Black strings in Gauss-Bonnet theory are unstable

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We report the existence of unstable s-wave modes for black strings in Gauss-Bonnet theory (which is quadratic in the curvature) in seven dimensions. This theory admits analytic uniform black strings that are, in the transverse section, black holes of the same Gauss-Bonnet theory in six dimensions. All the components of the perturbation can be written in terms of a single component and its derivatives. For this, we find a master equation that admits bounded solutions provided the characteristic time of the exponential growth of the perturbation is related to the wave number along the extra direction, as in general relativity. It is known that these configurations suffer from a thermal instability; therefore, the results presented here provide evidence for the Gubser-Mitra conjecture in the context of Gauss-Bonnet theory. Because of the nontriviality of the curvature of the background, all of the components of the metric perturbation appear in the linearized equations. Similar to spherical black holes, the black strings should be obtained as the short-distance limit $r \ll \alpha^{1/2}$ of the black-string solution of Einstein-Gauss-Bonnet theory (which is not known analytically), where α is the Gauss-Bonnet coupling.

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I. INTRODUCTION

Gravity in higher dimensions has been an important scenario to test to what degree the ideas we have gained from four-dimensional gravity are generic. Motivated also by string theory and supergravity, many results have been obtained in recent decades concerning gravity in dimensions higher than four—for example, the existence of the asymptotically flat black ring and all its extensions (for reviews, see [1] and [2]). These objects were conjectured to be unstable for large angular momentum, as they inherit the Gregory-Laflamme instability [3] of nonextremal black strings and black p-branes [4,5]. Indeed, the black-ring instability has been confirmed in Refs. [6–8]. The Gregory-Laflamme instability can be guessed from thermodynamical arguments since, as a function of the mass, the entropies of the black hole and the black string cross at a given critical mass M_c. This can be seen from the fact that the entropy of the black hole grows as $S_{\rm BH} \sim M^{\frac{D-2}{D-3}}$, while the entropy of the black string grows as $S_{\rm BS} \sim M^{\frac{D-3}{D-4}}$. For masses below M_c, the black hole is thermally favored; above M_c, the black-string solution is the one with greater entropy and, therefore, is the most favored. This relation between thermal and perturbative instabilities led Gubser and Mitra to conjecture that both kinds of instabilities always appear together for black-hole configurations with extended directions [9], which was recently proven in [10] for general relativity (GR) in vacuum. To understand the complete evolution of the unstable mode, a nonlinear analysis is required. Recent outstanding numerical results in five dimensions seem to indicate that the black string evolves toward a nonhomogenous configuration with sections in which the size of the string eventually shrinks to zero, generating a null singularity and providing a counterexample of the cosmic censorship conjecture [11] (for a historical review on this problem, see chap. 2 of [2]).

An interesting problem is whether higher-curvature corrections may modify this scenario. In the particular case of higher-curvature Lovelock theories [12], it is difficult to construct analytic homogenous black strings due to the fact that the new dimensionful coupling constants introduce a length scale that induces the existence of a cosmological constant. Numerical and approximate results in this context have been reported in [13–17]. Specifically, in [14], static uniform and nonuniform black strings were constructed. The latter were constructed along the lines of [18], i.e., perturbatively in a nonhomogeneity parameter, and therefore can be considered as a static perturbation of the uniform black string. Comparing the entropies of these configurations, the authors provided evidence for the Gubser-Mitra conjecture in the context of Einstein-Gauss-Bonnet theory.

The situation in theories that have a single Lovelock term is much more like the one in general relativity, since, as shown in Ref. [19], homogeneous black strings and black p-branes can be constructed analytically. These solutions are also important, since they should be obtained as the short-distance configuration ($r \ll \alpha^{1/2}$) of the black-string solution of Einstein-Gauss-Bonnet theory, which is not known analytically. Here, α is the Gauss-Bonnet coupling. This is what occurs, for example, with the "healthy branch" of the spherically symmetric black hole in Einstein-Gauss-Bonnet gravity, which is defined by the following action:

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$$I_{\mathrm{EGB}}[g] = \frac{1}{16\pi G}\int d^D x\sqrt{-g}[R + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})]. \tag{1}$$

This theory admits the following black-hole solution [20]:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, \tag{2}$$

where

$$f(r) = 1 + \frac{r^2}{(D-3)(D-4)\alpha}\left[1 - \sqrt{1 + \frac{64\pi G(D-3)(D-4)\alpha}{(D-2)V(S^{D-2})}\frac{M}{r^{D-1}}}\right], \tag{3}$$

where the integration constant M is the mass. Here α has dimensions of length squared; we can analyze the behavior of this metric function for $r \gg \sqrt{\alpha}$ and $r \ll \sqrt{\alpha}$ which, respectively, read

$$f(r) \underset{r\gg\sqrt{\alpha}}{\approx} 1 - \frac{32\pi G}{(D-2)V(S^{D-2})}\frac{M}{r^{D-3}} + \cdots, \tag{4}$$

$$f(r) \underset{r\ll\sqrt{\alpha}}{\approx} 1 - \left(\frac{64\pi G M}{(D-3)(D-4)(D-2)\alpha V(S^{D-2})}\right)^{\frac{1}{2}}\frac{1}{r^{\frac{D-5}{2}}} + \cdots. \tag{5}$$

In the former case the solution reduces to the Schwarzschild-Tangherlini black hole, while in the latter it reduces to the solution found in [21]. Therefore, we have that for large distances, the effects of the quadratic curvature term are subleading, whereas for short distance (as compared with $\sqrt{\alpha}$) the quadratic terms dominate and one recovers a solution of Gauss-Bonnet theory.

As shown in [19], the asymptotically flat black holes constructed in [21] can be oxidated to construct homogenous black-string and black p-brane solutions. These spacetimes are solutions of the theory that contains only the kth-order term in the Lovelock theory, where the case $k = 1$ is the solution of general relativity. For simplicity, let us consider only the quadratic Gauss-Bonnet term in seven dimensions,

$$I_{\mathrm{EGB}}[g] = \frac{\alpha}{16\pi G}\int d^7 x\sqrt{-g}[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}]. \tag{6}$$

This theory has the following two solutions:

$$ds^2 = -\left(1 - \frac{\mu}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{\mu}{r}} + r^2 d\Omega_5^2 \tag{7}$$

and

$$ds^2 = -\left(1 - \frac{m}{r^{1/2}}\right)dt^2 + \frac{dr^2}{1 - \frac{m}{r^{1/2}}} + r^2 d\Omega_4^2 + dz^2, \tag{8}$$

which correspond to a spherically symmetric black hole and a black string, respectively. The constants m and μ determine the masses of the configurations, while $d\Omega_n$ stands for the line element of the n-sphere, S^n. From the experience gained from the spherically symmetric black hole, one can expect that these black strings should be obtained as the short-distance limit of the black string of Einstein-Gauss-Bonnet theory in seven dimensions, which is not known analytically.

The black strings and black p-branes constructed in this way were proven to be thermally unstable [19] in exactly the same manner as the black strings in general relativity, since the entropies, as a function of the mass for Eqs. (7) and (8), read $S_{\mathrm{BH}}^{\mathrm{GB}} \sim M^{\frac{3}{2}}$ and $S_{\mathrm{BS}}^{\mathrm{GB}} \sim M^2$, respectively, crossing at a critical mass M_c^{GB}. The heat capacities of the black hole [Eq. (7)] and the black hole in the transverse section of Eq. (8) are negative (see [21]); therefore, both black objects (7) and (8) are locally thermally unstable.[1]

A natural question is whether such thermal instability has a perturbative counterpart. In this paper we show this is indeed the case. In the next section we show that the black strings of Gauss-Bonnet theory [Eq. (8)] are unstable under the s-wave mode, and that such instability disappears for compactified black strings that are sufficiently short.

[1]The temperature of the black string [Eq. (8)] is the same as that of the black hole on its transverse section, and the mass of such a black string corresponds to the mass of the black hole multiplied by the extension of the extended direction. Therefore, the sign of the heat capacity remains the same after the oxidation.

II. THE PERTURBATIVE INSTABILITY

Here we will be concerned with gravitational perturbations in the context of Gauss-Bonnet theory. The field equations are, therefore, given by

$$E_{\mu\nu} \coloneqq 2RR_{\mu\nu} - 4R_{\mu\rho\nu\sigma}R^{\rho\sigma} + 2R_{\mu\rho\sigma\tau}R_{\nu}{}^{\rho\sigma\tau} - 4R_{\mu\rho}R_{\nu}{}^{\rho} - \frac{1}{2}g_{\mu\nu}(R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}) = 0. \tag{9}$$

For simplicity we will focus on the seven-dimensional case. As mentioned above, this theory admits the homogenous black-string solution (8). The radius of the horizon reads $r_+ = m^2$. In order to work with a finite range of parameters, let us consider the change in the radial coordinate given by

$$r = \left(\frac{m}{1-x}\right)^2 \tag{10}$$

that maps the region outside the event horizon $r \in [m^2, +\infty[$ to $x \in [0, 1[$. In these new coordinates the metric 8)) reads

$$ds^2_{\mathrm{BS}_7} = -xdt^2 + \frac{4m^4dx^2}{x(1-x)^6} + \left(\frac{m}{1-x}\right)^2 d\Omega_4^2 + dz^2. \tag{11}$$

The s-wave perturbation on the background black-string metric (11) reads

$$h_{\mu\nu}(t,x,z) = e^{\Omega t}e^{ikz}\left(\begin{array}{cc|c|c} H_{tt}(x) & H_{tx}(x) & 0 & 0 \\ H_{tx}(x) & H_{xx}(x) & 0 & 0 \\ \hline 0 & 0 & H(x)\sigma_{S^4} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

where σ_{S^4} is the metric of the four-sphere and k is the wave number along the z direction. An unstable mode is defined as a bounded solution of the linearized Gauss-Bonnet equations (9) with positive Ω. It is easy to show that the linearized field equations imply that the components of the perturbation can be written in terms of $H_{tx}(x)$ in the following manner:

$$H_{tt}(x) = \frac{(1-x)^6x^2}{4m^4\Omega^4}H'_{tx} + \frac{x(1-x)^6}{4m^4\Omega}H_{tx}, \tag{12}$$

$$H_{xx}(x) = -\frac{x}{\Omega}H''_{tx} - \frac{2(1-4x)}{(1-x)\Omega}H'_{tx} + \left(\frac{(3k^2x+4\Omega^2)m^4}{x(1-x)^6\Omega} + \frac{6}{\Omega(1-p)}\right)H_{tx}, \tag{13}$$

$$H(x) = \frac{x^2(1-x)^2}{6\Omega}H''_{tx} + \frac{(1-3x)(1-x)x}{2\Omega}H'_{tx} + \left(\frac{(1-x)(1-7x)}{6\Omega} - \frac{m^4(3k^2x+4\Omega^2)}{6(1-x)^4\Omega}\right)H_{tx}, \tag{14}$$

where the prime ($'$) denotes differentiation with respect to x. The component $H_{tx}(x)$ fulfils the following linear second-order master equation:

$$A(x)H''_{tx} + B(x)H'_{tx} + C(x)H_{tx} = 0, \tag{15}$$

with

$$A(x) = (1-x)^6x^2((1-x)^6 - (12k^2x + 16\Omega^2)m^4), \tag{16}$$

$$B(x) = 3x(1-x)^5((32k^2x^2 + 48x\Omega^2 - 8k^2x - 16\Omega^2)m^4 + (1-x)^7), \tag{17}$$

$$C(x) = 4(4\Omega^2 + 3k^2x)^2m^8 + (1-x)^5(45k^2x^2 + 164x\Omega^2 + 3k^2x - 20\Omega^2)m^4 + (1-x)^{12}. \tag{18}$$

Then, one can see that all the linearized field equations are solved provided that Eqs. (12)–(14) and (15) hold. Note that the master equation is invariant under

$$m \to \gamma m, \qquad \Omega \to \gamma^{-2}\Omega, \qquad k \to \gamma^{-2}k \tag{19}$$

for an arbitrary constant γ. Therefore, it is enough to study the existence of unstable modes for a fixed value of the horizon radius $r_+ = m^2$, since the other values of r_+ can be obtained by applying the scaling symmetry (19).

We are then left with finding a well-behaved solution of the master equation (15). This equation implies that the solution $H_{tx}(x)$ admits the following asymptotic behaviors at the horizon $(x \to 0)$ and at infinity $(x \to +1)$, respectively:

$$H_{tx} \underset{x\to 0}{\to} C_{\pm} x^{-1\pm 2m^2\Omega}(1+\mathcal{O}(x)), \tag{20}$$

$$H_{tx} \underset{x\to 1}{\to} E_{\pm}(1-x)^{\alpha_{\pm}} e^{\mp\frac{m^2\sqrt{3k^2+4\Omega^2}}{2(1-x)^2}\mp\frac{(8\Omega^2+3k^2)m^2\sqrt{3k^2+4\Omega^2}}{2(3k^2+4\Omega^2)(1-x)}}(1+\mathcal{O}(1-x)), \tag{21}$$

with

$$\alpha_{\pm} = \frac{1}{8}\frac{-12(3k^2+4\Omega^2)^2 \pm m^2(144k^2\Omega^2+128\Omega^4+27k^4)\sqrt{3k^2+4\Omega^2}}{(3k^2+4\Omega^2)^2}. \tag{22}$$

Since we are looking for unstable modes, we need to find a numerical solution that interpolates between the plus sign in Eq. (20) and the minus sign in Eq. (21). It is natural to think that in order to have a well-posed behavior at the horizon we need to impose $\Omega > \Omega_{\mathrm{c,GB}} \coloneqq \frac{1}{2m^2}$; this is as it was originally considered in [22], where it was proven that in the five-dimensional black string in general relativity there is no nonsingular, single, unstable mode in this family (in GR in five dimensions, $\Omega_{c,\mathrm{GR}} = \frac{1}{r_+}$). Nevertheless, in general relativity, in the range $0 < \Omega < \Omega_{\mathrm{c,GR}}$ one can construct a perturbation that is a composition of single divergent modes at the horizon in such a manner that the divergences cancel; it occurs in this manner for the instabilities in some colored black holes [23], which was originally observed by Vishveshwara in [24]. This can also be seen considering the fact that a $t = \mathrm{const}$ surface intersects the bifurcation surface rather than the future horizon. It is, therefore, necessary to consider Kruskal-like coordinates, where the $T = \mathrm{const}$ surfaces do indeed intersect the future horizon. Then, by going to Kruskal coordinates, it is easy to see that the unstable modes we find below are regular at the future horizon provided that we choose the branch with the plus sign in Eq. (20), even if $\Omega < (2m^2)^{-1}$.

In order to find whether the master equation (15) admits a bounded solution for some positive values of Ω, we will follow the approach developed in [25] for quasinormal modes. Briefly, the method consists of proposing a power series solution around the horizon, then selecting the well-behaved branch, and, finally, truncating the power series to some order N. Then, using the fact that, due to the pole structure of the equation, such a power series has a convergence radius that includes at least $x = 1$, we can therefore request for the truncated series to vanish at infinity ($x = 1$). Such an equation provides for the spectrum of unstable modes. For details we refer to the original work, Ref. [25].

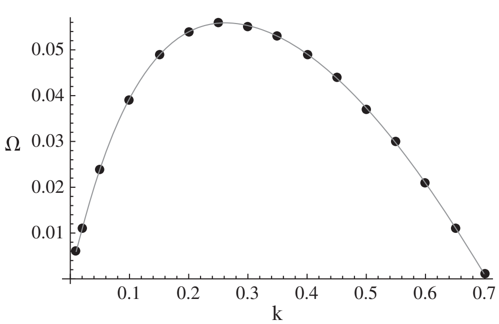

FIG. 1. Ω vs k for the homogeneous black string in Gauss-Bonnet in $D = 7$. The parameter m in the solution has been fixed to 1, and any value for the mass can be obtained by applying the scaling transformation in Eq. (19). The numerical precision is such that four digits in the values of Ω are stable (the continuous curve has been included to facilitate the visualization).

The results of the previous analysis are depicted in Fig. 1. From these results we see that there is a minimum wavelength $\lambda_{\min}$ and, by implication, the existence of a critical length for the string, above which instability occurs.

Similar to the black string in general relativity [5], provided that $k \neq 0$, one can show that the perturbation cannot be gauged away. Another straightforward method to check that these perturbations are physical and cannot be gauged away is to consider the following scalar invariant:

$$A = 881 R_{abcd}R^{cd}{}_{ef}R^{efab} + 2428 R^{ab}{}_{cd}R^{ce}{}_{bf}R^{df}{}_{ae}, \tag{23}$$

which vanishes identically on the unperturbed metric, but is nonvanishing for the perturbed black string.

We have, then, found a set of physical s-wave modes on the black string in Gauss-Bonnet theory [Eq. (8)], which drive the instability of the background; therefore, black strings in Gauss-Bonnet theory are unstable.

III. CONCLUSIONS

In this paper we have shown that the black strings in Gauss-Bonnet theory are unstable under gravitational perturbations. Following the arguments in [5], one can prove that the instability we have found cannot be gauged away; therefore, it represents a truly physical instability. Because the field equations are quadratic in the curvature, the linearization around the maximally symmetric Minkowski vacua does not provide any equation at all[2];

[2] As an example of how to deal with the phase-space structure of such degenerate systems see, e.g., [26].

therefore, in order to study the perturbative properties of the solutions of Gauss-Bonnet gravity, one needs to perturb around solutions that have a nontrivial Riemann tensor, as in the case of the black string. As mentioned above, the linearized equations around such a background are nondegenerate, since all the perturbed metric components appear in the linearized equations. In order for the black strings to be unstable, the wavelength of the perturbation along the extended direction has to be above some minimum critical value λ_c. This critical value tends to zero in the large-D limit in general relativity [27,28], and, since the large-D behavior of GR is qualitatively similar to that in gravity theories with a single Lovelock term [29], one may also expect $\lambda_c \to 0$ as D grows for the black strings and black p-branes constructed in [19]. Given the results presented in this work, it is natural to expect that the black-string solution of the full Einstein-Gauss-Bonnet theory will suffer from the Gregory-Laflamme instability, which will induce an instability for large angular momentum in the rotating version of the static black string constructed numerically in [30].

For Einstein-Gauss-Bonnet gravity, different stability analysis of black holes have been performed in [31–37]; it would be interesting to extend such analysis to the whole family of black holes in [21], which are the black holes in the transverse section of the black strings we have considered here.[3]

It is also worth exploring whether the results presented here can be extended to all the black strings and black p-branes obtained in [19], or even to the compactifications with Einstein manifold in four dimensions that were obtained in [43] for Lovelock theories. Work along these lines is in progress.

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[3]It would be also interesting to extend these results to cases with more general asymptotic behaviors, e.g., [38–42].

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