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Recursion relations for graviton scattering amplitudes from Bose symmetry and bonus scaling laws

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Modern on-shell S-matrix methods may dramatically improve our understanding of perturbative quantum gravity, but current foundations of on-shell techniques for general relativity still rely on off-shell Feynman diagram analysis. Here, we complete the fully on-shell proof of Schuster and Toro J. High Energy Phys. 06 (2009) 079] that the recursion relations of Britto, Cachazo, Feng, and Witten (BCFW) apply to relativity tree amplitudes. We do so by showing that the surprising requirement of "bonus" z^{-2} scaling under a BCFW shift directly follows from Bose symmetry. Moreover, we show that amplitudes in generic theories subjected to BCFW deformations of identical particles necessarily scale as z^{even}. When applied to the color ordered expansions of Yang-Mills, this directly implies the improved behavior under nonadjacent gluon shifts. Using the same analysis, three-dimensional gravity amplitudes scale as z^{-4}, compared to the z^{-1} behavior for conformal Chern-Simons matter theory.

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Mysteries abound at the interface between general relativity and quantum field theory. Particularly, graviton scattering amplitudes in maximally supersymmetric $\mathcal{N}=8$ supergravity have surprisingly soft behavior in the deep ultraviolet (UV). To four loops, it has been shown that the critical dimension of supergravity is the same as $\mathcal{N}=4$ super Yang-Mills, a conformally invariant theory free of UV divergences [2]. This result was obtained through the peculiar BCJ duality between color and kinematics, which relates graviton amplitudes to the squares of gluon amplitudes [3,4]. Other arguments, based the nonlinearly realized $E_{7(7)}$ symmetry of $\mathcal{N}=8$ supergravity, predict UV finiteness to six loops [5]. Yet others hint at a full finiteness (see e.g. [6]).

Standard perturbative techniques, i.e. Feynman diagrams, lead to incredibly complicated expressions, and obfuscate general features of the theory. Reframing the discussion in terms of the modern analytic S-matrix has so far proven incredibly useful for discussing Yang-Mills theory (for example, in Ref. [7]), and may provide crucial insights into quantum gravity as well. The on-shell program offers a different perspective on the principles of locality and unitarity, and their powerful consequences [1,8]. It also provides a computational powerhouse, the Britto, Cachazo, Feng, and Witten (BCFW) on-shell recursion relation [9].

Briefly, if two external momenta in the amplitude A_n are subjected to the on-shell BCFW shift,

$$p_1^\mu \to p_1^\mu + zq^\mu \qquad p_2^\mu \to p_2^\mu - zq^\mu, \tag{1}$$

and $A_n(z) \to 0$ for large z, then $A_n(z=0)$ can be recursively constructed from lower-point on-shell amplitudes:

$$A_n = \oint \frac{dz}{z} A_n(z) = \sum_{\{L\}} \frac{A_L(\hat{1},\{L\},\hat{P})A_R(\hat{P},\{R\},\hat{n})}{P^2}. \tag{2}$$

Initial proofs required sophisticated Feynman diagram analyses, and found that gluon amplitudes have the minimum scaling of z^{-1}, but that graviton amplitudes have a "bonus," seemingly unnecessary, scaling of z^{-2} [9–15]. Surprisingly, Ref. [1] found that a fully on-shell proof of BCFW constructability actually requires this improved scaling for gravitons, in order for Eq. (2) to satisfy unitarity. The bonus scaling is not just a "bonus," but a critical property of general relativity. This z^{-2} scaling, also present in the case of nonadjacent gluon shifts [16], implies new residue theorems,

$$0 = \oint A_n(z)dz = \sum_{\{L\}} z_P \frac{A_L(\hat{1},\{L\},\hat{P})A_R(\hat{P},\{R\},\hat{n})}{P^2}, \tag{3}$$

i.e., new relations between terms in Eq. (2)—the bonus relations. The bonus scaling and the bonus relations have a number of important implications. In [17], it was shown that BCJ relations can be extracted from bonus relations. In the case of gravity, bonus relations have been used to simplify tree level calculations [18]. At loop level, the large-z scaling of the BCFW shift corresponds to the high-loop momenta limit; not surprisingly, improved scaling implies improved UV behavior [15,19,20].

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In this paper, we prove that the inherent Bose symmetry between gravitons directly implies this improved bonus scaling, completing the arguments of Ref. [1]. Bose symmetry in general relativity endows it with a purely on-shell description and constrains its UV divergences.[1] We further apply the same argument to gauge theories and gravity in various dimensions.

I. COMPLETING ON-SHELL CONSTRUCTABILITY

Reference [1] first assumes n-point and lower amplitudes scale as z^{-1}—thereby ensuring Eq. (2) holds—and then checks if the BCFW expansion of the $(n+1)$-point amplitude factorizes correctly on all channels. Factorization on all channels is taken to define the amplitude. Correct factorization in most channels requires z^{-1} scaling of lower point amplitudes. However, some channels do not factor correctly without improved z^{-2} scaling, as well as a z^6 scaling on the "bad" shifts. In the following, we present a proof for both of these scalings.

Essentially, the argument rests on a very simple observation: any symmetric function $f(i,j)$, under deformations $i \to i + zk$, $j \to j - zk$, must scale as an even power of z. In particular, any function with a strictly better than $\mathcal{O}(1)$ large-z behavior (no poles at infinity), is automatically guaranteed to decay at least as z^{-2}.

Although straightforward, this is not manifest when constructing the amplitude. BCFW terms typically scale as z^{-1}, but only specific pairs have canceling leading z^{-1} pieces. Similarly, the bad BCFW shift behavior of z^6 is only obtained when the leading z^7 pieces cancel in pairs.

Consider the five point amplitude in $\mathcal{N}=8$ SUGRA, exposed by the $[1,5\rangle$ BCFW shift, where $|1] \to |1] - z|5]$ and $|n\rangle \to |n\rangle + z|1\rangle$,

$$\begin{aligned} M_5 &= M(123P)\times M(P45)/P_{123}^2 + (4\leftrightarrow 3) + (4\leftrightarrow 2) \\ &= \frac{[23][45]}{\langle 12\rangle\langle 13\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle\langle 45\rangle\langle 15\rangle^2} \\ &\quad + \frac{[24][35]}{\langle 12\rangle\langle 14\rangle\langle 24\rangle\langle 23\rangle\langle 43\rangle\langle 35\rangle\langle 15\rangle^2} \\ &\quad + \frac{[43][25]}{\langle 14\rangle\langle 13\rangle\langle 43\rangle\langle 42\rangle\langle 32\rangle\langle 25\rangle\langle 15\rangle^2}, \end{aligned} \tag{4}$$

with the SUSY-conserving delta-function stripped out.

Under a $[2,3\rangle$ shift, the first term scales as z^{-2}, while the other two scale as z^{-1}. However, their sum (now symmetric in 2 and 3) scales as z^{-2}: the whole amplitude has the correct scaling. This pattern holds true in general. Where present, z^{-1}'s cancel between pairs of BCFW terms, $M_L(K_L,i,P)\times M_R(-P,j,K_R)$ and $M_L(K_L,j,P)\times M_R(-P,i,K_R)$. Further, terms without pairs over saturate the bonus scaling.

One such example is $M(1^-2^-3^+P^+)M(P^-4^-5^+6^+)$, appearing in M_6^{NMHV}. Under a $[4,3\rangle$ shift, it has no corresponding pair: $M(1^-2^-4^-P)M(P3^+5^+6^+)$ vanishes for all helicities h_P. Luckily, it turns out these types of terms have a surprisingly improved scaling of z^{-9}. Hence, they never spoil the scaling of the full amplitude.

In the next section we classify and prove the scalings of all possible BCFW terms. Following this, we demonstrate how leading z pieces cancel between BCFW terms.

II. BCFW TERMS UNDER SECONDARY z SHIFTS

Consider the $[1,n\rangle$ BCFW expansion of an n-point GR tree amplitude M_n (where $\tilde{\lambda}_1 \to \tilde{\lambda}_1 - w\tilde{\lambda}_n$, $\lambda_n \to \lambda_n + w\lambda_1$):

$$M_n = \sum_{L,R} \frac{M_L(\hat{1},\{L\},\hat{P})M_R(-\hat{P},\{R\},\hat{n})}{P^2}. \tag{5}$$

We would like to understand how BCFW terms in $\mathcal{M}_n$ scale under secondary $[i,j\rangle$ z shifts,

$$\tilde{\lambda}_i \to \tilde{\lambda}_i - z\tilde{\lambda}_j \qquad \lambda_j \to \lambda_j + z\lambda_i. \tag{6}$$

We recall two features of these terms as they appear in Eq. (5). First, the value of the primary deformation parameter $w = w_P$, which accesses a given term, is

$$w_P = \frac{P^2}{\langle 1|P|n]}, \tag{7}$$

and, on this pole, the intermediate propagator factorizes as

$$\hat{P}^{\alpha\dot{\alpha}} = \frac{\{[\tilde{\lambda}_n|P\}^{\alpha}\{\langle\lambda_1|P\}^{\dot{\alpha}}}{\langle\lambda_1|P|\tilde{\lambda}_n]} = \frac{|\lambda_P\rangle[\tilde{\lambda}_P|}{\langle 1|P|n]} \equiv |\hat{P}\rangle[\hat{P}|. \tag{8}$$

The little-group ambiguity amounts to associating the denominator with either λ_P, $\tilde{\lambda}_P$, or some combination of them. In what follows, we find it easiest to associate it entirely with the antiholomorphic spinor, $|\hat{P}] = |\tilde{\lambda}_P]/\langle 1|P|n]$—see Eq. (9), below.

With this in hand, we now turn to the large-z scalings of the various BCFW terms, subjected to the secondary z shifts in Eq. (6). There will be two different types of BCFW terms: those with both i and j within the same subamplitude, and those with i and j separated by the propagator. The former inherit all z dependence from the lower point amplitudes in the theory, since the secondary shift acts like a usual BCFW shift on the subamplitude. The latter are more complicated, since the z shift affects the subamplitudes in several ways besides the simple shifts on i and j.

[1] The better than expected UV behavior was also at least partially understood from the "no-triangle" hypothesis of $\mathcal{N}=8$ supergravity as a consequence of crossing symmetry and the colorless nature of gravitons in Ref. [21].

Specifically, both w_P and the factorized form of the internal propagator acquire z dependence:

$$\begin{aligned} w_P &= \frac{P^2}{\langle 1|P|n]} \longrightarrow \frac{P^2 + z\langle i|P|j]}{\langle 1|P|n] + z\langle 1i\rangle[jn]}, \\ |\hat{P}\rangle^{\alpha} &\equiv \frac{|([n|P)\rangle^{\alpha}}{[jn]} \longrightarrow |\hat{P}\rangle^{\alpha} + z|i\rangle^{\alpha}, \\ |\hat{P}]^{\dot{\alpha}} &\equiv \frac{|(\langle 1|P)]^{\dot{\alpha}}}{\langle 1|P|n]/[jn]} \longrightarrow \frac{|\tilde{\lambda}_P]^{\dot{\alpha}} - z\langle 1i\rangle|j]^{\dot{\alpha}}}{\langle 1|P|n]/[jn] - z\langle 1i\rangle}. \end{aligned} \tag{9}$$

With this factorized form of the propagator, it turns out that the left- and right-hand subamplitudes have well-defined individual z scalings, which depend only on the helicity choices for i^{h_i}, j^{h_j} and P^h:

$$\begin{aligned} &M_L(i^- P^-) \sim z^{-2} && M_R(j^- P^-) \sim z^{+2} \\ &M_L(i^- P^+) \sim z^{-2} && M_R(j^- P^+) \sim z^{+2} \\ &M_L(i^+ P^-) \sim z^{+6} && M_R(j^+ P^-) \sim z^{+2} \\ &M_L(i^+ P^+) \sim z^{-2} && M_R(j^+ P^+) \sim z^{-6}. \end{aligned} \tag{10}$$

The scaling of a full BCFW term $M_L M_R / P^2$ can then be easily determined from these values, which we prove in two steps.

First, note that the large-z scalings on the left of Eq. (10) match the familiar BCFW scalings of full amplitudes. We prove this by showing that the large-z behavior of the left-hand subamplitude maps isomorphically onto a BCFW shift of M_L. Looking at Eq. (9), we see that, in the large-z limit, the spinors of i and P become

$$\begin{aligned} &\lambda_i \longrightarrow \lambda_i && \lambda_P \longrightarrow z\lambda_i \\ &\tilde{\lambda}_i \longrightarrow -z\tilde{\lambda}_j && \tilde{\lambda}_P \longrightarrow \tilde{\lambda}_j, \end{aligned} \tag{11}$$

which is just a regular BCFW $[i, P\rangle$ shift within the left-hand subamplitude.

Now we turn to the slightly unusual scalings on the right-hand side of Eq. (10). With the little-group choice in Eq. (9), the left-hand subamplitude has exactly the correct spinor variables to map onto the usual BCFW shift. Now observe that, starting with the other little-group choice for the spinors on the z-shifted internal propagator, we obtain the usual BCFW scalings on this side:

$$\begin{aligned} &M_R(j^- P^-) \sim z^{-2} \\ &M_R(j^- P^+) \sim z^{+6} \\ &M_R(j^+ P^-) \sim z^{-2} \\ &M_R(j^+ P^+) \sim z^{-2}. \end{aligned} \tag{12}$$

Proving these results is identical to the previous reasoning for the left-hand subamplitude.

It becomes clear now that to get the other half of the scalings, we need only account for the change in z scaling when switching the $1/\langle 1|P(z)|n]$ factor between λ_P and $\tilde{\lambda}_P$. Assume that the spinors of the propagator appear with weights,[2]

$$M_R \propto (|P\rangle)^a (|P])^b, \tag{13}$$

where $-a + b = 2h_P$, and h_P is the helicity of the internal propagator as it enters the right-hand subamplitude. Now, in the limiting cases where $1/\langle 1|P(z)|n]$ is entirely associated with $|\lambda_P\rangle$ or $|\tilde{\lambda}_P]$ the amplitude scales as:

$$M_R \propto \left(\frac{|\lambda_P\rangle}{\langle 1|P|n]}\right)^a (|\tilde{\lambda}_P])^b \to z^s, \quad \text{or} \tag{14}$$

$$M_R \propto (|\lambda_P\rangle)^a \left(\frac{|\tilde{\lambda}_P]}{\langle 1|P|n]}\right)^b \to z^t, \tag{15}$$

where s is the BCFW large-z scaling exponent, obtained in Eq. (12), and t is the related scaling, for the other internal little-group choice. It follows that $s - t = b - a = 2h_P$, and so the t scalings can be easily derived as $t = s \pm 4$, depending on the helicity of the propagator.

Having proven all eight scaling relations in Eq. (10), we can classify the scaling behavior of all possible types of BCF terms with i and j in different subamplitudes. For these terms the propagator contributes a z^{-1} to each term, and so from Eq. (10) we obtain the following eight possible types of terms:

$$\text{(i) } M_L(i^+ P^-) M_R(j^- P^+)/P^2 \text{ scales as } z^{+7}, \tag{16}$$

$$\text{(ii) } M_L(i^- P^-) M_R(j^+ P^+)/P^2 \text{ scales as } z^{-9}, \tag{17}$$

$$\text{(iii) The other six BCFW terms scale as } z^{-1}. \tag{18}$$

In the next section we will see how pairing terms improves these scalings by one power of z, such that we recover the required z^{-2} and z^6 scalings.

Finally, while the individual scalings in Eq. (10) are not invariant under z dependent little-group rescalings on the internal line $\hat{P}(z)$, the above results for full BCFW terms are invariant under these rescalings.

III. IMPROVED BEHAVIOR FROM SYMMETRIC SUMS

We first study $[+, +\rangle$ and $[-, -\rangle$ shifts, with scalings in Eq. (18). Define $M_L(K_L, i, P) \times M_R(-P, j, K_R)/P^2 \equiv M(i|j)$, where K_L is the momenta from the other external

[2] In general, the spinors need not appear with uniform homogeneity. The analysis below still holds, but must be applied term by term. The same caveat applies to Eqs. (24) and (26).

states on the left-hand subamplitude. We wish to show that in the large-z limit,

$$M(i|j) = -M(j|i). \tag{19}$$

so the leading z^{-1} pieces cancel in the symmetric sum of BCFW terms, $M(i|j) + M(j|i)$.

Because i and j have the same helicity, $M(j|i)$ is obtained directly from $M(i|j)$ by simply swapping labels:

$$M(i|j) = M(\lambda_i, \tilde{\lambda}_i, \lambda_j, \tilde{\lambda}_j) \tag{20}$$

$$M(j|i) = M(\lambda_j, \tilde{\lambda}_j, \lambda_i, \tilde{\lambda}_i) \tag{21}$$

In the large-z limit, these become

$$M(i|j) = M(\lambda_i, -z\tilde{\lambda}_j, z\lambda_i, \tilde{\lambda}_j) \tag{22}$$

$$M(i|j) = M(z\lambda_i, \tilde{\lambda}_j, \lambda_i, -z\tilde{\lambda}_j). \tag{23}$$

The two have equal z scaling, and so can only differ by a relative sign. The spinors appear with weights,

$$M(i|j) \propto \langle ij\rangle^F [ij]^G (\lambda_i)^a (\tilde{\lambda}_i)^b (\lambda_j)^c (\tilde{\lambda}_j)^d$$
$$M(j|i) \propto \langle ji\rangle^F [ji]^G (\lambda_j)^a (\tilde{\lambda}_j)^b (\lambda_i)^c (\tilde{\lambda}_i)^d, \tag{24}$$

while in the large-z limit, the leading terms are

$$M(i|j) \propto z^{b+c} (\langle ij\rangle^F [ij]^G (\lambda_i)^a (-\tilde{\lambda}_j)^b (\lambda_i)^c (\tilde{\lambda}_j)^d)$$
$$M(j|i) \propto z^{a+d} (\langle ji\rangle^F [ji]^G (\lambda_i)^a (\tilde{\lambda}_j)^b (\lambda_i)^c (-\tilde{\lambda}_j)^d). \tag{25}$$

These cancel if and only if $F + G + b + d = \text{odd}$. First, from Eq. (18), $M(a|b)$'s scale as z^{odd}. So $b + c = a + d = \text{odd}$. Second, by helicity counting in Eq. (24), we know $-F + G - c + d = 2h_j = \text{even}$. Therefore, we obtain the required result, and the leading z^{-1} pieces cancel.

For the $[-,+\rangle$ and $[+,-\rangle$ shifts a simple modification of the above argument is required. This is because we now expect the cancellation to occur between the pair terms $M_L(K_L, i^-, P^+) \times M_R(-P^-, j^+, K_R)/P^2$ and $M_L(K_L, j^+, P^-) \times M_R(-P^+, i^-, K_R)/P^2$. Switching different helicity particles requires us to flip the propagator's helicity as well. It can be shown that, in the large-z limit, $M_L(K_L, i^-, P^+) = M_L(K_L, j^+, P^-)$; likewise for the right-hand subamplitude. Note that switching i^- and j^+ requires more care now: functionally, the correct label swaps for M_L are $i \to P$, $P \to j$ while for M_R $j \to P$ and $P \to i$. Therefore, we can write, as above,

$$M_L(i^-, P^+) \propto \langle iP\rangle^F [iP]^G (\lambda_i)^a (\tilde{\lambda}_i)^b (\lambda_P)^k (\tilde{\lambda}_P)^l$$
$$M_L(j^+, P^-) \propto \langle Pj\rangle^F [Pj]^G (\lambda_P)^a (\tilde{\lambda}_P)^b (\lambda_j)^k (\tilde{\lambda}_j)^l. \tag{26}$$

Crucially, the large-z limit is also different for the two subamplitudes, since the limits (11) were obtained with $i \in P$. The second subamplitude instead has $j \in P$, and in this case the limits are $\lambda_P \to -z\lambda_i$ and $\tilde{\lambda}_P \to \tilde{\lambda}_j$. In the large-$z$ limit then identical counting as above shows that $a + b = \text{even}$, and the same will hold for M_R. The propagator is antisymmetric in the large-z limit under swapping i and j, and therefore the leading z pieces cancel as expected. This cancellation reduces the leading z^{-1} and z^{+7} scalings for the opposite helicity-shifted BCF terms in the previous section, down to the well-known z^{-2} and z^{+6} BCFW scalings for GR. This completes the proof of the bonus scaling for GR and closes the final gap in the on-shell proof of BCFW in GR Ref. [1].

IV. ANALYSIS OF THE FULL AMPLITUDE

The simple argument we used above can be applied directly to the whole amplitude, if we restrict to like-helicity shifts. Consider

$$A_n(i,j) \propto \langle ij\rangle^F [ij]^G (\lambda_i)^a (\tilde{\lambda}_i)^b (\lambda_j)^c (\tilde{\lambda}_j)^d. \tag{27}$$

If this amplitude is manifestly symmetric under exchange of two (bosonic) particle labels, then $A_n(i,j) = A_n(j,i)$, which fixes $a = c$, $b = d$, and $F + G = \text{even}$. By helicity counting, $-F + G - a + b = 2h_i = \text{even}$, and then $a + b = \text{even}$. So, under a $[i, j\rangle$ shift,

$$A_n(i(z), j(z)) \sim z^{b+c} = z^{a+b} = z^{\text{even}}. \tag{28}$$

This same logic holds in Eq. (27), even if the shifted lines are identical fermions. Permuting labels i and j again forces $a = c$, and $b = d$, and $F + G = \text{odd}$. But so must $2h_i = -F + G - a + b$. Hence $a + b$ remains even. BCFW shifts of identical particles, bosons or fermions, fix z^{even} scaling at large z.

To understand the opposite-helicity shifts, we are led to consider pure GR as embedded within maximal $\mathcal{N} = 8$ SUGRA. Amplitudes in maximal supergravity do not distinguish between positive and negative helicity graviton states. Using the methods of [22] to truncate to pure GR, we recover the usual BCFW scalings.

As an interesting corollary of our four-dimensional analysis, the large-z scaling of gravity amplitudes in three dimensions is drastically improved to z^{-4}. Due to the fact that the little group in three dimensions is a discrete group, the BCFW deformation is nonlinear. In particular the three-dimensional spinors shift as [23]

$$\lambda_i(z) = \text{ch}(z)\lambda_i + \text{sh}(z)\lambda_j, \qquad \lambda_j(z) = \text{sh}(z)\lambda_i + \text{ch}(z)\lambda_j, \tag{29}$$

where $\mathrm{ch}(z) = (z + z^{-1})/2$ and $\mathrm{sh}(z) = (z - z^{-1})/2i$. Thus, momenta shift as

$$p_i(z) = \overline{P_{ij}} + yq + \frac{1}{y}\tilde{q}, \qquad p_j(z) = \overline{P_{ij}} - yq - \frac{1}{y}\tilde{q} \tag{30}$$

where $\overline{P_{ij}} = \frac{p_i+p_j}{2}$, $y = z^2$, and $q, \tilde{q}$ can be read off from Eq. (29). Now let us consider three-dimensional gravity amplitudes that arise from the dimension reduction of four-dimensional gravity theory. The degrees of freedom are given by a dilaton and a scalar. Since both are bosons, little group dictates that one must have even power of λ_i. Thus the large-z behavior of gravity amplitudes is completely dictated by Eq. (30). Permutation invariance then requires the function to be symmetric under $y \leftrightarrow -y$, and so must be an even power of y. Thus if gravity amplitudes can be constructed via BCFW shift, the large-z asymptotic behavior must be at most $y^{-2} = z^{-4}$. Indeed it is straightforward to check that the four-point $\mathcal{N} = 16$ supergravity amplitude behaves as z^{-4} under a super-BCFW shift. This is to be compared with the z^{-1} scaling of superconformal Chern-Simons theory [23].

More generally, BCFW shifts in $d \geq 4$ take the form

$$p_i^\mu(z) = p_i^\mu + zq^\mu \qquad p_j^\mu(z) = p_j^\mu - zq^\mu, \tag{31}$$

where q is null and orthogonal to p_i and to p_j. External wave-functions of shifted boson lines also shift [13]. For identical bosons, Bose symmetry disallows z^{odd} scaling, as it would introduce a sign change under label swaps. Identical fermions shift similarly; here the antisymmetric contraction of the identical spinor wave functions absorbs their exchange-sign. BCFW shifts of identical particles must scale as z^{even} for large z in dimensions $d \geq 4$.

Symmetry between identical particles is crucial for these cancellations to occur. Gluon partial amplitudes are not permutation invariant: distinct gluons generally have different colors. This spoils the permutation invariance—as is clear from z^{-1} dropoff of adjacent shifts of a color-ordered tree amplitude in Yang-Mills. Gravitons, however, are unique: they cannot have different "colors" [24]. Thus graviton amplitudes are invariant under permutations from the outset: the discrete symmetry group of graviton amplitudes is larger than for gluon amplitudes. Consequently, gravity amplitudes are softer in the deep-UV than Yang-Mills amplitudes.

V. BOSE SYMMETRY AND COLOR IN YANG-MILLS

Finally, we explore the interplay between color and the large-z structure of Yang-Mills amplitudes. For ease, we focus on $A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$. It can be written in terms of color-ordered partial amplitudes as

$$\frac{A_4(1^-2^-3^+4^+)}{\langle 12\rangle^2[34]^2} = \frac{\mathrm{Tr}(1234)}{st} + \frac{\mathrm{Tr}(1243)}{su} + \frac{\mathrm{Tr}(1324)}{tu}. \tag{32}$$

Under a $[1, 2\rangle$ shift, only t and u shift, and in opposite directions: $\hat{t}(z) = t + z\langle 1|4|2]$, and $\hat{u}(z) = u - z\langle 1|4|2]$. The term proportional to Tr(1324) scales as z^{-2}, while the other two scale as z^{-1}. The leading z terms,

$$\frac{A_4(\hat{1}^-, \hat{2}^-, 3^+, 4^+)}{\langle 12\rangle^2[34]^2} \sim \frac{\mathrm{Tr}(1234) - \mathrm{Tr}(1243)}{z\langle 1|4|2]s} + \cdots, \tag{33}$$

cancel when gluons 1 and 2 are identical, and $T_1 = T_2$.

Cancellation of z^{-1} terms must hold for general tree amplitudes when the gluons have the same color labels. However, only BCFW shifts of lines that are *adjacent* in color-ordering cancel pairwise as in Eq. (33). For color-orderings where this shift is nonadjacent, there are no pairs of BCF terms with canceling z^{-1}-terms. This implies that good non-adjacent BCFW shifts in gluon partial amplitudes must scale as z^{-2}.

VI. FUTURE DIRECTIONS AND CONCLUDING REMARKS

We have shown that the z^{-2} bonus scalings and relations, crucial for consistent on-shell contraction of gravitational S matrices, follow from Bose symmetry. Similar z^{-1} cancellations occur in QED and GR [25]. Further, Bose symmetry alone implies z^{-2} dropoff of nonadjacent BCFW shifts in Yang-Mills. More broadly, BCFW shifts of identical particles—bosons and fermions—must scale as z^{even} in general settings, beyond $d = 4$.

Graviton amplitudes in Refs. [26–28], which manifest permutation symmetry, also manifest z^{-2} dropoff. This is not a coincidence: permutation symmetry automatically implies bonus behavior. A better understanding of gravity should be tied to more natural manifestations of permutation invariance. However, not all improved scalings obviously come from permutation invariance. Notably, Hodges's observation that BCFW-terms, built from "bad" "opposite helicity" z^{-1} $\mathcal{N} = 7$ SUGRA shifts, term-by-term scale as z^{-2} [29]. As the legs are not identical, permutation invariance is not prominent in the proof [30].

Permutation invariance has unrecognized and powerful consequences even at tree level. Do new constraints appear when accounting for it in other shifts? Does it have nontrivial consequences at high-loop orders in $\mathcal{N} = 8$ SUGRA or $\mathcal{N} = 4$ SYM? Would mandating it expose new facets of the "Amplituhedron" of Ref. [7]?

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