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Pragmatic mode-sum regularization method for semiclassical black-hole spacetimes

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Computation of the renormalized stress-energy tensor is the most serious obstacle in studying the dynamical, self-consistent, semiclassical evaporation of a black hole in 4D. The difficulty arises from the delicate regularization procedure for the stress-energy tensor, combined with the fact that in practice the modes of the field need to be computed numerically. We have developed a new method for numerical implementation of the point-splitting regularization in 4D, applicable to the renormalized stress-energy tensor as well as to $\langle\phi^2\rangle_{\rm ren}$, namely the renormalized $\langle\phi^2\rangle$. So far we have formulated two variants of this method: *t-splitting* (aimed for stationary backgrounds) and *angular splitting* (for spherically symmetric backgrounds). In this paper we introduce our basic approach, and then focus on the t-splitting variant, which is the simplest of the two (deferring the angular-splitting variant to a forthcoming paper). We then use this variant, as a first stage, to calculate $\langle\phi^2\rangle_{\rm ren}$ in Schwarzschild spacetime, for a massless scalar field in the Boulware state. We compare our results to previous ones, obtained by a different method, and find full agreement. We discuss how this approach can be applied (using the angular-splitting variant) to analyze the dynamical self-consistent evaporation of black holes.

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I. INTRODUCTION

After the discovery of Hawking radiation [1] in 1975, it was widely anticipated that the semiclassical approach to gravity based on quantum field theory (QFT) in curved spacetime would open the opportunity to explore various interesting physical problems, in which a quantum field interacts with the curved spacetime where it resides. Among these problems, of special interest is the self-consistent dynamical evaporation of a black hole (BH). Another outstanding problem in this class is the evolution of quantum fluctuations in the very early Universe, and the resulting cosmological structure formation.

Indeed, in the latter problem of cosmological quantum-field perturbations remarkable progress has been achieved over the last few decades [2,3]. However, the problem of analyzing the semiclassical evolution of an evaporating BH still remains a serious challenge.

To understand the difficulties in analyzing this interesting problem of self-consistent BH evaporation, let us briefly review the basic structure of semiclassical gravity. The metric $g_{\alpha\beta}(x)$ is treated as a classical field, and it is assumed to satisfy the *semiclassical Einstein equation*

$$G_{\alpha\beta} = 8\pi\langle T_{\alpha\beta}\rangle_{\rm ren}, \tag{1.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor associated with $g_{\alpha\beta}(x)$.[1] The source term $\langle T_{\alpha\beta}\rangle_{\rm ren}$ is the regularized expectation value of the stress-energy tensor associated with a quantum field $\phi(x)$. For the sake of simplicity we shall take ϕ here to be a scalar field. It is supposed to satisfy the field equation

$$(\Box - m^2 - \xi R)\phi = 0, \tag{1.2}$$

where m and ξ respectively denote the mass and coupling constant of the scalar field. Note that the theory is semiclassical, as the field ϕ is quantized but the metric $g_{\alpha\beta}$ is classical; nevertheless the two are coupled.

The major difficulty in solving (or even analyzing) the field equation (1.1) has to do with the regularization of the divergent quantity $\langle T_{\alpha\beta}\rangle$.[2] As opposed to QFT in flat spacetime in which one can use the normal-ordering procedure, in curved spacetime the outcome of this procedure depends on the choice of time slicing which is completely arbitrary. There exists a regularization method named *point splitting* (PS), also known as covariant point separation, which gives a general prescription for how to regularize quantities which are quadratic in the field and its derivatives such as $\langle T_{\alpha\beta}\rangle$. Alas, implementing this prescription in situations where the solution of the field equation (1.2) is known only numerically turns out to be a surprisingly difficult problem.

In this paper we shall focus on the regularization of $\langle\phi^2\rangle$ instead of $\langle T_{\alpha\beta}\rangle$. The quantity $\langle\phi^2\rangle$ is also divergent, but not as strong as $\langle T_{\alpha\beta}\rangle$. In addition, the scalar character of $\langle\phi^2\rangle$ (as opposed to the tensorial $\langle T_{\alpha\beta}\rangle$) makes it easier to regularize. These properties make $\langle\phi^2\rangle$ a convenient tool to

[1] Throughout this paper we use relativistic units $c = G = 1$ and the $(-+++)$ signature.

[2] There are also other difficulties, e.g. the runaway problem, as noted by Wald [4].

examine and explain new ideas concerning regularization. In order to calculate $\langle \phi^2(x) \rangle_{\rm ren}$ [i.e. the regularized $\langle \phi^2(x) \rangle$] using the PS method we split the point x and write it as a product of ϕ at two different points, namely $\langle \phi(x)\phi(x') \rangle$. This is known as the *two-point function* (TPF). We then subtract from the TPF a known *counterterm* and take the limit $x' \to x$. This limit, however, is what makes the numerical implementation so hard.

In 1984 Candelas and Howard [5] developed a method to numerically implement PS if a high-order WKB approximation for the field modes is known. Using this method, they calculated $\langle \phi^2 \rangle_{\rm ren}$ in Schwarzschild spacetime, and subsequently this method was used by Howard [6] to calculate $\langle T_{\alpha\beta} \rangle_{\rm ren}$ in Schwarzschild. Later, this method was extended to a general static spherically symmetric background, first for $\langle \phi^2 \rangle_{\rm ren}$ by Anderson [7], and subsequently to $\langle T_{\alpha\beta} \rangle_{\rm ren}$ by Anderson *et al.* [8].

The limitation of the method proposed by Candelas and Howard is that it requires a high-order WKB approximation for the field modes: at least second order for $\langle \phi^2 \rangle_{\rm ren}$ and fourth order for $\langle T_{\alpha\beta} \rangle_{\rm ren}$. In the ordinary (Lorentzian-signature) Schwarzschild metric this task of high-order WKB expansion is very difficult, especially because of the presence of a turning point: For typical modes of large ω and l, there is a turning point on the r axis, at a value $r_{\rm turn}(\omega, l)$ where the effective potential $V_l(r)$ equals ω^2.[3] The mode's radial function is essentially oscillatory at $r > r_{\rm turn}$ and exponential at $r < r_{\rm turn}$. Both these basic WKB approximations—the oscillatory approximation at $r > r_{\rm turn}$ and the exponential approximation at $r < r_{\rm turn}$—break down and actually diverge at $r \to r_{\rm turn}$. To correctly match the two approximations, one has to use another, intermediate approximation valid in the neighborhood of $r = r_{\rm turn}$. This *turning-point approximation* is based on the Airy function. Whereas the leading-order matching is manageable, it becomes exceedingly hard to go to higher-order WKB, because each succeeding order will now require its own turning-point matching. Furthermore, to the best of our understanding, the series of powers involved in the Airy-based turning-point expansion proceeds in powers of $\omega^{-1/3}$ (rather than ω^{-1}). Correspondingly, we may expect that in the presence of a turning point, to implement the WKB-based expansion to order ω^{-4} (required for calculating $\langle T_{\alpha\beta} \rangle_{\rm ren}$), one would have to carry matched asymptotic expansion up to *12th* order in $\omega^{-1/3}$ (or 6 such orders for $\langle \phi^2 \rangle_{\rm ren}$)—a formidably difficult task.

To overcome these difficulties, Candelas and Howard [5] and several others [6–8] used an elegant trick: They used Wick rotation to analytically extend the background metric to the Euclidean sector. Any static spacetime is guaranteed to have such a real-metric Euclidean sector. In the latter, the radial equation does not admit a turning point. This way, it was possible to carry the WKB analysis to the desired order and to implement the above regularization scheme—for $\langle \phi^2 \rangle_{\rm ren}$ as well as for $\langle T_{\alpha\beta} \rangle_{\rm ren}$.

Our ultimate goal, however, is to develop a regularization scheme for $\langle \phi^2 \rangle$ and $\langle T_{\alpha\beta} \rangle$, applicable to *time-dependent* backgrounds as well. Such a time-dependent metric (even if spherically symmetric) does *not* generically admit a Euclidean sector. We are thus led to carry the analysis directly in the Lorentzian sector, which in turn implies the presence of a turning point in the radial equation, hampering any attempt to carry high-order WKB expansion. We shall therefore refrain from establishing our regularization scheme on the WKB analysis.

There is another obvious reason for avoiding WKB analysis: Consider a time-dependent spherically symmetric background. The field equation for a given mode may still be expressed as a one-dimensional (namely $1 + 1$) wave equation with an effective potential, but now the potential will be time dependent. In such a situation, even the leading-order WKB (and even if we forget for the moment about the turning point) becomes a nontrivial task, let alone higher-order WKB analysis.

For these reasons, we shall not base our regularization scheme on high-order WKB expansion. Instead, in our method we extract the required information concerning the high-frequency field's modes directly from the well-known counterterm (2.6) for $\langle \phi^2 \rangle_{\rm ren}$ (and, for computing $\langle T_{\alpha\beta} \rangle_{\rm ren}$, from the counterterm developed by Christensen [9]).

We point out that this approach, namely extraction of the high-frequency asymptotic behavior of the modes from a known local counterterm, was actually initiated by Candelas [10]—already before he and Howard resorted to the Euclidean sector [5]. However, Candelas' analysis was restricted to the Schwarzschild case (which in particular means restriction to staticity, spherical symmetry, and to vacuum[4]). We should also comment that even in that case the analysis in Ref. [10] was not completed, because the required integral of the regularized mode contribution over ω was not carried out. When we attempted to implement this integral over ω, we found that it actually fails to converge in the usual sense, due to growing oscillations (see below), which led us to introduce the notion of a generalized integral. Our approach is in this sense a completion of Candelas's method, as well as its generalization beyond the vacuum case (and with the scope of further extending it to dynamical backgrounds).

Our method requires the field modes to admit a trivial decomposition in at least one of the coordinates (e.g. through $e^{-i\omega t}$ or spherical harmonics). This usually

[3] In Schwarzschild spacetime (and any other eternal BH spacetimes), for given ω and l there are usually *two* such roots. Here we shall explicitly refer to the larger one, but the same complications arise also at the smaller root.

[4] In particular, the logarithmic counterterm is not encountered in the vacuum case.

corresponds to having a Killing field in spacetime.[5] The splitting is then done in that trivial coordinate—which enables us to treat the coincidence limit analytically. So far we have developed two different variants of our method: (i) the *t-splitting* variant, which requires a time-translation symmetry; (ii) the *angular-splitting* variant, which requires spherical symmetry. We are also exploring a third variant, *azimuthal splitting* (which would only require axial symmetry), but this one is still in progress. In all these variants we assume for simplicity that the background is asymptotically flat, although this requirement can probably be relaxed.

Since our ultimate goal is to analyze $\langle T_{\alpha\beta}\rangle_{\rm ren}$ on the time-dependent background of an evaporating BH, the t-splitting method is insufficient, and we shall actually need the angular-splitting variant. It turns out, however, that the t-splitting variant is in some sense conceptually simpler and easier to present at first stage, because certain additional complications arise in the angular-splitting method. Although we know how to address these complications, they make the method's logical structure a bit more obscure and harder to explain. For this reason, for the sake of introducing our basic regularization strategy we choose to present here the t-splitting variant, which is logically simpler. And for exactly the same reason, in this paper we shall display the regularization of $\langle\phi^2\rangle$ rather than $\langle T_{\alpha\beta}\rangle$. Then in the next paper we plan to present the angular-splitting variant, which is going to be our main tool for analyzing dynamical BH evaporation.

The TPF diverges for any pair of points connected by a null geodesic, even if they are far from each other [11]. As it turns out, this long-distance divergence of the TPF leads to undamped oscillations in the mode contributions at large ω. To address this issue we use the concept of a *generalized integral*, in which these oscillations are properly damped upon integration over ω, which fully cures the oscillations problem. The origin of this complication (the presence of connecting null geodesics) is discussed in Sec. II B and also in Appendix B; and the resolution of the oscillations problem by means of a generalized integral (and particularly the so-called "self-cancellation integral") is described in Sec. II B and further in Appendix A.

In the description of the t-splitting method in Sec. III we assume a spherically symmetric static background for the sake of simplicity. However, as was discussed above, the t-splitting method does not require spherical symmetry, and in principle it may be applied to a generic (asymptotically flat) stationary spacetime. We outline this generalization of the method to stationary backgrounds in Sec. III B. We point out, however, that some completion is still required in the case of a stationary eternal BH (see therein).

[5]Having at least one trivial coordinate is a necessary condition for the applicability of our method, but we do not claim that it is also a sufficient condition.

Next we apply our method explicitly to the Schwarzschild case, computing $\langle\phi^2\rangle_{\rm ren}$ in the Boulware state. We compare our results to those obtained previously by Anderson (using the Euclidean sector), and find full agreement.

This article is divided as follows: In Sec. II we present the basic PS method, and then briefly outline the procedure developed by Candelas and Howard [5]. Note that Sec. II B discusses certain subtleties of the TPF in some detail, and may be skipped in first reading. In Sec. III we present our t-splitting method. We first describe it for a spherically symmetric static background, and then outline its generalization to a generic stationary background. In Sec. IV we harness this method for the calculation of $\langle\phi^2\rangle_{\rm ren}$ in the Schwarzschild metric. Finally, in Sec. V we discuss the implications of our new method and try to pave the path towards our ultimate goal of investigating self-consistent BH evaporation.

II. BASIC POINT-SPLITTING METHOD AND ITS NUMERICAL IMPLEMENTATION

We start by sketching the basic PS regularization method. This method is aimed to regularize the expectation value of various quantities which are quadratic in the field operator (and its derivatives). Among these quantities, the most important one is probably the energy-momentum tensor $\langle T_{\alpha\beta}\rangle$. However, in this first paper we shall consider $\langle\phi^2\rangle$ as a simpler example (although we shall occasionally remark on the analogous calculation of $\langle T_{\alpha\beta}\rangle$).

For simplicity we consider here a quantum scalar field $\phi(x)$ living in a static, spherically symmetric, asymptotically flat spacetime with metric

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$. The field operator may then conveniently be expressed as

$$\phi(x) = \int_0^\infty d\omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(f_{\omega lm}(x)a_{\omega lm} + f^*_{\omega lm}(x)a^\dagger_{\omega lm}). \tag{2.2}$$

Here, $a^\dagger_{\omega lm}$ and $a_{\omega lm}$ are the creation and annihilation operators of the field's ωlm mode, $f_{\omega lm}(x)$ is a complete, orthonormal family of modes[6] taking the form

$$f_{\omega lm}(x) = e^{-i\omega t}Y_{lm}(\theta,\varphi)\bar{\psi}_{\omega l}(r), \tag{2.3}$$

and $Y_{lm}(\theta,\varphi)$ are the usual spherical harmonics. The radial functions $\bar{\psi}_{\omega l}(r)$ are obtained by solving the field equation

[6]By "orthonormal" we mean that the inner product of two mode functions $f_{\omega lm}$ and $f_{\omega' l' m'}$ is $\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')$.

for $\phi(x)$ with the decomposition (2.3) and with appropriate boundary conditions.

The operation of summation over m, l and integration over ω repeats many times in the analysis below. We shall generally refer to this operation as the "mode sum" (despite the slight abuse of terminology). We point out, however, that the decomposition (2.2) applies as-is in the case of an asymptotically flat background spacetime with simple asymptotic structure (like e.g. Minkowski or a star). But if the background spacetime is an eternal BH with a past horizon, then for each ωlm combination there are actually two orthonormal modes, namely the "in" and "up" modes. The in modes are those described above (namely monochromatic waves propagating from past null infinity), and the up modes describe monochromatic waves that emerge from the past horizon. In this case of eternal BH, the mode sum should also include a summation over the contributions of these two independent modes for each ωlm (as explicitly described in Sec. IV for the Schwarzschild case).

The quantity $\langle\phi^2\rangle$ obviously depends on the quantum state. Naturally, one would like to evaluate it in the vacuum state. The latter is defined to be the quantum state annihilated by each of the above $a_{\omega lm}$ operators.[7]

Trying to naively calculate $\langle\phi^2(x)\rangle$ in the vacuum state yields the divergent expression

$$\langle\phi^2(x)\rangle_{\text{naive}} = \hbar\int_0^\infty d\omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |Y_{lm}(\theta,\varphi)|^2 |\bar{\psi}_{\omega l}(r)|^2. \tag{2.4}$$

Although the sum over m, l does converge for a given ω, the integral over ω diverges. In fact, one can easily check that already in Minkowski spacetime the integrand is $\propto \omega$ (and $\propto \omega^3$ for $\langle T_{\alpha\beta}\rangle$), and the same divergence occurs in the Schwarzschild case as well. If one tries to integrate over ω before the summation, one finds that the integral over ω again diverges (for a given l, m).

One therefore needs to somehow regularize the expression for $\langle\phi^2\rangle$. In flat spacetime this regularization may be easily achieved by normal ordering, but in curved spacetime this yields a slicing-dependent, nonunique result. This is where point splitting comes into play. DeWitt [12] proposed that $\langle\phi^2(x)\rangle$ (and, more generally, quantities quadratic in the field operator and its derivatives) can be treated by taking the product of the field operators in two separate points x, x' and then considering the coincidence limit $x \to x'$. More specifically, he showed that the regularized expectation value of ϕ^2 can be defined as

$$\langle\phi^2(x)\rangle_{\text{ren}} = \lim_{x'\to x}[\langle\phi(x)\phi(x')\rangle - G_{\text{DS}}(x,x')]. \tag{2.5}$$

Here $G_{\text{DS}}(x,x')$ is the DeWitt-Schwinger *counterterm*, namely a local term which fully captures the singular piece of the TPF. For a scalar field with mass m and coupling constant ξ it takes the form [8]

$$\frac{1}{\hbar}G_{\text{DS}}(x,x') = \frac{1}{8\pi^2\sigma} + \frac{m^2+(\xi-1/6)R}{8\pi^2}\left[\gamma+\frac{1}{2}\ln\left(\frac{\mu^2\sigma}{2}\right)\right] - \frac{m^2}{16\pi^2} + \frac{1}{96\pi^2}R_{\alpha\beta}\frac{\sigma^{;\alpha}\sigma^{;\beta}}{\sigma}. \tag{2.6}$$

Here R and $R_{\alpha\beta}$ are respectively the Ricci scalar and tensor, γ denotes the Euler constant, and $\sigma(x,x')$ is the biscalar associated with the short geodesic connecting x and x'. The value of σ is half the geodesic distance squared (see Ref. [9]). More specifically, for a timelike separation $\sigma = -\tau^2/2$, where τ denotes the proper time between x and x'. The parameter μ is unknown and it corresponds to the well-known ambiguity in the regularization procedure [4].[8]

A. Previous numerical implementations of point splitting

It is not easy to directly implement the point-splitting method, especially because in the cases of interest (e.g. black-hole backgrounds) the radial functions $\bar{\psi}_{\omega l}$ are only known from numerics. Furthermore, in Eq. (2.5) the TPF $\langle\phi(x)\phi(x')\rangle$, which is to be computed numerically, diverges as τ^{-2} (and as τ^{-4} for $\langle T_{\alpha\beta}\rangle$) as x' approaches x, where τ denotes the geodesic distance between x and x'.

This problem was addressed by Candelas, Howard and later Anderson and collaborators (see Refs. [5–8]) a long time ago. They developed a calculation scheme that allows regularization of $\langle\phi^2\rangle$ or $\langle T_{\alpha\beta}\rangle$ numerically, provided that an analytic approximation for the field is known, up to a sufficiently high order.[9]

In that scheme, one first analytically constructs the approximate singular piece of the field, which we denote $\phi_{\text{sing}}(x)$. It is composed of the contribution of the modes of large ω and l, up to second order in $1/\omega$ and $1/l$ (fourth order for $\langle T_{\alpha\beta}\rangle$), which is usually computed using WKB analysis. From this quantity one then constructs the approximate TPF $\langle\phi_{\text{sing}}(x)\phi_{\text{sing}}(x')\rangle$, and recasts Eq. (2.5) in the form

[7]If a past horizon exists, then one needs to further specify this vacuum state, e.g. by prescribing the outcome of the action of the up annihilation operators on that state. In the analysis of the Schwarzschild case in Sec. IV we shall consider the Boulware vacuum state.

[8]In the case of a massive scalar field some authors took μ to be the field's mass m [8,9].

[9]In general it would require a second-order WKB approximation in order to compute $\langle\phi^2\rangle_{\text{ren}}$, and a fourth-order approximation to compute $\langle T_{\alpha\beta}\rangle_{\text{ren}}$.

$$\langle\phi^2(x)\rangle_{\rm ren} = \lim_{x'\to x}[\langle\phi(x)\phi(x')\rangle - \langle\phi_{\rm sing}(x)\phi_{\rm sing}(x')\rangle] + \lim_{x'\to x}[\langle\phi_{\rm sing}(x)\phi_{\rm sing}(x')\rangle - G_{\rm DS}(x,x')]. \quad (2.7)$$

Since $\langle\phi_{\rm sing}(x)\phi_{\rm sing}(x')\rangle$ contains the entire singular piece of the TPF, both limits at the rhs are well defined. We denote the second limit by $\langle\phi^2(x)\rangle_{\rm Analytic}$. This quantity is well-defined and regular, and it is computed analytically by summing/integrating over the WKB expressions for the large ω, l modes which comprise $\phi_{\rm sing}$. The first term in the rhs is now expressed as a mode sum, and owing to its regularity the limit may trivially be taken by replacing x' by x. Putting it all together one obtains

$$\langle\phi^2(x)\rangle_{\rm ren} = \hbar\int_0^\infty d\omega\sum_{l=0}^\infty\sum_{m=-l}^l |Y_{lm}(\theta,\varphi)|^2(|\bar\psi_{\omega l}(r)|^2 - |\bar\psi^{\rm sing}_{\omega l}(r)|^2) + \langle\phi^2(x)\rangle_{\rm Analytic}. \quad (2.8)$$

The mode sum should now converge (even though it is taken in coincidence), owing to the subtraction of the singular piece.

Using this method with splitting in the t direction, $\langle\phi^2\rangle_{\rm ren}$ (as well as $\langle T_{\alpha\beta}\rangle_{\rm ren}$) was calculated for the Schwarzschild case [5,6], and later also for a generic static spherically symmetric spacetime [7,8].

Besides the need to numerically compute the various mode functions $\bar\psi_{\omega l}(r)$, this method also includes a challenging analytical component: It requires a high-order WKB analysis. As was mentioned in the Introduction, the presence of a turning point makes this an extremely difficult task. To overcome these difficulties, Candelas and Howard [5] and others [7,8] actually carried the analysis in the Euclidean sector.

B. Remarks about the TPF and its mode sum[10]

1. Caveats concerning the regularity of the TPF itself

The point-splitting method is based on the presumption that the two-point function is well behaved as long as the two points are separated. A few caveats are associated with this issue: First, even in flat spacetime, the two-point function diverges when the separation is in a null direction. Let us therefore assume, for the sake of simplicity, that the points are separated in a timelike (or possibly spacelike) direction. The second caveat is that in curved spacetime, assuming that the two points are indeed separated by a timelike geodesic, if the (proper-time) distance between x and x' is sufficiently large, there may also be a null geodesic connecting these two points. For example, consider the Schwarzschild spacetime and an approximately static timelike geodesic Γ located in the asymptotic region very far from the BH. Let x be a point on Γ. There is a null geodesic which emanates from x and moves towards the BH (but with an appropriate miss), makes a turn around the BH, then returns to Γ and hits it at a point x_1. It turns out that $\langle\phi(x)\phi(x')\rangle$ develops a singularity at $x' = x_1$ [11]. In fact, along the timelike geodesic Γ there is an infinite discrete set of points x_n which are connected to x by null geodesics that make n turns around the BH before returning to Γ, and $\langle\phi(x)\phi(x')\rangle$ diverges at all points $x' = x_n$.

Note, however, that the problematic points x_n are all located far away from x, outside the normal neighborhood; and for the point-splitting procedure only points x' in the immediate neighborhood of x are relevant. We shall thus restrict our attention now to points x' in the close neighborhood of x, with a timelike (or alternatively spacelike) separation between x and x'. Then the TPF should be well behaved.

2. Caveats concerning the convergence of the mode sum

Naively one might expect that since the TPF is well behaved for a short timelike or spacelike separation, the mode sum associated to it should converge. It turns out, however, that the situation is more subtle: The involved sum/integral usually fails to converge (in the literal sense). Typically this failure to converge is associated with undamped oscillations.

This phenomenon, the failure of strict convergence, occurs already in flat spacetime. The (non)convergence situation may depend on the direction of splitting, on the order of operations (summation over l, m and integration over ω), and on the specific quantity being calculated (whether it is the TPF, or a component of $\langle T_{\alpha\beta}\rangle$). As an example, consider the calculation of $\langle\phi(x)\phi(x')\rangle$ in Minkowski spacetime, using standard mode decomposition in spherical harmonics and temporal modes $e^{-i\omega t}$. The mode functions are given by

$$\bar\psi_{\omega l}(r) = \frac{1}{\sqrt{2r}}J_{l+1/2}(\omega r),$$

where J is the Bessel function of the first kind. Consider now a point x located at some $r > 0$, and a point x' displaced in the t direction by an amount $t' - t = \Delta t$. The sum over l, m (for a given ω) then converges, and one finds that

$$\sum_{l=0}^\infty\sum_{m=-l}^l |Y_{lm}(\theta,\varphi)|^2|\bar\psi_{\omega l}(r)|^2 = \frac{1}{4\pi^2}\omega.$$

The mode-sum expression for the TPF then becomes

[10] This subsection is somewhat remote from the main line of this paper, and can be skipped at first reading.

$$\langle\phi(x)\phi(x')\rangle = \frac{\hbar}{4\pi^2}\int_0^\infty \omega e^{i\omega\Delta t}d\omega. \qquad (2.9)$$

The problem is that this integral does not converge in the usual sense (even conditionally), due to the growing oscillations at large ω.

In this example of pure t-splitting, in which we first sum over m, l, it was the integral over ω that failed to converge. In another application of point splitting, in which the splitting is in both t and θ, and in which one first integrates over ω and sums over m, l afterward, one finds that this time the sum over l fails to converge, again due to growing oscillations.[11]

This situation, of nonconverging oscillatory integrals (or sums over l), is fairly common in various QFT calculations. The common practice (which may be justified by several arguments) is to contend that the large-ω oscillations should be damped in some appropriate manner.

3. Generalized integral

This situation motivates us to introduce the notion of generalized integral, which properly incorporates oscillation damping. We should emphasize that all the ω integrals in this paper are in fact generalized integrals.

We shall consider here two specific procedures of oscillation damping, which yield two (mutually consistent) definitions of generalized integral.

Abel-summation integral: This is a commonly used method for giving a meaning for such oscillatory integrals. The Abel-summed integral is defined as

$$\int_0^{\infty(A)} h(\omega)d\omega \equiv \lim_{\epsilon\to 0_+}\int_0^\infty e^{-\epsilon\omega}h(\omega)d\omega, \qquad (2.10)$$

provided of course that the integral in the rhs is well defined for $\epsilon > 0$, and the limit $\epsilon \to 0_+$ exists. For example, with Abel summation, the integral in Eq. (2.9) reads $-1/\Delta t^2$, yielding the standard expression $-\hbar/(4\pi^2\Delta t^2)$ for the flat-space TPF [cf. Eq. (2.6)].

A crucial property of the Abel-summation integral is that it is consistent with the standard integral. Namely, whenever the function $h(\omega)$ is integrable in the strict sense, its Abel-summation integral coincides with the standard integral of this function.

Self-cancellation integral: For the calculation of $\langle\phi^2\rangle_{\rm ren}$ in a black-hole spacetime we shall have to carry a (generalized) integral of oscillatory functions $h(\omega)$ that we determine numerically. In such a case, this procedure of Abel summation—which must now be implemented numerically—is fairly inconvenient. Furthermore, in practice we only determine $h(\omega)$ in a restricted range $0 < \omega < \omega_{\rm max}$, which makes the Abel-summation integral even harder to implement.[12] We therefore find it much more convenient to use another concept of generalized integral, which we name *self-cancellation* (of the oscillations). This type of generalized integral is applicable whenever the oscillations have well-defined frequencies—which is indeed the situation in our problem (see Appendix B). To formulate this concept, we first define the integral function

$$H(\omega) \equiv \int_0^\omega h(x)dx. \qquad (2.11)$$

The standard integral may then be expressed as

$$\int_0^\infty h(\omega)d\omega = \lim_{\omega\to\infty} H(\omega)$$

(whenever this limit exists). Instead, our self-cancellation generalized integral is defined as

$$\int_0^{\infty(sc)} h(\omega)d\omega \equiv \lim_{\omega\to\infty}\left[\frac{H(\omega)+H(\omega+\lambda/2)}{2}\right]. \qquad (2.12)$$

Here, λ denotes the "wavelength" of the oscillation in $h(\omega)$ (which is also inherited by H). For example, in Eq. (2.9) the oscillatory factor is $e^{i\omega\Delta t}$; hence the period of oscillation is $\lambda = 2\pi/\Delta t$. The idea is simple: The nonoscillating piece of $H(\omega)$ is unaffected by this averaging, but the oscillatory piece will be very effectively annihilated by such averaging with half-wavelength shift.

As a simplest example, consider the case $h(\omega) = \sin(\omega L)$. Then $H(\omega) = [1 - \cos(\omega L)]/L$, and obviously $\lambda = 2\pi/L$. Clearly $H(\omega)$ fails to have a limit $\omega \to \infty$. Yet the self-cancellation integral is perfectly well defined: The term in squared brackets in Eq. (2.12) is simply $1/L$, entirely independent of ω. This example demonstrates the potential of the self-cancellation method to yield extremely fast convergence in ω. This last property is important, especially because in an actual calculation we have to determine $h(\omega)$ numerically, and we do so in a restricted range of ω.

For later convenience we reformulate this notion of self-cancellation integral as follows:

$$\int_0^{\infty(\rm sc)} h(\omega)d\omega \equiv \lim_{\omega\to\infty} T_\lambda[H(\omega)], \qquad (2.13)$$

where T_λ is the "self-cancellation operation" defined by

[11]In addition, the integral over ω converges for the TPF but only conditionally, and fails to converge even conditionally for T_{tt}, due to growing oscillations.

[12]To this end one would have to generalize the definition of the Abel-summation integral, so as to combine the two (noncommuting) limits $\epsilon \to 0$ and $\omega_{\rm max} \to \infty$ in an appropriate manner. And it turns out that the convergence of this generalized Abel integral with increasing $\omega_{\rm max}$ is rather slow.

$$T_\lambda[f(\omega)] \equiv \frac{f(\omega)+f(\omega+\lambda/2)}{2}.$$

In the actual calculation of $\langle\phi^2\rangle$ we shall have to repeat the self-cancellation operation several times (for several different oscillation frequencies). In Appendix A we shall introduce the "multiple self-cancellation operation" T_*, formed by combining several T_λ operations.

It is easy to show that the notion of a self-cancellation integral is fully consistent with the standard integral—whenever the latter is well defined. Furthermore, the self-cancellation integral is also fully consistent with the Abel-summation integral, in the following sense: If the self-cancellation integral converges, then the Abel integral converges too, and the two generalized integrals yield the same result. Note, however, that the Abel summation method is more general than self-cancellation. Namely, there are functions $h(\omega)$ for which the Abel-summed integral is well defined but the self-cancellation integral is not [that is, the limit in Eq. (2.12) is nonexistent]. Nevertheless, for the functions $h(\omega)$ involved in the analysis below, the self-cancellation integral is well defined and extremely powerful.

III. OUR NEW METHOD: THE *t*-SPLITTING VARIANT

As was already noted in the Introduction, since our ultimate goal is to address dynamical background metrics as well, we shall carry the analysis directly in the Lorentzian sector. Hence, due to the inevitable presence of a turning point (and also due to the partial differential equation (PDE) nature of the time-dependent mode equation), we shall avoid using WKB expansion in our method. Instead we extract the required information about the high-frequency modes of the field directly from the counterterm (2.6).

We shall present here the t-splitting variant, which requires a time-translation Killing field. Although this variant should be applicable to a rather generic stationary asymptotically flat background, we shall restrict our attention first to the more specific case of static spherically symmetric background, for the sake of simplicity. Then in Sec. III B we outline the generalization of the t-splitting formulation beyond spherical symmetry (and beyond staticity), to a more generic stationary asymptotically flat background.

A. Spherically symmetric static background

We split the points in the t direction, namely

$$x=(t,r,\theta,\varphi), \qquad x'=(t+\varepsilon,r,\theta,\varphi). \tag{3.1}$$

The TPF then takes the form

$$\langle\phi(x)\phi(x')\rangle = \hbar\int_0^\infty d\omega e^{i\omega\varepsilon}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|Y_{lm}(\theta,\varphi)|^2|\bar\psi_{\omega l}(r)|^2. \tag{3.2}$$

As was already mentioned above, the sum over l,m converges, and we denote it by $F(\omega,r)$. In fact, the sum of $|Y_{lm}|^2$ over m yields $(2l+1)/4\pi$, and therefore

$$F(\omega,r)=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}|\bar\psi_{\omega l}(r)|^2. \tag{3.3}$$

This function is to be computed numerically. The TPF now reduces to

$$\langle\phi(x)\phi(x')\rangle = \hbar\int_0^\infty F(\omega,r)e^{i\omega\varepsilon}d\omega. \tag{3.4}$$

(Note that in the coincidence limit $\varepsilon\to 0$ this integral would diverge. However, the oscillatory factor $e^{i\omega\varepsilon}$ regularizes it.) Equation (2.5) now reads

$$\langle\phi^2(x)\rangle_{\rm ren}=\lim_{\varepsilon\to0}\left[\hbar\int_0^\infty F(\omega,r)e^{i\omega\varepsilon}d\omega - G_{\rm DS}(\varepsilon)\right], \tag{3.5}$$

where by $G_{\rm DS}(\varepsilon)$ we refer to $G_{\rm DS}(x,x')$ with x' given by Eq. (3.1). Note that for a given x, σ is uniquely determined by ε. By a fairly straightforward Taylor expansion of the geodesic equation (and σ) in ε, one finds that $G_{\rm DS}$ in Eq. (2.6) takes the general form

$$\frac{1}{\hbar}G_{\rm DS}(x,x') = a(r)\varepsilon^{-2}+c(r)\left[\ln(\varepsilon\mu)+\gamma-\frac{i\pi}{2}\right] + d(r)+O(\varepsilon), \tag{3.6}$$

where $a(r), c(r), d(r)$ are certain (real) functions that depend on the background metric and the parameters of the field.[13] The explicit form of these functions is not important in the present discussion (although it is certainly needed for the actual calculation of $\langle\phi^2\rangle_{\rm ren}$).

To proceed, we now decompose the ε-dependent terms in $G_{\rm DS}$ using the Laplace transform. We have the following identities:

$$\varepsilon^{-2} = -\int_0^\infty \omega e^{i\omega\varepsilon}d\omega, \tag{3.7}$$

$$\ln(\varepsilon\mu) = -\int_0^\infty\frac{1}{\omega+\mu}e^{i\omega\varepsilon}d\omega+\left(\frac{i\pi}{2}-\gamma\right)+O(\varepsilon\ln\varepsilon). \tag{3.8}$$

[13] The term $-i\pi/2$ appears in the brackets because σ is negative.

Inserting these identities into Eqs. (3.5), (3.6) we obtain

$$\langle\phi^2(x)\rangle_{\rm ren} = \hbar\lim_{\varepsilon\to 0}\int_0^\infty F_{\rm reg}(\omega,r)e^{i\omega\varepsilon}d\omega - \hbar d(r) \quad (3.9)$$

where

$$F_{\rm reg}(\omega,r) \equiv F(\omega,r) - F_{\rm sing}(\omega,r), \quad (3.10)$$

and

$$F_{\rm sing}(\omega,r) \equiv -a(r)\omega - c(r)\frac{1}{\omega+\mu}. \quad (3.11)$$

Consider now the rhs of Eq. (3.9). If the integral of $F_{\rm reg}(\omega,r)$ converges (in either the strict sense or the generalized sense), we can interchange the limit and integration, and get rid of the $\varepsilon\to 0$ limit altogether. We should expect the convergence of the integral of $F_{\rm reg}(\omega,r)$, because the singular piece $F_{\rm sing}(\omega,r)$ has already been removed from $F(\omega,r)$. In fact, we find that in the Schwarzschild case the integral of $F_{\rm reg}(\omega,r)$ indeed converges, although only in the generalized sense due to oscillations (see next section). We were unable to *prove* the (even generalized) convergence of the integral of $F_{\rm reg}(\omega,r)$, but nevertheless since this convergence is naturally expected, and since the Schwarzschild example confirms this expectation, we shall hereafter *assume* that this integral indeed converges in the generalized sense.[14]

We therefore write our final result as

$$\langle\phi^2(x)\rangle_{\rm ren} = \hbar\int_0^\infty F_{\rm reg}(\omega,r)d\omega - \hbar d(r) \quad (3.12)$$

where, recall, the integral over ω is a generalized one (as defined in Sec. II B). The implementation of this generalized integral is demonstrated in Sec. IV for the Schwarzschild case.

B. The general stationary case

As was already pointed out above, our t-splitting method does not require the background metric to be spherically symmetric or even static: It should be applicable to a generic stationary asymptotically flat background. Here we outline this extension.

On account of asymptotic flatness, we now choose our coordinates (t,r,θ,φ) such that the large-r asymptotic metric still takes its standard Minkowski form $-dt^2+dr^2+r^2d\Omega^2$.[15] Due to a lack of spherical symmetry, the expression (2.3) for the field modes is now replaced by

$$f_{\omega lm}(x) = e^{-i\omega t}\tilde\psi_{\omega lm}(r,\theta,\varphi),$$

where $\tilde\psi_{\omega lm}(r,\theta,\varphi)$ is a set of solutions to the (ω-dependent) spatial part of the field equation, which is now a PDE (in r,θ,φ) rather than an ordinary differential equation (ODE). These solutions are required to be regular everywhere, and to satisfy the large-r boundary condition

$$\tilde\psi_{\omega lm}(r,\theta,\varphi) = (r\sqrt{4\pi\omega})^{-1}e^{-i\omega r_*}Y_{lm}(\theta,\varphi) + \tilde\psi^{\rm out}_{\omega lm}(r,\theta,\varphi), \qquad (r\to\infty)$$

where $\tilde\psi^{\rm out}_{\omega lm}$ denotes the reflected field which is $e^{i\omega r_*}/r$ times some function of θ and φ.[16] Once the modes $f_{\omega lm}(x)$ were defined, the expression (2.2) for the field operator is unchanged.

The TPF now takes the form

$$\langle\phi(x)\phi(x')\rangle = \hbar\int_0^\infty d\omega e^{i\omega\varepsilon}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|\tilde\psi_{\omega lm}(r,\theta,\varphi)|^2. \quad (3.13)$$

It is important to recall that even though the metric is not spherically symmetric, the sums over m,l should still converge (for a given ω): Due to asymptotic flatness, at large r there will be a centrifugal barrier $\approx l(l+1)/r^2$ in the effective potential, just like in Minkowski, preventing the penetration of modes with too large l. Thus we can again define

$$F(\omega,r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}|\tilde\psi_{\omega lm}(r,\theta,\varphi)|^2, \quad (3.14)$$

and the TPF still takes the form (3.4) [although now with $F(\omega,r,\theta,\varphi)$]. The calculation now proceeds just as in the spherically symmetric case—except that all the quantities in Sec. (III A) that were dependent on r only now depend on θ and φ as well. The final result is

$$\langle\phi^2(x)\rangle_{\rm ren} = \hbar\int_0^\infty F_{\rm reg}(\omega,r,\theta,\varphi)d\omega - \hbar d(r,\theta,\varphi) \quad (3.15)$$

with

[14]Note that there is not much risk in making such an assumption, because if for a certain background metric this assumption turns out to be false, then the attempt to integrate $F_{\rm reg}$ will demonstrate this nonconvergence right away.

[15]Furthermore we choose our coordinates such that g_{tt} takes the standard weak-field form $-1+2M/r+O(1/r)^2$, where here M denotes the system's asymptotic mass.

[16]Here r_* is to be regarded as the standard function of r given in Eq. (4.4).

$$F_{\rm reg}(\omega,r,\theta,\varphi) \equiv F(\omega,r,\theta,\varphi) - F_{\rm sing}(\omega,r,\theta,\varphi), \quad (3.16)$$

and

$$F_{\rm sing}(\omega,r,\theta,\varphi) \equiv -a(r,\theta,\varphi)\omega - c(r,\theta,\varphi)\frac{1}{\omega+\mu}. \quad (3.17)$$

It should be pointed out that although the choice of the coordinates r,θ,φ is certainly nonunique (due to lack of spherical symmetry), the resultant mode decomposition is still unique. This is because of asymptotic flatness (and the standard weak-field metric that we require our t,r,θ,φ coordinates to satisfy at large r), and because the mode functions $\tilde{\psi}_{\omega lm}(r,\theta,\varphi)$ are defined through their asymptotic form at $r\to\infty$. The unique mode decomposition in turn leads to a unique Fock space associated to it, and to a well-defined vacuum state.

Note, however, that in the above formulation we implicitly assumed a stationary background metric with no past horizon (e.g. a spinning star); hence it was sufficient to construct the in modes. In the case of eternal nonspherical BH the situation becomes more subtle, because now we also need to specify the up modes. Here asymptotic flatness will not be of much help, because these modes are to be defined by boundary conditions at the past horizon. With the lack of spherical symmetry and staticity, one still needs to figure out how to make a unique mode decomposition and to obtain a "natural" vacuum state.[17]

The implementation of this method in the nonspherical case is of course technically more challenging (even for a non-BH background), because now the mode functions $\tilde{\psi}_{\omega lm}$ which comprise $F(\omega,r,\theta,\varphi)$ are to be obtained by numerically solving PDEs rather than just ODEs.

IV. CALCULATION OF $\langle\phi^2\rangle_{\rm ren}$ IN SCHWARZSCHILD

Using the method presented in the last section, we compute $\langle\phi^2\rangle_{\rm ren}$ in the exterior region of Schwarzschild spacetime, in the Boulware vacuum state, for a minimally coupled massless scalar field. The Schwarzschild metric is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2. \quad (4.1)$$

[17]Furthermore, with an arbitrary choice of r,θ,φ coordinates, and with a corresponding arbitrary construction of the set of up modes, we have no guarantee that the resultant "vacuum state" would at all be a well-defined Hadamard state. In this regard, we should mention the observation that there is no Hadamard state that respects the symmetries of Kerr spacetime and is regular everywhere [13,14].

In this metric the quantum field $\phi(x)$ can be expanded in the form presented in Eqs. (2.2)–(2.3), where $\bar{\psi}_{\omega l}(r)$ is conveniently recast as

$$\bar{\psi}_{\omega l}(r) = \frac{1}{r\sqrt{4\pi\omega}}\psi_{\omega l}(r), \quad (4.2)$$

and $\psi_{\omega l}(r)$ satisfies the radial equation

$$\frac{d^2\psi_{\omega l}(r)}{dr_*^2} = -[\omega^2 - V_l(r)]\psi_{\omega l}(r). \quad (4.3)$$

Henceforth, r_* will denote the tortoise coordinate given by

$$r_* = r + 2M\ln\left(\frac{r}{2M}-1\right), \quad (4.4)$$

and

$$V_l(r) = \left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]$$

is the effective potential.

The general solution of Eq. (4.3) (for given ω, l) is spanned by two basic solutions, that we denote $\psi^{\rm in}_{\omega l}(r)$ and $\psi^{\rm up}_{\omega l}(r)$. The boundary conditions for these two basic solutions are taken to be

$$\psi^{\rm in}_{\omega l}(r) = \begin{cases} \tau_{\omega l}e^{-i\omega r_*}, & r_*\to-\infty \\ e^{-i\omega r_*}+\rho_{\omega l}e^{i\omega r_*}, & r_*\to\infty \end{cases}$$
$$\psi^{\rm up}_{\omega l}(r) = \begin{cases} e^{i\omega r_*}+\tilde{\rho}_{\omega l}e^{-i\omega r_*}, & r_*\to-\infty \\ \tau_{\omega l}e^{i\omega r_*}, & r_*\to\infty \end{cases} \quad (4.5)$$

where $\tau_{\omega l}$, $\rho_{\omega l}$, and $\tilde{\rho}_{\omega l} = -\rho^*_{\omega l}\tau_{\omega l}/\tau^*_{\omega l}$ represent the transmission and reflection amplitudes. These two basic solutions are properly normalized and mutually orthogonal.

The presence of two independent modes for each ωlm (as opposed to a single such mode in e.g. Minkowski) requires a slight modification of the formalism above: We now have *two* sets of annihilation operators, $a^{\rm in}_{\omega lm}$ and $a^{\rm up}_{\omega lm}$, as well as their conjugate operators $a^{\dagger{\rm in}}_{\omega lm}$, $a^{\dagger{\rm up}}_{\omega lm}$. Correspondingly, in Eq. (2.2), in addition to the integral over ω and the sum over l and m, we also have to sum over the separate contributions of the in and up modes. We shall consider here the *Boulware vacuum*, namely the quantum state annihilated by all the operators $a^{\rm up}_{\omega lm}$ as well as $a^{\rm in}_{\omega lm}$. Revisiting the analysis of the previous sections, one finds that everything remains intact, except that all the equations that involve summation over l,m should now also include a summation over the in and up contributions. Correspondingly, Eq. (3.3) is now replaced by

$$F(\omega, r) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} (|\bar{\psi}^{\text{in}}_{\omega l}(r)|^2 + |\bar{\psi}^{\text{up}}_{\omega l}(r)|^2), \qquad (4.6)$$

but otherwise the results of the previous section, and in particular Eqs. (3.10), (3.11), (3.12), are unaffected.

Since the Schwarzschild metric is a vacuum solution, and since we are dealing with a massless field, the counterterm (2.6) now reduces to

$$G_{\text{DS}}(x, x') = \frac{\hbar}{8\pi^2 \sigma}. \qquad (4.7)$$

σ is related to $\varepsilon \equiv t' - t$ through the proper-time τ of the short geodesic connecting x to x', via $\sigma = -\tau^2/2$. By conducting a second-order expansion of the geodesic equation we obtain

$$\sigma(\varepsilon) = -\frac{1 - 2M/r}{2} \varepsilon^2 - \frac{M^2(1 - 2M/r)}{24 r^4} \varepsilon^4 + O(\varepsilon^5), \qquad (4.8)$$

and correspondingly

$$\frac{1}{\hbar} G_{\text{DS}}(x, x') = -\frac{1}{4\pi^2 (1 - 2M/r)} \varepsilon^{-2} + \frac{M^2}{48 \pi^2 r^4 (1 - 2M/r)} + O(\varepsilon). \qquad (4.9)$$

Comparing this to Eq. (3.6) we find that

$$a(r) = -\frac{1}{4\pi^2 (1 - 2M/r)},$$
$$d(r) = \frac{M^2}{48\pi^2 r^4 (1 - 2M/r)}, \qquad c(r) = 0. \qquad (4.10)$$

Therefore in the Schwarzschild case Eqs. (3.10), (3.11), (3.12) reduce to

$$\langle \phi^2(x) \rangle_{\text{ren}} = \hbar \int_0^\infty F_{\text{reg}}(\omega, r) d\omega - \hbar d(r), \qquad (4.11)$$

with

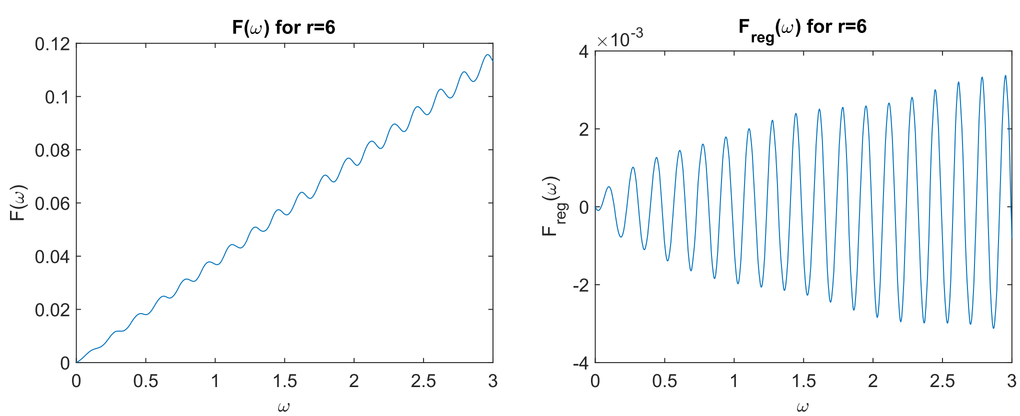
$$F_{\text{reg}}(\omega, r) = F(\omega, r) + a(r)\omega. \qquad (4.12)$$

Summarizing the analytical part of the calculation, $\langle \phi^2(x) \rangle_{\text{ren}}$ is given by Eq. (4.11) along with Eqs. (4.12), (4.10), and (4.6). Note that basically this expression for Schwarzschild was already obtained by Candelas [10], but here we also complete the calculation by implementing the numerical part as well (and by doing so, we encounter the oscillations problem and address it).

A. Numerical implementation

We have numerically solved the radial equation for $\psi^{\text{in}}_{\omega l}(r)$ and $\psi^{\text{up}}_{\omega l}(r)$, using the standard MATHEMATICA numerical ODE solver, in the domain $0 < \omega < 3$, at a set of ω values with a uniform separation $d\omega = 1/300$. Hereafter, we use units in which the BH mass is $M = 1$ (in addition to $C = G = 1$); hence ω and r are dimensionless.

For a given ω, the contribution of the different l modes to $F(\omega, r)$ starts to decay exponentially fast beyond a certain l value, typically of order $\sim \omega r (1 - 2M/r)^{-1/2}$. (This decay



(a) (b)

FIG. 1 (color online). (a) The numerically calculated $F(\omega, r = 6)$ in the Schwarzschild case. (b) $F_{\text{reg}}(\omega, r = 6)$, which is the result of subtracting the linearly diverging piece $a(r)\omega$ from $F(\omega, r)$.

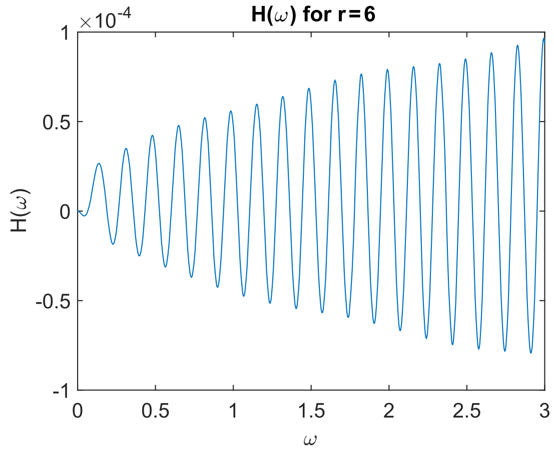


FIG. 2 (color online). The integral function $H(\omega, r=6)$ (that is, the integral of $F_{\rm reg}$ from zero to ω).

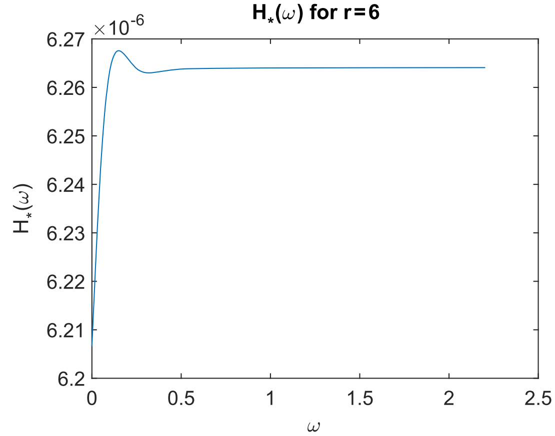


FIG. 3 (color online). The function $H_*(\omega, r=6)$ (obtained from H after self-cancellation of the oscillations). Notice the quick convergence.

may be interpreted as tunneling into the potential barrier.) Correspondingly, for each ω value, we truncate the sum (4.6) at an l value where the contribution becomes negligible ($< 10^{-10}$). Then from $F(\omega, r)$ we construct the regularized function $F_{\rm reg}(\omega, r)$ according to Eq. (4.12).

Figure 1(a) displays $F(\omega, r)$ for $r = 6$, which we choose here as our representative r value. Then Fig. 1(b) displays the regularized function $F_{\rm reg}(\omega, r)$, obtained from $F(\omega, r)$ by removing the linear piece $a(r)\omega$. Clearly, the linear divergence has been removed, but there remain oscillations that grow as $\omega^{1/2}$, which we address below. The origin of these oscillations (the aforementioned connecting null geodesics), and the determination of their frequencies, are discussed in Appendix B.

To calculate $\langle\phi^2(x)\rangle_{\rm ren}$ we need the generalized integral of $F_{\rm reg}(\omega, r)$ from $\omega = 0$ to infinity; see Eq. (4.11). To this end we define the integral function

$$H(\omega) \equiv \int_0^{\omega} F_{\rm reg}(\omega', r)d\omega'. \tag{4.13}$$

The strict integral of $F_{\rm reg}$ would of course correspond to the limit $\omega \to \infty$ of $H(\omega)$. Figure 2 displays $H(\omega)$, and makes it clear that this limit does not exist, due to the growing oscillations (which were inherited directly from $F_{\rm reg}$). We therefore have to resort to the generalized integral instead, as discussed in Sec. II B. The Abel-summation integral (2.10) is well defined in this case. However, since we know

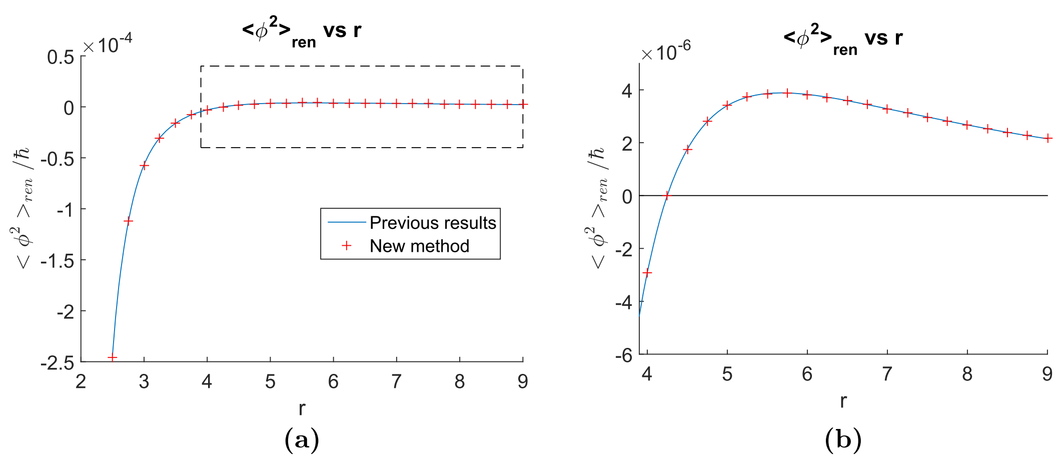


FIG. 4 (color online). (a) Comparing $\langle\phi^2\rangle_{\rm ren}$ calculated using our new method (the red pluses) to the previous calculation by Anderson using the Euclidean sector and the WKB expansion. (b) Zoom-in on the box in Fig. 4(a) for the values close to zero.

the precise frequency of oscillations, it is much more convenient and more efficient to employ the self-cancellation integral (which is fully consistent with the Abel integral). As it turns out, there are multiple oscillation frequencies in $F_{\rm reg}(\omega)$ (see Appendix B), and we cancel each of the four dominant ones by a fourth-order self-cancellation operation. Adapting the notation of Appendix A to the present specific context (integration of $F_{\rm reg}$), the desired generalized integral is

$$\int_0^{\infty(sc*)} F_{\rm reg}(\omega)d\omega = \lim_{\omega\to\infty} H_*(\omega), \tag{4.14}$$

where $H_*(\omega) \equiv T_*[H(\omega)]$, $H(\omega)$ is given in Eq. (4.13), and T_* denotes the multiple self-cancellation operation as generally defined in Eq. (A6) and detailed in Eq. (B1).

Figure 3 displays the function $H_*(\omega)$. It is remarkable that after the oscillations have been removed the function converges very quickly. This allows a fairly precise determination of the limit $\omega\to\infty$ of this function, which constitutes the generalized integral in Eq. (4.14). We then substitute this integral, as well as $d(r)$ of Eq. (4.10), in Eq. (4.11). In Figs. 4(a) and 4(b) we present our results for $\langle\phi^2\rangle_{\rm ren}$ as a function of r, and compare them to results obtained previously by Anderson [15] using a very different method (analytic extension to the Euclidean sector). The differences are typically of order a few parts in 10^3, consistent with the estimated numerical errors.

V. DISCUSSION

We presented here a new approach for implementing point-splitting regularization numerically, for the computation of $\langle\phi^2\rangle_{\rm ren}$ in various asymptotically flat spacetimes. Our main motivation in developing this approach is to allow systematic investigation of self-consistent semiclassical evaporation of BHs. This would require the calculation of $\langle T_{\alpha\beta}\rangle_{\rm ren}$ in a time-dependent BH background.

So far we developed two variants of our basic approach (both for a quantum scalar field in asymptotically flat background): (i) t-splitting, applicable to stationary spacetimes, and (ii) angular splitting, applicable to spherically symmetric spacetimes. In this first paper we focused on the simplest of the two, the t-splitting variant. In presenting this method, we restricted our attention to static spherically symmetric backgrounds, and to $\langle\phi^2\rangle_{\rm ren}$ rather than $\langle T_{\alpha\beta}\rangle_{\rm ren}$, for the sake of simplicity. But we also described the extension of the method to more generic, nonspherical, stationary, asymptotically flat backgrounds. This extension suffices for the non-BH case, e.g. for a rotating star. In the case of a stationary BH which is not a spherically symmetric static one, our method still needs a completion: We still have to formulate the construction of an appropriate up state at the past horizon, which would constitute a physically meaningful (and properly Hadamard) “vacuum” state.

We then implemented the t-splitting variant to the specific case of the Boulware state in Schwarzschild spacetime (for a minimally coupled massless scalar field). The analytical part of the regularization procedure coincides in this case with the one developed by Candelas [10]. However, here we also implemented the numerical part, which involves the numerical solution of the radial equation for the various modes, and the summation/integration over the mode contributions. Doing so, we found that the regularized function $F_{\rm reg}(\omega,r)$, which was naively expected to be well behaved at large ω, actually suffers from growing oscillations, which make the ω-integral nonconvergent. As it turns out, this phenomenon has little to do with the short-distance behavior of the TPF in the coincidence limit $x'\to x$. Instead, the oscillations originate from divergences of the TPF at *remote* points x' which are connected to x by null geodesics. We used the notion of a generalized integral—and particularly the pragmatic method of the self-cancellation integral—in order to handle these oscillations and to carry the desired integration over ω. Doing so, we found excellent agreement with previous results obtained by a different method, the Euclidean extension [15].

Putting aside for a moment the ultimate goal of analyzing the time-dependent evaporation process, we wish to emphasize that even the simplest version of the method, the t-splitting variant presented here, makes it possible to do PS regularization in nonspherical stationary spacetimes, e.g. that of a strong-field spinning star (although in the generic stationary BH case our construction still needs a completion). This was not possible so far, due to the difficulties in conducting high-order WKB expansion in such spacetimes.

However, to achieve our primary goal of self-consistent semiclassical evaporation, we must deal with time-dependent backgrounds. To this end we shall need the angular-splitting variant, which is slightly more complicated than t-splitting. We shall describe this variant in a separate paper. We already applied the angular-splitting method to $\langle\phi^2\rangle_{\rm ren}$ and also to $\langle T_{\alpha\beta}\rangle_{\rm ren}$ in the Boulware state in Schwarzschild, and again we found very good agreement with previous calculations [15]. These results will be presented elsewhere [16].

ACKNOWLEDGMENTS

We are grateful to Paul Anderson and Bob Wald for many interesting and fruitful discussions. In addition A. O. thanks both Paul Anderson and Bob Wald for kind hospitality in their institutions, where many of these discussions took place. In particular the understanding of the nature of the oscillations, and their relation to connecting null geodesics, emerged in a discussion with Bob Wald during a visit to the University of Chicago. We also wish to thank Paul Anderson for sharing his unpublished data with us.

APPENDIX A: MULTIPLE SELF-CANCELLATION

For a given integrand function $h(\omega)$, the self-cancellation integral was defined in Sec. II B as

$$\int_0^{\infty(\mathrm{sc})} h(\omega)d\omega \equiv \lim_{\omega\to\infty} T_\lambda[H(\omega)], \tag{A1}$$

where

$$H(\omega) \equiv \int_0^\omega h(x)dx,$$

T_λ is defined by

$$T_\lambda[f(\omega)] \equiv \frac{f(\omega)+f(\omega+\lambda/2)}{2},$$

and λ is the oscillation's wavelength.

Consider next the situation in which $h(\omega)$ contains oscillations in two different frequencies (say, with ω-independent amplitudes). Then $H(\omega)$ will take the asymptotic form

$$H(\omega) \approx A_1 e^{i\omega L_1} + A_2 e^{i\omega L_2} + \mathrm{const} \tag{A2}$$

with two different wavelengths $\lambda_{1,2} = 2\pi/L_{1,2}$. If we apply a self-cancellation with respect to (say) λ_1, then we will be left with a function $T_{\lambda_1}[H(\omega)]$ in which the const is of course unaffected, and the term $\propto e^{i\omega L_1}$ has entirely been annihilated. The other oscillatory term $\propto e^{i\omega L_2}$ is still present in $T_{\lambda_1}[H(\omega)]$, although its amplitude will decrease by the factor $\cos[(L_2/L_1)\pi/2]$, as one can easily verify. The self-cancellation integral (associated to T_{λ_1}) is thus still nonconvergent. A second self-cancellation, this time with respect to λ_2, will yield a convergent self-cancellation integral, which may be formulated as

$$\lim_{\omega\to\infty} T_{\lambda_2} T_{\lambda_1}[H(\omega)].$$

The order of the two operations T_{λ_1} and T_{λ_2} is unimportant, as one can easily verify. With both orderings, the double self-cancellation integral yields the "const" in Eq. (A2).[18]

This process is straightforwardly generalized to functions $H(\omega)$ with any number k of oscillation frequencies that need to be annihilated. To handle this situation, we define the multiple self-cancellation operator

$$T_* \equiv T_{\lambda_1} T_{\lambda_2} \ldots T_{\lambda_k}. \tag{A3}$$

Then the multiple self-cancellation integral may be expressed as

$$\int_0^{\infty(\mathrm{sc}*)} h(\omega)d\omega \equiv \lim_{\omega\to\infty} H_*(\omega), \tag{A4}$$

where

$$H_*(\omega) \equiv T_*[H(\omega)]. \tag{A5}$$

As was demonstrated in Sec. II B, if $H(\omega)$ contains an oscillation with fixed amplitude, then the T_λ operation fully nullifies this oscillation. [This was demonstrated there for the case $h(\omega) = \sin(\omega L)$, but it equally applies to the more general cases $h(\omega) = e^{\pm i\omega L}$.] However, if the oscillation's amplitude varies with ω, the cancellation is not complete. Consider the case $H(\omega) = g(\omega)e^{i\omega L}$ where $g(\omega)$ is some slowly varying function—namely, a function whose typical length of variation ℓ becomes $\gg \lambda$ at large ω. To be more specific, let us further assume that ℓ diverges as $\omega \to \infty$.[19] Then one can easily show that

$$T_\lambda[H(\omega)] \simeq -\frac{\lambda}{2}\frac{dg}{d\omega} e^{i\omega L}$$

at large ω. In particular, if $H(\omega) = \omega^p e^{i\omega L}$ then

$$T_\lambda[H(\omega)] \propto p\omega^{p-1} e^{i\omega L}.$$

Consider now the case $H(\omega) \approx \omega^p e^{i\omega L}$ for some $1 \le p < 2$. Then $T_\lambda[H(\omega)]$ still does not converge as $\omega \to \infty$. Nevertheless, $T_\lambda(T_\lambda[H(\omega)])$ is $\propto \omega^{p-2} e^{i\omega L}$ and hence it converges. This illustrates that in certain circumstances one may want to apply the same self-cancellation operation several times, say n times, an operation which we shall denote as $(T_\lambda)^n$.

Quite generally, in the case $H(\omega) \approx \omega^p e^{i\omega L}$ one finds that[20]

$$(T_\lambda)^n[H(\omega)] \propto \omega^{p-n} e^{i\omega L}.$$

Therefore, for this class of $H(\omega)$ functions, the convergence criterion for the (single-frequency) multiple self-cancellation integral is simple: It converges if and only if $n > p$.

[18]To avoid confusion, we point out that an application of self-cancellation operation $T_{\lambda'}$, but with a "mistaken" wavelength parameter λ' that differs from the true oscillation wavelength λ, does not "spoil" the generalized integral in any way: The mismatch in λ does not lead to any new oscillatory terms, nor does it modify the value of the generalized integral. The effect of the mismatch in λ is merely to limit the efficiency of the self-cancellation operation: It decreases the oscillation's amplitude by the factor $\cos[(\lambda'/\lambda)\pi/2]$ instead of fully annihilating it.

[19]This includes for example all powers ω^p, because in this class $\ell \propto \omega$. [But it also includes much more general classes of functions, e.g. $\omega^p(\ln\omega)^q$ for any p, q.]

[20]The exception is the case of natural p with $n > p$, in which $(T_\lambda)^n[H(\omega)]$ strictly vanishes.

In the case $0 < p < 1$ a single self-cancellation operation would be sufficient for achieving convergence. However, one would then be left with a slowly decaying amplitude $\propto \omega^{p-1}$. To speed the large-ω convergence, one may repeat T_λ several times. This is in fact the situation in our specific problem, where $h(\omega) \propto \omega^{1/2} e^{i\omega L}$ and hence $H(\omega) \propto \omega^{1/2} e^{i\omega L} + \text{const}$: The single self-cancellation integral converges, but rather slowly, with amplitude $\propto \omega^{-1/2}$, and we therefore repeat T_λ several times to achieve faster convergence (which is crucial for numerical implementation).

If there are several different oscillation frequencies that need to be annihilated, one can freely choose how many times to repeat the T_λ operation for each frequency. We therefore generalize the above expression (A3) for the multiple self-cancellation operator:

$$T_* \equiv (T_{\lambda_1})^{n_1} (T_{\lambda_2})^{n_2} \ldots (T_{\lambda_k})^{n_k}. \tag{A6}$$

The multiple self-cancellation integral (sc$*$) is thus obtained by using this T_* operator in Eqs. (A4), (A5).

Finally we point out that this multiple self-cancellation integral is fully consistent with the Abel-summation integral—in the same sense discussed in Sec. II B (concerning the single self-cancellation integral).

APPENDIX B: THE OSCILLATIONS IN $F_{\mathrm{reg}}(\omega,r)$

As can be clearly seen in Fig. 1(b), the function $F_{\mathrm{reg}}(\omega, r)$ admits growing oscillations in ω. These oscillations are of course inherited directly from $F(\omega, r)$, as can be seen in Fig. 1(a) (although in the latter the oscillations are overshadowed by the linearly growing term). In this appendix we shall discuss the origin of these oscillations, their frequencies, and their amplitudes. Then at the end we shall specify the self-cancellation operator that we apply in order to practically remove these oscillations.

1. Origin and nature of oscillations

Owing to spherical symmetry and staticity, in t-splitting the TPF may depend only on r and on $\varepsilon = t' - t$. Let us then introduce the abbreviated notation for the TPF

$$P(\varepsilon, r) \equiv \langle \phi(t, r, \theta, \varphi) \phi(t + \varepsilon, r, \theta, \varphi) \rangle.$$

Equation (3.4) actually tells us that $F(\omega, r)$ is the Fourier transform of $P(\varepsilon, r)$.[21] If $P(\varepsilon, r)$ were regular and smooth for all ε, then its Fourier transform $F(\omega, r)$ would decay quickly at large ω, faster than any power of the latter. The undamped oscillations in $F(\omega, r)$ must therefore indicate some irregularity in $P(\varepsilon, r)$. We still need to understand the nature of this singularity, and its location on the ε axis.

The singularity of the TPF at $\varepsilon \to 0$ indeed leads to a divergence of $F(\omega, r)$ at large ω [this is the linearly growing term shown in Fig. 1(a)], but not to oscillations; and this linear singularity is no longer present in $F_{\mathrm{reg}}(\omega, r)$. The oscillations in Fig. 1(b) must then indicate another singularity, located at some point $\varepsilon = \varepsilon_s \neq 0$. To illustrate this, consider for example the delta-function case: The Fourier transform of $\delta(\varepsilon - \varepsilon_s)$ is $e^{-i\omega\varepsilon_s}$. It is oscillatory if and only if $\varepsilon_s \neq 0$. The frequency of the ω-oscillations is just ε_s; namely, it directly tells us the distance of the singularity from the point $\varepsilon = 0$.

One more example is the singularity $|\varepsilon - \varepsilon_s|^{-\beta}$, whose transform is $\propto \omega^{\beta-1} e^{-i\omega\varepsilon_s}$. But this phenomenon is of course more general: If a certain function $P(\varepsilon)$ admits a Fourier transform $F(\omega)$, then the transform of $P(\varepsilon - \varepsilon_s)$ is $F(\omega) e^{-i\omega\varepsilon_s}$.

The oscillations seen in e.g. Fig. 1(b) have a certain "ω-wavelength" $\lambda_1 \approx 0.17$ (in units in which the BH mass is $M = 1$). They must therefore correspond to a singularity in the TPF, located at a distance $\varepsilon_s \equiv t' - t = 2\pi/\lambda_1 \approx 37$ from point x. What is the nature of this nonlocal singularity of the TPF? As already pointed out in Sec. II B, the function $\langle \phi(x)\phi(x') \rangle$ admits a singularity whenever a null geodesic exists which connects x and x'. In the present context of Schwarzschild background and t-splitting, we are dealing here with a null geodesic which emanates from a certain spatial point (r, θ, φ), makes a round trip around the BH, and then returns to that same spatial point, but obviously with a certain delay in t, which should correspond to the shift parameter ε_s.

2. Spectrum of oscillations

In fact there is not only one but an *infinite*, discrete set of such connecting null geodesics (for each r). This is because a null geodesic emanating from a spatial point (r, θ, φ) can make any integer number of revolutions around the BH before returning to that spatial point. Therefore, the TPF will actually admit an infinite number of singular points at a discrete set of values $t' - t = \varepsilon_n$, one for each integer n.

Correspondingly, there will be a discrete spectrum of oscillation modes in $F_{\mathrm{reg}}(\omega, r)$, with ($r$-dependent) ω-frequencies ε_n and corresponding wavelengths $\lambda_n = 2\pi/\varepsilon_n$. We point out, however, that the dominant mode is always $n = 1$, and the oscillation's amplitude quickly decays with n (see next subsection).

To perform the self-cancellation of oscillations, we shall need to know the spectrum of frequencies ε_n, at any desired r value. This requires integration of the null geodesic equation, to find the connecting null geodesics. For $r = 3M$ the situation is especially simple, because in that case the connecting null geodesic is circular. One then finds that

[21] In this context we should regard $F(\omega, r)$ as a function that vanishes for all $\omega < 0$. Note also that in the present context one should not think of ε as a small parameter. Instead, it is allowed to take all real values.

$\varepsilon_n = (2\pi\sqrt{27})n$. At other r values the spectrum becomes more complicated, and needs to be calculated numerically. The null orbits in Schwarzschild are characterized by a single constant of motion, namely the angular momentum per unit energy (or the "impact parameter"). The calculation of ε_n involves (i) numerical integration of the null geodesic equation (for prescribed values of that constant of motion), and (ii) using the Newton-Raphson method for adjusting this constant, in order to find the connecting geodesics, which return to the original spatial point after n rounds. The required parameters ε_n are the t-duration of these connecting null geodesics. Overall, this is an easy numerical procedure.

In our representative case $r = 6M$, the first few oscillation frequencies are found to be

$$\varepsilon_1 \simeq 37.50, \qquad \varepsilon_2 \simeq 70.17, \qquad \varepsilon_3 \simeq 102.8, \qquad \varepsilon_4 \simeq 135.5.$$

In Fig. 1(b) we predominantly see the basic oscillation $n = 1$, and the frequency agrees very well with this value of ε_1. We can also notice the residual effect of the $n = 2$ oscillation, which causes the small distortion (i.e. small deviation from the smooth $\propto \omega^{1/2}$ envelope) in the pattern of peaks of $F_{\rm reg}(\omega, r)$, seen in Fig. 1(b). After the basic oscillation $n = 1$ is removed by self-cancellation, the next one ($n = 2$) dominates and becomes very clear, and again, its frequency is found to agree very well with the above value of ε_2. By this procedure it is possible to expose a few more modes, and to confirm their agreement with the above ε_n values, that were obtained from the connecting null geodesics.

As was already mentioned above, for $r = 3M$ the frequencies ε_n form an arithmetic sequence with a common difference $\Delta\varepsilon_0 \equiv 2\pi\sqrt{27} \simeq 32.65$. For other r values this is no longer the case. Still, the difference $\varepsilon_{n+1} - \varepsilon_n$ quickly approaches the standard spacing $\Delta\varepsilon_0$. For example, from the four ε_n values specified above for the $r = 6M$ case, one sees that $\varepsilon_2 - \varepsilon_1 \simeq 32.67$, and then $\varepsilon_3 - \varepsilon_2 \simeq \varepsilon_4 - \varepsilon_3 \simeq 32.65$. This is simply because for any r, a connecting null geodesic with large n makes most of the revolutions around the BH along an orbit very close to the circle $r = 3M$.

3. Amplitude of oscillations

The numerical data indicate that the oscillation's amplitude grows as $\omega^{1/2}$. This in turn implies that the divergence of the TPF at $\varepsilon \to \varepsilon_s$ should be $\propto |\varepsilon - \varepsilon_s|^{-3/2}$. We shall not address this issue here in detail, but we point out that this $-3/2$ power is just what one would expect from simple arguments. To this end one has to recall that since the background is spherically symmetric, and since the splitting is only in the t direction (implying that $\theta' = \theta$ and $\varphi' = \varphi$), whenever x' is located on a null geodesic emanating from x, it is placed *exactly* at a *caustic point* of that null geodesic. Qualitative arguments suggest that on crossing such a caustic point, the TPF should indeed diverge as $|\varepsilon - \varepsilon_s|^{-3/2}$. But the discussion of this issue is far beyond our present scope.

The $n > 1$ oscillations, too, grow as $\omega^{1/2}$ at large ω (the numerics confirms this, at least for the first few n values). This is for the same reason as that described above for $n = 1$. We thus express the large-ω amplitudes of the various ε_n modes as $\approx A_n \omega^{1/2}$, where A_n is a set of (r-dependent) amplitude parameters.

It is important to explore how A_n behaves with increasing n, in order to control the possible effect of the infinite number of oscillating modes. The simplest case to analyze is again $r = 3M$, because the connecting null geodesic is then the circular geodesic at $r = 3M$. Simple analytical considerations suggest that in this case A_n should form an almost-exact geometric sequence with $A_n/A_{n+1} \cong e^{\pi} \simeq 23.14$.[22] Our numerical results for $F_{\rm reg}(\omega, r = 3M)$ allow reliable evaluation of the first three amplitudes, and the calculated ratios A_1/A_2 and A_2/A_3 agree very well with e^{π}, to about one part in 10^3. (For the $n > 3$ modes the oscillations are too weak to reliably measure their A_n.)

For other r values the situation is more complicated, and we do not expect to find such a well-approximated geometric sequence; yet, we still expect that as n increases, A_n/A_{n+1} should quickly approach the above "canonical" value e^{π}. The reason is that, for large n, the connecting null geodesic makes most of the n revolutions around the BH along an orbit very close to the circle $r = 3M$. Hence, the decrease in the Van Vleck determinant at each revolution is approximately the same as in the analogous $r = 3M$ case (an approximation that ever improves with increasing n). At very small n, however, the ratio between two successive amplitudes may slightly differ from e^{π}. We numerically find that the ratio A_1/A_2 ranges from $e^{\pi} \simeq 23.1$ at $r = 3M$ to ≈ 23.7 at $r = 9M$.

Overall, at least in the range $3M \le r \le 9M$ that we have numerically explored, the numerical data as well as the theoretical considerations are all consistent with an almost-geometric sequence (even for small n), with A_n/A_{n+1} ranging between 23 and 24. In turn this implies that for practical computation of the generalized integral of $F_{\rm reg}(\omega, r)$, we shall have to cancel the first few n modes, but the contribution of large-n modes may be neglected.

[22] A simple (though still unproved) analytical argument, based on evaluating the Van Vleck determinant along the $r = 3M$ geodesic, suggests that in this case A_n should be exactly proportional to $[2\sinh(2\pi n)]^{-1/2}$. This expression deviates from the geometric sequence $e^{-\pi n}$ by a tiny relative amount $\cong e^{-4\pi n}/2$. This deviation is smaller than one part in 10^5 even for $n = 1$, too small to be detected by our numerics, but nevertheless our numerical results are fully consistent with that expression.

4. Self-cancellation operator

We self-cancel the first four frequencies $n = 1\ldots4$. The higher modes $n > 4$ are too weak to notice.[23] The values of the frequencies ε_n are numerically obtained from the connecting null geodesics, for any desired r, as explained above. The corresponding wavelengths are then given by $\lambda_n = 2\pi/\varepsilon_n$. For each frequency, we apply a fourth-order self-cancellation. Thus, in the terminology of Eq. (A6), our actual multiple self-cancellation operator is

$$T_* \equiv (T_{\lambda_1})^4 (T_{\lambda_2})^4 (T_{\lambda_3})^4 (T_{\lambda_4})^4. \tag{B1}$$

[23] We point out that although the amplitude formally diverges at large ω for *any* n, because after all we only integrate along a finite interval of ω [which is in turn allowed due to the fast convergence of $H_*(\omega)$], and because of the fast decay of A_n with n, the large-n oscillation terms do not have any noticeable effect on the integral.

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