Rényi entropy of a free compact boson on a torus

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(Received 4 March 2015; published 18 May 2015)

In this paper, we reconsider the single-interval Rényi entropy of a free compact scalar on a torus. In this case, the contribution to the entropy could be decomposed into a classical part and a quantum part. The classical part includes the contribution from all the saddle points, while the quantum part is universal. After considering a different monodromy condition from the one in the literature, we reevaluate the classical part of the Rényi entropy. Moreover, we expand the entropy in the low-temperature limit and find the leading thermal correction term, which is consistent with the universal behavior suggested in [J. Cardy and C. P. Herzog, Phys. Rev. Lett. 112, 171603 (2014)]. Furthermore, we investigate the large-interval behavior of the entanglement entropy and show that the universal relation between the entanglement entropy and thermal entropy holds in this case.

DOI: 10.1103/PhysRevD.91.105013 PACS numbers: 11.10.Kk, 11.25.Tq

I. INTRODUCTION

Entanglement entropy is an important notion in many-body quantum systems [1,2]. For a bipartite system, the entanglement entropy of the subsystem A is defined to be the von Neumann entropy of the reduced density matrix

$$S_{EE} = -\mathrm{Tr}\rho_A \log \rho_A, \tag{1.1}$$

where the reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$ is obtained by tracing out the degrees of freedom in B complementary to A. When the total system is in the vacuum $\rho = |0\rangle\langle 0|$, the entanglement entropy of A equals that of its complement:

$$S_{EE}(A) = S_{EE}(B). \tag{1.2}$$

At a finite temperature, however, such equality does not hold anymore. The entanglement entropy has been studied in various condensed matter systems [see for example Ref. [3] (also the nice reviews [4–6])], and also in the context of AdS/CFT correspondence [7,8] (see Refs. [9,10] for nice reviews).

The computation of the entanglement entropy from its definition becomes a formidable task when the number of degrees of freedom in the system is huge. Especially in quantum field theory with infinite degrees of freedom, it is more convenient to compute the Rényi entropy via the replica trick [11]. The Rényi entropy is defined to be

$$S_n = -\frac{1}{n-1} \log \mathrm{Tr}\rho_A^n. \tag{1.3}$$

It is related to the entanglement entropy by the relation

$$S_{EE} = \lim_{n\to 1} S_n \tag{1.4}$$

if the analytic continuation of the limit is available. In field theory, the entropy is defined with respect to a spatial submanifold at a fixed time. In two-dimensional spacetime, the submanifold could be interval(s). However, after Wick rotation, the Euclideanized field theory is defined in a complex plane, and the Rényi entropy becomes

$$S_n = -\frac{1}{n-1} \log\left(\frac{Z_n}{Z_1^n}\right), \tag{1.5}$$

in which Z_n is the partition function on an n-sheeted Riemann surface resulting from the pasting of n complex planes along the branch cuts (intervals).

There has been a long history to study the entanglement entropy in quantum field theory. In dimensions higher than 2, the entanglement entropy has been found to obey the area law (for a nice review, see Ref. [12]). But we are only allowed to compute the entanglement entropy analytically in very restricted situations—for example, the one for a sphere in free field theory [13]. In $(1+1)$ dimensions, we can do better, especially for the field theory with conformal symmetry. For a 2D CFT on the complex plane, the Rényi entropy for one interval of length ℓ is universal and only depends on the central charge [14]

$$S_n = c\frac{n+1}{6n} \log \frac{l}{\epsilon}, \tag{1.6}$$

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where c is the central charge and ϵ is the UV cutoff. For more complicated cases—for example, the multi-interval at zero temperature or single interval on a circle at finite temperature—the entanglement entropy and Rényi entropy depend on the details of the CFT. In Refs. [15,16], the double-interval Rényi entropies for free bosons and the Ising model have been computed. In Refs. [17,18], the finite-temperature Rényi entropy for free fermions has been discussed. In Ref. [19], the finite-temperature Rényi entropy for free bosons has been studied. Moreover, the treatment on the W functions in the partition functions of free bosons has been improved in Ref. [20].

In this paper, we reconsider the Rényi entropy of a free compact scalar on a torus. Our motivation is twofold. First of all, in our study on the free noncompact scalar case [20], we noticed at least two novel features, originating from the continuous spectrum of the theory. One is that the leading thermal correction at low temperature in this case takes a form different from the one suggested in Ref. [21]. This is because the noncompact scalar has a degenerate vacuum, while the universal thermal correction found in Ref. [21] was based on the assumption that the CFT has a mass gap. The other one is that the universal relation between the large-interval entanglement entropy and thermal entropy does not hold anymore [20,22]. Since the noncompact free scalar could be taken as the large-radius limit of the compact free scalar, it would be interesting to study the Rényi entropy of a free compact scalar, which has a discrete spectrum. On the other hand, even though the Rényi entropy of a free compact scalar has been computed in Ref. [19], the detailed discussion on the low-temperature or large-interval expansion has not been worked out. Moreover, we find that the classical part of the partition function actually depends on a different monodromy condition from the one in Ref. [19]. With the corrected classical partition function, we compute the Rényi entropy and do expansion in several limits, and rediscover the expected universal behaviors.

In the next section, we revaluate the Rényi entropy for free compact bosons. We consider a slightly different monodromy condition to read the classical part of the partition function. Then, in Sec. III, we discuss a low-temperature limit, and expand the Rényi entropy with respect to $e^{-2\pi\beta}$. We find that the leading order is now consistent with the universal thermal correction suggested in Ref. [21]. In Sec. IV, we investigate the small-interval and large-interval limits of the Rényi entropy, and prove the universal relation between entanglement and thermal entropies. We collect some technical details on the W functions and the theta functions in the Appendix.

II. COMPACT BOSON RÉNYI ENTROPY

For a free boson, the partition function on a Riemann surface can be decomposed into classical and quantum parts,

$$Z = Z_{\text{quantum}} Z_{\text{classical}}. \tag{2.1}$$

The classical part gets contributions from all the saddle points carrying different monodromy conditions:

$$Z_{\text{classical}} = \sum_{\mathcal{M}} e^{-S_E(\mathcal{M})}. \tag{2.2}$$

For the quantum correction, we need to consider the perturbations around the classical saddle point and evaluate their contribution to the partition function. In general, for different classical solutions their quantum corrections are different, but in the case of free bosons the quantum correction is universal, so the partition function can be decomposed into the classical and quantum parts as in (2.1).

In this section, we compute the compact scalar partition functions on an n-sheeted torus connected by a branch cut. The free complex scalar is compactified on a square torus of radius R. It obeys the boundary condition

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = X(z,\bar{z}) + 2\pi R(m_1 + i m_2),\quad m_1, m_2 \in Z. \tag{2.3}$$

The quantum part of the partition function is equal to [19,20]

$$Z_{n,\text{quantum}} = \frac{1}{|\eta(\tau)|^{4n}} \prod_{k=0}^{n-1} \frac{1}{|W_1^{1(k)} W_2^{2(k)}|} \left(\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z_1 - z_2|\tau)}\right)^{\frac{1}{6}n(1-\frac{1}{n^2})} \times \left(\frac{\bar{\vartheta}_1'(0|\bar{\tau})}{\bar{\vartheta}_1'(\bar{z}_1 - \bar{z}_2|\bar{\tau})}\right)^{\frac{1}{6}n(1-\frac{1}{n^2})}, \tag{2.4}$$

where we have already used the modular symmetry to simplify the expression. The definitions of the W functions and the theta function can be found in the Appendix. This result could be derived by using the Ward identity on the correlation functions involving the twist operators [23].

For the classical part, we need to find all of the classical solutions and calculate their action. For convenience, we redefine the fields as

$$X^{(t,k)}(z,\bar{z}) = \sum_{j=0}^{n-1} e^{\frac{2\pi i}{n} jk} X^{(j)}(z,\bar{z}), \tag{2.5}$$

where $0 \le k < n$. For each redefined field $X^{(t,k)}(z,\bar{z})$, when the argument goes around the branch point z_1 or z_2, it gets an extra phase $e^{2\pi i \frac{k}{n}}$ or $e^{-2\pi i \frac{k}{n}}$, respectively. Moreover, the boundary condition (2.3) changes to

$$X^{(t,k)}(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = e^{2\pi i \frac{k}{n}} X^{(t,k)}(z,\bar{z}) + v^{(k)}, \tag{2.6}$$

where $v^{(k)}$ is a vector in the lattice $\Lambda_{\frac{k}{n}}$ defined by

$$\Lambda_{\frac{k}{n}} \equiv \left\{ 2\pi R \sum_{j=0}^{n-1} e^{\frac{2\pi i}{n} jk} (m_{j,1} + i m_{j,2}),\quad m_{j,1}, m_{j,2} \in Z \right\}. \tag{2.7}$$

These boundary conditions induce the following monodromy conditions:

$$\oint_{\gamma_a} dz\partial X^{(t,k)}(z) + \oint_{\gamma_a} d\bar{z}\,\bar{\partial}\,X^{(t,k)} = v_a^{(k)},$$
$$\oint_{\gamma_a} dz\partial \bar{X}^{(t,k)}(z) + \oint_{\gamma_a} d\bar{z}\,\bar{\partial}\,\bar{X}^{(t,k)} = \bar{v}_a^{(k)}, \tag{2.8}$$

where the γ_a's are the two cycles of worldsheet torus. In terms of cut differentials

$$\omega_1^{(k)}(z) = \vartheta_1(z-z_1|\tau)^{-(1-\frac{k}{n})}\vartheta_1(z-z_2|\tau)^{-\frac{k}{n}}\vartheta_1 \times\left(z-\left(1-\frac{k}{n}\right)z_1-\frac{k}{n}z_2|\tau\right),$$
$$\omega_2^{(k)}(z) = \vartheta_1(z-z_1|\tau)^{-\frac{k}{n}}\vartheta_1(z-z_2|\tau)^{-(1-\frac{k}{n})}\vartheta_1 \times\left(z-\frac{k}{n}z_1-\left(1-\frac{k}{n}\right)z_2|\tau\right), \tag{2.9}$$

the classical solutions can be written as

$$\partial X^{(t,k)} = a^{(k)}\omega_1^{(k)}(z),$$
$$\bar{\partial} X^{(t,k)} = b^{(k)}\bar{\omega}_2^{(k)}(\bar{z}),$$
$$\partial \bar{X}^{(t,k)} = \tilde{a}^{(k)}\omega_2^{(k)}(z),$$
$$\bar{\partial} \bar{X}^{(t,k)} = \tilde{b}^{(k)}\bar{\omega}_1^{(k)}(\bar{z}). \tag{2.10}$$

Solving the monodromy conditions, we get

$$a^{(k)} = \frac{W_2^{2(k)}v_1 - W_1^{2(k)}v_2}{\det W^{(k)}}, \qquad b^{(k)} = \frac{-W_2^{1(k)}v_1 + W_1^{1(k)}v_2}{\det W^{(k)}},$$
$$\tilde{a}^{(k)} = \frac{\bar{W}_1^{1(k)}\bar{v}_2 - \bar{W}_2^{1(k)}\bar{v}_1}{\det \bar{W}^{(k)}}, \qquad \tilde{b}^{(k)} = \frac{\bar{W}_2^{2(k)}\bar{v}_1 - \bar{W}_1^{2(k)}\bar{v}_2}{\det \bar{W}^{(k)}}, \tag{2.11}$$

where the W functions are defined to be the integral of the cut differentials along different cycles. The definition and properties of the W functions can be found in the Appendix. With these results, the classical action for $X^{(t,k)}$ is just

$$S^{(k)} = \frac{1}{4\pi n\alpha'}\frac{1}{|W_1^{1(k)}W_2^{2(k)}|}(|v_1^{(k)}|^2|W_2^{2(k)}|^2 + |v_2^{(k)}|^2|W_1^{1(k)}|^2). \tag{2.12}$$

The above discussion is the same as the one in Ref. [19]. However, in Ref. [19], the classical action is further simplified using the relation between the W functions. Actually, it was suggested that the W functions are related by

$$W_2^2 = \tau W_1^1 = i\beta W_1^1.$$

This relation turns out to be incorrect, as shown by the direct expansions in the short-interval and large-interval limits in Ref. [20]. Furthermore, the lattice translations v_1 and v_2 were determined in Ref. [19] along the way in Ref. [15]. But notice that we are considering an n-sheeted torus, which is very different from the n-sheeted Riemann surface gotten from a two-interval complex plane. Actually, the translation vectors are much simpler in the n-sheeted torus case. On this Riemann surface, the spatial cycles and time cycles on n replicas build the canonical cycles, and all of the cycles on the Riemann surface can be generated by these cycles. Once we fix the monodromy around the canonical cycles, we can get all of the monodromy on the Riemann surface.

For the n-sheeted torus, the monodromy condition can be fixed as

$$\Delta_1 X = 2\pi R m_j, \qquad \Delta_2 X = 2\pi R n_j, \tag{2.13}$$

where

$$m_j = m_j^{(1)} + i m_j^{(2)}, \qquad n_j = n_j^{(1)} + i n_j^{(2)} \tag{2.14}$$

are complex integers. If we transform into $X^{(t,k)}$ bases, then

$$v_1^{(k)} = 2\pi R\sum_{j=0}^{n-1} e^{2\pi i j\frac{k}{n}} m_j,$$
$$v_2^{(k)} = 2\pi R\sum_{j=0}^{n-1} e^{2\pi i j\frac{k}{n}} n_j. \tag{2.15}$$

Taking these relations into (2.12) and summing over all the saddle points, we get

$$S_{\rm cl} = \sum_k S^{(k)} = \sum_{r=1,2}\frac{\pi R^2}{n\alpha'}\left(\sum_k\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot m_j^{(r)}m_{j'}^{(r)} + \sum_k\left|\frac{W_1^{1(k)}}{W_2^{2(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot n_j^{(r)}n_{j'}^{(r)}\right). \tag{2.16}$$

For convenience, let us define

$$A_{jj'} = \sum_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\cos 2\pi(j-j')\frac{k}{n},$$
$$B_{jj'} = \sum_{k=0}^{n-1}\left|\frac{W_1^{1(k)}}{W_2^{2(k)}}\right|\cos 2\pi(j-j')\frac{k}{n}. \tag{2.17}$$

We can diagnose the two matrices with the same matrix

$$U_{jk} = e^{2\pi i j \frac{k}{n}} \tag{2.18}$$

and find their different eigenvalues

$$\begin{aligned} U^{-1}\cdot A\cdot U &= \mathrm{diag}\left(n\left|\frac{W_2^{2(0)}}{W_1^{1(0)}}\right|, n\left|\frac{W_2^{2(1)}}{W_1^{1(1)}}\right|, \ldots n\left|\frac{W_2^{2(n-1)}}{W_1^{1(n-1)}}\right|\right), \\ U^{-1}\cdot B\cdot U &= \mathrm{diag}\left(n\left|\frac{W_1^{1(0)}}{W_2^{2(0)}}\right|, n\left|\frac{W_1^{1(1)}}{W_2^{2(1)}}\right|, \ldots n\left|\frac{W_1^{1(n-1)}}{W_2^{2(n-1)}}\right|\right), \end{aligned} \tag{2.19}$$

so

$$B^{-1} = \frac{1}{n^2}A. \tag{2.20}$$

We can calculate the classical contribution to the partition function:

$$\begin{aligned} Z_{\mathrm{classical}} = &\sum_{m_j^{(r)},n_j^{(r)}} \exp\left[-\frac{\pi R^2}{\alpha' n}\sum_{r=1,2}\sum_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot m_j^{(r)}m_{j'}^{(r)}\right. \\ &\left.-\frac{\pi R^2}{\alpha' n}\sum_{r=1,2}\sum_{k=0}^{n-1}\left|\frac{W_1^{1(k)}}{W_2^{2(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot n_j^{(r)}n_{j'}^{(r)}\right] \\ = &\left(\sum_{m_j,n_j}\exp\left[-\frac{\pi R^2}{\alpha' n}\sum_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot m_j m_{j'}\right.\right. \\ &\left.\left.-\frac{\pi R^2}{\alpha' n}\sum_{k=0}^{n-1}\left|\frac{W_1^{1(k)}}{W_2^{2(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot n_j n_{j'}\right]\right)^2 \\ = &\frac{\alpha'^n}{R^{2n}}\prod_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\left(\sum_{m_j,p_j}\exp\left\{-\frac{\pi R^2}{\alpha' n}\sum_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot m_j m_{j'}\right.\right. \\ &\left.\left.-\frac{\alpha'\pi}{R^2 n}\sum_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot p_j p_{j'}\right\}\right)^2. \end{aligned} \tag{2.21}$$

For the last equation, we have used higher-dimensional Poisson resummation. It is remarkable that the classical partition function depends explicitly on the W functions and then on the interval. This is very different from the result in Ref. [19]. Combined with the quantum part, the full partition function reads

$$\begin{aligned} Z_n &= Z_{\mathrm{classical}}Z_{\mathrm{quantum}} \\ &= c_n\frac{1}{|\eta(\tau)|^{4n}}\prod_{k=0}^{n-1}\frac{1}{|W_1^{1(k)}|^2}\left(\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z_1-z_2|\tau)}\right)^{\frac{1}{6}n(1-\frac{1}{n^2})}\left(\frac{\bar\vartheta_1'(0|\bar\tau)}{\bar\vartheta_1(\bar z_1-\bar z_2|\bar\tau)}\right)^{\frac{1}{6}n(1-\frac{1}{n^2})} \\ &\cdot\left(\sum_{m_j,p_j}\exp\left\{-\frac{\pi R^2}{\alpha' n}\sum_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot m_j m_{j'}\right.\right. \\ &\left.\left.-\frac{\alpha'\pi}{R^2 n}\sum_{k=0}^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right|\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\cos 2\pi(j-j')\frac{k}{n}\cdot p_j p_{j'}\right\}\right)^2, \end{aligned} \tag{2.22}$$

where other coefficients have been absorbed into c_n, and the function η is defined as

$$\eta(\tau) = e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}). \tag{2.23}$$

This is the main result of this paper.

III. LOW-TEMPERATURE LIMIT

For the single-interval Rényi entropy at finite temperature, there is a universal thermal correction coming from the lowest excitation [21]

$$\delta S_n = \frac{gn}{1-n}\left(\frac{\sin\frac{\pi l}{L}}{n\sin\frac{\pi l}{nL}}\right)^{2\Delta} e^{-2\pi\Delta\beta/L} + \cdots, \tag{3.1}$$

where Δ is the scaling dimension of the excitations and g is their degeneracy. This relation has been checked to be true for the vacuum module in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [24]. However, it has been shown to break down for the noncompact free scalar, which has a continuous spectrum. In this section, we study the low-temperature limit of the partition function and Rényi entropy of the compact free scalar, and we check this relation.

For simplicity, we assume $\frac{\alpha'}{R^2} < 1$, which means that the momentum mode has lower dimension than the descendants of the vacuum module. We expand the results with respect to $q = e^{-2\pi\beta}$:

$$\eta(\tau) = q^{\frac{1}{24}}(1+O(q)), \tag{3.2}$$

$$\vartheta_1'(0) = 2\pi q^{\frac{1}{8}}(1+O(q)), \tag{3.3}$$

$$\vartheta_1(z_1 - z_2|\tau) = 2q^{\frac{1}{8}}\sin\pi(z_1-z_2)(1+O(q)), \tag{3.4}$$

$$\frac{W_2^{2(k)}}{W_1^{1(k)}} = i\beta + \int_{z_1}^{z_2} dt \frac{2i(\sin\frac{\pi k}{n}(t-z_1))(\sin\pi(1-\frac{k}{n})(t-z_1))}{\sin\pi(t-z_1)} + O(q). \tag{3.5}$$

For the summation in (2.22), the leading and next-leading contributions come from $m_j = p_j = 0$ and $m_j = 0$, $p_i = \pm 1$, $p_j = 0$ for $j \neq i$. When all of m_j and p_j are zero, then the exponential is 1. Only when $p_i = \pm 1$, the terms on the exponential contribute

$$-\frac{\alpha'\pi}{R^2 n}\sum_k^{n-1}\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right| = -\frac{\alpha'\pi}{R^2 n} \times \sum_k^{n-1}\left(\beta + \int_{z_1}^{z_2} dt \frac{2\sin\frac{\pi k}{n}(t-z_1)\sin\pi(1-\frac{k}{n})(t-z_1)}{\sin\pi(t-z_1)}\right) + O(q)$$
$$= -\frac{\alpha'\pi}{R^2}\beta - \frac{\alpha'}{R^2}\log\frac{n\sin\frac{\pi}{n}(z_2-z_1)}{\sin\pi(z_2-z_1)} + O(q). \tag{3.6}$$

Thus, the leading and subleading contributions in the summation give

$$1 + 2ne^{-\frac{\pi\alpha'\beta}{R^2}}\left(\frac{\sin\pi(z_2-z_1)}{n\sin\frac{\pi}{n}(z_2-z_1)}\right)^{\frac{\alpha'}{R^2}} + O(e^{\frac{2\pi\alpha'\beta}{R^2}}), \tag{3.7}$$

and the partition function is approximately

$$Z_n = c_n\frac{1}{q^{\frac{n}{6}}}\left(\frac{\pi}{\sin\pi l}\right)^{\frac{1}{3}n(1-\frac{1}{n^2})} \times \left(1 + 4ne^{-\frac{\pi\alpha'\beta}{R^2}}\left(\frac{\sin\pi l}{n\sin\frac{\pi}{n}l}\right)^{\frac{\alpha'}{R^2}} + O(e^{\frac{2\pi\alpha'\beta}{R^2}})\right) \tag{3.8}$$

and

$$S_n = -\frac{1}{n-1}\log c_n + \frac{n+1}{3n}\log\left(\frac{\pi}{\sin\pi l}\right) - \frac{1}{n-1}4n\left[\left(\frac{\sin\pi l}{n\sin\frac{\pi}{n}l}\right)^{\frac{\alpha'^2}{R}} - 1\right]e^{-\frac{\pi\alpha'\beta}{R^2}} + O(e^{-\frac{2\pi\alpha'\beta}{R^2}}). \tag{3.9}$$

This result is in consistent with the universal behavior suggested in Ref. [21]. For the complex scalar, the central charge is 2. There are fourfold degeneracies for the lowest excitation states, with the conformal dimension $(\frac{\alpha'}{4R^2}, \frac{\alpha'}{4R^2})$.

IV. RELATION BETWEEN THERMAL ENTROPY AND ENTANGLEMENT ENTROPY

In Ref. [17], it was suggested that when the interval becomes very large, the entanglement entropy and thermal entropy could be related by[1]

$$S_{th} = \lim_{l\to 0}(S_{EE}(1-l) - S_{EE}(l)). \tag{4.1}$$

In Refs. [20,22], this relation has been proved for a general CFT with a discrete spectrum. For the free compact scalar case at hand, it has a discrete spectrum, so it should satisfy the relation. We can prove this directly by expanding the W functions.

[1]We have set the spatial length of the worldsheet to be unit.

Since it is not obvious how to take the $n \to 1$ limit on the Rényi entropy, we do not have an explicit form of the entanglement entropy. Instead, we first try to study the $l \to 0$ limit and then take the $n \to 1$ limit. We assume that taking the limits of $n \to 1$ and $l \to 0$ is commutable.

For a small interval, we have

$$W_1^{1(k)} = 1 + O(z_1 - z_2), \qquad W_2^{2(k)} = i\beta + O(z_1 - z_2), \tag{4.2}$$

and thus

$$\begin{aligned} Z_n &= c_n l^{-\frac{1}{3}n(1-\frac{1}{n^2})} \frac{1}{|\eta(\tau)|^{4n}} \cdot \left(\sum_{m_j, p_j} \exp\left[-\frac{\pi R^2}{\alpha'} m_j^2 - \frac{\alpha' \pi}{R^2} p_j^2 \right] + O(z_1 - z_2) \right)^2 \\ &= c_n l^{-\frac{1}{3}n(1-\frac{1}{n^2})} \left[\left(\frac{1}{|\eta(\tau)|^2} \sum_{m,p} \exp\left[-\frac{\pi R^2}{\alpha'} m^2 - \frac{\alpha' \pi}{R^2} p^2 \right] \right)^{2n} + O(l) \right] \\ &= c_n l^{-\frac{1}{3}n(1-\frac{1}{n^2})} [Z_1^n + O(l)]. \end{aligned} \tag{4.3}$$

For a large interval, we first analyze the summation terms in the exponential. We only extract the leading terms with respect to $z_1 - z_2$. Since the eigenvalues for the matrix A are $n|\frac{W_2^{2(k)}}{W_1^{1(k)}}|, k = 0, \ldots n-1$, in the limit $z_1 \to z_2$, all the terms are suppressed but the terms with $m_1 = m_2 = \ldots m_n = m, p_1 = p_2 = \ldots p_n = p$. This fact is consistent with the observation in Refs. [20,22] that the excitations on different replicas should be the same in order to give nonvanishing contributions in the large-interval limit. Therefore, the summation yields

$$\sum_{m,p} \exp\left[-\frac{\pi R^2 n\beta}{\alpha'} m^2 - \frac{\alpha' n\beta}{R^2} p^2 \right]. \tag{4.4}$$

In this case, when $z_1 \to z_2$, the prefactor before the exponential goes to

$$c_n l^{-\frac{1}{3}n(1-\frac{1}{n^2})} \frac{1}{|\eta(\tau)|^{4n}} \prod_{k=0}^{n-1} \left| \frac{2\sin\frac{\pi k}{n} \eta(\tau)^3}{\vartheta(-\frac{k}{n}|\tau)} \right|^2. \tag{4.5}$$

This factor could be simplified more:

$$\begin{aligned} &\frac{1}{|\eta(i\beta)|^{4n}} \prod_{k=1}^{n-1} \left| \frac{2\sin\frac{\pi k}{n} \eta(i\beta)^3}{\vartheta_1(-\frac{k}{n}|i\beta)} \right|^2 \\ &= \prod_{k=1}^{n-1} \left(2\sin\frac{\pi k}{n} \right)^2 \left(\beta^{-\frac{1}{2}} \eta\left(\frac{i}{\beta}\right) \right)^{2n-6} \\ &\quad \times \prod_{k=1}^{n-1} \left| \frac{1}{\beta^{-\frac{1}{2}} e^{-\frac{k^2}{n^2\beta}} \vartheta_1(-\frac{k}{i\beta n}|\frac{i}{\beta})} \right|^2 \\ &= \prod_{k=1}^{n-1} \left(2\sin\frac{\pi k}{n} \right)^2 \beta^2 \frac{1}{\eta(\frac{i}{n\beta})^4} \\ &= \frac{1}{n^2} \prod_{k=1}^{n-1} \left(2\sin\frac{\pi k}{n} \right)^2 \frac{1}{\eta(in\beta)^4}. \end{aligned} \tag{4.6}$$

As

$$\frac{1}{n^2} \prod_{k=1}^{n-1} \left(2\sin\frac{\pi k}{n} \right)^2 = 1, \tag{4.7}$$

we find that for the large interval

$$Z_n = c_n l^{\frac{1}{3}n(1-\frac{1}{n^2})} (Z_1[n\beta] + O(l^\lambda)), \tag{4.8}$$

where $\lambda < 1$ and $O(l^\lambda)$ terms come from the contributions of the primary fields in the operator product expansion of two twist operators. Now it is easy and straightforward to prove the relation (4.1) between the thermal entropy and the entanglement entropy [20,22]. Actually,

$$\begin{aligned} &\lim_{l\to 0} (S_{EE}(1-l) - S_{EE}(l)) \\ &= -\lim_{n\to 1} \frac{1}{n-1} (\log Z_1[n\beta] - n\log Z_1[\beta]) \\ &= \log Z[\beta] - \frac{1}{\beta} \frac{Z'[\beta]}{Z[\beta]} \\ &= S_{\mathrm{th}}. \end{aligned}$$

This relation should still hold in the large-volume limit $R \to \infty$, under which the theory becomes a noncompact free scalar. In Ref. [20], we showed that the left-hand side of the relation (4.1) for a noncompact free scalar involves a log-logarithmic term by expanding the partition function directly. Such divergence comes from the continuous spectrum of the theory. The discrepancy between the two different methods is due to a subtle order-of-limits issue. If we regularize the noncompact theory by taking it to be the large-volume limit of the compact scalar, then the relation (4.1) is recovered. On the contrary, if we take the large-volume limit first and then discuss the large-interval limit, we find the log-logarithmic divergence.

ACKNOWLEDGMENTS

The work was in part supported by NSFC Grants No. 11275010, No. 11335012, and No. 11325522.

APPENDIX: *W* FUNCTIONS

In this appendix, we list some useful results for the W functions and study their properties in various limits. The W functions are defined as the line integral of the cut differentials:

$$\begin{aligned} W_1^{1(k)} &= \int_0^1 dz \vartheta_1(z-z_1|\tau)^{-(1-\frac{k}{n})}\vartheta_1(z-z_2|\tau)^{-\frac{k}{n}}\vartheta_1\left(z-\left(1-\frac{k}{n}\right)z_1-\frac{k}{n}z_2|\tau\right), \\ W_1^{2(k)} &= \int_0^1 d\bar{z}\bar{\vartheta}_1(\bar{z}-\bar{z}_1|\tau)^{-\frac{k}{n}}\bar{\vartheta}_1(\bar{z}-\bar{z}_2|\tau)^{-(1-\frac{k}{n})}\bar{\vartheta}_1\left(\bar{z}-\frac{k}{n}\bar{z}_1-\left(1-\frac{k}{n}\right)\bar{z}_2|\tau\right), \\ W_2^{1(k)} &= \int_0^\tau dz \vartheta_1(z-z_1|\tau)^{-(1-\frac{k}{n})}\vartheta_1(z-z_2|\tau)^{-\frac{k}{n}}\vartheta_1\left(z-\left(1-\frac{k}{n}\right)z_1-\frac{k}{n}z_2|\tau\right), \\ W_2^{2(k)} &= \int_0^{\bar{\tau}} d\bar{z}\bar{\vartheta}_1(\bar{z}-\bar{z}_1|\tau)^{-\frac{k}{n}}\bar{\vartheta}_1(\bar{z}-\bar{z}_2|\tau)^{-(1-\frac{k}{n})}\bar{\vartheta}_1\left(\bar{z}-\frac{k}{n}\bar{z}_1-\left(1-\frac{k}{n}\right)\bar{z}_2|\tau\right). \end{aligned} \tag{A1}$$

As we have chosen the modular parameter to be purely imaginary $\tau = i\beta$, the W functions are related by

$$W_1^{1*} = W_1^1 = W_1^2, \qquad W_2^{1*} = -W_2^1 = W_2^2. \tag{A2}$$

The theta function is defined by

$$\vartheta_1(u|\tau) \equiv 2e^{\frac{1}{4}\pi i\tau}\sin\pi u\prod_{m=1}^{\infty}(1-q^m)(1-e^{2\pi iu}q^m)(1-e^{-2\pi iu}q^m), \tag{A3}$$

where

$$q = e^{2\pi i\tau}. \tag{A4}$$

1. Low-temperature expansion

The low-temperature expansion for $W_1^{1(k)}$ is

$$\begin{aligned} W_1^{1(k)} &= \left(\int_0^{z_1}+\int_{z_1}^{z_2}+\int_{z_2}^{1}\right)du\vartheta_1(z-z_1|\tau)^{-(1-\frac{k}{n})}\vartheta_1(z-z_2|\tau)^{-\frac{k}{n}}\vartheta_1\left(z-\left(1-\frac{k}{n}\right)z_1-\frac{k}{n}z_2|\tau\right) \\ &= \int_{z_2-1}^{z_1} dz\vartheta_1(z-z_1|\tau)^{-(1-\frac{k}{n})}(\vartheta_1(z-(z_2-1)|\tau)e^{\pi i})^{-\frac{k}{n}}\vartheta_1\left(z-\left(1-\frac{k}{n}\right)z_1-\frac{k}{n}z_2|\tau\right) \\ &= -\frac{e^{-\frac{\pi ik}{n}}}{1-e^{-\frac{2\pi ik}{n}}}\oint_A dz(\sin\pi(z-z_1))^{-(1-\frac{k}{n})}(\sin\pi(z-(z_2-1)))^{-\frac{k}{n}}\sin\pi\left(z-\left(1-\frac{k}{n}\right)z_1-\frac{k}{n}z_2\right)+O(q) \\ &= \frac{e^{-\pi i\frac{k}{n}}}{-1+e^{-2\pi i\frac{k}{n}}}\left(\oint_\infty \frac{du}{2\pi iu}e^{-\frac{k}{n}\pi i}(u-u_1)^{-(1-\frac{k}{n})}(u-u_2)^{-\frac{k}{n}}(u-u_1^{(1-\frac{k}{n})}u_2^{\frac{k}{n}})\right. \\ &\quad \left.-\oint_0 \frac{du}{2\pi iu}e^{-\frac{k}{n}\pi i}(u-u_1)^{-(1-\frac{k}{n})}(u-u_2)^{-\frac{k}{n}}(u-u_1^{(1-\frac{k}{n})}u_2^{\frac{k}{n}})\right)+O(q) \\ &= \frac{e^{-\pi i\frac{k}{n}}}{-1+e^{-2\pi i\frac{k}{n}}}\left(\oint_\infty e^{-\frac{k}{n}\pi i}(1-(e^{\pi i}e^{2\pi iz_1})^{-(1-\frac{k}{n})}(e^{\pi i}e^{2\pi i(z_2-1)})^{-\frac{k}{n}})\right)+O(q) \\ &= 1+O(q). \end{aligned} \tag{A5}$$

In the third equation, the contour integral is along the A cycle in Fig. 1(a). In the fourth equation, we have used a conformal transformation

$$u = e^{2\pi iz}, \tag{A6}$$

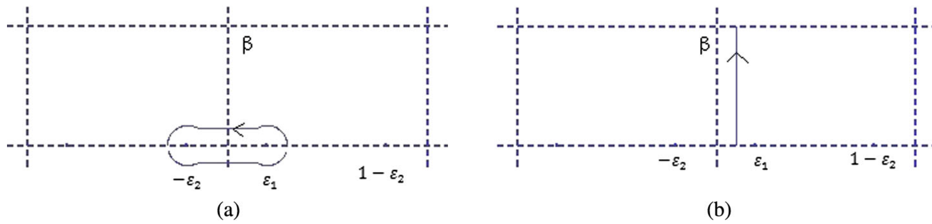

FIG. 1 (color online). There are two contour integral paths. (a) A cycle. (b) B cycle.

and in the fifth equation we have changed the integral contour to the one surrounding the infinity and the origin. For $W_2^{2(k)}$, we have the expansion

$$\begin{aligned}\bar{W}_2^{2(k)} &= \int_0^{i\beta} dz \vartheta(z-z_1|\tau)^{-\frac{k}{n}}\vartheta(z-z_2|\tau)^{-(1-\frac{k}{n})}\vartheta\left(z-\left(\frac{k}{n}z_1+\left(1-\frac{k}{n}\right)z_2\right)|\tau\right)\\ &= \int_0^{i\beta} dz((1-e^{2\pi i(z-z_1)})(1-qe^{-2\pi i(z-z_1)}))^{-\frac{k}{n}}((1-e^{2\pi i(z-z_2)})(1-qe^{-2\pi i(z-z_2)}))^{-(1-\frac{k}{n})}\\ &\quad\times((1-e^{2\pi i(z-(\frac{k}{n}z_1+(1-\frac{k}{n})z_2))})(1-qe^{-2\pi i(z-(\frac{k}{n}z_1+(1-\frac{k}{n})z_2))}))+O(q),\end{aligned} \tag{A7}$$

where the integral should be along the B cycle in Fig. 1(b). We can analytically expand the equations with respect to $e^{2\pi iz}$, and the summations in the expansions still converge. There appear several kinds of terms in the expansion:

$$1, \qquad e^{2\pi miz}, \qquad q^m e^{-2\pi miz}, \qquad q^m e^{2\pi i(-m+n)z}. \tag{A8}$$

After being integrated, the first term gives $i\beta$. The second and third terms give

$$\int_0^{i\beta} e^{2\pi miz} = \frac{1}{2\pi mi} e^{2\pi miz}|_0^{i\beta} = \frac{1}{2\pi mi}(q^m-1), \tag{A9}$$

$$\begin{aligned}\int_0^{i\beta} q^m e^{-2\pi miz} &= \frac{1}{-2\pi mi} q^m e^{-2\pi miz}|_0^{i\beta}\\ &= \frac{1}{-2\pi mi} q^m(q^{-m}-1),\end{aligned} \tag{A10}$$

$$\int_0^{i\beta} q^m e^{2\pi i(-m+n)z} = O(q). \tag{A11}$$

If we are interested in the leading contribution with respect to q, we only need to consider the first three kinds of terms. In the end, we have

$$\begin{aligned}\bar{W}_2^{2(k)} &= i\beta + \int_0^{i\beta} dz[(1-e^{2\pi i(z-z_1)})^{-\frac{k}{n}}(1-e^{2\pi i(z-z_2)})^{-(1-\frac{k}{n})}(1-e^{2\pi i(z-\frac{k}{n}z_1-(1-\frac{k}{n})z_2)})-1]\\ &\quad+\int_0^{i\beta} dz[(1-qe^{-2\pi i(z-z_1)})^{-\frac{k}{n}}(1-qe^{-2\pi i(z-z_2)})^{-(1-\frac{k}{n})}(1-e^{-2\pi i(z-\frac{k}{n}z_1-(1-\frac{k}{n})z_2)})-1]+O(q)\\ &= i\beta+\int_{-i\infty}^{i\infty} dz\left[(\sin\pi(z-z_1))^{-\frac{k}{n}}(\sin\pi(z-z_2))^{-(1-\frac{k}{n})}\left(\sin\pi\left(z-\frac{k}{n}z_1-\left(1-\frac{k}{n}\right)z_2\right)\right)-1\right]+O(q).\end{aligned} \tag{A12}$$

To deal with the integral in the last relation, we define

$$\begin{aligned}F(z_1,z_2) = \int_{-i\infty}^{i\infty} dz\bigg[&(\sin\pi(z-z_1))^{-\frac{k}{n}}(\sin\pi(z-z_2))^{-(1-\frac{k}{n})}\\ &\times\left(\sin\pi\left(z-\frac{k}{n}z_1-\left(1-\frac{k}{n}\right)z_2\right)\right)-1\bigg],\end{aligned} \tag{A13}$$

which only depends on z_1-z_2. Considering

$$\begin{aligned}\frac{\partial_{z_1}F(z_1,z_2)}{\sin\pi(1-\frac{k}{n})(z_1-z_2)} &= \pi\frac{k}{n}\int_{-i\infty}^{i\infty}(\sin\pi(z-z_1))^{-\frac{k}{n}-1}\\ &\quad\times(\sin\pi(z-z_2))^{-(1-\frac{k}{n})},\end{aligned} \tag{A14}$$

$$\begin{aligned}&\partial_{z_2}\frac{\partial_{z_1}F(z_1,z_2)}{\sin\pi(1-\frac{k}{n})(z_1-z_2)}\\ &\quad= \pi^2\frac{k}{n}\left(1-\frac{k}{n}\right)\int_{-i\infty}^{i\infty}(\sin\pi(z-z_1))^{-\frac{k}{n}-1}\\ &\quad\times(\sin\pi(z-z_2))^{-(2-\frac{k}{n})}\cos\pi(z-z_2),\end{aligned} \tag{A15}$$

we find

$$\partial_{z_2}F = -\frac{1-\frac{k}{n}}{\frac{k}{n}}\sin\frac{\pi k}{n}(z_2-z_1)\cos\pi(z_2-z_1)\frac{\partial_{z_2}F}{\sin\pi(1-\frac{k}{n})(z_1-z_2)} - \frac{1}{\pi\frac{k}{n}}\sin\frac{\pi k}{n}(z_2-z_1)\sin\pi(z_2-z_1)\partial_{z_2}\left(\frac{\partial_{z_2}F}{\sin\pi(1-\frac{k}{n})(z_1-z_2)}\right). \tag{A16}$$

Defining

$$T = \partial_{z_2}F(z_1,z_2), \tag{A17}$$

we have the equation

$$\frac{\partial_{z_2}T}{T} = \pi\frac{k}{n}\frac{\cos\pi\frac{k}{n}(z_2-z_1)}{\sin\pi\frac{k}{n}} + \pi\frac{n-k}{n}\frac{\cos\pi(1-\frac{k}{n})(z_2-z_1)}{\sin\pi(1-\frac{k}{n})(z_2-z_1)} - \pi\frac{\cos\pi(z_2-z_1)}{\sin\pi(z_2-z_1)}, \tag{A18}$$

which has the solution

$$T = C\cdot\frac{\sin\frac{\pi k}{n}(z_2-z_1)\sin\pi(1-\frac{k}{n})(z_2-z_1)}{\sin\pi(z_2-z_1)}. \tag{A19}$$

Comparing with the direct evaluation of $\partial_{z_2}F(z_1,z_2)$ for $z_2\to z_1$, we have

$$C = 2i. \tag{A20}$$

Therefore,

$$F(z_1,z_2) = \int_{z_1}^{z_2}dt\frac{2i\sin\frac{\pi k}{n}(t-z_1)\sin\pi(1-\frac{k}{n})(t-z_1)}{\sin\pi(t-z_1)}, \tag{A21}$$

and

$$\bar{W}_2^{2(k)} = i\beta + \int_{z_1}^{z_2}dt\frac{2i\sin\frac{\pi k}{n}(t-z_1)\sin\pi(1-\frac{k}{n})(t-z_1)}{\sin\pi(t-z_1)} + O(q). \tag{A22}$$

2. Small- and large-interval limits

To study the relation between thermal entropy and entanglement entropy, we need to study the small-interval and large-interval limits of the W functions. For small intervals, it is simple to calculate. When $z_1 = z_2$, the integrand is 1, so

$$W_1^{1(k)} = 1, \qquad \bar{W}_2^{2(k)} = i\beta. \tag{A23}$$

For large intervals, we set $z_1\to 0$, $z_2\to 1$:

$$\begin{aligned} W_1^{1(k)} &= -\frac{e^{-\frac{\pi ik}{n}}}{1-e^{-\frac{2\pi ik}{n}}}\oint_A dz\vartheta_1(z-z_1|\tau)^{-(1-\frac{k}{n})}\vartheta_1(z-(z_2-1)|\tau)^{-\frac{k}{n}}\vartheta_1\left(z-\left(1-\frac{k}{n}\right)z_1-\frac{k}{n}z_2|\tau\right) \\ &= -\frac{e^{-\frac{\pi ik}{n}}}{1-e^{-\frac{2\pi ik}{n}}}\oint_A dz\vartheta_1(z)^{-1}\vartheta_1\left(z-\frac{k}{n}|\tau\right) + O(z_1-(z_2-1)) \\ &= -\frac{1}{2\sin\pi\frac{k}{n}}\eta(\tau)^{-3}\vartheta_1\left(-\frac{k}{n}|\tau\right) + O(z_1-(z_2-1)). \end{aligned} \tag{A24}$$

Let us evaluate the most singular term in $W_2^{2(k)}$ when $z_1\to 0$ and $z_2\to 1$:

$$\bar{W}_2^{2(k)} = e^{-(1-\frac{k}{n})\pi i}\times\int_0^{i\beta}dz\vartheta_1(z-z_1|\tau)^{-\frac{k}{n}}\vartheta_1(z-(z_2-1)|\tau)^{-(1-\frac{k}{n})}\vartheta_1\times\left(z-\frac{k}{n}z_1-\left(1-\frac{k}{n}\right)z_2|\tau\right).$$

There is a $(z-z_1)$ term in $\vartheta_1(z-z_1|\tau)$, and a $(z-(z_2-1))$ term in $\vartheta(z-(z_2-1)|\tau)$. The most singular terms in the integral come from the integral range near the origin. In this limit,

$$\vartheta_1(z-z_1|\tau)\sim 2\pi(z-z_1)\eta(\tau) \tag{A25}$$

$$\vartheta_1(z-(z_2-1)|\tau)\sim 2\pi(z-(z_2-1))\eta(\tau) \tag{A26}$$

$$\vartheta_1\left(z-\frac{k}{n}z_1-\left(1-\frac{k}{n}\right)z_2|\tau\right)\sim\vartheta_1\left(-\left(1-\frac{k}{n}\right)|\tau\right), \tag{A27}$$

and then

$$\bar{W}_2^{2(k)} \sim e^{-(1-\frac{k}{n})\pi i} \times \int_{-iM}^{iM} dz \frac{1}{2\pi} (z-z_1)^{-\frac{k}{n}} (z-(z_2-1))^{-(1-\frac{k}{n})} \eta(\tau)^{-3} \vartheta_1 \times \left(-\left(1-\frac{k}{n}\right)|\tau\right), \tag{A28}$$

where $(z_2 - z_1) \ll M \ll \beta$. Since

$$\int_{-iM}^{iM} dz (z-z_1)^{-\frac{k}{n}} (z-(z_2-1))^{-(1-\frac{k}{n})} \sim -(1-e^{-2\pi i \frac{k}{n}}) \log(z_1-(z_2-1)), \tag{A29}$$

the leading singular term in $W_2^{2(k)}$ is

$$\bar{W}_2^{2(k)} \sim i \frac{\sin \frac{\pi k}{n}}{\pi} \eta(\tau)^{-3} \vartheta_1 \left(-\left(1-\frac{k}{n}\right)|\tau\right) \log(z_1-(z_2-1)), \tag{A30}$$

and

$$\left|\frac{W_2^{2(k)}}{W_1^{1(k)}}\right| \sim \frac{2 \sin \frac{\pi k}{n}}{\pi} (-\log(z_1-(z_2-1))). \tag{A31}$$

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