



Derivation of the exact expression for the D function in $\mathcal{N} = 1$ SQCD

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We discuss various details of our derivation of the exact expression for the Adler D function in $\mathcal{N} = 1$ supersymmetric QCD. This exact formula relates the D function to anomalous dimensions of the matter superfields. Our perturbative derivation refers to the D function defined in terms of the bare coupling constant in the case of using the higher covariant derivative regularization. The exact expression for this function is obtained by direct summation of supergraphs to all orders in the non-Abelian coupling constant. As we argued previously, our formula should be valid beyond perturbation theory too. The perturbative result we present here coincides with the general formula order by order. We discuss consequences for $\mathcal{N} = 1$ supersymmetric QCD in the conformal window. It is noted that our exact relation can allow one to determine the (infrared) critical anomalous dimension of the Seiberg M field present in the dual theory.

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I. INTRODUCTION

In our brief publication [1] we reported a new exact relation between the Adler function D in supersymmetric QCD (SQCD) with the gauge group $SU(N)$ on the one hand, and the anomalous dimensions of the matter superfields on the other,

$$D(Q^2) = \frac{3}{2} N_c \sum_f q_f^2 [1 - \gamma(\alpha_s(Q^2))], \quad (1)$$

where N_c is the number of colors, f is the flavor index, and q_f is the corresponding electric charge (in units of e). Equation (1) assumes that all matter fields are in the fundamental representation of $SU(N)$, although their electric charges can be different. In calculating $\gamma(\alpha_s(Q^2))$ one should remember that $\alpha_s(Q^2)$ runs according to the Novikov-Vainshtein-Shifman-Zakharov (NSVZ) β function [2–4].

The basic idea of our derivation was the same as that of the NSVZ β functions. The key object of our consideration was the effective Lagrangian for the external electromagnetic field, with two terms in it: the $U(1)$ gauge kinetic term and the matter term. The latter is related to the former through the exact Konishi anomaly [5], and, therefore, brings in γ 's. Although our terminology was perturbative, the result (1), being based on the exact anomaly relations, should be exact too. Then we verified that it is valid to all orders in the $SU(N)$ coupling constant α_s by a direct supergraph calculation.

From previous works [6–8] one could find that a relation between the two constants

$$D_* = \frac{3}{2} N_c \sum_f q_f^2 [1 - \gamma(\alpha_s^*)] \quad (2)$$

is valid in the (super)conformal points, i.e. at the points at which $\beta_{\text{NSVZ}} = 0$. After our publication [1] it was argued [9] that the above conformal relation can be generalized for the renormalization group (RG) flow by considering R charges as functions of the running α_s .

Our task in this paper is twofold. First, we will present a detailed account of our supergraph calculation, which *a priori* seems quite nontrivial. Second, we will consider implications of (1) in Seiberg's conformal window [10,11].

Note, that the Adler function D is directly related to the celebrated ratio R defined as

$$R(s) = \frac{\sigma(e^+e^- \rightarrow (s)\text{quarks, gluons(inos)} \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}, \quad (3)$$

where s is the (center-of-mass) total energy squared. R plays a very important role in the QCD-based phenomenology [12]. For example, it is used for a precise determination of the strong coupling α_s from precision data on $e^+e^- \rightarrow \text{hadrons}$ in an appropriate range of energy. The relation between D and R is as follows [12] (see also [13–15]):

$$D(\alpha_s(P^2)) \equiv -12\pi^2 \frac{d\Pi(P^2)}{d \log P^2} = P^2 \int_0^\infty ds \frac{R(s)}{(s + P^2)^2}, \quad (4)$$

where Π denotes the photon polarization operator and P is the Euclidian momentum. (Throughout this paper Euclidian momenta are denoted by capital letters.) In our notation Π is related to the inverse invariant charge d^{-1} by the equation

$$d^{-1} = \alpha_0^{-1} + 4\pi\Pi, \quad (5)$$

where α_0 is the bare electromagnetic coupling constant.

To find the Adler function D in SQCD one must omit all terms proportional to the electromagnetic coupling constant in calculating the polarization operator Π . This implies that the electromagnetic field is considered as the *external field*. The D function encodes QCD corrections to the photon propagator. In this sense, this function is similar to ordinary β functions. In QCD the Adler D function was calculated up to the order $O(\alpha_s^4)$ [16,17]. For $\mathcal{N} = 1$ SQCD the two-loop expression for the Adler function was obtained in [18].

Similarly to the β function, the Adler D function depends on the subtraction scheme (beyond two loops). In this paper we work with the D function defined in terms of the bare coupling constant, namely,

$$D(\alpha_{0s}) \equiv -\frac{3\pi}{2} \frac{d}{d \log \Lambda} \alpha_0^{-1}(\alpha, \alpha_s, \Lambda/\mu)|_{\alpha, \alpha_s = \text{const}}, \quad (6)$$

where the derivative is calculated at fixed values of the renormalized coupling constants $\alpha(\alpha_0, \alpha_{0s}, \Lambda/\mu)$ and $\alpha_s(\alpha_{0s}, \Lambda/\mu)$. Parameter Λ denotes the ultraviolet cutoff which (within the higher derivative regularization, see below) can be identified with the dimensionful parameter in the higher derivative regularizing term.

The difference between the definitions of the RG functions in terms of the bare coupling constant versus renormalized coupling constant is discussed in detail in [19]. In particular, it was demonstrated that the RG functions defined in terms of the bare coupling constant depend on the regularization, but are independent of the subtraction scheme for a fixed regularization.

The Adler D function consists of two distinct contributions,

$$D(\alpha_s) = \sum_f q_f^2 D_1(\alpha_s) + \left(\sum_f q_f \right)^2 D_2(\alpha_s), \quad (7)$$

where q_f denotes the ‘‘electric charge’’ of the flavor f . The first contribution D_1 comes from diagrams in which the external photon lines are attached to the same matter loop, and the second one D_2 (the so-called ‘‘singlet contribution’’) comes from diagrams in which the external lines are attached to different matter loops. The diagrams contributing to both parts of the function D are sketched in Fig. 1.

In this paper we consider $\mathcal{N} = 1$ SQCD with matter superfields which, in principle, can be in any representation of the $SU(N)$ gauge group, although Eq. (1) is given for (anti)fundamental (s)quarks. The matter superfields interact with the external Abelian gauge superfield (which is a supersymmetric generalization of the photon field).

Our supergraph derivation of the exact relation for the D function (1) in terms of γ 's uses the higher covariant

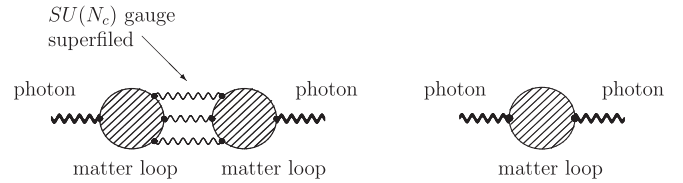


FIG. 1. Diagrams contributing to the singlet part of the function D are schematically presented in the left-hand side, and those contributing to the nonsinglet part are presented in the right-hand side. In the former case the external photon lines are attached to different loops of the matter superfields, and the number of internal lines joining the circles should be more than 1. In the latter case the external lines are attached to a single matter loop.

derivative regularization. In terms of the bare coupling α_{0s} for each given value of the momentum Q we have

$$D(\alpha_{0s}) = \frac{3}{2} N_c \sum_{f=1}^{N_f} q_f^2 (1 - \gamma(\alpha_{0s})), \quad (8)$$

where N_c is a number of colors, N_f is a number of flavors, and $\gamma(\alpha_{0s})$ is the anomalous dimension (of a single chiral matter superfield) defined in terms of the bare coupling constant

$$\gamma(\alpha_{0s}) \equiv -\frac{d}{d \log \Lambda} \log Z(\alpha_s, \Lambda/\mu)|_{\alpha_s = \text{const}}. \quad (9)$$

In the two-loop approximation this equation is in agreement with the result of [18]

$$D(\alpha_s) = \frac{3}{2} N_c \sum_f q_f^2 \left[1 + \frac{N_c^2 - 1}{2N_c} \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right], \quad (10)$$

if we take into account the fact that the one-loop anomalous dimension is given by the expression

$$\gamma(\alpha_s) = -\frac{N_c^2 - 1}{2N_c} \frac{\alpha_s}{\pi} + O(\alpha_s^2). \quad (11)$$

(The two-loop contribution to the Adler function and the one-loop contribution to the anomalous dimension are scheme independent. Consequently, they are the same for the RG functions defined in terms of the bare coupling constant and the RG functions defined in terms of the renormalized coupling constant.)

Certainly, the D function in the supersymmetric case cannot be applied for the phenomenological purposes, because supersymmetry is not observed at low energies. Nevertheless, investigation of SQCD can be useful for a better understanding of gauge theories dynamics. Therefore, our result is promising for investigating the quantum structure in supersymmetric gauge theories.

As was mentioned in [1], the exact relation (1) is closely related to the exact NSVZ β function [2–4,20]

$$\beta(\alpha) = -\frac{\alpha^2(3C_2 - T(R) + C(R)_i^j \gamma_j^i(\alpha)/r)}{2\pi(1 - C_2\alpha/2\pi)}, \quad (12)$$

where

$$\begin{aligned} \text{tr}(T^A T^B) &\equiv T(R)\delta^{AB}; & (T^A)_i^k (T^A)_k^j &\equiv C(R)_i^j; \\ f^{ACD} f^{BCD} &\equiv C_2\delta^{AB}; & r &\equiv \delta_{AA}. \end{aligned} \quad (13)$$

The exact NSVZ β function relates the renormalization of the coupling constant in $\mathcal{N} = 1$ supersymmetric theories to the renormalization of the matter superfields. In the particular case of $SU(N_c)$ gauge theory with N_f flavors in the fundamental representation (each flavor gives one Dirac fermion in components) Eq. (12) gives

$$\beta(\alpha_s) = -\frac{\alpha_s^2(3N_c - N_f + N_f\gamma(\alpha_s))}{2\pi(1 - N_c\alpha_s/2\pi)}, \quad (14)$$

where $\gamma(\alpha_s)$ is the anomalous dimension of the chiral superfields.

The NSVZ β function was originally derived from the analysis of the structure of instanton corrections, namely, by requiring their invariance under the renormalization group [2,3,21]. Another possibility is to use the structure of the anomaly supermultiplet [4,20,22]. Yet another (albeit related) derivation of the NSVZ β function was based on the nonrenormalization theorem for the topological term [23].

Explicit perturbative calculations carried out in dimensional reduction [24] in the $\overline{\text{DR}}$ subtraction scheme up to the three [25,26] and four loops [27,28] agree with the NSVZ expression only in the one- and two-loop approximation. In higher loops the scheme dependence of the RG functions becomes essential, and to obtain the NSVZ β function one has to make a specially tuned finite renormalization. It was verified in the three- and four-loop orders that such a finite renormalization exists [26,29–31]. According to Ref. [26] its existence is a nontrivial fact. The reason is that the NSVZ relation leads to some scheme-independent consequences which should be valid in *all* subtraction schemes [32,33]. (The general equations which describe how the NSVZ β function is changed under finite reparametrizations of the gauge coupling and finite rescalings of the matter superfields are presented in [33,34].)

In the Abelian case the NSVZ scheme (in which the NSVZ relation is valid in all orders) was constructed in [19] by imposing some simple boundary conditions on the renormalization constants for $\mathcal{N} = 1$ supersymmetric theories regularized by higher derivatives. The higher covariant derivative regularization [35,36] turns out to be very convenient for investigating quantum corrections in

supersymmetric theories. It is mathematically consistent in contrast with the dimensional reduction [37] and can be formulated in a manifestly supersymmetric way [38,39]. It can be used in $\mathcal{N} = 2$ supersymmetric theories too [40,41].

The NSVZ β function naturally occurs in $\mathcal{N} = 1$ supersymmetric theories, regularized by higher covariant derivatives, because momentum integrals for the β function defined in terms of the bare coupling constant are integrals of total derivatives [42] and even integrals of double total derivatives [43]. This allows one calculating them analytically. Consequently, one obtains the NSVZ relation for the RG functions defined in terms of the bare coupling constant.

In the Abelian case this was proved in all loops [44,45] and confirmed by an explicit three-loop calculation [19,32,44,46]. For a generic non-Abelian gauge theory the factorization of relevant integrals into integrals of double total derivatives was verified only in the two-loop approximation [47–51].

For proving Eq. (8) we note that the momentum integrals for the D function [defined in terms of the bare coupling constant by Eq. (6)] in $\mathcal{N} = 1$ SQCD are also integrals of double total derivatives. In this paper we prove this in all orders using a method similar to the one proposed in [44]. These integrals do not vanish because of singularities which occur due to the identity

$$\frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \frac{1}{Q^2} = -4\pi^2 \delta^4(Q). \quad (15)$$

(The above equation is written in Euclidian space.)

Calculating contributions of all singularities we obtain that the singlet contribution to D symbolically depicted in the left-hand side of Fig. 1 automatically vanishes once all relevant supergraphs are summed, while the remaining part of the D function satisfies the relation (8). Thus, highly nontrivial calculation of the Adler function and the anomalous dimension and their comparison fully confirms Eqs. (1) or (8).

The study of the consequences following from Eq. (1) in the conformal window apparently paves the way to subsequent intriguing explorations.

This paper is organized as follows: In Sec. II we explain how one can calculate the Adler D function using the higher covariant derivative regularization. In particular, we find a relation between the D function and a part of the effective action corresponding to the two-point Green function of the Abelian gauge superfield. The exact formula for this part of the effective action is constructed in Sec. III. This formula consists of three parts: the singlet contribution (which comes from diagrams in which external lines are attached to different matter loops), the contribution of diagrams in which external lines are attached to a single matter loop, and a noninvariant contribution (due to which the two-point Green function of the Abelian gauge

superfield turns out to be transversal). Derivation of the exact expression for the D function by direct summation of supergraphs is presented in Sec. IV. First, in Sec. IV A we find a sum of certain subdiagrams. Then we substitute this sum to the expression for the D function using the results of the Appendix. In particular, in Sec. IV B we prove that the singlet contribution to the D function is given by integrals of double total derivatives and vanishes in all orders. In Sec. IV C we prove that the nonsinglet contribution is also given by integrals of double total derivatives in the momentum space. However, unlike the integrals for the singlet contribution, these integrals do not vanish, because the integrands have singularities. The sum of these singularities is calculated in Sec. IV D, where we relate it to the anomalous dimension of the matter superfields. Finally, in Sec. V we discuss some consequences of the exact relation (1) for SQCD in the conformal window.

II. D FUNCTION AND THE HIGHER COVARIANT DERIVATIVE REGULARIZATION

Let us consider $\mathcal{N} = 1$ SQCD interacting with the external Abelian gauge superfield V . This theory can be described by the action

$$\begin{aligned}
S = S_{\text{gauge}} + S_{\text{matter}} &= \frac{1}{2g_0^2} \text{trRe} \int d^4x d^2\theta W^a W_a \\
&+ \frac{1}{4e_0^2} \text{Re} \int d^4x d^2\theta W^a W_a \\
&+ \sum_{f=1}^{N_f} \left[\frac{1}{4} \int d^4x d^4\theta (\Phi_f^+ e^{2q_f V + 2V} \Phi_f + \tilde{\Phi}_f^+ e^{-2q_f V - 2V'} \tilde{\Phi}_f) \right. \\
&\left. + \left(\frac{1}{2} \int d^4x d^2\theta m_{0f} \tilde{\Phi}_f^t \Phi_f + \text{c.c.} \right) \right], \quad (16)
\end{aligned}$$

where e_0 is the bare coupling constant for the group $U(1)$ and $q_f e_0$ is a charge of the superfield with respect to $U(1)$. The sum runs over all flavors, a is the spinor index, g_0 is the bare coupling constant for the non-Abelian gauge group G , which corresponds to the real gauge superfield V . (Also we use the notation $\alpha_0 = e_0^2/4\pi$, $\alpha_{0s} = g_0^2/4\pi$.) The corresponding field strength is

$$W_a \equiv \frac{1}{8} \bar{D}^2 (e^{-2V} D_a e^{2V}). \quad (17)$$

The Abelian gauge field strength W_a is defined as

$$W_a = \frac{1}{4} \bar{D}^2 D_a V. \quad (18)$$

m_{0f} denotes the bare mass of the matter superfields. Below we consider only the limit $m_0 \rightarrow 0$. For simplicity, we will omit the flavor index f . The Abelian gauge superfield V is treated as an external field and is present only in the

external lines. Due to loop corrections both coupling constants of the theory are running. In this paper we investigate the renormalization of the coupling constant corresponding to the group $U(1)$ and the exact expression for the corresponding RG function, which is the Adler D function.

Certainly, for calculating quantum corrections one should regularize the theory. We are certain that in the Abelian case the NSVZ β function is obtained exactly in all loops for the RG functions defined in terms of the bare coupling constant if the theory is regularized by higher derivatives. For non-Abelian theories the corresponding analysis is not yet fully completed, but there are multiple indications that is the case too. That is why we use the higher covariant derivative regularization [35,36] in this paper, as well as the definition of the RG functions in terms of the bare coupling constant.

The main idea of the higher derivative regularization is adding to the classical action a term with the higher derivatives, which increases a degree of momentum in the propagator. In supersymmetric theories such terms can be easily constructed by using $\mathcal{N} = 1$ superfields [38,39,41]. The argumentation of this paper does not depend on a particular form of the higher derivative regularization.¹

For definiteness, we will stick to one possible variant of the higher derivative term. In order to construct it we use the superfield Ω which is related to the gauge superfield V as follows:

$$e^{2V} \equiv e^{\Omega^+} e^{\Omega}. \quad (19)$$

Under the gauge transformations this superfield transforms as

$$e^{\Omega} \rightarrow e^{iK} e^{\Omega} e^{i\lambda}, \quad (20)$$

where λ is a chiral superfield, a parameter of ordinary gauge transformations, and K is a real superfield, which reflects an arbitrariness of constructing Ω from V . Using the superfield Ω one can construct the gauge covariant supersymmetric derivatives

$$\nabla_a = e^{-\Omega^+} D_a e^{\Omega^+}; \quad \bar{\nabla}_{\dot{a}} = e^{\Omega} \bar{D}_{\dot{a}} e^{-\Omega}. \quad (21)$$

(Acting on a superfield S which transforms as $S \rightarrow e^{iK} S$ these derivatives transform in the same way, $\nabla_a S \rightarrow e^{iK} \nabla_a S$.) Then the possible higher derivative term is

¹Even in the Abelian case there are different versions of the higher derivative regularization, see, e.g., [47–49], which lead to the same structure of quantum corrections.

$$S_\Lambda = \frac{1}{2g_0^2} \text{trRe} \int d^4x d^2\theta (e^{\Omega} W^a e^{-\Omega}) \times \left[R \left(-\frac{\bar{\nabla}^2 \nabla^2}{16\Lambda^2} \right) - 1 \right] (e^{\Omega} W_a e^{-\Omega}), \quad (22)$$

where the parameter Λ (with dimension of mass) plays a role of the ultraviolet cutoff. The function R is the ultraviolet regulator such that $R(0) - 1 = 0$ and $R(x) \rightarrow \infty$ for $x \rightarrow \infty$. For example, it is convenient to choose

$$R(x) = 1 + x^n. \quad (23)$$

In order to fix a gauge it is necessary to add the term S_{gf} to the action. Also one should introduce the corresponding ghosts with the action S_{ghosts} . Here we will not concretize these expressions. We only assume that they do not include matter superfields Φ and $\tilde{\Phi}$.

It is well known that by introducing the higher derivative term we regularize all divergences beyond the one-loop approximation [52]. The remaining one-loop divergences (and the one-loop subdivergencies) should be regularized by inserting the Pauli-Villars determinants² into the generating functional [53],

$$\Gamma[\mathbf{V}] = -i \log \int D\mathbf{V} D\Phi D\tilde{\Phi} \prod_{I=1}^m \det(\mathbf{V}, \mathbf{V}, M_I)^{c_I} \times \exp(i(S + S_\Lambda + S_{\text{gf}} + S_{\text{ghosts}})), \quad (24)$$

where $M_I = a_I \Lambda$ and a_I do not depend on α_{0s} and α_0 . We are interested in the case $m_{0f} = 0$, in which, for simplicity, it is possible to assume that the parameters M_I do not depend on the flavor f . Note that sources are not included into this expression, because we consider only diagrams with the external lines corresponding to the Abelian superfield \mathbf{V} . In order to cancel the remaining one-loop divergences, the coefficients c_I should satisfy the constraints

$$\sum_{I=1}^m c_I = 1; \quad \sum_{I=1}^m c_I M_I^2 = 0. \quad (25)$$

The Pauli-Villars determinants can be presented in the form of functional integrals over the corresponding Pauli-Villars superfields

$$\det(\mathbf{V}, \mathbf{V}, M_I)^{-1} = \int D\Phi_I D\tilde{\Phi}_I e^{iS_I}, \quad (26)$$

where

²The Pauli-Villars determinants should be also introduced for ghosts, but in this paper they are not essential and we do not write them explicitly.

$$S_I = \sum_f \left[\frac{1}{4} \int d^4x d^4\theta (\Phi_I^+ e^{2qV+2V} \Phi_I + \tilde{\Phi}_I^+ e^{-2qV-2V} \tilde{\Phi}_I) + \left(\frac{1}{2} \int d^4x d^2\theta M_I \tilde{\Phi}_I^+ \Phi_I + \text{c.c.} \right) \right]. \quad (27)$$

Let us note that the functional integral over the usual matter fields Φ and $\tilde{\Phi}$ can be also written as a determinant with $M_0 = 0$ and $c_0 = -1$. This allows one to treat the usual fields and the Pauli-Villars fields in a similar manner and rewrite Eq. (25) in a simpler form

$$\sum_{I=0}^m c_I = 0; \quad \sum_{I=0}^m c_I M_I^2 = 0. \quad (28)$$

The two-point Green function of the Abelian gauge superfield \mathbf{V} is transversal

$$\Delta\Gamma^{(2)} = -\frac{1}{16\pi} \int \frac{d^4p}{(2\pi)^4} d^4\theta \mathbf{V}(\theta, -p) \partial^2 \Pi_{1/2} \times \mathbf{V}(\theta, p) (d^{-1}(\alpha_0, \alpha_{0s}, \Lambda/p) - \alpha_0^{-1}), \quad (29)$$

because of the $U(1)$ background gauge invariance. (In this equation

$$\partial^2 \Pi_{1/2} = -D^a \bar{D}^2 D_a / 8$$

denotes the supersymmetric transversal projection operator.) We will calculate the function

$$D(\alpha_{0s}) = 6\pi^2 \frac{d}{d \log \Lambda} \Pi(\alpha_{0s}(\alpha_s, \Lambda/\mu), \Lambda/p)|_{p=0} = \frac{3\pi}{2} \frac{d}{d \log \Lambda} (d^{-1}(\alpha_0, \alpha_{0s}, \Lambda/p) - \alpha_0^{-1})|_{p=0} = \frac{3\pi}{2\alpha_0^2} \frac{d\alpha_0}{d \log \Lambda}, \quad (30)$$

where the differentiation is performed at fixed values of the renormalized coupling constants α_s and α . Writing the last identity we take into account that the function d^{-1} expressed in terms of the renormalized coupling constant should be finite. Possible finite terms proportional to p/Λ vanish in the limit $p \rightarrow 0$. In order to extract the expression (30) from the effective action, we differentiate $\Delta\Gamma = \Gamma - S$ with respect to $\log \Lambda$ and then make a substitution

$$\mathbf{V}(x, \theta) \rightarrow \theta^4. \quad (31)$$

Then we easily obtain

$$\frac{1}{3\pi^2} \mathcal{V}_4 \cdot D(\alpha_{0s}) = \left. \frac{d(\Delta\Gamma^{(2)})}{d \log \Lambda} \right|_{\mathbf{V}=\theta^4}, \quad (32)$$

where $\mathcal{V}_4 \rightarrow \infty$ is the (properly regularized) space-time volume [45].

III. EXACT EQUATION FOR THE TWO-POINT FUNCTION OF THE ABELIAN GAUGE SUPERFIELD

Let us derive the exact expression for the function D . First, we prove that this function is given by integrals of double total derivatives in the momentum space. This can be done using the argumentation similar to the one proposed in Ref. [44]. First, it is necessary to calculate formally the integral over the matter superfields. This can be done, because the action is quadratic in them. However, this calculation should be carried out very carefully, because the matter superfields satisfy the chirality constraint. The result can be written in the following form [44]:

$$B \equiv \begin{pmatrix} 0 & (e^{-2qV+2V'} - 1) & 0 & 0 \\ (e^{2qV+2V} - 1) & 0 & 0 & 0 \\ 0 & 0 & 0 & (e^{-2qV-2V} - 1) \\ 0 & 0 & (e^{-2qV-2V'} - 1) & 0 \end{pmatrix} \quad (35)$$

encodes vertices of the theory, and the matrix

$$P \equiv \begin{pmatrix} 0 & \frac{\bar{D}^2 D^2}{16(\partial^2 + M^2)} & \frac{M \bar{D}^2}{4(\partial^2 + M^2)} & 0 \\ \frac{D^2 \bar{D}^2}{16(\partial^2 + M^2)} & 0 & 0 & \frac{M D^2}{4(\partial^2 + M^2)} \\ \frac{M \bar{D}^2}{4(\partial^2 + M^2)} & 0 & 0 & \frac{\bar{D}^2 D^2}{16(\partial^2 + M^2)} \\ 0 & \frac{M D^2}{4(\partial^2 + M^2)} & \frac{D^2 \bar{D}^2}{16(\partial^2 + M^2)} & 0 \end{pmatrix} \quad (36)$$

contains propagators of various matter superfields. In our notation the strings and rows of these matrices correspond to the following sequence of the matter superfields:

$$(\Phi, \Phi^*, \tilde{\Phi}, \tilde{\Phi}^*). \quad (37)$$

The expression \star encodes a sequence of vertices and matter propagators. In the case $V = 0$ we use the notations

$$\star \equiv \star|_{V=0}; \quad I_0 \equiv I_0|_{V=0}; \quad B \equiv B|_{V=0}. \quad (38)$$

These expressions correspond to the diagrams without the external lines of the Abelian gauge superfield. In particular, the operator

$$\star = \frac{1}{1 - I_0} = 1 + BP + BPBP + BPBPBP + \dots \quad (39)$$

encodes chains of vertices B (in which the external gauge superfield V is set to zero, and only V is kept) and the matter

$$\exp(i\Gamma[V]) = \int DV \prod_{I=0}^m \prod_{f=1}^{N_f} \det(\star_{I,f})^{c_I/2} \times \exp\{i(S_{\text{gauge}} + S_\Lambda + S_{\text{gf}} + S_{\text{ghosts}})\}, \quad (33)$$

where we use the notation

$$\star \equiv \frac{1}{1 - I_0}; \quad I_0 = BP. \quad (34)$$

(For simplicity we omit the subscripts I and f for \star , I_0 , and P , which mark the dependence on I via the mass M_I and on f via the electric charge q_f .) The matrix

propagators P . Graphically this equation is presented in Fig. 2.

Let us recall that the case $I = 0$ corresponds to the original theory in the massless limit, so that $c_0 = -1$ and $M_0 = 0$. Below we will also use the following operators:

$$(I_1)_a \equiv [I_0, \theta_a]; \quad (\bar{I}_1)_{\dot{a}} \equiv [I_0, \bar{\theta}_{\dot{a}}]. \quad (40)$$

It is important that all these operators do not manifestly depend on θ and $\bar{\theta}$. (By other words, θ and $\bar{\theta}$ are present only inside the supersymmetric covariant derivatives.)

In order to calculate the two-point Green function of the background Abelian gauge superfield V we should find terms quadratic in V in Eq. (33). The result has the form

$$\star = 1 + BP + BPBP + BPBPBP + \dots$$

FIG. 2. Graphical interpretation of the operator \star .

$$\begin{aligned} \Delta\Gamma^{(2)} = & -\frac{i}{2} \left(\sum_{f=1}^{N_f} q_f \right)^2 \cdot \left\langle \left(\sum_{I=0}^m c_I \text{Tr}(\mathbf{V}QJ_0\star)_I \right)^2 \right\rangle_{\text{1PI}} \\ & + i \sum_{f=1}^{N_f} q_f^2 \cdot \sum_{I=0}^m c_I \langle \text{Tr}(\mathbf{V}QJ_0\star\mathbf{V}QJ_0\star) + \text{Tr}(\mathbf{V}^2J_0\star) \rangle_{I,\text{1PI}}, \end{aligned} \quad (41)$$

where the symbol 1PI implies that it is necessary to omit all diagrams except for the one-particle irreducible (1PI) ones,

$$Q \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad J_0 \equiv \begin{pmatrix} 0 & e^{2V'} & 0 & 0 \\ e^{2V} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2V} \\ 0 & 0 & e^{-2V'} & 0 \end{pmatrix} P, \quad (42)$$

and

$$\langle A[V] \rangle \equiv \frac{\int DVA[V] \prod_{I=0}^m \det(\star_I)^{N_f c_I/2} \exp\{i(S_{\text{gauge}} + S_\Lambda + S_{\text{gf}} + S_{\text{ghosts}})\}}{\int DV \prod_{I=0}^m \det(\star_I)^{N_f c_I/2} \exp\{i(S_{\text{gauge}} + S_\Lambda + S_{\text{gf}} + S_{\text{ghosts}})\}}. \quad (43)$$

The term in Eq. (41) proportional to $(\sum_f q_f)^2$ corresponds to attaching the external lines to different loops of the matter superfields. The terms in Eq. (41) proportional to $\sum_f q_f^2$ encode diagrams in which two external lines are attached to a single matter loop. The exact Adler function D defined in terms of the bare coupling constant can be found from Eq. (41) by the following prescription:

$$D(\alpha_{0s}) = \frac{3\pi^2}{\mathcal{V}_4} \cdot \left. \frac{d\Delta\Gamma^{(2)}}{d \log \Lambda} \right|_{V=\theta^4}. \quad (44)$$

IV. RELATION BETWEEN THE FUNCTION D AND THE ANOMALOUS DIMENSION OF THE MATTER SUPERFIELDS

A. Summation of subdiagrams

We calculate the expression (41) after the substitution $V \rightarrow \theta^4$. This expression encodes the sum of all supergraps with two external V lines. Summing these supergraps we encounter certain sequences of subdiagrams presented in Fig. 3, in which the wavy lines correspond to $V = \theta^4$. The left and right dots correspond to vertices to which an arbitrary number of V lines can be attached. The middle dot (if it is present) corresponds to the vertex *without* any V lines (and with a single V line). Formally, the subdiagrams presented in Fig. 3 can be constructed by transforming the expression $\star V J_0$. The details of this calculation are presented in the Appendix.

To find these sums of subdiagrams the external line is commuted (with the operators corresponding to the propagators) to the left. Then we obtain two vertices

joined by the matter line. This matter line corresponds to a certain operator which is obtained after the commutation and summing the results. (This operator replaces the ordinary propagator.) θ 's are attached to the left vertex. This procedure is qualitatively illustrated in Fig. 4 for the case corresponding to the first string in Fig. 3.

The results for the sums of the subdiagrams presented in Fig. 3 were found in Ref. [44]. (Note that the external line corresponds to the Abelian field, so that it is possible to use the result obtained earlier for the Abelian case.) It is convenient to present the sum of subdiagrams in the matrix form, namely,

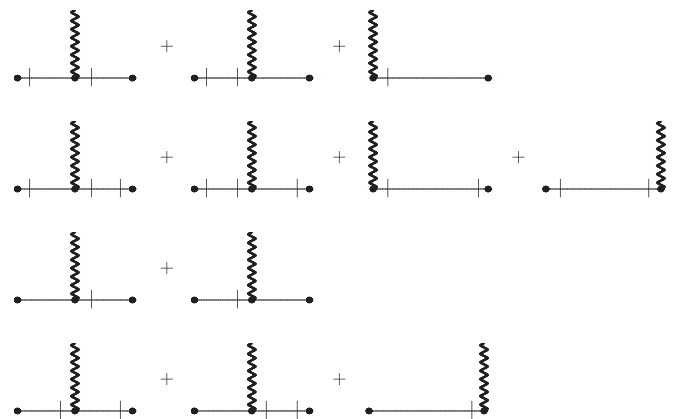


FIG. 3. Summation of subdiagrams. The wavy lines correspond to the Abelian background gauge superfield $V = \theta^4$. To the left and right vertices one can attach an arbitrary number ≥ 1 of the internal lines of the non-Abelian gauge superfield. The middle vertex (if it is present) does not contain such lines at all.

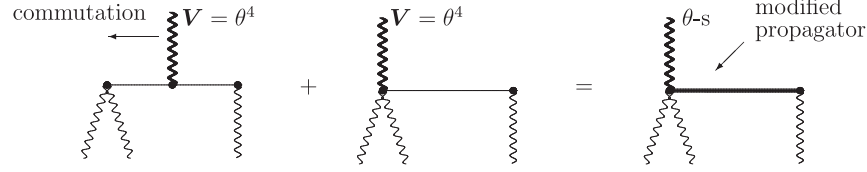


FIG. 4. An example illustrating summation of the effective diagrams, which gives the modified propagator. (This modified propagator is denoted by the bold line.)

$$i\bar{\theta}^{\dot{a}}(\gamma^{\mu})_{\dot{a}}{}^b\theta_b[y_{\mu}^*, \tilde{Q}I_0] - 2\theta^a\theta_a\bar{\theta}^{\dot{b}}[\bar{\theta}_{\dot{b}}, \tilde{Q}I_0] + \tilde{Q}\bar{\theta}^{\dot{a}}B \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{i(\gamma^{\mu})_{\dot{a}}{}^b D_b \bar{D}^2 \partial_{\mu}}{4(\partial^2 + M^2)^2} + \frac{\bar{D}_{\dot{a}}}{\partial^2 + M^2} & 0 & 0 & \frac{M\bar{D}_{\dot{a}}D^2}{4(\partial^2 + M^2)^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{M\bar{D}_{\dot{a}}D^2}{4(\partial^2 + M^2)^2} & \frac{i(\gamma^{\mu})_{\dot{a}}{}^b D_b \bar{D}^2 \partial_{\mu}}{4(\partial^2 + M^2)^2} + \frac{\bar{D}_{\dot{a}}}{\partial^2 + M^2} & 0 \end{pmatrix} + \text{terms without } \bar{\theta}, \quad (45)$$

where $y_{\mu}^* = x_{\mu} - i\bar{\theta}^{\dot{a}}(\gamma_{\mu})_{\dot{a}}{}^b\theta_b$ and

$$\tilde{Q} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (46)$$

Equation (45) contains exponents corresponding to the left vertices and the (modified) propagators which are attached to these vertices from the right. Below we will use the following properties of the matrix (46):

$$[\tilde{Q}, I_0] = 0; \quad [\tilde{Q}, \star] = 0; \quad \tilde{Q}^2 = 1. \quad (47)$$

B. External lines are attached to different matter loops

Let us consider the term in Eq. (41) proportional to $(\sum_f q_f)^2$. It corresponds to the case in which the external lines are attached to different closed loops of the matter superfields and, therefore, contributes to the singlet part of the function D . Such diagrams are sketched in the left-hand side of Fig. 1. The term under consideration contains the expression

$$\text{Tr}(VQJ_0\star) \quad (48)$$

in the second power. In the Appendix we prove that this expression can be presented as

$$\text{Tr}(VQJ_0\star) = \text{Tr}\{\star(BP(VQ)B_0P + B(VQ)(\Pi_+P) + B(P\Pi_-)(VQ))\}, \quad (49)$$

where Π_{\pm} are chiral projection operators [given by Eq. (A4)] and B_0 [given by Eq. (A2)] corresponds to the vertex with a single external V line and no V lines. The

expression in the round brackets coincides with the sum of subdiagrams presented in Fig. 3, which was calculated in Sec. IV A and is given by Eq. (45).

It is well known that any supergraph does not vanish only if it contains θ^4 . Therefore, if we multiply two expressions (49), the nontrivial contributions come only from terms which are linear in $\bar{\theta}$. [There are no terms quadratic in $\bar{\theta}$ in Eq. (45).] Let us consider the sum of the diagrams in which the matter loop under consideration [corresponding to the expression (48)] has n vertices with the internal gauge lines. (Vertices with a single external V line are not summed.) We will denote the corresponding contribution as

$$\text{Tr}(VQJ_0\star)_n. \quad (50)$$

Using Eqs. (49) and (45) after the substitution $V \rightarrow \theta^4$ this expression can be presented as

$$\begin{aligned} \text{Tr}(\theta^4 QJ_0\star)_n &= \text{Tr}(i\bar{\theta}^{\dot{c}}(\gamma^{\mu})_{\dot{c}}{}^d\theta_d[y_{\mu}^*, \tilde{Q}I_0]\star \\ &\quad - 2\theta^c\theta_c\bar{\theta}^{\dot{d}}[\bar{\theta}_{\dot{d}}, \tilde{Q}I_0]\star + \bar{\theta}^1\text{terms})_n \\ &= \text{Tr}(i\bar{\theta}^{\dot{c}}(\gamma^{\mu})_{\dot{c}}{}^d\theta_d[y_{\mu}^*, \tilde{Q}I_0](\star)_{n-1} \\ &\quad - 2\theta^c\theta_c\bar{\theta}^{\dot{d}}[\bar{\theta}_{\dot{d}}, \tilde{Q}I_0](\star)_{n-1} + \bar{\theta}^1\text{terms}). \end{aligned} \quad (51)$$

(One vertex is written explicitly and corresponds to B inside I_0 . Therefore, \star should give $n - 1$ vertices.)

Comparing the Taylor expansions one can easily see that

$$(\star^2)_n = (n + 1)(\star)_n. \quad (52)$$

Therefore, after a cyclic permutation the expression under consideration can be written as follows:

$$\begin{aligned}
& \frac{1}{n} \text{Tr} \{ \tilde{Q}(-2\theta^c \theta_c \tilde{\theta}^d \star [\bar{\theta}_d, I_0] \star + i\tilde{\theta}^c (\gamma^\mu)_{\dot{c}}{}^d \theta_d \star [y_\mu^*, I_0] \star) \\
& \quad + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{terms} \}_n \\
& = \frac{1}{n} \text{Tr} \{ \tilde{Q}(-2\theta^c \theta_c \tilde{\theta}^d [\bar{\theta}_d, \star] + i\tilde{\theta}^c (\gamma^\mu)_{\dot{c}}{}^d \theta_d [y_\mu^*, \star]) \\
& \quad + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{terms} \}_n. \tag{53}
\end{aligned}$$

Again comparing the Taylor expansions we obtain

$$(\log \star)_n = \frac{1}{n} (\star)_n. \tag{54}$$

Therefore, Eq. (50) gives

$$\begin{aligned}
& \text{Tr} \{ \tilde{Q}(-2\theta^c \theta_c \tilde{\theta}^d [\bar{\theta}_d, \log \star] + i\tilde{\theta}^c (\gamma^\nu)_{\dot{c}}{}^d \theta_d [y_\nu^*, \log \star]) \\
& \quad + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{terms} \}_n \\
& = \text{Tr} \{ i\tilde{\theta}^c (\gamma^\nu)_{\dot{c}}{}^d \theta_d \tilde{Q}[x_\nu, \log \star] + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{terms} \}_n, \tag{55}
\end{aligned}$$

where we take into account the fact that the trace of θ commutators always gives 0. Again using this fact and calculating the square of the last expression we see that the remaining terms proportional to $\theta^2, \bar{\theta}^1, \theta^1$, or θ^0 do not give θ^4 and can be omitted. Therefore (summing contributions for all n) we arrive at

$$\text{Tr}(\mathbf{V}QJ_0 \star) |_{V=\theta^4} \rightarrow i \text{Tr} \tilde{\theta}^c (\gamma^\nu)_{\dot{c}}{}^d \theta_d \tilde{Q}[x_\nu, \log \star], \tag{56}$$

where the symbol \rightarrow means that in the right-hand side we omit terms which give vanishing contribution to the whole supergraph.

The total singlet contribution to the effective action can be written as

$$\begin{aligned}
\Delta\Gamma^{(2)} & = \frac{i}{2} \left(\sum_{f=1}^{N_f} q_f \right)^2 \\
& \quad \cdot \left\langle \left(\sum_{I=0}^m c_I \text{Tr} \tilde{\theta}^{\dot{a}} (\gamma^\nu)_{\dot{a}}{}^b \theta_b \tilde{Q}[x_\nu, \log(\star_I)] \right)^2 \right\rangle. \tag{57}
\end{aligned}$$

The commutator with x^μ in the momentum space gives an integral of a total derivative. Taking into account that these commutators enter Eq. (57) in the second power, we see that the contributions of the diagrams under consideration (in which external lines are attached to distinct loops of the matter superfields) are given by integrals of double total derivatives.

Each of these total derivatives is taken with respect to the momentum of its closed matter loop. It is important that in this case no singularities appear in the integrand, because there are only factors Q^{-2} in the numerator, and there are no

factors Q^μ/Q^4 . Therefore, all integrals of total derivatives vanish.

Thus, the class of diagrams considered in this section gives vanishing contribution to the Adler D function.

Let us also note that if the matter loop corresponds to the usual superfields Φ and $\tilde{\Phi}$ (for which $M = 0$), Eq. (56) can be equivalently rewritten as

$$\text{Tr}(\mathbf{V}QJ_0 \star) |_{V=\theta^4} \rightarrow 2i \text{Tr} \tilde{\theta}^c (\gamma^\nu)_{\dot{c}}{}^d \theta_d [x_\nu, \log(\ast) - \log(\tilde{\ast})], \tag{58}$$

where

$$\begin{aligned}
\ast & \equiv \frac{1}{1 - (e^{2V} - 1)\bar{D}^2 D^2 / 16\partial^2}, \\
\tilde{\ast} & \equiv \frac{1}{1 - (e^{-2V'} - 1)\bar{D}^2 D^2 / 16\partial^2}. \tag{59}
\end{aligned}$$

Therefore, taking into account that $c_0 = -1$ the result can be also presented in the form

$$\begin{aligned}
\Delta\Gamma^{(2)} & = 2i \left(\sum_{f=1}^{N_f} q_f \right)^2 \cdot \langle [\text{Tr} \tilde{\theta}^c (\gamma^\nu)_{\dot{c}}{}^d \theta_d [x_\nu, \log(\ast) - \log(\tilde{\ast})] \\
& \quad + (PV)]^2 \rangle = 0, \tag{60}
\end{aligned}$$

which was used in [1]. [Here (PV) denotes contributions of diagrams with the loops of the Pauli-Villars superfields.] As was discussed above, the sum of the diagrams with the Pauli-Villars loop(s) is also given by a vanishing integral of a double total derivative.

C. External lines are attached to a single matter loop

In this case we should consider the terms in Eq. (41) proportional to $\sum_f q_f^2$, which give the nonsinglet contribution. The term containing V^2 vanishes after the substitution $V \rightarrow \theta^4$. The remaining term

$$i \sum_{f=1}^{N_f} q_f^2 \cdot \sum_{I=0}^m c_I \langle \text{Tr}(\mathbf{V}QJ_0 \star \mathbf{V}QJ_0 \star) \rangle_{I, \text{1PI}} \tag{61}$$

gives six effective diagrams with two external lines corresponding to the Abelian background superfield. These diagrams are presented in Fig. 5. They can be obtained from Eq. (61) using the results of the Appendix. The first five diagrams (a)–(e) contain two group of subdiagrams presented earlier in Fig. 3. Each bold line corresponds to the modified propagator which appears after summation of the corresponding group of subdiagrams (see Fig. 4). Expressions for the external lines are obtained by extracting the terms proportional to the corresponding θ structure from Eq. (45). There are no other relevant diagrams, because a nonvanishing supergraph should contain the second power of θ and the second

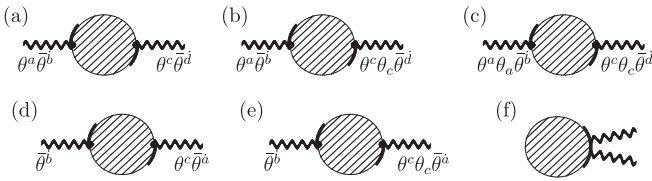


FIG. 5. Effective diagrams which appear if the external V lines are attached to a single matter loop.

power of $\bar{\theta}$. [The diagram (d) contains θ^2 , because one θ comes from the commutator with y_μ^* .]

The last effective diagram (f) contains contribution of those graphs in which external lines are close to each other. In the Appendix we explain why they should be added. The vertex in the diagram (f) corresponds to the sum of many subdiagrams. All of them are collected in Ref. [44]. Their structure is described in the Appendix.

Calculation of the graphs (a)–(f) exactly repeats the corresponding calculation that was carried out in Ref. [44] in the Abelian case. The only difference is the presence of the charges q_f^2 and the sum over all flavors. All non-Abelian effects are encoded in the angular brackets, which are now defined in a different way (due to the different form of the gauge part of the action). The result can be written in the following form³:

$$\begin{aligned}
& i \frac{d}{d \log \Lambda} \sum_f q_f^2 \sum_{I=0}^m c_I \langle \text{Tr}(\mathbf{V} Q J_0 \star \mathbf{V} Q J_0 \star) \rangle_I \Big|_{\mathbf{v}=\theta^4} \\
&= \text{One-loop result} - \frac{i}{2} \frac{d}{d \log \Lambda} \\
& \quad \times \sum_f q_f^2 \sum_{I=0}^m c_I \text{Tr} \langle \theta^4 [y_\mu^*, [(y^\mu)^*, \log(\star)]] \rangle_I \\
& \quad - \text{singular terms containing } \delta\text{-functions,} \tag{62}
\end{aligned}$$

where it is necessary to subtract singularities of the expression in the second string of this equation. Taking into account that the trace of the θ commutators always vanishes, we can replace y_μ^* with x_μ in this expression.

The term with $I = 0$ ($c_0 = -1$, $M_0 = 0$) corresponds to the case in which the external lines are attached to the loop of ordinary superfields Φ and $\tilde{\Phi}$, while for $I \geq 1$ they are attached to the Pauli-Villars loop. Therefore, it is possible to present Eq. (62) in the form [1]

$$\begin{aligned}
& \text{one-loop result} + i \frac{d}{d \log \Lambda} \sum_f q_f^2 \text{Tr} \langle \theta^4 [x_\mu, [x^\mu, \log(\star) \\
& \quad + \log(\tilde{\star})]] \rangle + (PV) \\
& \quad - \text{singular terms containing } \delta\text{ functions,} \tag{63}
\end{aligned}$$

³Unlike Ref. [44] here we treat the usual matter superfields and the Pauli-Villars superfields in the same way.

where (PV) denotes the contributions of the Pauli-Villars loops.

According to Eq. (62) the Adler D function (defined in terms of the bare coupling constant) is given by integrals of double total derivatives exactly in the same way as the β function (defined in terms of the bare coupling constant) in the Abelian case [43,44,49]. This structure allows us to calculate one of the loop integrals analytically and relate the result to the anomalous dimensions of the matter superfields. To this end it is convenient to rewrite Eq. (62) in a different form.

Let us consider the sum of the diagrams with n vertices on the matter loop to which external lines are attached. As in the previous section this sum is denoted by the subscript n . Then, after simple transformations (see Ref. [44] for details), the right-hand side of Eq. (62) (for simplicity, without the one-loop contribution) can be equivalently presented in the form

$$\begin{aligned}
& - \frac{d}{d \log \Lambda} \sum_f q_f^2 \sum_{I=0}^m c_I \text{Tr} \left\langle \theta^4 \left[y_\mu^*, \frac{i}{2} [(y^\mu)^*, I_0] \star \right. \right. \\
& \quad \left. \left. - \sum_{a+b+2=n} \frac{(b+1)(\gamma^\mu)^{cd}}{n} (I_1)_c(\star)_a (\bar{I}_1)_d(\star)_b \right] \right\rangle_{I,n} \\
& \quad - \text{singular terms containing } \delta\text{ functions,} \tag{64}
\end{aligned}$$

where $(\star)_a$ and $(\star)_b$ (with $a + b + 2 = n$) denote the a th and b th terms in the Taylor expansion of \star , respectively. The traces of the commutators in Eqs. (62) and (64) evidently vanish. Therefore, the result for the Adler function is completely determined by the singular contributions. In the next section we will calculate them starting from Eq. (64).

D. Summation of singularities

We have proved that the Adler function (defined in terms of the bare coupling constant) is given by integrals of double total derivatives in the momentum space. It is important to note that these integrals do not vanish due to singularities, which originate from the identity

$$\frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \frac{1}{Q^2} = -2 \frac{\partial}{\partial Q^\mu} \frac{Q^\mu}{Q^4} = -4\pi^2 \delta^4(Q) \tag{65}$$

(which is written in the Euclidean space after the Wick rotation). Indeed, for a nonsingular function $f(Q^2)$, with a sufficiently rapid falloff at infinity,

$$\begin{aligned}
& \int \frac{d^4 Q}{(2\pi)^4} \frac{Q^\mu}{Q^4} \frac{\partial f}{\partial Q^\mu} = \frac{1}{8\pi^2} \int_0^\infty dQ^2 \frac{df(Q^2)}{dQ^2} = -\frac{1}{8\pi^2} f(0) \\
& \quad = -\frac{1}{8\pi^2} \int d^4 Q \delta^4(Q) f \\
& \quad = - \int \frac{d^4 Q}{(2\pi)^4} \frac{\partial}{\partial Q^\mu} \frac{Q^\mu}{Q^4} \cdot f. \tag{66}
\end{aligned}$$

Beyond the one-loop approximation all diagrams in which the external lines are attached to the closed loops of the Pauli-Villars fields give a vanishing contribution, because there are no such singularities. One loop in this approach should be considered separately.

Thus, we must consider only diagrams in which two external lines are attached to a single loop of the (non-regulator) superfields Φ and $\tilde{\Phi}$. This case corresponds to $I = 0$. (Let us recall that $c_0 = -1$, $M_0 = 0$.) The $I = 0$ part of Eq. (64) can be reduced to a simpler expression

$$\begin{aligned} & \sum_f q_f^2 \frac{d}{d \log \Lambda} \text{Tr} \left\langle \theta^4 \left[y_{\mu}^*, i(e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_{\mu}}{8\partial^4} * \right. \right. \\ & - \sum_{a+b+2=n} \frac{2(b+1)(\gamma^{\mu})^{cd}}{n} (e^{2V} - 1) \frac{\bar{D}^2 D_c}{8\partial^2} (*)_a \\ & \times (e^{2V} - 1) \frac{\bar{D}_d D^2}{8\partial^2} (*)_b + i(e^{-2V'} - 1) \frac{\bar{D}^2 D^2 \partial_{\mu}}{8\partial^4} \tilde{*} \\ & - \sum_{a+b+2=n} \frac{2(b+1)(\gamma^{\mu})^{cd}}{n} (e^{-2V'} - 1) \frac{\bar{D}^2 D_c}{8\partial^2} \\ & \left. \left. \times (\tilde{*})_a (e^{-2V'} - 1) \frac{\bar{D}_d D^2}{8\partial^2} (\tilde{*})_b \right] \right\rangle_n \\ & - \text{singular terms containing } \delta \text{ functions.} \end{aligned} \quad (67)$$

It is easy to see that the contributions corresponding to the superfield Φ (which contain e^{2V} and $*$) are equal to the ones corresponding to the superfield $\tilde{\Phi}$ (which contain $e^{-2V'}$ and $\tilde{*}$). To prove this statement we make the substitution $V \rightarrow -V'$ in the functional integrals in Eq. (43). Then

$$\Omega \rightarrow -\Omega'; \quad W_a \rightarrow -(W_a)'; \quad V_{\text{Adj}} \rightarrow V_{\text{Adj}}, \quad (68)$$

where (for arbitrary X which belongs to the Lie algebra of the gauge group) $V_{\text{Adj}} X \equiv [V, X]$. This implies that S_{gauge} , S_{Λ} , S_{gf} , and S_{ghosts} remain unchanged. As a consequence, we obtain the required statement. Thus, the expression under consideration can be written as

$$\begin{aligned} & \sum_f q_f^2 \frac{d}{d \log \Lambda} \text{Tr} \left\langle \theta^4 \left[y_{\mu}^*, i(e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_{\mu}}{4\partial^4} * \right. \right. \\ & - \sum_{a+b+2=n} \frac{4(b+1)(\gamma^{\mu})^{cd}}{n} (e^{2V} - 1) \frac{\bar{D}^2 D_c}{8\partial^2} (*)_a \\ & \left. \left. \times (e^{2V} - 1) \frac{\bar{D}_d D^2}{8\partial^2} (*)_b \right] \right\rangle_n \\ & - \text{singular terms containing } \delta \text{ functions.} \end{aligned} \quad (69)$$

Singular contributions appear if the derivative $\partial/\partial Q^{\mu}$ acts on Q^{μ}/Q^4 . Such terms can come both from the first

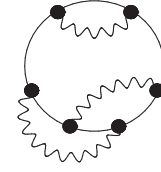


FIG. 6. This diagram contains $p = 2$ coinciding momenta in the matter loop, which can lead to a singular contribution. This is related to the fact that two cuts of the matter line can make this diagram disconnected.

term of Eq. (69) and from the second term. In the former case the derivative with respect to the momentum Q^{μ} corresponds to the commutator with y_{μ}^* , and Q^{μ}/Q^4 is obtained from $\partial_{\mu}/\partial^4$. The latter possibility requires the existence of the coinciding momenta in the matter loop. Let us denote as p the number of the coinciding momenta in the matter loop. An example of a diagram with $p = 2$ is presented in Fig. 6.

Repeating the calculations of Ref. [44], the contribution of the first term in Eq. (69) can be rewritten as

$$\begin{aligned} & - \sum_f q_f^2 \frac{d}{d \log \Lambda} \text{Tr} \left\langle \frac{\pi^2}{2} \theta^4 * (e^{2V} - 1) \bar{D}^2 D^2 \delta^4(\partial) \right\rangle \\ & = \frac{1}{2\pi^2} \mathcal{V}_4 \sum_f q_f^2 \cdot \text{tr} \frac{d}{d \log \Lambda} G^{-1} |_{Q=0}, \end{aligned} \quad (70)$$

where tr denotes the conventional matrix trace. By definition, the function $G(\partial^2)$ is related to the two-point Green function of the matter superfields as

$$\frac{\delta^2 \Gamma}{\delta \Phi_j(x) \delta \Phi^{*i}(y)} = \frac{\bar{D}_x^2 D_x^2}{16} G_i^j \delta^4(x-y) \delta^4(\theta_x - \theta_y), \quad (71)$$

where Γ denotes the effective action of the non-Abelian theory.

The expression (70) is written formally, because it gives integrals which are not well defined. The well-defined integrals are obtained after adding contributions of the remaining singularities. They appear from the second term in Eq. (69) if the matter loop has p coinciding momenta ($p \geq 2$). They are situated between p 1PI subdiagrams (see an example in Fig. 6 corresponding to $p = 2$). Let us denote the numbers of vertices at the matter line in these 1PI subdiagrams by k_1, k_2, \dots, k_p . The expression Q^{μ}/Q^4 appears if there is the only 1PI subdiagram between D_c and \bar{D}_d in the second term of Eq. (69); see Fig. 7 in which the disks denote ($p = 6$) 1PI subdiagrams. This follows from the identities

$$\begin{aligned} & \frac{\bar{D}^2 D_c}{8\partial^2} \cdot \frac{\bar{D}_d D^2}{8\partial^2} = i(\gamma^{\mu})_{cd} \frac{\partial_{\mu}}{32\partial^4} \bar{D}^2 D^2; \\ & \frac{\bar{D}_d D^2}{8\partial^2} \cdot \frac{\bar{D}^2 D_c}{8\partial^2} = -i(\gamma^{\mu})_{cd} \frac{\partial_{\mu}}{32\partial^4} D^a \bar{D}^2 D_a. \end{aligned} \quad (72)$$

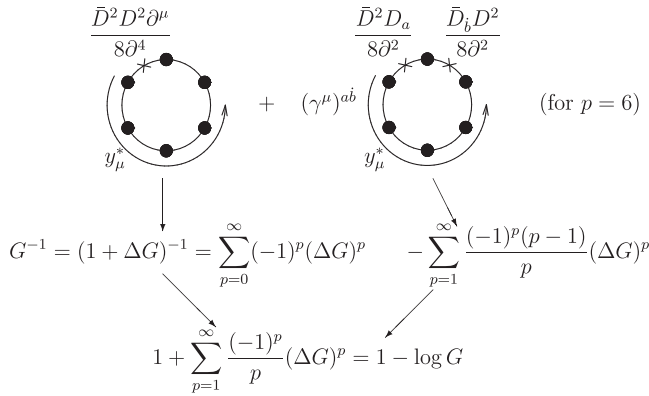


FIG. 7. Obtaining the exact expression for the Adler function by summation of singular contributions. The example presented in this figure corresponds to the case $p = 6$.

Any other possibilities are excluded due to the equalities

$$\frac{\bar{D}^2 D^2}{16\partial^2} \cdot \frac{\bar{D}_b D^2}{8\partial^2} = 0; \quad \frac{\bar{D}^2 D_a}{8\partial^2} \cdot \frac{\bar{D}^2 D^2}{16\partial^2} = 0. \quad (73)$$

Therefore, for $p \geq 2$ we obtain p singular contributions inside the second term in Eq. (69), which correspond to the following values of a and b :

$$\begin{aligned} a+1 &= k_1; & b+1 &= k_2 + k_3 + \dots + k_p; \\ a+1 &= k_2; & b+1 &= k_1 + k_3 + \dots + k_p; \\ & \dots & & \\ a+1 &= k_p; & b+1 &= k_1 + k_2 + \dots + k_{p-1}, \end{aligned} \quad (74)$$

where $n = k_1 + k_2 + \dots + k_p$. Then one can obtain (see Ref. [44] for more details) that the corresponding singular contribution differs from the contribution of the first term in Eq. (69) by the factor

$$\begin{aligned} & -\frac{1}{p} \left(\frac{k_1 + \dots + k_{p-1}}{k_1 + k_2 + \dots + k_p} + \frac{k_1 + \dots + k_{p-2} + k_p}{k_1 + k_2 + \dots + k_p} \right. \\ & \left. + \dots + \frac{k_2 + \dots + k_p}{k_1 + k_2 + \dots + k_p} \right) = -\frac{p-1}{p}. \end{aligned} \quad (75)$$

Therefore, the coefficients in the sum of all singularities differ from the coefficients corresponding to Eq. (70) by

$$1 - \frac{p-1}{p} = \frac{1}{p}. \quad (76)$$

The whole contribution of diagrams with p coinciding momenta is proportional to $(\Delta G)^p$, where $\Delta G \equiv G - 1$. (ΔG corresponds to the sum of 1PI diagrams starting from one loop.) Then we compare the Taylor expansions

$$\log G = \log(1 + \Delta G) = -\sum_{p=1}^{\infty} \frac{(-1)^p}{p} (\Delta G)^p \quad \text{and}$$

$$G^{-1} = \frac{1}{1 + \Delta G} = \sum_{p=0}^{\infty} (-1)^p (\Delta G)^p. \quad (77)$$

We see that after adding the singular contributions coming from the second term in Eq. (69) the result can be written as

$$\begin{aligned} \left. \frac{d\Delta\Gamma^{(2)}}{d\log\Lambda} \right|_{v=\theta^4} &= \text{One loop} - \frac{1}{2\pi^2} \mathcal{V}_4 \sum_f q_f^2 \cdot \text{tr} \left. \frac{d\log G}{d\log\Lambda} \right|_{Q=0} \\ &= \frac{1}{2\pi^2} \mathcal{V}_4 \sum_f q_f^2 \cdot \text{tr} \left(1 - \left. \frac{d\log G}{d\log\Lambda} \right|_{Q=0} \right). \end{aligned} \quad (78)$$

Obtaining $\log G$ in this way is illustrated in Fig. 7. Unlike Eq. (70), Eq. (78) leads to the well-defined integrals.

The derivative of $\log G$ with respect to $\log\Lambda$ (which should be calculated at a fixed value of the renormalized coupling constant α_s) in the vanishing momentum limit gives the anomalous dimension defined in terms of the bare coupling constant,

$$\begin{aligned} \left. \frac{d\log G}{d\log\Lambda} \right|_{Q=0} &= \frac{d}{d\log\Lambda} (\log(ZG) - \log Z)|_{Q=0} \\ &= -\frac{\log Z}{d\log\Lambda} = \gamma(\alpha_{0s}) \cdot 1, \end{aligned} \quad (79)$$

where 1 is written in order to stress that in the non-Abelian case the anomalous dimension is $N_c \times N_c$ matrix, where N_c is a number of colors. The matrix trace gives the factor N_c , so that according to Eq. (32) we finally arrive at

$$D(\alpha_{0s}) = \frac{3}{2} \sum_f q_f^2 \cdot N_c (1 - \gamma(\alpha_{0s})). \quad (80)$$

V. $\mathcal{N} = 1$ SQCD IN THE CONFORMAL WINDOW

In this section we will study some consequences ensuing from the exact formulas (1) and (14) in the conformal window [10,11]

$$\frac{3}{2} N_c < N_f < 3N_c. \quad (81)$$

Inside this window SQCD flows to the conformal points: $\gamma = 0$ in the ultraviolet (asymptotic freedom) and

$$\gamma_* = -\frac{3N_c - N_f}{N_f} \quad (82)$$

in the infrared. Equation (1) then implies that the Adler function reads

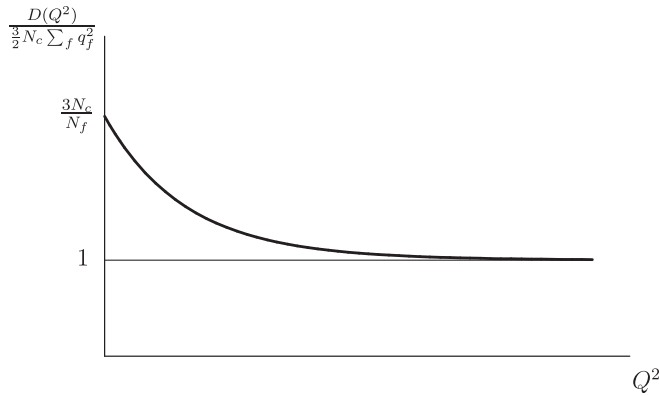


FIG. 8. $D(Q^2) \cdot (\frac{3}{2} N_c \sum_f q_f^2)^{-1}$ versus Q^2 . The horizontal lines correspond to $N_f = 3N_c$, i.e. the right edge of the conformal window.

$$D(Q^2) \longrightarrow \frac{3}{2} N_c \sum_f q_f^2 \times \begin{cases} 1, & Q^2 \rightarrow \infty, \\ \frac{3N_c}{N_f}, & Q^2 \rightarrow 0. \end{cases} \quad (83)$$

The Q^2 evolution of the Adler function in the conformal window is sketched in Fig 8.

The next interesting question is as follows: what new information can be obtained by combining our exact formula (1) with the Seiberg duality? We recall that the Seiberg duality connects with each other two distinct SQCD theories which flow to each other in the infrared where their respective β functions have fixed points. If the number of colors in the original theory is N_c the number of colors in its dual is $\tilde{N}_c = N_f - N_c$. The number of flavors in the original theory and its dual is the same, and both have one and the same (global) flavor symmetry, including the $SU(N_f)_L \times SU(N_f)_R$ factor. To introduce our background $U(1)$ one can gauge a vector subgroup of the above factor, for instance a diagonal subgroup of $SU(2)$. Then the electric charges q_f can be chosen as follows:

$$q(\Phi_1) = q(\tilde{\Phi}_2) = \frac{1}{\sqrt{2}}, \quad q(\tilde{\Phi}_1) = q(\Phi_2) = -\frac{1}{\sqrt{2}}, \quad (84)$$

with vanishing charges for all other matter superfields. In this case

$$\sum_f q_f^2 = 1, \quad \sum_f q_f = 0. \quad (85)$$

The dual theory has two couplings rather than one. In addition to the dual gauge coupling it has a superpotential term

$$\mathcal{W} = \lambda M_{ff'} \Phi_f \tilde{\Phi}_{f'}, \quad (86)$$

where $M_{ff'}$ is the meson Seiberg field [10,11], and λ is a super-Yukawa constant. Both theories are superconformal

in the limits $Q^2 \rightarrow \infty$ or 0. This means that the β function for λ must have zero at nonvanishing values of the coupling constants.

Now, our exact formula gives $D(Q^2 \rightarrow 0)$ both in the original and dual theories, and since they are equivalent in the infrared, this should be one and the same value. Equation (83) gives $D(Q^2 \rightarrow 0)$ in the original theory. What about its dual?

Equation (1) is incomplete in the case of the dual theory. Indeed, it takes into account only the Φ and $\tilde{\Phi}$ matter fields. The M field, being neutral with regards to the gauge group $SU(\tilde{N}_c)$, is not neutral with regards to the chosen flavor $SU(2)$ to which the background photon is coupled. Thus, its loop must be included in an analog of Eq. (1). Moreover, its superpotential interaction (86) will generate the anomalous dimension γ_M of the M superfield. The anomalous dimensions of the Φ and $\tilde{\Phi}$ superfields acquire a matrix (in flavor) structure. Their critical values are still given by (82) with the replacement $N_c \rightarrow \tilde{N}_c$.

Now, requiring that the infrared limit of the Adler functions in the original and dual theories is the same, one should be able to determine the critical value of γ_M .

A few words are in order here to explain why we think that our analysis goes beyond perturbation theory. We are interested in the effective Lagrangian for SQCD with photons expressed in terms of various possible operators. The first operator is $F_{\mu\nu}^2$. It appears perturbatively in one loop; nonperturbatively it could appear as $f(\Phi^2/\Lambda^2)F_{\mu\nu}^2$. However, any function $f \neq 1$ would break an anomaly-free generalized R symmetry which is present in massless SQCD.

Nonperturbative effects can show up in the Kähler potential $K(\Phi^+\Phi)$ in the kinetic term for the matter fields. In the effective Lagrangian it will be converted into a nonperturbative term in γ . The relation between β_γ and γ will remain the same as in (1), given the fact that the conversion of $\Phi^+\Phi$ into $F_{\mu\nu}^2$ is an exact one-loop Konishi anomaly.

Let us mention another general consequence from Eq. (1). The Adler function $D(Q^2)$ is a physical quantity and, as such, satisfies an appropriate dispersion relation. Equation (1) implies that the anomalous dimension $\gamma(Q^2)$ must satisfy the same dispersion relation.

VI. CONCLUSIONS

Building on our previous publication [1] we expand our ideas on the exact Adler function in $\mathcal{N} = 1$ SQCD. We present a very detailed proof of the master formula (1) in perturbation theory. More exactly, we relate the Adler function (defined in terms of the bare coupling constant) with the anomalous dimension of the matter superfields (which is also defined in terms of the bare coupling constant) to all orders in the case of using the higher covariant derivative regularization.

In essence, our relation is very similar to the NSVZ β function and has a similar origin: all integrals for the D function in the momentum space are, in fact, integrals of double total derivatives. As a consequence, the singlet contribution, which is proportional to $(\sum_f q_f)^2$, vanishes. The remaining contribution does not vanish due to the existence of the integrand singularities. We proved that the sum of these singularities gives the contribution proportional to the anomalous dimension.

It should be noted that the result obtained in this paper is scheme independent, because we consider the RG functions defined in terms of the bare coupling constant. (These RG functions depend on the regularization but do not depend on the subtraction scheme for a fixed regularization.) For the (scheme-dependent) RG functions defined in terms of the renormalized coupling constant, the relation proposed in this paper is valid only in a special subtraction scheme, which may be possibly constructed similarly to the NSVZ scheme in $\mathcal{N} = 1$ supersymmetric electrodynamics by imposing certain boundary conditions on the renormalization constants.

Our general arguments, both in [1] and in this paper (see also [9]; the consideration in this paper is complementary to ours), tell us that the relations between the Adler function and the anomalous dimensions of the type (1) are valid beyond perturbation theory too. We discussed some consequences of these relations in the conformal window. We demonstrated that they allow one to determine the critical value of the anomalous dimension γ_M of the Seiberg M field which is present in the dual theory.

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APPENDIX: HOW TO OBTAIN SUBDIAGRAMS AND EFFECTIVE DIAGRAMS FROM THE FORMAL EXPRESSIONS

Let us obtain the subdiagrams presented in Fig. 3 starting from the formal expressions entering Eq. (41). First, we consider the expression

$$\begin{aligned} \star VQJ_0 &= \frac{1}{1-I_0} VQ(I_0 + B_0P) \\ &= VQB_0P + \frac{1}{1-I_0} I_0 VQB_0P + \frac{1}{1-I_0} VQI_0 \\ &= VQB_0P + \star B(PVQB_0P + VQP), \end{aligned} \quad (\text{A1})$$

where we use the notation

$$B_0 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (\text{A2})$$

for the vertices with a single external V -line and no V lines. Making simple algebraic operations one can easily verify the identity

$$P(VBQ)P = P(VBQ)(\Pi_+P) + (P\Pi_-)(VBQ)P, \quad (\text{A3})$$

where the chiral projection operators Π_{\pm} are defined by

$$\Pi_+ \equiv -\frac{\bar{D}^2 D^2}{16\partial^2}; \quad \Pi_- \equiv -\frac{D^2 \bar{D}^2}{16\partial^2}. \quad (\text{A4})$$

The equality (A3) has a simple interpretation: one matter line coming from the vertex is chiral and the second one is antichiral. In the first term the chiral end of the right matter line is attached to the considered vertex, and the chiral end of the left line is attached to the vertex in the second term. Using Eq. (A3) we rewrite the expression (A1) in the form

$$\begin{aligned} \star VQJ_0 &= VQB_0P + \star BPVQB_0P + BVQP \\ &+ (\star - 1)((BVQ)\Pi_+P + \Pi_-(BVQ)P), \end{aligned} \quad (\text{A5})$$

where we take into account that

$$\star - 1 = \left(\sum_{k=0}^{\infty} (I_0)^k \right) BP = \star BP. \quad (\text{A6})$$

Because the propagators are chiral (or antichiral),

$$P = \Pi_+P + \Pi_-P. \quad (\text{A7})$$

Therefore, again using Eq. (A6) for transforming the last term and taking into account that Q and B commute we obtain

$$\begin{aligned} \star VQJ_0 &= VQB_0P + \star (BPVQB_0P + (BVQ)\Pi_+P) \\ &+ BVQ\Pi_-P + \star B\Pi_-(VQ)BP. \end{aligned} \quad (\text{A8})$$

Presenting BP in the last term as $(I_0 - 1) + 1$, the considered vertex can be finally written in the following form:



FIG. 9. Topology of subdiagrams which correspond to the contribution A_1 in Eq. (A12) and form the effective vertex in the diagram (f) in Fig. 5. In order to obtain all subdiagrams it is necessary to take into account the projection operators Π_{\pm} in Eq. (A14).

$$\begin{aligned} \star VQJ_0 &= VQB_0P + VQB\Pi_-P + \star B\Pi_-VQ(I_0 - 1) \\ &+ \star(BP(VQ)B_0P + B(VQ)(\Pi_+P) + B(P\Pi_-)(VQ)). \end{aligned} \quad (\text{A9})$$

Two first terms in the first string contain a single vertex. In the third one the factor $I_0 - 1$ cancels \star in expressions like (41). These terms should be considered separately. The terms in the second string give the subdiagrams presented in Fig. 3. More exactly, the subdiagrams in Fig. 3 correspond to terms in the brackets. Really, the first of these terms contains the left vertex B attached to the sequence of the propagator P , the vertex with a single external line VQB_0 and the propagator P . Thus, this term encodes the subdiagrams in the first two columns of Fig. 3. The second term

[in the brackets in the second string of Eq. (A9)] consists of the left vertex BQ to which the external line V is attached and the propagators with the left chiral end (due to the projection operator Π_+). The last (third) term gives the left vertex B and the propagators $P\Pi_-$, which have the right chiral end to which the external line (VQ) is attached.

Let us apply Eq. (A9) to the calculation of various parts of Eq. (41). Let us start with the diagrams in which the external lines are attached to different matter loops. As we discuss in Sec. IV B, they include the expression

$$\begin{aligned} \text{Tr}(\star VQJ_0) &= \text{Tr}\{VQB_0P + VQB\Pi_-P - B\Pi_-VQ \\ &+ \star(BP(VQ)B_0P + B(VQ)(\Pi_+P) + B(P\Pi_-)(VQ))\}, \end{aligned} \quad (\text{A10})$$

where we use Eq. (A9), make a cyclic permutation in the third term, and take into account that $(I_0 - 1)\star = -1$. The terms in the second string of this equation are discussed in Sec. IV B. Here we consider only the terms in the first string of Eq. (A10). Multiplying the matrixes and calculating the trace one can easily see that they give the vanishing contribution. Therefore, in Sec. IV B the terms in the first string of Eq. (A9) are not essential.

Let us proceed to the diagrams considered in Sec. IV C. They are encoded in the expression⁴

$$\text{Tr}(\star VQJ_0 \star VQJ_0), \quad (\text{A11})$$

in which we substitute Eq. (A9). Taking into account that $(I_0 - 1)\star = -1$ we obtain the following result:

$$\text{Tr}(\star VQJ_0 \star VQJ_0) = A_2 + A_1 + A_0, \quad (\text{A12})$$

where A_2 contains two operators \star , A_1 contains one operator \star , and A_0 does not contain \star at all. It is easy to see that

⁴For simplicity we omit the numerical factor and the sum over I .

$$\begin{aligned} A_2 &= \text{Tr}\{\star(BP(VQ)B_0P + B(VQ)(\Pi_+P) \\ &+ B(P\Pi_-)(VQ))\star(BP(VQ)B_0P + B(VQ)(\Pi_+P) \\ &+ B(P\Pi_-)(VQ))\} \end{aligned} \quad (\text{A13})$$

includes two copies of subdiagrams presented in Fig. 3. This contribution corresponds to diagrams (a)–(e) in Fig. 5. The contribution A_1 after some simple transformations can be written as

$$\begin{aligned} A_1 &= 2 \cdot \text{Tr}\{\star BP(VQB_0)P(VQB_0)P \\ &+ \star(BVQ)(\Pi_+P)(VQB_0)P \\ &+ \star BP(VQB_0)(P\Pi_-)(VQ) \\ &+ \star(BVQ)(\Pi_+P\Pi_-)(VQ)\}. \end{aligned} \quad (\text{A14})$$

Subdiagrams (many) which correspond to this term are presented in Ref. [44] in Figs. 7–10 [together with subdiagrams proportional to V^2 which comes from the last term in Eq. (41).] These diagrams have structure presented in Fig. 9. However, it is also necessary to point out chiral and antichiral ends of the propagators. This should be made taking into account positions of the projection operators Π_{\pm} in Eq. (A14).

The contribution A_1 is graphically denoted by the diagram (f) in Fig. 5. The last contribution A_0 (that does not contain the operator \star) after some transformations can be reduced to the one-loop expression

$$A_0 = \text{Tr}((VQB_0)P(VQB_0)P). \quad (\text{A15})$$

Certainly, the diagrams containing the V^2 vertex are not present in this expression, because they come from the last term in Eq. (41). Evidently, the angular brackets in this case can be omitted due to the absence of internal gauge lines.

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