge currents (including the energy-momentum tensor) at their vertices, and fermions circulating inside. It is well known [3] that the contributions of such diagrams vanish at $\varepsilon = 0$ in the standard model coupled to quantum gravity.

Next, consider the terms $\mathcal{A}_{\rm nt}$ of class (iii). They are anomalies of the global $U(1)_Y$ symmetry. To prove that they are absent, it is sufficient to choose a regularization technique that is globally $U(1)_Y$ invariant. Again, the standard dimensional regularization has this property, while the CD technique does not [because of the terms (2.13), which are of the Majorana type]. Finally, formula (5.11) ensures that the terms of class (iv) are not generated, because they depend on the sources K.

This proves that $\mathcal{A}_{\rm nt} = 0$; i.e. the basic action $S_{d\rm SMG}$ of the standard model coupled to quantum gravity satisfies assumption (IV). We also note that the arguments of Sec. VII A imply that the action S_d of the standard model coupled to quantum gravity, which is equal to $S_{d\rm SMG}$ plus corrections multiplied by powers of $1/\Lambda_-$, is also

cohomologically complete and satisfies the physical Kluberg-Stern–Zuber conjecture (7.8).

The absence of the terms of class (iv) is a general fact, not tied to the particular model we are considering. It can also be proved by lifting the discussion to $\check{S}_{d\mathrm{SMG}}$, where all accidental symmetries are broken. The one-loop anomaly functional $\check{\mathcal{A}}_{\mathrm{b}}^{(1)}$ of the theory with action $\check{S}_{d\mathrm{SMG}}$ satisfies $(\check{S}_{d\mathrm{SMG}}, \check{\mathcal{A}}_{\mathrm{b}}^{(1)}) = 0$ and can be decomposed as $\check{\mathcal{A}}_{\mathrm{b}}^{(1)} = \check{\mathcal{A}}_{\mathrm{nt}} + (\check{S}_{d\mathrm{SMG}}, \check{\mathcal{X}})$, where the nontrivial anomalous terms $\check{\mathcal{A}}_{\mathrm{nt}}$ can only belong to the classes (i)–(iii), and $\check{\mathcal{X}}$ is a local functional of Φ and K. The functional $\mathcal{A}_{\mathrm{b}}^{(1)}$ can be retrieved from $\check{\mathcal{A}}_{\mathrm{b}}^{(1)}$ by switching off the coefficients $\check{\lambda}$ and $\check{\eta}$ of the terms that break the nonanomalous accidental symmetries. This operation gives a result of the form (8.3), where $\mathcal{A}_{\mathrm{nt}}$ and $\mathcal{X}$ are equal to $\check{\mathcal{A}}_{\mathrm{nt}}$ and $\check{\mathcal{X}}$ at $\check{\lambda} = \check{\eta} = 0$, respectively. If, in addition, we average on the group G_{nas}, we can assume that $\mathcal{X}$ is invariant under G_{nas}. It follows that $\mathcal{A}_{\mathrm{nt}}$ is a linear combination of terms belonging to the classes (i)–(iii).

It remains to study assumption (V) of Sec. II C. If a functional $\mathcal{F}(\kappa\Phi)$ of ghost number one can be written in the form $(S_{d\mathrm{b}}, \mathcal{X})$, it clearly satisfies $(S_{d\mathrm{b}}, \mathcal{F}) = 0$. Then it also satisfies $(S_K, \mathcal{F}) = 0$, since $\mathcal{F}$ is K independent. We want to show that $\mathcal{F}$ can be written as (S_K, χ), where $\chi(\kappa\Phi)$ is a local functional of the fields Φ.

The most general solution of the problem $(S_K, \mathcal{F}) = 0$, when the gauge symmetries are diffeomorphisms, local Lorentz symmetry, and Abelian and non-Abelian Yang-Mills symmetries, is worked out in Ref. [28]. The functional $\mathcal{F}$ is the sum of nontrivial terms $\mathfrak{A}_{\mathrm{nt}}$ belonging to the classes (i)–(iii) listed above, plus trivial terms of the correct form $(S_K, \chi(\kappa\Phi))$. Combining this fact with $\mathcal{F} = (S_{d\mathrm{b}}, \mathcal{X})$, we obtain

$$\mathcal{F} = (S_{d\mathrm{b}}, \mathcal{X}) = \mathfrak{A}_{\mathrm{nt}} + (S_K, \chi).$$

Turning this equation around, we also get $\mathfrak{A}_{\mathrm{nt}} = (S_{d\mathrm{b}}, \mathcal{X}'')$, with $\mathcal{X}'' = \mathcal{X} - \chi$. In other words, the functional $\mathfrak{A}_{\mathrm{nt}}$ is trivial in the $S_{d\mathrm{b}}$ cohomology and nontrivial in the S_K cohomology. The results of Ref. [18] ensure that in four-dimensional Einstein–Yang-Mills theories that have an action of the form (8.1), this is impossible, unless $\mathfrak{A}_{\mathrm{nt}}$ vanishes. Thus, the standard model coupled to quantum gravity satisfies assumption (IV).

We stress again that assumptions (IV) and (V) are just needed to prove that the one-loop anomalies (4.3) of the HD theory are trivial in the S_K cohomology, which means that they have the form (5.14). The same result is more quickly implied by assumption (IV′) of Sec. II C. In several practical cases, it may be simpler to prove assumption (IV′), rather than assumptions (IV) and (V).

We conclude that the standard model coupled to quantum gravity satisfies all the assumptions made in this paper. Therefore, it is free of gauge anomalies to all orders in perturbation theory. In a generic framework, the Adler-Bardeen Theorem 1 of the Introduction tells us that the cancellation of gauge anomalies is nonmanifest, and can be enforced by fine-tuning finite local counterterms order by order. If we use the framework elaborated in this paper, Theorem 3 tells us that the cancellation is manifest within any given truncation and nonmanifest outside.

The arguments of this section apply with simple modifications to most standard model extensions, irrespectively of their gauge groups and accidental symmetries. When the other assumptions are met, it is sufficient to check that the gauge anomalies are trivial at one loop to infer that they can be canceled to all orders. It is also clear how to generalize the analysis of this section to theories living in spacetime dimensions different than four.

IX. CONCLUSIONS

In this paper we proved the Adler-Bardeen theorem for the cancellation of gauge anomalies in nonrenormalizable theories, which is the statement that there exists a subtraction scheme where the gauge anomalies cancel to all orders, when they are trivial at one loop. We assumed that the gauge symmetries are diffeomorphisms, local Lorentz symmetry, and Yang-Mills symmetries, and that the local functionals of vanishing ghost number satisfy a variant of the Kluberg-Stern–Zuber conjecture. In our approach, the cancellation is "almost manifest," which means that, given a truncation of the theory, once the gauge anomalies are canceled at one loop, they manifestly vanish from two loops onwards within the truncation, while outside the truncation their cancellation can be achieved by fine-tuning finite local counterterms. The truncation can contain arbitrarily many terms.

Although some arguments of the proof are technically involved, the key ideas are actually intuitive. The hardest part of the job is building the right framework. We used a regularization technique that combines a modified version of the dimensional regularization with a suitable higher-derivative gauge invariant regularization. This trick allows us to isolate the sources of potential anomalies, which are just one loop, from the nonanomalous sector of the theory. When the HD energy scale Λ is kept fixed, we have a superrenormalizable theory that satisfies the manifest Adler-Bardeen theorem to all orders in $\hbar$ by simple power counting arguments. When Λ is taken to infinity, the Λ divergences are subtracted by means of canonical transformations and redefinitions of parameters. At each step, the HD theory must be re-renormalized at Λ fixed, to subtract the newly generated divergences in ε. While doing so, it is possible to enforce the cancellation of gauge anomalies again by fine-tuning finite local counterterms.

The standard model coupled to quantum gravity satisfies the assumptions we have made, so it is free of gauge

anomalies to all orders. The theorem we have proved also applies to most extensions of the standard model, coupled to quantum gravity or not, and to a variety of other theories, including higher-derivative and Lorentz violating theories, in arbitrary dimensions.

Among the prospects for the future, we mention the generalization of the proof to supergravity. The complexity of local supersymmetry makes this task quite challenging, especially in the presence of scalar multiplets and when it is not known how to achieve closure off shell.

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