Critical exponents of $O(N)$ models in fractional dimensions

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We compute critical exponents of $O(N)$ models in fractional dimensions between $d = 2$ and 4, and for continuous values of the number of field components N, in this way completing the RG classification of universality classes for these models. These curves represent nonperturbative approximation to the exact results, they respect all the qualitative features expected from such quantities conciliating previously known perturbative results in three dimensions with exact results in two dimensions and giving a strong indication of what could be the exact behavior of such curves. We also report critical exponents for some multicritical universality classes in the cases $N \geq 2$ and $N = 0$. Finally, in the large-N limit our critical exponents correctly approach those of the spherical model, allowing us to set $N \sim 100$ as the threshold for the quantitative validity of leading order large-N estimates.

I. INTRODUCTION

The understanding of universality—namely, the independence of the critical properties of a system from its microscopic details—by means of the renormalization group (RG) has been one emblematic example of the twist of paradigm that such a technique has brought to modern physics. In Wilson's general framework [1], the way physics changes with respect to the energy scale is represented by a flow along a trajectory in a generalized *theory space*, which is the space of all theories describing fluctuations of a given set of degrees of freedom. Critical phenomena arising in a physical system are understood as described by theories that are fixed points of its RG flow [1]. In this way different trajectories, corresponding to different microscopic theories, which lie in the same basin of attraction of a given fixed point, will describe the same critical properties. Universality then tells us that these are determined by a few parameters, such as the dimensionality, the symmetry group of the system and the order of criticality. Each value of these parameters defines a different universality class; classifying them is tantamount to classifying all possible continuous phase transitions that can occur in nature.

Among the universal quantities characterizing a phase transition, a set of parameters which acts as a bridge between theory and experiment is that of critical exponents, which parametrize how certain measurable quantities (such as specific heat, density, susceptibility and so on) depend on temperature near a critical point. Being universal observables, critical exponents are both a test ground for theoretical methods and possible predictions for, yet unobserved, phase transitions. Having a simple mathematical tool to compute

and predict these exponents is thus an important theoretical and phenomenological task.

In this paper we compute the critical exponents of $O(N)$ models in fractional dimensions between $d = 2$ and 4 and for continuous values of the number of field components N, starting from the basic principles of Wilsonian RG in its modern functional realization [2,3]. $O(N)$ models have many applications to low dimensional systems: they can describe long polymer chains ($N = 0$), liquid–vapor ($N = 1$), superfluid helium ($N = 2$), ferromagnetic ($N = 3$) and QCD chiral ($N = 4$) phase transitions [3,4].

The present work complements and completes the analysis and classification of universality classes of $O(N)$ models made in [5] with the dependence of critical exponents $\nu, \alpha, \beta, \gamma, \delta$ on d and N and shows how, even at this approximate level, the numerical results strictly follow the prescription of the Mermin–Wagner–Hohenberg (MWH) theorem [6,7]. We also compute the critical exponents for many new $N \geq 2$ universality classes describing multicritical models in fractional dimension $2 \leq d \leq 3$. We also complement the analysis of the possible multicritical phases of polymeric systems, as found in our previous work, by giving the critical exponents associated to these phase transitions. Thus, if these phases can be realized in some system, these can be seen as predictions for parameters yet to be measured.

II. SCALING SOLUTIONS AND η

Our tool will be the running effective potential $U_k(\rho)$, which is a function of the $O(N)$ invariant $\rho = \frac{1}{2}\varphi^2$, for a constant field φ. It represents an effective Hamiltonian for the model where all the excitation with momentum greater

than k have been integrated out [5]. In terms of dimensionless variables $\tilde{U}_k(\tilde{\rho}) = k^{-d}U_k(\rho)$, with $\tilde{\rho} = k^{-(d-2+\eta)}\rho$, a scaling solution $\partial_t \tilde{U}_*(\tilde{\rho}) = 0$, where $t = \log(k/\Lambda)$ and Λ is some ultraviolet cutoff, satisfies the following ordinary differential equation [2],

$$-(d-2+\eta)\tilde{\rho}\tilde{U}'_* + d\tilde{U}_*$$
$$= c_d(N-1)\frac{1-\frac{\eta}{d+2}}{1+\tilde{U}'_*} + c_d\frac{1-\frac{\eta}{d+2}}{1+\tilde{U}'_* + 2\tilde{\rho}\tilde{U}''_*}, \quad (1)$$

where $c_d^{-1} = (4\pi)^{d/2}\Gamma(d/2+1)$. The anomalous dimension η fixes the scaling properties of the field at a particular fixed point; to lowest order its value is given by [3]

$$\eta = 4c_d\frac{\tilde{\rho}_0\tilde{U}''_*(\tilde{\rho}_0)^2}{[1+2\tilde{\rho}_0\tilde{U}''_*(\tilde{\rho}_0)]^2}, \quad (2)$$

with $\tilde{\rho}_0$ the absolute minimum of the fixed point potential $\tilde{U}'_*(\tilde{\rho}_0) = 0$. The approximation scheme described by Eqs. (1) and (2) is often called LPA′ (since it is the simplest improvement of the linear potential approximation [3]).

Every solution of (1), together with its domain of attraction, represents a different $O(N)$ universality class [5]. For every d and N one finds a discrete set of solutions corresponding to multicritical potentials of increasing order, i.e. with i minima, which describe multicritical phase transitions (in which one needs to tune multiple parameters to reach the critical point). For each of these it is possible to obtain the anomalous dimension $\eta_i(d, N)$ (we define $\eta \equiv \eta_2$) as a function of d and N, by means of which we can follow the evolution through theory space of the fixed point representing the ith multicritical potential [8].

The analysis presented in [5] revealed that for $d > 4$ and for any N, in accordance with the Ginzburg criterion, one finds only the Gaussian fixed point ($i = 1$). (See [9] for a discussion on the possible existence of nontrivial universality classes in $d \geq 4$ raised by [10]). Starting at $d = 4$, the upper critical dimension for $O(N)$ models, the Wilson-Fisher (WF) fixed points ($i = 2$) branch away from the Gaussian fixed point. When $d = 3$ these fixed points describe the known universality classes of the Ising, XY, Heisenberg and other models.

Approaching $d = 2$ one clearly observes that only the $N = 1$ anomalous dimension continues to grow: for all other values of $N \geq 2$ the anomalous dimension bends downward to become zero exactly when $d = 2$. As explained in [5], this nontrivial fact, not evident from the structure of Eq. (1) alone, is a manifestation of the MWH theorem. We now complement this analysis with the results for the correlation length critical exponents $\nu_i(d, N)$ as a function of d and N. We obtained results for the first several multicritical universality classes $i = 2, 3, 4, 5, \ldots$. Here we will only report in detail the analysis for the WF and tricritical cases ($i = 2, 3$) and briefly comment on the
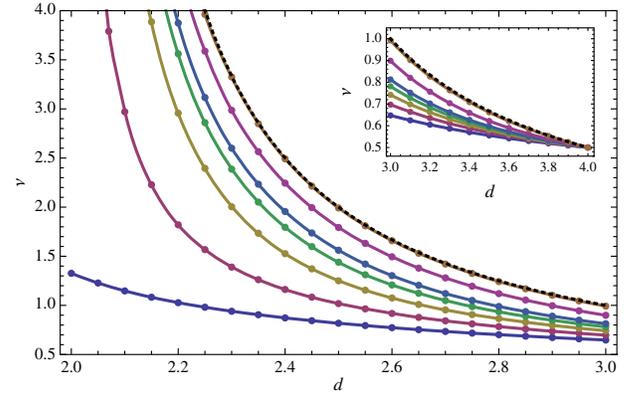
FIG. 1 (color online). Correlation length critical exponent ν, for the WF universality class, as a function of d between two and three for $N = 1, 2, 3, 4, 5, 10, 100$, from bottom to top. In the inset we show the critical exponent in the range $3 \leq d \leq 4$.

other multicritical cases. See Fig. 1 for results on $\nu = \nu_2$ as a function of d for various values of N.

III. EIGENPERTURBATIONS AND ν

The correlation length exponent ν_i is related to the greatest negative (infrared repulsive) eigenvalue $y_{1,i}$ of the linearized RG transformation by $\nu_i = 1/y_{1,i}$ (we define $\nu \equiv \nu_2$). In order to calculate it, we will use the eigenperturbation method described in [11]. As a starting point, we expand the dimensionless effective potential as follows:

$$\tilde{U}_k = \tilde{U}_*(\tilde{\rho}) + \epsilon\tilde{u}_k(\tilde{\rho})e^{yt}, \quad (3)$$

where $\tilde{U}^*(\tilde{\rho})$ is a solution of the fixed point equation (1) and $\tilde{u}_k(\tilde{\rho})$ is a perturbation around the solution whose eigenvalue is y. Substituting this expression into the flow equation, and considering only terms of first order in ϵ, we obtain an equation for the perturbation:

$$(d+y)\tilde{u}_k(\tilde{\rho}) - (d-2+\eta)\tilde{\rho}\tilde{u}'_k(\tilde{\rho})$$
$$= -c_d(N-1)\left(1-\frac{\eta}{d+2}\right)\frac{\tilde{u}'_k(\tilde{\rho})}{(1+\tilde{U}'_*(\tilde{\rho}))^2}$$
$$- c_d\left(1-\frac{\eta}{d+2}\right)\frac{\tilde{u}'_*(\tilde{\rho}) + 2\tilde{\rho}\tilde{u}''_*(\tilde{\rho})}{(1+\tilde{U}'_*(\tilde{\rho}) + 2\tilde{\rho}\tilde{U}''_*(\tilde{\rho}))^2}. \quad (4)$$

In order to solve this equation we need two initial conditions. The first is obtained by noting that the perturbation equation (4) is linear, so we can require the normalization condition $\tilde{u}_k(0) = 1$, while the second one is imposed on $\tilde{u}'_k(0)$ from continuity at zero field:

$$(y + d)\tilde{u}_k(0) = -c_d\frac{(1-\frac{\eta}{d+2})N}{(1+\tilde{U}'_*(0))^2}\tilde{u}'_k(0). \quad (5)$$

It should be noted that in the special case $N = 0$ the continuity at zero field is given by $\tilde{u}_k(0) = 0$ and then the

normalization condition should be imposed on the first derivative of the perturbation $\tilde{u}'_k(0) = 1$.

Now we can identify a single solution for any value of y, but we know that just a discrete set of y values will be the ensemble of the physical eigenvalues of Eq. (1). The supplementary condition to identify this discrete set is given on the shape of the solutions.

A generic solution of Eq. (4) in the $\rho \to \infty$ limit behaves at leading order as

$$\tilde{u}_k(\tilde{\rho}) = a(y)\rho^{\frac{(d-y)}{(d-2+\eta)}} + b(y)e^{C\rho^{\frac{2d}{d-2+\eta}-1}}, \qquad (6)$$

where $a(y), b(y)$ are two functions of the eigenvalue y and C is a constant depending on d and η. This shows that in the infinite field limit, the solution is a linear combination of power-law and exponential diverging parts [11]. In order to find the discrete set of eigenvalues that we need, we have to require the solution to grow no faster than a power-law, so the condition is just $b(y) = 0$.

Using this condition we found just one infrared (IR) repulsive eigenvalue for the WF fixed point, two for the tricritical fixed point, three for the tetracritical fixed point and so on. In this way we were able to construct the curves shown in Figs. 1, 2, 4 and 5.

The proliferation of eigenvalues is due to the fact that the ith universality class has $i - 1$ IR repulsive directions in theory space, and thus we have $i - 1$ solutions with negative eigenvalue in the perturbation equation (4). In the following we will denote as $y_{j,i}$ the jth eigenvalue of the ith universality class.

As was already observed in [5], the vanishing of the anomalous dimension, when combined with the behavior of the ν_i exponents, implies that there are no continuous phase transitions for $N > 2$ in $d = 2$. The case $N = 2$ is peculiar due to the presence of the Kosterlitz-Thouless phase transition [12]; our method is not able to recover this
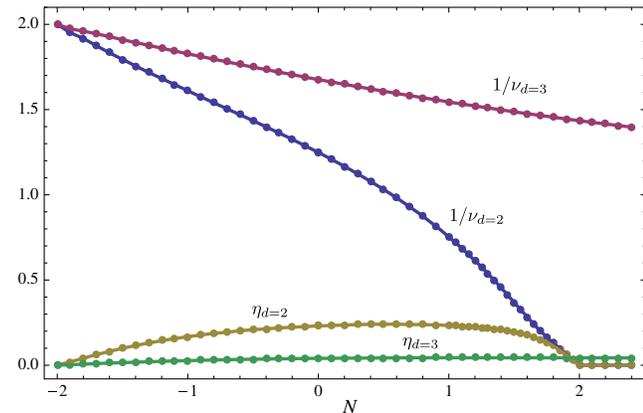
FIG. 2 (color online). Critical exponents η and $y_{1,2} = 1/\nu$ as a function of N in two and three dimensions for the WF universality class. The fact that the two dimensional curves are zero for $N \geq 2$ is a manifestation of the MWH theorem.

result because of its topological nature. Note that this will apply also to the following, and the case $N = 2$ is to be understood in light of the previous remark. Consistent with this argument, we find that only the $N = 1$ model has a finite correlation length exponent in two dimensions; in all other cases, $N \geq 2$, ν diverges as $d \to 2$. This allows us to distinguish the spherical model, related to the $N \to \infty$ limit [13], from the Gaussian model, both having $\eta = 0$. In the $N \to \infty$ limit, we instead recover the known exact relation $\nu(d, \infty) = \frac{1}{d-2}$ [14].

Figure 2 shows η and $y_{1,2} = 1/\nu$ as function of N in the interval between $-2 \leq N \leq 2.5$, for the two cases $d = 2$ and $d = 3$. The critical exponents are continuous in the whole range and in particular around $N = 0$; this is an indication that the $N \to 0$ limit, relevant to the problem of self avoiding random walks (SAW) [15], is well defined.

These curves strictly follow the prescription of the MWH theorem: for $N \geq 2$ both $\eta(2, N)$ and $y_{1,2}(2, N)$ vanish, while in $d = 3$ they have finite values; thus $O(N)$ models with continuous symmetries cannot have a spontaneous symmetry breaking in two dimensions. We remark that both exponents are necessary to distinguish between the case of no phase transition, where we have seen both exponents vanish, and the $N = \infty$ case where, for example $\eta(3, \infty)$ vanishes but $y_{1,2}(3, \infty)$ attains a finite non-mean-field value. Our computation of $\nu(d, N)$ thus completes the RG derivation of the MWH theorem started in [5] with the analysis of $\eta(d, N)$. In the limit $N \to -2$, both exponents attain their mean-field values (namely $\eta = 0$ and $\nu = 1/2$), where indeed the model is know to have Gaussian critical exponents in both dimensions [16].

Our functions $\eta(d, 1)$ and $\nu(d, 1)$ can be compared with results from the bootstrap (BS) approach [17]. The anomalous dimension compares fairly well considering that our computation is based on the solution of a single ODE, while the correlation length critical exponent is slightly overestimated for d in the proximity of two. It will be interesting to have BS results for the $N > 1$ cases in dimension other than three [18] and in particular to see the emergence of the MWH theorem in this approach.

Finally, to our knowledge, our results are the only ones available in the literature regarding the full form of the functions $\eta(d, N)$ and $\nu(d, N)$, functions that are both universal and, in principle, experimentally accessible.

IV. SCALING RELATIONS AND $\alpha, \beta, \gamma, \delta$

Having obtained ν and η as a function of d and N, we can now use the standard scaling relations to obtain the other critical exponents:

$$\alpha = 2 - \nu d \qquad \beta = \nu \frac{d-2+\eta}{2}$$
$$\gamma = \nu(2-\eta) \qquad \delta = \frac{d+2-\eta}{d-2+\eta}. \qquad (7)$$
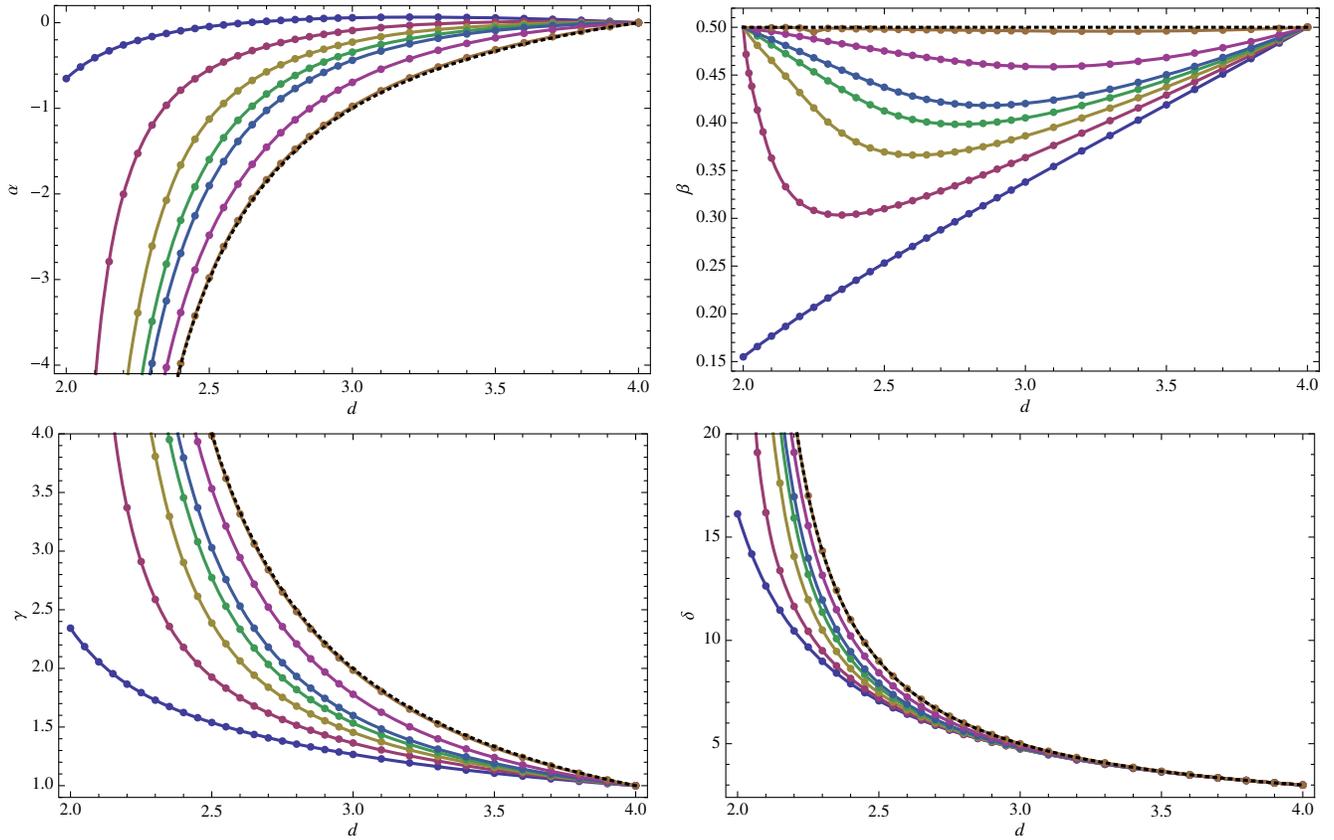
FIG. 3 (color online).   Critical exponents $\alpha, \beta, \gamma, \delta$ of the WF universality class for $N = 1, 2, 3, 4, 5, 10, 100$. The $N = 1$ curves reach a distinguished value at $d = 2$, while the $N = 100$ curve is practically equivalent to the exact large–N spherical model limit (represented by dashed lines). The curves for all the other values of N interpolate between the two.

Our results are shown in Fig. 3 for $2 \leq d \leq 4$ and for $N = 1, 2, 3, 4, 5, 10, 100$.

The first thing we notice is that in the large-N limit we smoothly recover the critical exponents of the spherical model [13] $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = \frac{2}{d-2}$ and $\delta = \frac{d+2}{d-2}$. Our results indicate that the $N = 100$ case is perfectly approximated by the spherical model, while already at $N = 10$ deviations from this limit are appreciable. This shows that, regarding critical exponents (or related) quantities, the leading large-N estimates are quantitatively good only for N of order 10^2 or larger [19].

For $N = 1$ and $d = 2$ our results can be compared with the known exact Ising critical exponents found by Onsager and others [20,21], the comparison can be found in Table I. Quantitative agreement is not excellent, as expected by the simplicity of our approach, based entirely on the solution of a single ODE (1) and the relative eigenvalue problem (4).

Also it should be noted that the errors of our method are found to be most relevant in this case, as it will be clear in the following. This is due to the fact that the error we commit is of the same order of the anomalous dimension of the model considered [24,25].

No other method, to our knowledge, has a similar versatility. In any case, once qualitative understanding

has been achieved, one can obtain arbitrarily good quantitative estimates by resorting to higher orders of derivative expansion [26], of which Eq. (1) represents just the first order.

It is possible to find a better ν value in the $N = 1$ case using a different definition for the anomalous dimension [8] rather than the one we used [5]. This definition, which is strictly valid only in the $N = 1$ case, gives a worse value for $\eta (\simeq 0.4)$, but a much better result for $\nu (= 1.01)$.

In Table I we show also the results obtained for the three dimensional Ising model, as expected, in this case the agreement is much better. This is again due to the fact that the derivative expansion can be considered as an expansion

TABLE I.   Ising Results.

Exp.	3d(BS[22])	3d(this work)	2d(exact[23])	2d(this work)
η	0.036	0.044	1/4	0.23
ν	0.63	0.65	1	1.33
α	0.11	0.050	0	−0.65
β	0.33	0.34	0.125	0.15
γ	1.24	1.27	1.75	2.34
δ	4.79	4.75	15	16.12

in terms of the anomalous dimension: the error we commit will then be of the order of the anomalous dimension, which is smaller in $d = 3$ than in $d = 2$.

In fact the ansatz used in this work (the LPA′ ansatz) for the effective action neglects the dependence of the wave function renormalization on momentum and field; this dependence is essentially governed by the anomalous dimension of the model and it is small when the anomalous dimension itself is small. Thus the LPA′ approximation turns out to be quite effective in these cases [24,25]. As a confirmation of this fact one should consider the good results obtained in the three-dimensional case (again Table I) and the fact that the results become exact in the $N \to \infty$ limit, where in fact the anomalous dimension vanishes. As N grows our quantitative estimates become better; we made comparisons in the cases $N = 2, 3, 4$ and higher and we found good agreement with best known values [27]. Also our predictions are found to be better in the case of multicritical universality classes, where the results for the anomalous dimension are smaller than the standard WF case.

V. TRICRITICAL UNIVERSALITY CASE

In this case we have two IR repulsive eigenvalues of the linearized flow, both shown in Fig. 4 for $2 \leq d \leq 3$, and $N = 1, 2, 3, 4$. The exponent $y_{1,3} = 1/\nu_3$ is the inverse correlation length exponent; indeed at the upper tricritical dimension, $d_{c,3} = 3$, it reaches its mean-field value $y_{1,3} = 2$. When $N = 1$ the exponent does not depart so much from the mean-field result as in the standard WF case. The $d = 2$ value we obtain is $y_{1,3} = 1.90$ to be compared with the exact result [28] $y_{1,3}^{ex} = 1.80$, both rather close to the mean-field value. In the case of continuous symmetries ($N \geq 2$) the tricritical universality class disappears in $d = 2$, and the $y_{1,3}$ exponents correctly return to their mean-field values for every N.
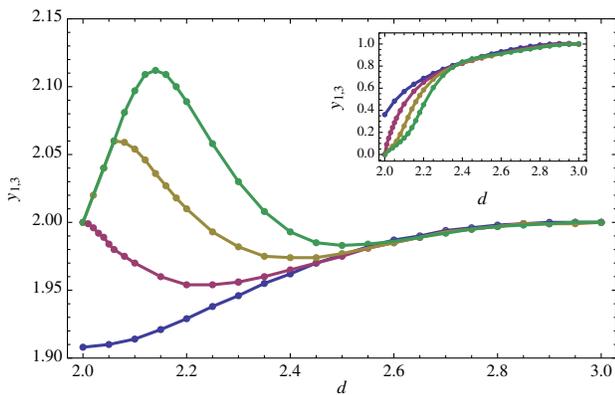
The $y_{2,3}$ exponent, instead, describes the divergence of the correlation length as a function of an additional critical parameter. At the upper tricritical dimension, the mean-field result is $y_{2,3} = 1$. When $N = 1$ we find in two dimensions $y_{2,3} \simeq 0.4$, which should be compared with the exact value [28] $y_{2,3}^{ex} = 0.8$. In this case the agreement is rather low, but this is not surprising since we know that the LPA′ approximation is rather inefficient in $d = 2$. However it should be noted that even if not quantitatively correct, these results can be used to evaluate the crossover exponent $\phi = \frac{y_{2,3}}{y_{1,3}}$. In $d = 2$ this gives $\phi \simeq 0.2$ which, despite the quantitative error, gives a much better estimation than the ϵ–expansion, which provides a negative value for this exponent at order ϵ^2 [29]. For continuous symmetries, $N \geq 2$, $y_{2,3}$ vanishes in $d = 2$, in the same way as the exponent $y_{1,2}$ does in the WF case.

VI. MULTICRITICAL UNIVERSALITY CLASSES

The behavior of the tricritical case can be generalized to the other multicritical universality classes. For these classes with $i > 3$, we have that at the upper critical dimension, $d_{c,i} = 2 + \frac{2}{i-1}$ [8], all the $i - 1$ IR repulsive eigenvalues attain their mean-field values. The largest one will always be $y_{1,i} = 2$, as in the standard WF case, with all the others having a mean-field value smaller than 2. For $N \geq 2$ all the exponents, but the lowest one, will have different values as a function of $d < d_{c,i}$, all remaining pretty close to the mean-field value, which is eventually recovered in $d = 2$. Conversely the lowest eigenvalue will decrease monotonically until it vanishes in $d = 2$. For $N = 1$ instead, all the multicritical universality classes will still exist in $d = 2$ and thus all the exponents will reach a finite non-mean-field value, which will be given by the relative conformal field theory (CFT) result [23].

FIG. 4 (color online). Critical exponents $y_{1,3}, y_{2,3}$ of the tricritical fixed points as a function of d for $N = 1, 2, 3, 4$. These exponents describe the divergence of the correlation length as a function of the two critical parameters of the tricritical universality class.
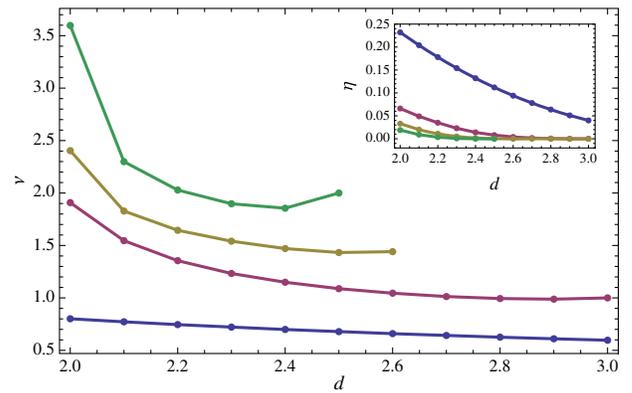
FIG. 5 (color online). Critical exponents in the $N = 0$ case. In the main plot are shown the values of ν_i in the range $2 \leq d \leq 3$ for the (from the bottom) WF, tricritical, tetracritical and pentacritical universality classes, corresponding, respectively, to $i = 2, 3, 4, 5$. In the inset the corresponding values of η_i are reported (in inverted order, from top to bottom).

TABLE II. SAW results.

Exp.	3d(MC[4])	3d(this work)	2d(exact[31])	2d(this work)
η	0.028	0.04	$5/24 \simeq 0.208$	0.232
ν	0.587	0.597	$3/4 = 0.75$	0.801
α	0.239	0.210	$1/2 = 0.5$	0.398
β	0.302	0.310	$5/64 \simeq 0.078$	0.093
γ	1.157	1.169	$43/32 \simeq 1.344$	1.416
δ	4.837	4.769	$91/5 = 18.2$	16.24

A. The $N = 0$ case

Multicritical scaling solutions are also found for $N = 0$, which survive in infinite number when $d \to 2$ [5]. A plot of η_i and ν_i for the first four universality classes $i = 2, 3, 4, 5$ is shown in Fig. 5; these are numerically very similar to those of the $N = 1$ cases (see [5] and Fig. 1 for the WF class). This was indeed expected, judging from Fig. 2.

In Table II we compare the $d = 2$ exact and the $d = 3$ Monte Carlo (MC) results for (WF) self-avoiding walks (SAW) [30], which correspond to the $N = 0$ limit of $O(N)$ models [15], with the results obtained from our analysis and using scaling relations: From these comparisons we see that the $N = 0$ estimates are better than the $N = 1$ estimates, since also the $N \geq 2$ estimates are so, this indicates that the (WF) Wilson-Fisher universality class is the one for which our estimates are poorer.

We are not aware of any known result regarding multicritical phase transitions of polymeric systems, or any other model that belongs to one of the $N = 0$ multicritical universality class. Our estimates for the critical exponents are given in Fig. 5, and to our knowledge these results are novel predictions: it will be interesting to find physical systems or theoretical models described by these universality classes to test them.

VII. CONCLUSIONS

In this paper we reported the computation of critical exponents of $O(N)$ universality classes as a function of the dimension and of the number of field components. The correlation length critical exponent ν was computed by studying the eigenvalue problem obtained linearizing the RG flow of the running effective potential around the scaling solutions found in [5], representing the $O(N)$ multicritical fixed point theories. From this and the previous knowledge of the anomalous dimensions, all the remaining exponents α, β, γ, δ were found using scaling relations.

In particular we displayed the critical exponents for the Wilson-Fisher and tricritical phase transitions for general d and N. Another result which is new to our knowledge are the critical exponents for the multicritical classes in the $N \to 0$ limit. These, via the De Gennes correspondence [15], are universal, observable quantities which can be associated to possible new phases of polymeric systems. To the best of our knowledge, this physics is yet to be observed.

One interesting feature which is worth mentioning is that there is a correspondence between critical exponents of models with short–range interactions in fractional dimension and models with long–range interactions in integer dimension [32]. This means that our curves $\eta(d, N)$ and $\nu(d, N)$ have direct physical interpretation, not only for systems in fractional dimensions, but as describing the critical behavior of models with long–range interactions in two or three dimensions. In this case our universal results could be indirectly tested in the near future, both by numerical simulations and laboratory experiments. Further details on this correspondence can be found in [32].

By computing the function $\nu(d, N)$ we provided the information necessary to complete the nonperturbative RG scenario of $O(N)$ models universality classes as put forward in our previous work [5]. This constitutes a first important example of how one can use RG equations to give precise statements on how universality classes depend on dimension and symmetry group parameters, a general and fundamental problem whose solution has important applications in physical model building in both condensed matter and high energy physics.

It is worth noting that the approach here presented makes a bridge between all the known features of the critical behavior of $O(N)$ models. In fact, epsilon expansion techniques while providing good numerical results close to four dimensions are unable to reproduce even the qualitative features of the models in $d \simeq 2$ [27]. In $d = 2$ it is necessary to use ad hoc methods as CFT to obtain exact quantities. These exact results are however difficult to connect with the $d > 2$ approximate results. Also other expansions based on the exact solution of the spherical model are difficult to calculate at high orders [19] and they fail both in quantitative and qualitative agreement for small N values.

In particular multicritical results are not available in $1/N$ expansions and are also qualitatively incorrect in ϵ–expansions [28], while the approach described here gives all the qualitatively correct results even for these models. The correctness of these findings is granted by the functional description of the theory space of $O(N)$ models which is developed here to full extent.

We conclude by stressing that here we explored just the simplest realization of our method and this alone allowed a complete qualitative understanding of $O(N)$ universality classes. We believe that its numerical results, where not fully satisfactory, can be fairly improved in future extensions along the lines explained in the text, and will ultimately lead to a definitive quantitative understanding of critical properties of $O(N)$ models.

ACKNOWLEDGMENTS

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