Perturbative calculations with the first order form of gauge theories

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(Received 23 March 2015; published 5 May 2015)

The first- and second-order forms of gauge theories are classically equivalent; we consider the consequence of quantizing the first-order form using the Faddeev-Popov approach. Both the Yang-Mills and the Einstein-Hilbert actions are considered. An advantage of this approach is that the interaction vertices are quite simple, being independent of momenta. However, it is necessary to consider the propagator for two fields (including a mixed propagator). We derive the Feynman rules for both models and consider the one-loop correction for the thermal energy momentum tensor.

DOI: 10.1103/PhysRevD.91.105006 PACS numbers: 11.15.-q

I. INTRODUCTION

The covariant quantization of the classical Yang-Mills (YM) field only became possible when it was realized that nonphysical modes of the vector field had to be canceled by contributions from so-called "ghost" fields that had non-trivial interactions [1–5]. Even then, computations are quite involved in large part because vertices arising from the classical second-order Yang-Mills Lagrangian

$$\mathcal{L}^{(2)}_{\mathrm{YM}} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)^2 \tag{1}$$

are quite complicated; there is a momentum-dependent three-point vertex as well as a four-point vertex.

The second-order Lagrangian of Eq. (1) is classically equivalent to the first-order Yang-Mills Lagrangian,

$$\mathcal{L}^{(1)}_{\mathrm{YM}} = -\frac{1}{2}F^a_{\mu\nu}(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g f^{abc} A^{b\mu} A^{c\nu}) + \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \tag{2}$$

as once the equation of motion for the independent field $F^a_{\mu\nu}$ is used to eliminate it from the Lagrangian of Eq. (2), the Lagrangian of Eq. (1) is recovered. The advantage of working directly with the Lagrangian of Eq. (2) is that there is now only a relatively simple three-point vertex $\mathrm{F}-\mathrm{A}-\mathrm{A}$. However, it is necessary to work with not only propagators $\mathrm{A}-\mathrm{A}$ and $\mathrm{F}-\mathrm{F}$ for the fields A^a_μ and $F^a_{\mu\nu}$, but also a mixed propagator $\mathrm{A}-\mathrm{F}$. This has been considered in Ref. [6] using background field quantization.

The second-order Einstein-Hilbert (EH) Lagrangian written in terms of the metric is

$$\mathcal{L}^{(2)}_{\mathrm{EH}} = -\kappa\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma), \tag{3}$$

where

$$R_{\mu\nu} = \Gamma^\rho_{\mu\rho,\nu} - \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho} \tag{4}$$

with

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}). \tag{5}$$

If we now set

$$\sqrt{-g}g^{\mu\nu} = h^{\mu\nu}, \tag{6}$$

$$G^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2}(\delta^\lambda_\mu\Gamma^\sigma_{\nu\sigma} + \delta^\lambda_\nu\Gamma^\sigma_{\mu\sigma}), \tag{7}$$

then Eq. (3) becomes

$$\mathcal{L}^{(2)}_{\mathrm{EH}} = \kappa h^{\mu\nu}\left(G^\lambda_{\mu\nu,\lambda} + \frac{1}{d-1}G^\lambda_{\mu\lambda}G^\sigma_{\nu\sigma} - G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda}\right), \tag{8}$$

where d is the space-time dimension.

The "Faddeev-Popov" (FP) quantization procedure of Refs. [2–5] has been applied to the action of Eq. (3) with either $g_{\mu\nu}$ [7–9] or $\sqrt{-g}g^{\mu\nu}$ [10,11] being treated as the independent field. (The FP procedure has to be extended to accommodate the "transverse-traceless" gauge [12]). Background field quantization is employed [13–15], with $g_{\mu\nu}$ being expanded about a classical background field such as the flat metric $\eta_{\mu\nu}$. This leads to exceedingly complicated vertices as g and $g^{\mu\nu}$ now both become infinite series in the quantum field. The part of $\mathcal{L}^{(2)}_{\mathrm{EH}}$ that is just bilinear in the quantum field is a free second-order spin-two Lagrangian.

If in Eqs. (3) and (4) $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ are taken to be independent fields, then for $d > 2$ the equation of motion for $\Gamma^\rho_{\mu\nu}$ results in Eq. (5) [16]. (This was noted by Einstein [17]; it is often a result credited to Palatini [18].) We will consider the first-order Einstein-Hilbert (1EH) Lagrangian $\mathcal{L}^{(1)}_{\mathrm{EH}}$ as being identical to $\mathcal{L}^{(2)}_{\mathrm{EH}}$ in Eq. (8) with $h^{\mu\nu}$ and $G^\lambda_{\mu\nu}$

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being taken as independent fields. We then have only one relatively simple momentum-independent vertex G − G − h, with the propagators h − h, G − G, and h − G.

In $d = 2$ dimensions, $\mathcal{L}^{(1)}_{\mathrm{EH}}$ and $\mathcal{L}^{(2)}_{\mathrm{EH}}$ are inequivalent; an extra vector field arises when solving the equation of motion for $\Gamma^{\lambda}_{\mu\nu}$ [19,20]. The Lagrangian $\mathcal{L}^{(2)}_{\mathrm{EH}}$ in $d = 2$ dimensions is not a total divergence although its equations of motion are trivial and the constraint structure reveals that the gauge invariance is simply $\delta g_{\mu\nu} = \epsilon_{\mu\nu}(x)$ for and arbitrary tensor $\epsilon_{\mu\nu}(x)$ [21]. This shows that no physical degrees of freedom reside in $\mathcal{L}^{(2)}_{\mathrm{EH}}$ when $d = 2$. When $d = 2$, a canonical analysis of $\mathcal{L}^{(1)}_{\mathrm{EH}}$ also possesses no physical degrees of freedom, but it does possess an unusual local gauge invariance that is distinct from the manifest diffeomorphism invariance [22,23]. Furthermore, upon quantizing $\mathcal{L}^{(1)}_{\mathrm{EH}}$ when $d = 2$ using the FP procedure, it can be shown that all perturbative radiative effects vanish [24].

We will now consider the quantization of $\mathcal{L}^{(1)}_{\mathrm{YM}}$ and $\mathcal{L}^{(1)}_{\mathrm{EH}}$ when $d > 2$.

II. FIRST-ORDER YANG-MILLS ACTION

The Lagrangian of Eq. (2) is invariant under an infinitesimal local gauge transformation,

$$\delta A^a_\mu = D^{ab}_\mu \theta^b \equiv (\partial_\mu \delta^{ab} + g f^{apb} A^p_\mu)\theta^b, \tag{9a}$$

$$\delta F^a_{\mu\nu} = a f^{apb} F^p_{\mu\nu} \theta^b, \tag{9b}$$

which necessitates the introduction of a gauge-fixing Lagrangian $\mathcal{L}_{\mathrm{gf}}$ and its associated ghost Lagrangian $\mathcal{L}_{\mathrm{gh}}$ [1–5]. Working with the covariant gauge-fixing Lagrangian

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\alpha}(\partial \cdot A^a)^2, \tag{10}$$

one has

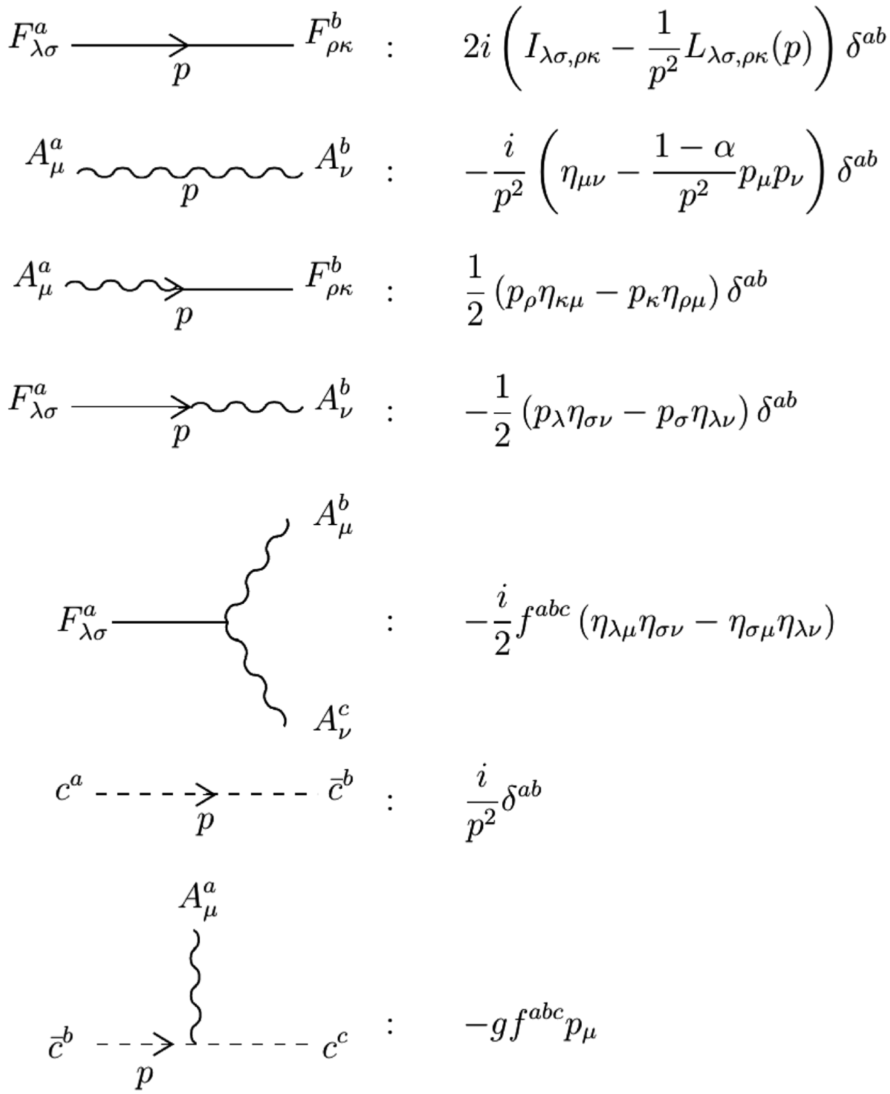

FIG. 1. Feynman rules for first-order Yang-Mills.

$$\mathcal{L}_{\rm gh} = \bar{c}^a \partial \cdot D^{ab} c^b, \tag{11}$$

where $\bar{c}^a$ and c^a are the usual Fermionic scalar ghost fields.

The terms in $\mathcal{L}^{(1)}_{\rm YM} + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh}$ that are bilinear in the fields A^a_μ and $F^a_{\lambda\sigma}$ are

$$\frac{1}{2}\left(A_\mu, F^a_{\lambda\sigma}\right)\begin{pmatrix} \frac{1}{\alpha}\partial^\mu\partial^\nu & \frac{1}{2}(\partial^\rho\eta^{\kappa\mu} - \partial^\kappa\eta^{\rho\mu}) \\ -\frac{1}{2}(\partial^\lambda\eta^{\sigma\nu} - \partial^\sigma\eta^{\lambda\nu}) & \frac{1}{4}(\eta^{\lambda\rho}\eta^{\sigma\kappa} - \eta^{\lambda\kappa}\eta^{\sigma\rho}) \end{pmatrix}\begin{pmatrix} A_\nu \\ F^a_{\rho\kappa} \end{pmatrix}. \tag{12}$$

The inverse of the matrix appearing in Eq. (12) is

$$\Delta(\partial) = \begin{pmatrix} \frac{1}{\partial^2}\left(\eta^{\mu\nu} - \frac{(1-\alpha)}{\partial^2}\partial^\mu\partial^\nu\right) & -\frac{1}{\partial^2}(\partial^\rho\eta^{\kappa\mu} - \partial^\kappa\eta^{\rho\mu}) \\ \frac{1}{\partial^2}(\partial^\lambda\eta^{\sigma\nu} - \partial^\sigma\eta^{\lambda\nu}) & 2\left(I^{\lambda\sigma,\rho\kappa} - \frac{1}{\partial^2}L^{\lambda\sigma,\rho\kappa}\right) \end{pmatrix}, \tag{13}$$

where

$$I^{\lambda\sigma,\rho\kappa} = \frac{1}{2}(\eta^{\lambda\rho}\eta^{\sigma\kappa} - \eta^{\lambda\kappa}\eta^{\sigma\rho}), \tag{14a}$$

$$L^{\lambda\sigma,\rho\kappa}(\partial) = \frac{1}{2}(\partial^\lambda\partial^\rho\eta^{\sigma\kappa} + \partial^\sigma\partial^\kappa\eta^{\lambda\rho} - \partial^\lambda\partial^\kappa\eta^{\sigma\rho} - \partial^\sigma\partial^\rho\eta^{\lambda\kappa}). \tag{14b}$$

The propagators are given by $i\Delta(ip)$ and the F − A − A vertex follows from the interacting part of $\mathcal{L}^{(1)}_{\rm YM}$,

$$-\frac{1}{2}gf^{abc}F^a_{\mu\nu}A^{b\mu}A^{c\nu}. \tag{15}$$

The Feynman rules appear in Fig. 1.

We now turn to examining the 1EH Lagrangian.

III. FIRST-ORDER EINSTEIN-HILBERT ACTION

It is tempting to consider directly applying the FP quantization procedure to the 1EH action of Eq. (8) when $h^{\mu\nu}$ and $G^\lambda_{\mu\nu}$ are treated as independent fields. However, it is soon discovered that no choice of gauge leads to bilinears in the effective Lagrangian that can be inverted so as to result in a suitable propagator. However, if we write $h^{\mu\nu} = \eta^{\mu\nu} + \phi^{\mu\nu}$, where $\eta^{\mu\nu} = \mathrm{diag}(+ - - - \cdots -)$ is a flat background and $\phi^{\mu\nu}$ is a quantum fluctuation, then Eq. (8) becomes (with $\kappa = 1/2$)

$$\begin{aligned}\mathcal{L}^{(1)}_{\rm EH} &= \frac{1}{2}\left[\phi^{\mu\nu}G^\lambda_{\mu\nu,\lambda} + \eta^{\mu\nu}\left(\frac{1}{d-1}G^\lambda_{\lambda\mu}G^\sigma_{\sigma\nu} - G^\lambda_{\sigma\mu}G^\sigma_{\lambda\nu}\right)\right] \\ &\quad + \frac{1}{2}\left[\phi^{\mu\nu}\left(\frac{1}{d-1}G^\lambda_{\lambda\mu}G^\sigma_{\sigma\nu} - G^\lambda_{\sigma\mu}G^\sigma_{\lambda\nu}\right)\right] \\ &\equiv \mathcal{L}^{(1)2}_{\rm EH} + \mathcal{L}^{(1)3}_{\rm EH}.\end{aligned} \tag{16}$$

The infinitesimal form of diffeomorphism invariance associated with the action of Eq. (8) is

$$\delta h^{\mu\nu} = h^{\mu\lambda}\partial_\lambda\theta^\nu + h^{\nu\lambda}\partial_\lambda\theta^\mu - \partial_\lambda(h^{\mu\nu}\theta^\lambda), \tag{17a}$$

$$\begin{aligned}\delta G^\lambda_{\mu\nu} &= -\partial^2_{\mu\nu}\theta^\lambda + \frac{1}{2}(\delta^\lambda_\mu\partial_\nu + \delta^\lambda_\nu\partial_\mu)\partial_\rho\theta^\rho - \theta^\rho\partial_\rho G^\lambda_{\mu\nu} \\ &\quad + G^\rho_{\mu\nu}\partial_\rho\theta^\lambda - (G^\lambda_{\mu\rho}\partial_\nu + G^\lambda_{\nu\rho}\partial_\mu)\theta^\rho,\end{aligned} \tag{17b}$$

which means that for $\mathcal{L}^{(1)}_{\rm EH}$ in Eq. (15) we have the gauge transformation of Eq. (17b), while Eq. (17a) now implies that

$$\begin{aligned}\delta\phi^{\mu\nu} &= \partial^\mu\theta^\nu + \partial^\nu\theta^\mu + \phi^{\mu\lambda}\partial_\lambda\theta^\nu + \phi^{\nu\lambda}\partial_\lambda\theta^\mu - \eta^{\mu\nu}\partial\cdot\theta \\ &\quad - \partial_\lambda(\phi^{\mu\nu}\theta^\lambda).\end{aligned} \tag{18}$$

(Indices are now raised using $\eta^{\mu\nu}$.)

If we now choose the gauge-fixing condition

$$\mathcal{L}_{\rm gf} = -\frac{1}{2\alpha}(\partial_\mu\phi^{\mu\nu})^2, \tag{19}$$

then the Faddeev-Popov ghost contribution to the effective Lagrangian would be [10,11]

$$\begin{aligned}\mathcal{L}_{\rm FP} = \bar{d}_\mu[&\partial^2\eta^{\mu\nu} + (\partial_\rho\phi^{\rho\sigma})\partial_\sigma\eta^{\mu\nu} - (\partial_\rho\phi^{\rho\mu})\partial^\nu \\ &+ \phi^{\rho\sigma}\partial_\rho\partial_\sigma\eta^{\mu\nu} - (\partial_\rho\partial^\nu\phi^{\rho\mu})]d_\nu.\end{aligned} \tag{20}$$

The terms bilinear in $\phi^{\mu\nu}$ and $G^\lambda_{\mu\nu}$ that follow from Eqs. (16) and (19) are

$$\mathcal{L}^{(2)}_{\rm eff} = \frac{1}{2}\left[\phi^{\mu\nu}, G^\lambda_{\alpha\beta}\right]\begin{bmatrix} A_{\mu\nu\rho\kappa} & B^{\gamma\delta}_{\mu\nu\sigma} \\ C^{\alpha\beta}_{\rho\kappa\lambda} & D^{\alpha\beta}_{\lambda}{}^{\gamma\delta}_{\sigma} \end{bmatrix}\begin{bmatrix} \phi^{\rho\kappa} \\ G^\sigma_{\gamma\delta} \end{bmatrix}, \tag{21}$$

where

$$A_{\mu\nu\rho\kappa} \equiv \frac{1}{4\alpha}(\partial_\mu\partial_\rho\eta_{\nu\kappa} + \partial_\nu\partial_\rho\eta_{\mu\kappa} + \partial_\mu\partial_\kappa\eta_{\nu\rho} + \partial_\nu\partial_\kappa\eta_{\mu\rho}), \tag{22a}$$

$$B^{\gamma\delta}_{\mu\nu\sigma} \equiv \frac{1}{4}(\delta^\gamma_\mu\delta^\delta_\nu + \delta^\gamma_\nu\delta^\delta_\mu)\partial_\sigma, \tag{22b}$$

$$C^{\alpha\beta}_{\rho\kappa\lambda} \equiv -\frac{1}{4}(\delta^\alpha_\rho\delta^\beta_\kappa + \delta^\beta_\rho\delta^\alpha_\kappa)\partial_\lambda, \tag{22c}$$

$$D_{\lambda}^{\alpha\beta}{}_{\sigma}^{\gamma\delta} \equiv \frac{1}{4}\left[\left(\frac{1}{d-1}\delta^\alpha_\lambda\delta^\gamma_\sigma\eta^{\beta\delta} - \delta^\alpha_\sigma\delta^\gamma_\lambda\eta^{\beta\delta} + \alpha\leftrightarrow\beta\right) + \gamma\leftrightarrow\delta\right]. \tag{22d}$$

Using the blockwise matrix inversion

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{X}^{-1} & -\mathbf{X}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{X}^{-1}\mathbf{D}^{-1} & +\mathbf{D}^{-1}\mathbf{C}\mathbf{X}^{-1}\mathbf{B}\mathbf{D}^{-1}\end{bmatrix}, \tag{23}$$

where

$$\mathbf{X} = \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} \tag{24}$$

and **A**, **B**, **C**, and **D** have tensor representations given by Eqs. (22), we can obtain the propagators in a straightforward way. (Some of the following steps were carried out using computer algebra.) First, we compute the inverse of $D_{\lambda}^{\alpha\beta}{}_{\sigma}^{\gamma\delta}$. Using Eq. (22d), we obtain

$$D^{-1}{}_{\alpha\beta}^{\lambda}{}_{\gamma\delta}^{\sigma} = \frac{1}{2}\eta^{\lambda\sigma}\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \frac{2}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right) - \frac{1}{2}(\delta^\lambda_\delta\delta^\sigma_\beta\eta_{\alpha\gamma} + \delta^\lambda_\gamma\delta^\sigma_\beta\eta_{\alpha\delta} + \delta^\lambda_\delta\delta^\sigma_\alpha\eta_{\gamma\beta} + \delta^\lambda_\gamma\delta^\sigma_\alpha\eta_{\beta\delta}). \tag{25}$$

Then, substituting Eqs. (22a), (22b), (22c), and (25) into the tensor form of Eq. (24), we obtain ($i\partial = p$)

$$X_{\mu\nu\rho\kappa} = \frac{p^2}{8}\left(\frac{2\eta_{\mu\nu}\eta_{\rho\kappa}}{d-2} - \eta_{\mu\rho}\eta_{\nu\kappa} - \eta_{\mu\kappa}\eta_{\nu\rho}\right) + \left(\frac{1}{8} - \frac{1}{4\alpha}\right)(p_\mu p_\rho\eta_{\nu\kappa} + p_\nu p_\rho\eta_{\mu\kappa}p_\mu p_\kappa\eta_{\nu\rho} + p_\nu p_\kappa\eta_{\mu\rho}). \tag{26}$$

Computing the inverse of this expression, we obtain

$$X^{-1\mu\nu\rho\kappa} = \frac{1}{p^2}[(4-\alpha)\eta^{\mu\nu}\eta^{\rho\kappa} - 2(\eta^{\mu\rho}\eta^{\nu\kappa} + \eta^{\mu\kappa}\eta^{\nu\rho})] + \frac{\alpha-2}{p^4}[2(p^\mu p^\nu\eta^{\rho\kappa} + p^\rho p^\kappa\eta^{\mu\nu}) - p^\mu p^\rho\eta^{\nu\kappa} - p^\nu p^\rho\eta^{\mu\kappa} - p^\mu p^\kappa\eta^{\nu\rho} - p^\nu p^\kappa\eta^{\mu\rho}] \equiv \mathcal{D}^{\phi^2}_{\mu\nu\rho\kappa}, \tag{27}$$

where we have identified the result with the graviton propagator $\mathcal{D}^{\phi^2}_{\mu\nu\rho\kappa}$ (notice that for $\alpha = 2$, $\mathcal{D}^{\phi^2}_{\mu\nu\rho\kappa}$ has the same structure as the DeDonder gauge propagator in the second-order formulation).

Substituting Eqs. (22b), (22c), (25), and (27) into the tensor form of the off-diagonal blocks of Eq. (23), we obtain

$$\mathcal{D}^{G\phi}{}_{\alpha\beta}^{\lambda}{}^{\rho\kappa} = \frac{i}{2p^2}[p_\alpha((\alpha-4)\delta^\lambda_\beta\eta^{\rho\kappa} + 2\delta^\rho_\beta\eta^{\lambda\kappa} + 2\delta^\kappa_\beta\eta^{\lambda\rho}) - 2p^\lambda\delta^\kappa_\alpha\delta^\rho_\beta + \alpha\leftrightarrow\beta] - \frac{i(\alpha-2)}{p^4}[p^\kappa p^\rho(p_\beta\delta^\lambda_\alpha + p_\alpha\delta^\lambda_\beta) - p_\alpha p_\beta(p^\rho\eta^{\kappa\lambda} + p^\kappa\eta^{\rho\lambda}) + p_\alpha p_\beta p^\lambda\eta^{\kappa\rho}], \tag{28}$$

$$\mathcal{D}^{\phi G\mu\nu}{}^{\sigma}_{\gamma\delta} = -\mathcal{D}^{G\phi\sigma}{}_{\gamma\delta}{}^{\mu\nu} \tag{29}$$

The propagator for the $G^\lambda_{\mu\nu}$ field can similarly be obtained by computing the second diagonal block of Eq. (23) with the help of Eqs. (22b), (22c), (25), and (27), which yields

$$\begin{aligned}\mathcal{D}^{G^2}{}_{\alpha\beta}^{\lambda}{}_{\gamma\delta}^{\sigma} = {}& \frac{\alpha-2}{2p^4}[p_\alpha p_\beta p^\lambda(p_\delta\delta^\sigma_\gamma + p_\gamma\delta^\sigma_\delta) + p_\gamma p_\delta p^\sigma(p_\alpha\delta^\lambda_\beta + p_\beta\delta^\lambda_\alpha) - 2p_\alpha p_\beta p_\gamma p_\delta\eta^{\lambda\sigma}] \\ & + \frac{1}{4p^2}\Bigg[2p^\lambda p^\sigma\left(\frac{2\eta_{\alpha\beta}\eta_{\gamma\delta}}{d-2} - \eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\beta\gamma}\eta_{\alpha\delta}\right) + 2p^\lambda(p_\gamma(\delta^\sigma_\alpha\eta_{\beta\delta} + \delta^\sigma_\beta\eta_{\alpha\delta}) + p_\delta(\delta^\sigma_\alpha\eta_{\beta\gamma} + \delta^\sigma_\beta\eta_{\alpha\gamma})) \\ & + 2p^\sigma(p_\alpha(\delta^\lambda_\gamma\eta_{\beta\delta} + \delta^\lambda_\delta\eta_{\gamma\beta}) + p_\beta(\delta^\lambda_\gamma\eta_{\delta\alpha} + \delta^\lambda_\alpha\eta_{\alpha\gamma})) - 2p_\alpha(p_\gamma(\eta^{\lambda\sigma}\eta_{\beta\delta} + \delta^\lambda_\delta\delta^\sigma_\beta) + p_\delta(\eta^{\lambda\sigma}\eta_{\beta\gamma} + \delta^\lambda_\gamma\delta^\sigma_\beta)) \\ & - 2p_\beta(p_\gamma(\eta^{\lambda\sigma}\eta_{\alpha\delta} + \delta^\lambda_\delta\delta^\sigma_\alpha) + p_\delta(\eta^{\lambda\sigma}\eta_{\alpha\gamma} + \delta^\lambda_\gamma\delta^\sigma_\alpha)) + (4-\alpha)(p_\gamma\delta^\sigma_\delta + p_\delta\delta^\sigma_\gamma)(p_\alpha\delta^\lambda_\beta + p_\beta\delta^\lambda_\alpha)\Bigg] \\ & - \frac{\eta^{\lambda\sigma}}{2}\left(\frac{2\eta_{\alpha\beta}\eta_{\gamma\delta}}{d-2} - \eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\beta\gamma}\eta_{\alpha\delta}\right) - \frac{\delta^\sigma_\alpha}{2}(\eta_{\beta\gamma}\delta^\lambda_\delta + \eta_{\beta\delta}\delta^\lambda_\gamma) - \frac{\delta^\sigma_\beta}{2}(\eta_{\alpha\gamma}\delta^\lambda_\delta + \eta_{\alpha\delta}\delta^\lambda_\gamma).\end{aligned} \tag{30}$$

There is only one interaction vertex which can be read from Eq. (16). The symmetrized result can be written as

$$\mathcal{V}_{\mu\nu}{}^{\lambda}_{\alpha\beta}{}^{\sigma}_{\gamma\delta} = \frac{1}{8}\left\{\left[\left(\frac{\eta_{\mu\beta}\eta_{\nu\delta}\delta^{\lambda}_{\alpha}\delta^{\sigma}_{\gamma}}{d-1} - \eta_{\mu\beta}\eta_{\nu\delta}\delta^{\sigma}_{\alpha}\delta^{\lambda}_{\gamma} + \mu \leftrightarrow \nu\right) + \alpha \leftrightarrow \beta\right] + \gamma \leftrightarrow \delta\right\}. \tag{31}$$

The ghost propagator and vertex, which can be read from Eq. (20), are given by

$$\mathcal{D}^{gh}_{\mu\nu} = -\frac{\eta_{\mu\nu}}{p^2} \tag{32}$$

and

$$\mathcal{V}^{gh}{}^{\mu\nu}_{\sigma\beta}(p_1,p_2,p_3) = \frac{\delta(p_1+p_2+p_3)}{2}[\eta_{\alpha\beta}(p_2^{\mu}p_3^{\nu} + p_2^{\nu}p_3^{\mu}) - p_{2\beta}(p_1^{\mu}\delta^{\nu}_{\alpha} + p_1^{\nu}\delta^{\mu}_{\alpha})]. \tag{33}$$

Using Eqs. (27)–(33), we put together in Fig. 2 all of the Feynman rules for first-order gravity.

As an example of the effectiveness of the perturbative first-order formalism, let us now consider an explicit perturbative calculation which makes use of the Feynman rules in Fig. 2. We will consider a simple one-loop calculation which takes into account the coupling of the graviton field to the energy-momentum tensor of a thermal gravitational plasma. Since this is a well-known result which has been obtained in the usual formulation of thermal gravity [25] as well as in the transverse-traceless gauge-fixing formulation [12], it provides a simple test of the consistency of the first-order formalism.

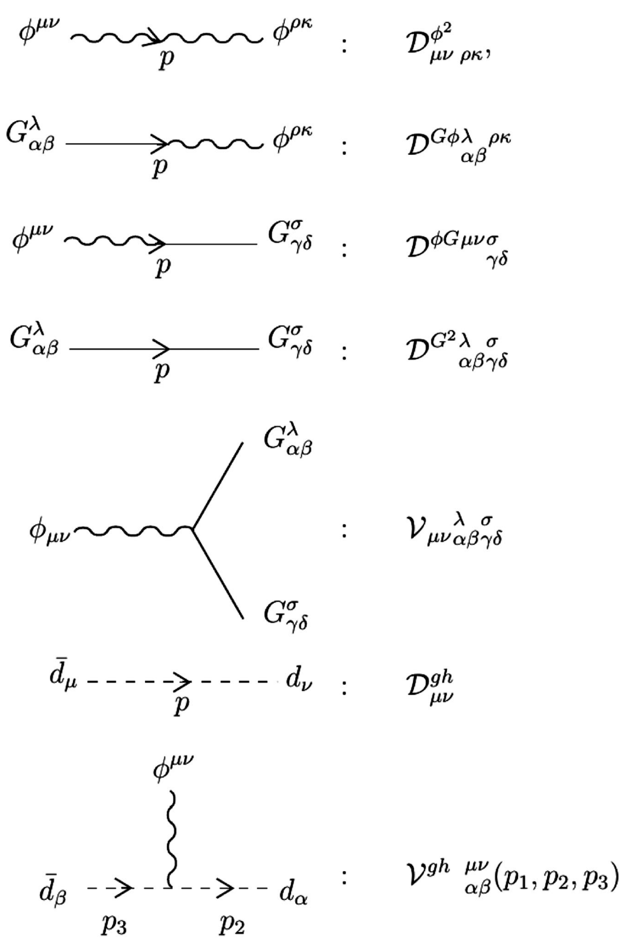

FIG. 2. Feynman rules for first-order gravity.

The energy-momentum tensor $T_{\mu\nu}$ and the one-graviton function $\Gamma_{\mu\nu}$ are related by

$$\Gamma_{\mu\nu} = \frac{\delta\Gamma}{\delta\phi^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}T_{\mu\nu}, \tag{34}$$

where Γ is the one-loop thermal effective action. In the Fig. 3 we shown the one-loop diagrams that contribute to $\Gamma^{\mu\nu}$. Using the imaginary-time formalism [26], the thermal part of each of these diagrams can be written as

$$\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\int_{-i\infty+\delta}^{i\infty+\delta}\frac{\mathrm{d}k_0}{2\pi i}N_B(k_0)[f^I_{\mu\nu}(k) + f^I_{\mu\nu}(-k)], \tag{35}$$

where $N_B(k_0)$ is the Bose-Einstein thermal distribution function. For convenience, we are considering the more general case of a d-dimensional space-time. The integrand of each contribution from Fig. 3 is denoted by $f^I_{\mu\nu}(k)(I = a,b,c)$.

Let us first consider the diagram with a mixed propagator, as shown in Fig. 3(a). Using the Feynman rules in

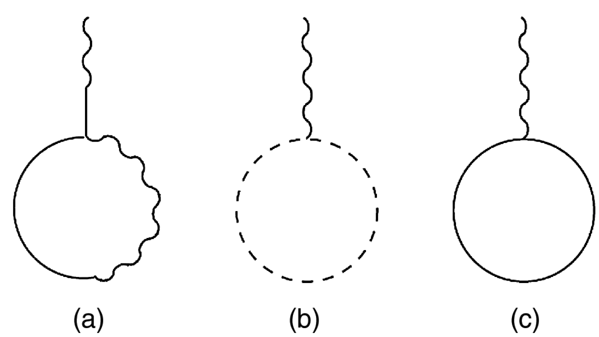

FIG. 3. Diagrams that contribute to the thermal energy-momentum tensor. The loops in diagrams (a), (b) and (c) represent the mixed, ghost and G field contributions, respectively.

Fig. 2, we obtain

$$f^a_{\mu\nu}(k) = -i\left[(\alpha-2)\frac{k^\lambda k_\alpha k_\beta}{k^4} - \frac{(3\alpha+2d-4)k^\lambda\eta_{\alpha\beta} - d(k_\alpha\delta^\lambda_\beta + k_\beta\delta^\lambda_\alpha)}{2k^2}\right](DB^{-1}X)^{\alpha\beta}_{\lambda\mu\nu}, \tag{36}$$

where the factor $(DB^{-1}X)^{\alpha\beta}_{\lambda\mu\nu}$ produces the corresponding amputated Green function. Since $f^a_{\mu\nu}(k)$ is an odd function of k the net result in Eq. (35) will vanish trivially.

We are then left with the ghost loop and the G-loop contributions. From Figs. 3(b) and 3(c) we obtain

$$f^b_{\mu\nu}(k) = -d\frac{k_\mu k_\nu}{k^2} \tag{37}$$

and

$$f^c_{\mu\nu}(k) = \frac{1}{4}\left[d(d-1)\eta_{\mu\nu} + d(d+1)\frac{k_\mu k_\nu}{k^2}\right]. \tag{38}$$

Since we are using dimensional regularization, the first term in Eq. (38) produces a vanishing contribution when inserted into Eq. (35). Adding the nonvanishing contribution from $f^b_{\mu\nu}(k)$ and $f^c_{\mu\nu}(k)$, and using Eq. (35), we obtain

$$\Gamma^{\mathrm{therm}}_{\mu\nu} = \frac{d(d-3)}{2}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\int_{-i\infty+\delta}^{i\infty+\delta}\frac{\mathrm{d}k_0}{2\pi i}N_B(k_0)\frac{k_\mu k_\nu}{k^2}, \tag{39}$$

where the factor $d(d-3)/2$ counts the degrees of freedom of a graviton in d dimensions. Closing the contour of integration on the right-hand side of the plane, the pole at $k_0 = |\vec{k}|$ gives the following contribution [there is a minus sign from the clockwise contour integration and the pole from $1/k^2$ at $k_0 = |\vec{k}|$ yields a factor $1/(2|\vec{k}|)$]:

$$\begin{aligned}\Gamma^{\mathrm{therm}}_{\mu\nu} &= -\frac{d(d-3)}{4}\int_0^\infty \mathrm{d}|\vec{k}|\frac{|\vec{k}|^{d-1}}{\mathrm{e}^{\frac{|\vec{k}|}{T}}-1}\int\frac{\mathrm{d}\Omega_{d-1}}{(2\pi)^{d-1}}\hat{k}_\mu\hat{k}_\nu \\ &= -\frac{d(d-3)}{4}\zeta(d)\Gamma(d)T^d\int\frac{\mathrm{d}\Omega_{d-1}}{(2\pi)^{d-1}}\hat{k}_\mu\hat{k}_\nu,\end{aligned} \tag{40}$$

where $\hat{k}_\mu = (1, \vec{k}/|\vec{k}|)$. This result can be expressed in terms of the heat bath four-velocity $u_\mu = (1, 0)$ as follows:

$$\begin{aligned}\Gamma^{\mathrm{therm}}_{\mu\nu} &= \frac{d(d-3)}{4(d-1)}\zeta(d)\Gamma(d)\left(\int\frac{\mathrm{d}\Omega_{d-1}}{(2\pi)^{d-1}}\right)T^d(\eta_{\mu\nu} - du_\mu u_\nu) \\ &= \frac{d(d-3)}{4(d-1)}\zeta(d)\Gamma(d)\frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\frac{T^d}{(2\pi)^{d-1}}(\eta_{\mu\nu} - du_\mu u_\nu).\end{aligned} \tag{41}$$

For $d = 4$ we obtain

$$\Gamma^{\mathrm{therm}}_{\mu\nu}|_{d=4} = \frac{\pi^2T^4}{90}(\eta_{\mu\nu} - 4u_\mu u_\nu), \tag{42}$$

which is in agreement with the known result obtained using the second-order formalism [25].

IV. DISCUSSION

We have examined how the first-order form of both the Yang-Mills and Einstein-Hilbert actions can be used to compute quantum effects. In both cases, using the first-order form of the action simplifies the vertices encountered when using the Faddeev-Popov quantization; unfortunately, the propagators become more involved.

The first- and second-order form of the actions can be shown to be classically equivalent by examining the classical equations of motion. To show that the path integrals associated with $\mathcal{L}^{(2)}_{\mathrm{YM}}$ and $\mathcal{L}^{(1)}_{\mathrm{YM}}$ are equivalent, we only need to take

$$\mathcal{L}^{(2)}_{\mathrm{eff}} = \mathcal{L}^{(2)}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} \tag{43}$$

[using Eqs. (1), (10), and (11)], and to insert into the path integral

$$Z^{(2)}_{\mathrm{eff}} = \int\mathcal{D}A^a_\mu\mathcal{D}c^a\mathcal{D}\bar{c}^a\exp i\int dx\mathcal{L}^{(2)}_{\mathrm{eff}} \tag{44}$$

the constant

$$\int\mathcal{D}F^a_{\mu\nu}\exp i\int dx\left(\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right). \tag{45}$$

Upon performing the shift

$$F^a_{\mu\nu} \to F^a_{\mu\nu} - (\partial_\mu A_\nu - \partial_\nu A_\mu + gf^{abc}A^b_\mu A^c_\nu), \tag{46}$$

we convert $Z^{(2)}_{\mathrm{eff}}$ into $Z^{(1)}_{\mathrm{eff}}$, where

$$Z^{(1)}_{\mathrm{eff}} = \int\mathcal{D}A^a_\mu\mathcal{D}F^a_{\mu\nu}\mathcal{D}c^a\mathcal{D}\bar{c}^a\exp i\int dx\mathcal{L}^{(1)}_{\mathrm{eff}}, \tag{47}$$

where $\mathcal{L}^{(1)}_{\mathrm{eff}}$ is identical to $\mathcal{L}^{(2)}_{\mathrm{eff}}$ of Eq. (43), except that now $\mathcal{L}^{(1)}_{\mathrm{YM}}$ of Eq. (2) replaces $\mathcal{L}^{(2)}_{\mathrm{YM}}$.

Unfortunately, it is not so straightforward to show that when the Faddeev-Popov quantization procedure is used in conjunction with $\mathcal{L}^{(1)}_{\mathrm{EH}}$, the same result is obtained as when

$\mathcal{L}_{\rm EH}^{(2)}$ is treated this way. In any case, it is not clear that the Faddeev-Popov procedure is appropriate for $\mathcal{L}_{\rm EH}^{(1)}$ as the constraint structure of this Lagrangian implies that the functional measure receives a nontrivial contribution from second-class constraints [27]. Such contributions also have a significant effect when quantizing a model with an antisymmetric tensor field that interacts with a non-Abelian vector field and that possesses a pseudoscalar mass [28].

The problem of renormalizing the divergences that arise when using the Faddeev-Popov approach to quantizing $\mathcal{L}_{\rm YM}^{(1)}$ and $\mathcal{L}_{\rm EH}^{(1)}$ is quite delicate on account of the presence of mixed propagators. We are currently considering this issue.

ACKNOWLEDGMENTS

We would like to thank CNPq and Fapesp (Brazil) for financial support. Roger Macleod helped with this work.

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