PHYSICAL REVIEW D **91,** 104027 (2015)

Entropy of a self-gravitating electrically charged thin shell and the black hole limit

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(Received 21 November 2014; published 21 May 2015)

A static self-gravitating electrically charged spherical thin shell embedded in a $(3+1)$-dimensional spacetime is used to study the thermodynamic and entropic properties of the corresponding spacetime. Inside the shell, the spacetime is flat, whereas outside it is a Reissner-Nordström spacetime, and this is enough to establish the energy density, the pressure, and the electric charge in the shell. Imposing that the shell is at a given local temperature and that the first law of thermodynamics holds on the shell one can find the integrability conditions for the temperature and for the thermodynamic electric potential, the thermodynamic equilibrium states, and the thermodynamic stability conditions. Through the integrability conditions and the first law of thermodynamics an expression for the shell's entropy can be calculated. It is found that the shell's entropy is generically a function of the shell's gravitational and Cauchy radii alone. A plethora of sets of temperature and electric potential equations of state can be given. One set of equations of state is related to the Hawking temperature and a precisely given electric potential. Then, as one pushes the shell to its own gravitational radius and the temperature is set precisely equal to the Hawking temperature, so that there is a finite quantum backreaction that does not destroy the shell, one finds that the entropy of the shell equals the Bekenstein-Hawking entropy for a black hole. The other set of equations of state is such that the temperature is essentially a power law in the inverse Arnowitt-Deser-Misner (ADM) mass and the electric potential is a power law in the electric charge and in the inverse ADM mass. In this case, the equations of thermodynamic stability are analyzed, resulting in certain allowed regions for the parameters entering the problem. Other sets of equations of state can be proposed. Whatever the initial equation of state for the temperature, as the shell radius approaches its own gravitational radius, the quantum backreaction imposes the Hawking temperature for the shell in this limit. Thus, when the shell's radius is sent to the shell's own gravitational radius the formalism developed allows one to find the precise form of the Bekenstein-Hawking entropy of the correlated black hole.

DOI: 10.1103/PhysRevD.91.104027 PACS numbers: 04.70.Bw, 04.20.Gz

I. INTRODUCTION

In general relativity, in $3+1$ dimensions, a black hole spacetime is characterized by its conserved charges and the fundamental constants. The conserved charges are for example the Arnowitt-Deser-Misner (ADM) mass m and the electric charge Q. The fundamental constants are the two constants of the theory, namely, the gravitational constant G, and the velocity of light (which is set to one). In an analysis of quantum aspects of a black hole, such as the black hole entropy and its inherent degrees of freedom, the other fundamental constant in physics, Planck's constant $\hbar$, also appears naturally. With these three constants one makes the Planck length $l_p = \sqrt{G\hbar}$ and the Planck area $A_p = l_p^2$. Also, m, Q, G, and the velocity of light give the horizon radius r_+ and so the horizon area $A_+ = 4\pi r_+^2$. Then, the Bekenstein-Hawking entropy of a black hole, given by $S_{\rm bh} = \frac{1}{4}\frac{A_+}{A_p}$ [1–3], where the Boltzmann constant is set to one, is a measure of how many Planck areas there are in the horizon area. It also shows that black hole quantum mechanics, and consequently black hole entropy, is in its essence and generality a process of pure quantum gravity, as no other constants besides the

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gravitational constant G, the velocity of light, and Planck's constant $\hbar$ enter, through the Planck area A_p, in the final process. In addition, it suggests that the ultimate degrees of freedom that inhabit the realm of quantum gravity are in the area of the enclosing region, rather than within the volume as it is the case for ordinary matter [4,5] (for a review see, e.g., Ref. [6]). However, since there is no quantum gravity theory at hand, black hole entropy is still an enigma although there has been progress in its understanding, especially through the resort to gravitational low-energy quantum theories.

Since black holes are vacuum solutions, and our primitive concepts of entropy are based on the quantum properties of matter, it would be useful to have a spacetime with matter and study its thermodynamic and entropic properties. One then can look for a limit where a black hole might emerge. In this way, one can have hints to how a black hole's entropy develops. We are thus interested in a system which contains both gravitational and material degrees of freedom but which does not introduce too many complexities due to the matter constitution.

The next simplest solution to a black hole solution, is a vacuum solution except for an infinitesimally thin region of spacetime where there is matter, i.e., a self-gravitating thin shell. As a thin shell is the nearest to a vacuum solution one can have, it is a very useful system that allows one to probe almost pure spacetime properties. A thin shell is defined as an infinitesimally thin surface which partitions spacetime into an interior region and an exterior region. Since it corresponds to some sort of matter and the spacetime properties must reflect it, the thin shell should satisfy some conditions in order for the entire spacetime to be a valid solution of the Einstein equations. Such conditions relate the stress-energy tensor of the shell to the extrinsic curvature of the spacetime. The stress-energy tensor yields the density and pressure, and in general the matter properties are also the equations of state, such as the temperature and possibly others, and the entropy. A particularly simple thin shell is one that is static and is spherically symmetric.

Suppose then a self-gravitating static spherical thin shell. We assume the simplest case: the inner spacetime is Minkowski and the outer spacetime is Schwarzschild. One can then work out its dynamics and thermodynamic properties, such as the temperature and entropy. In an elegant work, by finding the surface energy density and pressure, and imposing that the shell is at a given local temperature T, and so using a canonical ensemble, Martinez [7] found those thermodynamic properties for the simplest shell, characterized by its rest mass M and radius R. In Ref. [7] only shells whose matter obeyed the dominant energy condition, and so the radius R greater than a given value, were considered. Martinez's approach [7] draws in many respects from York's work [8] where the thermodynamic properties of a pure Schwarzschild black hole were treated using a canonical ensemble, i.e., imposing a fixed temperature on some fictitious massless shell at a definite radius outside the event horizon. Another reason that motivates the use of thin shells is the fact that they can be taken with some ease to their own gravitational radius, i.e., to the black hole limit. If one does that, as was done in Ref. [9], one recovers the black hole entropy, i.e., the entropy S of the shell at its own gravitational radius is $S = S_{\rm bh} = \frac{1}{4}\frac{A_+}{A_p}$ for such a matter configuration at the black hole limit. Such a configuration is called a quasiblack hole. Thus, the black hole thermodynamic properties can be studied by a direct computation if thin shells are used.

It is important to generalize Martinez's work [7] for electrically charged shells, which we will do here. We consider that the shell has an electric charge Q. In this case, the inner spacetime is Minkowski and the outer spacetime is Reissner-Nordström. One can then work out the shell's dynamics and thermodynamic properties, such as the energy density, the pressure, the electric potential temperature, and the entropy. Due to the introduction of a new state variable in the thermodynamic system, namely, the additional thermodynamic electric potential, the calculations become considerably more complex. At the same time the richness of the physical results increases as well. We take the shell to its own gravitational radius, the black hole end point, which is meaningful in the calculation of the shell's entropy S, and find that the entropy is equal to the Bekenstein-Hawking entropy. The extremal $\sqrt{G}m = Q$ limit can then be taken which gives the same expression for the entropy, i.e., $S = S_{\rm bh} = \frac{1}{4}\frac{A_+}{A_p}$; see, however, Refs. [10,11] for a discussion of the entropy of extremal black holes taken from extremal shells. Electrically charged black holes were studied in Refs. [12,13], where the thermodynamic properties of a pure Reissner-Nordström black hole were treated using a grand canonical ensemble, i.e., imposing a fixed temperature and electric potential at some definite radius outside the event horizon.

There are other works that used thin shells to understand the thermodynamics and the evolution of the entropy in certain spacetimes. In Refs. [14,15] the formalism of Martinez [7] was used to study three-dimensional thin shells including the thermodynamics of a thin shell with a static Bañados-Teitelboim-Zanelli outer spacetime. In Refs. [16,17] thin shells with a black hole inside were used to understand how the entropy of the spacetime evolves as the shell approaches its own event horizon.

We analyze static thin shells using the junction condition formalism established in Ref. [18] with the complement to electrically charged shells developed in Ref. [19]. Our thermodynamic approach, follows the general approach for thermodynamic systems given in Ref. [20], as does the approach of Ref. [7].

We will adopt the following line of work. In Sec. II we study a static spherical symmetric thin shell whose interior is Minkowski and exterior is Reissner-Nordström. We find the main properties of the global spacetime as well as the rest energy density, and thus the rest mass, the pressure in the shell, and the shell's electric charge. In Sec. III we exhibit the first law of thermodynamics, find the generic integrability conditions and the stability conditions. In Sec. IV we use the spherical shell whose dynamics is displayed in Sec. II. We present the three independent thermodynamic variables (M, R, Q) and then through the integrability conditions find the functional dependence for the temperature T and the thermodynamic electric potential Φ on those variables. Then in Sec. V the differential for the entropy S of the shell is obtained as a differential on the gravitational radius r_+ and the Cauchy horizon radius r_- and up to two functions which depend on r_+ and r_-. Those functions are essentially the inverse of the temperature and the electric potential of the shell if it were located at infinity. Moreover, it is shown that the two functions are related by a specific differential equation, and that the entropy of the shell is a function of r_+ and r_- alone, which themselves are functions of (M, R, Q). In Sec. VI, to advance further, one needs to specify the form of the equations of state. We give a particular set of equations of state that will lead us with some ease to the black hole entropy when the shell is taken to its own gravitational radius. Indeed, by choosing the Hawking temperature due to quantum-mechanical arguments and a precise electric potential, the entropy of a charged black hole will naturally emerge. We then compare our approach with the usual thermodynamic approach for black holes. In Sec. VII we give another simple set of phenomenological equations of state for the temperature and the electric potential, where free parameters encoding the details of the matter fields will naturally appear. This set of equations of state allows us to also find the entropy and study analytically the stability conditions. In Sec. VIII we briefly discuss other interesting equations of state. Finally, in Sec. IX we conclude. We leave for Appendix A a study of the dominant energy condition of the matter fields in the shell which is not important in the thermodynamic study, but which is interesting to have. In Appendix B we derive the equations of thermodynamic stability for a system with three independent variables.

II. THE THIN-SHELL SPACETIME

A. The Einstein-Maxwell equations

We start with the Einstein-Maxwell equations in $3+1$ dimensions

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \tag{1}$$

$$\nabla_\beta F^{\alpha\beta} = 4\pi J^\alpha. \tag{2}$$

$G_{\alpha\beta}$ is the Einstein tensor, built from the spacetime metric $g_{\alpha\beta}$ and its first and second derivatives, $8\pi G$ is the coupling, with G being the gravitational constant in $3+1$ dimensions and we are using units in which the velocity of light is one, and $T_{\alpha\beta}$ is the energy-momentum tensor. $F_{\alpha\beta}$ is the Faraday-Maxwell tensor, J_α is the electromagnetic four-current and ∇_β denotes covariant derivative. The other Maxwell equation $\nabla_{[\gamma} F_{\alpha\beta]} = 0$, where $[\ldots]$ means anti-symmetrization, is automatically satisfied for a properly defined $F_{\alpha\beta}$. Greek indices will be used for spacetime indices and run as $\alpha, \beta = 0, 1, 2, 3$, with 0 being the time index.

B. The thin-shell gravitational junction conditions

We consider now a two-dimensional timelike massive electrically charged shell with radius R, which we will call Σ. The shell partitions spacetime into two parts, an inner region $\mathcal{V}_i$ and an outer region $\mathcal{V}_o$. In order to find a global spacetime solution for the Einstein equation, Eq. (1), we will use the thin-shell formalism developed in Ref. [18].

First, we specify the metrics on each side of the shell. In the inner region $\mathcal{V}_i$ $(r < R)$ we assume the spacetime is flat, i.e.

$$ds_i^2 = g^i_{\alpha\beta} dx^\alpha dx^\beta = -dt_i^2 + dr^2 + r^2 d\Omega^2, \quad r < R, \tag{3}$$

where t_i is the inner time coordinate, polar coordinates (r, θ, ϕ) are used, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. In the outer region $\mathcal{V}_o$ $(r > R)$, the spacetime is described by the Reissner-Nordström line element

$$ds_o^2 = g^o_{\alpha\beta} dx^\alpha dx^\beta = -\left(1 - \frac{2Gm}{r} + \frac{GQ^2}{r^2}\right) dt_o^2 + \frac{dr^2}{1 - \frac{2Gm}{r} + \frac{GQ^2}{r^2}} + r^2 d\Omega^2, \quad r > R, \tag{4}$$

where t_o is the outer time coordinate, and again (r, θ, ϕ) are polar coordinates, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The constant m is to be interpreted as the ADM mass, or energy, and Q as the electric charge. Finally, on the hypersurface itself, $r = R$, the metric h_{ab} is that of a 2-sphere with an additional time dimension, such that,

$$ds_\Sigma^2 = h_{ab} dy^a dy^b = -d\tau^2 + R^2(\tau) d\Omega^2, \qquad r = R, \tag{5}$$

where we have chosen $y^a = (\tau, \theta, \phi)$ as the time and spatial coordinates on the shell. We have adopted the convention to use latin indices for the components on the hypersurface. The time coordinate τ is the proper time for an observer located at the shell. The shell radius is given by the parametric equation $R = R(\tau)$ for an observer on the shell. On each side of the hypersurface, the parametric equations for the time and radial coordinates are denoted by $t_i = T_i(\tau)$, $r_i = R_i(\tau)$, and $t_o = T_o(\tau)$, $r_o = R_o(\tau)$. The metric h_{ab} is also called the induced metric and can be written in terms of the $3+1$-dimensional spacetime metric $g_{\alpha\beta}$. In particular, viewed from each side of the shell, the induced metric is given by

$$h^i_{ab} = g^i_{\alpha\beta} e^\alpha_{i\,a} e^\beta_{i\,b}, \qquad h^o_{ab} = g^o_{\alpha\beta} e^\alpha_{oa} e^\beta_{ob}, \tag{6}$$

where $e^\alpha_{i\,a}$ and e^α_{oa} are tangent vectors to the hypersurface viewed from the inner and outer regions, respectively. With these last expressions, we have all the necessary information to employ the formalism developed in Ref. [18]. We will also apply this formalism to electrically charged systems which was displayed first in Ref. [19].

The thin-shell formalism states that two junction conditions are needed in order to have a smooth change across the hypersurface. The first junction condition is expressed by the relation

$$[h_{ab}] = 0, \tag{7}$$

where the parentheses symbolize the jump in the quantity across the hypersurface, which in this case is the induced metric. This condition immediately implies that $h^i_{ab} = h^o_{ab} = h_{ab}$, or explicitly

$$-\left(1 - \frac{2Gm}{r} + \frac{GQ^2}{r^2}\right)\dot{T}_o^2 + \frac{\dot{R}_o^2}{(1 - \frac{2Gm}{r} + \frac{GQ^2}{r^2})} = -\dot{T}_i^2 + \dot{R}_i^2 = -1, \tag{8}$$

where a dot denotes differentiation with respect to τ. The second junction condition is related to the inner and outer extrinsic curvature $K^a_{i\,b}$ and K^a_{ob}, respectively, defined as

$$K^a_{i\,b} = \left(\nabla_\beta n^i_\alpha\right) e^\alpha_{i\,c} e^\beta_{i\,b} h^{ca}_i, \qquad K^a_{ob} = \left(\nabla_\beta n^o_\alpha\right) e^\alpha_{oc} e^\beta_{ob} h^{ca}_o, \tag{9}$$

where n^i_α and n^o_α, are the inner and outer normals to the shell, respectively. The second junction condition then says $[K^a{}_b] = 0$ if the metric is to be smooth across the hypersurface. However, this condition can be violated, in which case it can be physically interpreted as the existence of a thin matter shell where the hypersurface is located. In addition, the shell's stress-energy tensor $S^a{}_b$ is related to the jump in the extrinsic curvature through the Lanczos equation, namely,

$$S^a{}_b = -\frac{1}{8\pi G}([K^a{}_b] - [K] h^a{}_b), \tag{10}$$

where $K = h^b{}_a K^a{}_b$. Proceeding then to the calculation of the extrinsic curvature components, one can show that they are given by the general expressions

$$K^\tau_{i\,\tau} = \frac{\ddot{R}}{\sqrt{1 + \dot{R}^2}}, \tag{11}$$

$$K^\tau_{o\tau} = \frac{-\frac{G\dot{m}}{R\dot{R}} - \frac{GQ^2}{R^3} + \frac{Gm}{R^2} + \ddot{R}}{\sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2} + \dot{R}^2}}, \tag{12}$$

$$K^\phi_{i\,\phi} = K^\theta_{i\,\theta} = \frac{1}{R}\sqrt{1 + \dot{R}^2}, \tag{13}$$

$$K^\phi_{o\phi} = K^\theta_{o\theta} = \frac{1}{R}\sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2} + \dot{R}^2}. \tag{14}$$

Using Eqs. (11)–(14) in Eq. (10), one can calculate the non-null components of the stress-energy tensor S_{ab} of the shell. In particular, we will assume a static shell, such that $\dot{R} = 0$, $\ddot{R} = 0$, and $\dot{m} = 0$. In that case, we are led to

$$S^\tau{}_\tau = \frac{\sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2}} - 1}{4\pi GR}, \tag{15}$$

$$S^\phi{}_\phi = S^\theta{}_\theta = \frac{\sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2}} - 1}{8\pi GR} + \frac{\frac{mG}{R} - \frac{GQ^2}{R^2}}{8\pi GR\sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2}}}. \tag{16}$$

To further advance, one needs to specify what kind of matter the shell is made of, which we will consider to be a perfect fluid with surface energy density σ and pressure p. This implies that the stress-energy tensor will be of the form

$$S^a{}_b = (\sigma + p) u^a u_b + p h^a{}_b, \tag{17}$$

where u^a is the three-velocity of a shell element. We thus find that

$$S^\tau{}_\tau = -\sigma, \tag{18}$$

$$S^\theta{}_\theta = S^\phi{}_\phi = p. \tag{19}$$

Combining Eqs. (18)–(19) with Eqs. (15)–(16) results in the equations

$$\sigma = \frac{1 - \sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2}}}{4\pi GR}, \tag{20}$$

$$p = \frac{\sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2}} - 1}{8\pi GR} + \frac{\frac{mG}{R} - \frac{GQ^2}{R^2}}{8\pi GR\sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2}}}. \tag{21}$$

Note that Eq. (21) is purely a consequence of the Einstein equation which is encoded in the junction conditions. Thus, although no information about the matter fields of the shell has been given, we know that they must have a pressure equation of the form (21), otherwise no mechanical equilibrium can be achieved.

It is useful to define the shell's redshift function k as

$$k = \sqrt{1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2}}. \tag{22}$$

Equation (22) allows Eqs. (20)–(21) to be written as

$$\sigma = \frac{1-k}{4\pi GR}, \tag{23}$$

$$p = \frac{R^2(1-k)^2 - GQ^2}{16\pi GR^3 k}. \tag{24}$$

From the energy density σ of the shell we can define the rest mass M through the equation

$$\sigma = \frac{M}{4\pi R^2}. \tag{25}$$

Note that from Eqs. (23) and (25) one has

$$M = \frac{R}{G}(1-k). \tag{26}$$

Using Eqs. (22) and (26), we are led to an equation for the ADM mass m,

$$m = M - \frac{GM^2}{2R} + \frac{Q^2}{2R}. \tag{27}$$

This equation is intuitive on physical grounds as it states that the total energy m of the shell is given by its mass M minus the energy required to built it against the action of gravitational and electrostatic forces, i.e., $-\frac{GM^2}{2R} + \frac{Q^2}{2R}$. For $Q = 0$, we recover the result derived in Ref. [7]. Note that Eq. (27) is also purely a consequence of the Einstein equation encoded in the junction conditions, i.e., although no information about the matter fields of the shell has been given, we know that they must have an ADM mass given by Eq. (27).

The gravitational radius r_+ and the Cauchy horizon r_- of the shell spacetime are given by the zeros of the g^o_{00} in Eq. (4). They are then

$$r_+ = Gm + \sqrt{G^2m^2 - GQ^2}, \tag{28}$$

$$r_- = Gm - \sqrt{G^2m^2 - GQ^2}, \tag{29}$$

respectively. The gravitational radius r_+ is also the horizon radius when the shell radius R is inside r_+, i.e., the spacetime contains a black hole. Although they have the same expression, conceptually, the gravitational and horizon radii are distinct. Indeed, the gravitational radius is a property of the spacetime and matter, independently of whether there is a black hole or not. On the other hand, the horizon radius exists only when there is a black hole. The gravitational radius r_+ and the Cauchy horizon r_- in Eqs. (28)–(29) can be inverted to give

$$m = \frac{1}{2G}(r_+ + r_-), \tag{30}$$

$$Q = \sqrt{\frac{r_+ r_-}{G}}. \tag{31}$$

From Eq. (28) one can define the gravitational area A_+ as

$$A_+ = 4\pi r_+^2. \tag{32}$$

This is also the event horizon area when there is a black hole. Using Eqs. (28)–(29) implies that k in Eq. (22) can be written as

$$k = \sqrt{\left(1 - \frac{r_+}{R}\right)\left(1 - \frac{r_-}{R}\right)}. \tag{33}$$

The area A of the shell, an important quantity, is from Eq. (5) given by

$$A = 4\pi R^2. \tag{34}$$

C. The thin-shell electromagnetic junction conditions

Now we have to deal with Eq. (2). The Faraday-Maxwell tensor $F_{\alpha\beta}$ is usually defined in terms of an electromagnetic four-potential A_α by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{35}$$

where ∂_β denotes partial derivative.

To use the thin-shell formalism related to the electric part we need to specify the vector potential A_α on each side of the shell. We assume an electric ansatz for the electromagnetic four-potential A_α, i.e.,

$$A_\alpha = (-\phi, 0, 0, 0), \tag{36}$$

where ϕ is thus the electric potential. In the inner region $\mathcal{V}_i$ $(r < R)$ the spacetime is flat. So the Maxwell equation $\nabla_\beta F^{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}F^{\alpha\beta}) = 0$ has as a constant solution for the inner electric potential ϕ_i which, for convenience, can be written as

$$\phi_i = \frac{Q}{R} + \text{constant}, \qquad r < R, \tag{37}$$

where Q is a constant, to be interpreted as the conserved electric charge. In the outer region $\mathcal{V}_o$ $(r > R)$, the spacetime is Reissner-Nordström and the Maxwell equation $\nabla_\beta F^{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}F^{\alpha\beta}) = 0$ now yields

$$\phi_o = \frac{Q}{r} + \text{constant}, \qquad r > R. \tag{38}$$

Due to the existence of electricity in the shell, another important set of restrictions must also be considered. These restrictions are related to the discontinuity present in the electric field across the charged shell. We are interested in the projection

$$A_a = A_\alpha e^\alpha_a \tag{39}$$

of the four-potential in the shell's hypersurface, since it will contain quantities which are intrinsic to the shell. Indeed, following Ref. [19],

$$[A_a] = 0, \tag{40}$$

with $A_{ia} = (-\phi_i, 0, 0)$, and $A_{oa} = (-\phi_o, 0, 0)$ being the vector potential at R, on the shell, seen from each side of it. Thus, the constants in Eqs. (37) and (38) are indeed the same and so at R

$$\phi_o = \phi_i, \qquad r = R. \tag{41}$$

Following Ref. [19] further, the tangential components F_{ab} of the electromagnetic tensor $F_{\alpha\beta}$ must change smoothly across Σ, i.e.

$$[F_{ab}] = 0, \tag{42}$$

with

$$F^i_{ab} = F^i_{\alpha\beta} e^\alpha_{i\,a} e^\beta_{i\,b}, \qquad F^o_{ab} = F^o_{\alpha\beta} e^\alpha_{o\,a} e^\beta_{o\,b}, \tag{43}$$

while the normal components $F_{a\perp}$ must change by a jump as,

$$[F_{a\perp}] = 4\pi\sigma_e u_a, \tag{44}$$

where

$$F^i_{a\perp} = F^i_{\alpha\beta} e^\alpha_{i\,a} n^\beta_i, \quad F^o_{a\perp} = F^o_{\alpha\beta} e^\alpha_{o\,a} n^\beta_o, \tag{45}$$

and $\sigma_e u_a$ is the surface electric current, with σ_e being the density of charge and u_a its 3-velocity, defined on the shell. One can then show that Eq. (42) is trivially satisfied, while Eq. (44) leads to the single nontrivial equation at R, on the shell,

$$\frac{\partial\phi_o}{\partial r} - \frac{\partial\phi_i}{\partial r} = -4\pi\sigma_e, \qquad r = R. \tag{46}$$

Then, from Eqs. (37), (38), and (46) one obtains

$$\frac{Q}{R^2} = 4\pi\sigma_e, \tag{47}$$

relating the total charge Q, the charge density σ_e, and the shell's radius R in the expected manner. This section with its equations forms the dynamical side of the electric thin-shell solution.

D. Restrictions on the thin-shell radius

A natural inequality that the shell should obey is to consider the shell to be outside its gravitational radius in all instances, so

$$R \geq r_+. \tag{48}$$

It is then clear that the physical allowed values for k in Eq. (33) are in the interval [0, 1]. It is also interesting to consider the restrictions imposed by the dominant energy condition. However, since it will not take part in our analysis we leave this discussion for Appendix A.

III. THERMODYNAMICS AND STABILITY CONDITIONS FOR THE THIN SHELL: GENERICS

A. Thermodynamics and integrability conditions for the thin shell

We now turn to the thermodynamic side and to the calculation of the entropy of the shell. We use units in which the Boltzmann constant is one. We start with the assumption that the shell in static equilibrium possesses a well-defined temperature T and an entropy S which is a function of three variables, call them M, A, Q, i.e.,

$$S = S(M, A, Q). \tag{49}$$

(M, A, Q) can be considered as three generic parameters. In our connection they are the shell's rest mass M, area A, and charge Q. The first law of thermodynamics can thus be written as

$$TdS = dM + pdA - \Phi dQ \tag{50}$$

where dS is the differential of the entropy of the shell, dM is the differential of the rest mass, dA is the differential of the area of the shell, dQ is the differential of the charge, and T, p and Φ are the temperature, the pressure, and the thermodynamic electric potential of the shell, respectively. In order to find the entropy S, one thus needs three equations of state, namely,

$$p = p(M, A, Q), \tag{51}$$

$$\beta = \beta(M, A, Q), \tag{52}$$

$$\Phi = \Phi(M, A, Q), \tag{53}$$

where

$$\beta \equiv \frac{1}{T} \tag{54}$$

is the inverse temperature.

It is important to note that the temperature and the thermodynamic electric potential play the role of integration factors, which implies that there will be integrability conditions that must be specified in order to guarantee the existence of an expression for the entropy, i.e. that the differential dS is exact. These integrability conditions are

$$\left(\frac{\partial\beta}{\partial A}\right)_{M,Q} = \left(\frac{\partial\beta p}{\partial M}\right)_{A,Q}, \tag{55}$$

$$\left(\frac{\partial\beta}{\partial Q}\right)_{M,A} = -\left(\frac{\partial\beta\Phi}{\partial M}\right)_{A,Q}, \tag{56}$$

$$\left(\frac{\partial\beta p}{\partial Q}\right)_{M,A} = -\left(\frac{\partial\beta\Phi}{\partial A}\right)_{M,Q}. \tag{57}$$

These equations enable one to determine the relations between the three equations of state of the system.

B. Stability conditions for the thin shell

With the first law of thermodynamics given in Eq. (50), one is able to perform a thermodynamic study of the local intrinsic stability of the shell. To have thermodynamic stability the following inequalities should hold:

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_{A,Q} \le 0, \tag{58}$$

$$\left(\frac{\partial^2 S}{\partial A^2}\right)_{M,Q} \le 0, \tag{59}$$

$$\left(\frac{\partial^2 S}{\partial Q^2}\right)_{M,A} \le 0, \tag{60}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial A^2}\right) - \left(\frac{\partial^2 S}{\partial M\partial A}\right)^2 \ge 0, \tag{61}$$

$$\left(\frac{\partial^2 S}{\partial A^2}\right)\left(\frac{\partial^2 S}{\partial Q^2}\right) - \left(\frac{\partial^2 S}{\partial A\partial Q}\right)^2 \ge 0, \tag{62}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial Q^2}\right) - \left(\frac{\partial^2 S}{\partial M\partial Q}\right)^2 \ge 0, \tag{63}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial Q\partial A}\right) - \left(\frac{\partial^2 S}{\partial M\partial A}\right)\left(\frac{\partial^2 S}{\partial M\partial Q}\right) \ge 0. \tag{64}$$

The derivation of these expressions follows the rationale presented in Ref. [20]; see Appendix B.

IV. THE THERMODYNAMIC INDEPENDENT VARIABLES AND THE THREE EQUATIONS OF STATE: EQUATIONS FOR THE PRESSURE, TEMPERATURE AND ELECTRIC POTENTIAL

A. The three independent thermodynamic variables (M, R, Q)

We will work from now onwards with the three independent variables (M, R, Q) instead of (M, A, Q). The rest mass M of the shell is from Eq. (25) given by

$$M = 4\pi R^2 \sigma, \tag{65}$$

where σ is given by Eq. (23) and R is the radius of the shell. The first law of thermodynamics written in generic terms is simpler when expressed using the area A of the shell, but here it is handier to use the radius R in this specific study. The radius R is related to the area A through Eq. (5), i.e.,

$$R = \sqrt{\frac{A}{4\pi}}. \tag{66}$$

As for the charge Q, using Eq. (47), it is given by

$$Q = 4\pi R^2 \sigma_e. \tag{67}$$

The three independent thermodynamic variables are thus (M, R, Q).

We should now envisage Eq. (27) and Eqs. (28)–(29) as functions of (M, R, Q), i.e.

$$m(M, R, Q) = M - \frac{GM^2}{2R} + \frac{Q^2}{2R}, \tag{68}$$

and

$$r_+(M, R, Q) = Gm(M, R, Q) + \sqrt{G^2 m(M, R, Q)^2 - GQ^2}, \tag{69}$$

$$r_-(M, R, Q) = Gm(M, R, Q) - \sqrt{G^2 m(M, R, Q)^2 - GQ^2}, \tag{70}$$

respectively. The function k in Eq. (33) is also a function of (M, R, Q),

$$k(r_+(M, R, Q), r_-(M, R, Q), R) = \sqrt{\left(1 - \frac{r_+(M, R, Q)}{R}\right)\left(1 - \frac{r_-(M, R, Q)}{R}\right)}. \tag{71}$$

B. The pressure equation of state

Expressing the pressure equation of state in the form of Eq. (51), we obtain from Eqs. (21) and (27) [or Eq. (68)],

$$p(M,R,Q) = \frac{GM^2 - Q^2}{16\pi R^2(R-GM)}, \tag{72}$$

or changing from the variables (M,R,Q) to (r_+,r_-,R) which is more useful, we find [see Eqs. (24) and (31)],

$$p(r_+,r_-,R) = \frac{R^2(1-k)^2 - r_+r_-}{16\pi G R^3 k}, \tag{73}$$

where k can be envisaged as $k = k(r_+,r_-,R)$ as given in Eq. (71) and r_+ and r_- are functions of (M,R,Q); see Eqs. (69)–(70). This reduces to the expression obtained in Ref. [7] in the limit $Q=0$ or $r_-=0$. This equation, Eq. (73), is a pure consequence of the Einstein equation, encoded in the junction conditions.

C. The temperature equation of state

Turning now to the temperature equation of state (52), we will need to focus on the integrability condition (55). Changing from the variables (M,R,Q) to (r_+,r_-,R), Eq. (55) becomes

$$\left(\frac{\partial\beta}{\partial R}\right)_{r_+,r_-} = \beta\frac{R(r_+ + r_-) - 2r_+r_-}{2R^3k^2} \tag{74}$$

which has the analytic solution

$$\beta(r_+,r_-,R) = b(r_+,r_-)k \tag{75}$$

where k is a function of r_+, r_-, and R, as given in Eq. (71), and $b(r_+,r_-) \equiv \beta(r_+,r_-,\infty)$ is an arbitrary function, representing the inverse of the temperature of the shell if its radius were infinite. Hence, in a sense, from Eq. (75), we recover Tolman's formula for the temperature of a body in curved spacetime. The arbitrariness of this function is due to the fact that the matter fields of the shell are not specified. Note that b and k are still functions of (M,R,Q) through the variables r_+ and r_-; see Eqs. (69)–(70) and Eq. (71).

D. The electric potential equation of state

The remaining equation of state to be studied is the electric potential. Using Eqs. (26) and (71), one can deduce $(\frac{\partial M}{\partial A})_{r_+,r_-} = -p$, i.e.,

$$\left(\frac{\partial M}{\partial R}\right)_{r_+,r_-} = -8\pi R\, p. \tag{76}$$

Then, it follows from Eqs. (55)–(57) and Eq. (76) that the differential equation

$$\left(\frac{\partial p}{\partial Q}\right)_{M,R} + \frac{1}{8\pi R}\left(\frac{\partial\Phi}{\partial R}\right)_{r_+,r_-} + \Phi\left(\frac{\partial p}{\partial M}\right)_{R,Q} = 0, \tag{77}$$

holds, where the second term has been expressed in the variables (r_+,r_-,R) and the other terms in the variables (M,R,Q) for the sake of computational simplicity. Then, after using Eq. (72) in Eq. (77), we obtain that Eq. (77) takes the form

$$R^2\left(\frac{\partial\Phi k}{\partial R}\right)_{r_+r_-} - \frac{\sqrt{r_+r_-}}{\sqrt{G}} = 0, \tag{78}$$

where k can be envisaged as $k = k(r_+,r_-,R)$ as given in Eq. (71). The solution of Eq. (78) is then

$$\Phi(r_+,r_-,R) = \frac{\phi(r_+,r_-) - \frac{\sqrt{r_+r_-}}{\sqrt{G}R}}{k} \tag{79}$$

where $\phi(r_+,r_-) \equiv \Phi(r_+,r_-,\infty)$ is an arbitrary function that corresponds physically to the electric potential of the shell if it were located at infinity. This thermodynamic electric potential Φ is the difference in the electric potential ϕ between infinity and R, blueshifted from infinity to R (see a similar result in Refs. [12,13] for an electrically charged black hole in a grand canonical ensemble). We also see that, once again, by changing to the variables (r_+,r_-,R) we are able somehow to reduce the number of arguments of the arbitrary function from three to two.

It is convenient to define a function $c(r_+,r_-)$ through $c(r_+,r_-) \equiv \frac{\phi(r_+,r_-)}{Q}$, i.e.,

$$c(r_+,r_-) \equiv \sqrt{G}\frac{\phi(r_+,r_-)}{\sqrt{r_+r_-}}, \tag{80}$$

where we have used $Q = \sqrt{r_+r_-/G}$ as given in Eq. (31). Then, Eq. (79) is written as

$$\Phi(r_+,r_-,R) = \frac{c(r_+,r_-) - \frac{1}{R}}{k}\sqrt{\frac{r_+r_-}{G}}, \tag{81}$$

where k can be envisaged as $k = k(r_+,r_-,R)$ as given in Eq. (71).

V. ENTROPY OF THE THIN SHELL

At this point we have all the necessary information to calculate the entropy S. By inserting the equations of state for the pressure, Eq. (73), for the temperature, Eq. (75), and for the electric potential, Eq. (81), as well as the differential of M given in Eq. (26) and the differential of the area A or of the radius R [see Eq. (66)] into the first law, Eq. (50), we arrive at the entropy differential

$$dS = b(r_+,r_-)\frac{1 - c(r_+,r_-)r_-}{2G}dr_+ + b(r_+,r_-)\frac{1 - c(r_+,r_-)r_+}{2G}dr_-. \tag{82}$$

Now, Eq. (82) has its own integrability condition if dS is to be an exact differential. Indeed, it must satisfy the equation

$$\frac{\partial b}{\partial r_-}(1 - r_- c) - \frac{\partial b}{\partial r_+}(1 - r_+ c) = \frac{\partial c}{\partial r_-} b r_- - \frac{\partial c}{\partial r_+} b r_+. \tag{83}$$

This shows that in order to obtain a specific expression for the entropy one can choose either b or c, and the other remaining function can be obtained by solving the differential equation (83) with respect to that function. Since Eq. (83) is a differential equation there is still some freedom in choosing the other remaining function. In the first examples we will choose to specify the function b first and from it obtain an expression for c. We also give examples where the function c is specified first.

From Eq. (82) we obtain

$$S = S(r_+, r_-), \tag{84}$$

so that the entropy is a function of r_+ and r_- alone. In fact S is a function of (M, R, Q), $S(M, R, Q)$, but the functional dependence has to be through $r_+(M, R, Q)$ and $r_-(M, R, Q)$, i.e., in full form

$$S(M, R, Q) = S(r_+(M, R, Q), r_-(M, R, Q)). \tag{85}$$

This result shows that the entropy of the thin charged shell depends on the (M, R, Q) through r_+ and r_- which themselves are specific functions of (M, R, Q).

It is also worth noting the following feature. From Eq. (85) we see that shells with the same r_+ and r_-, i.e., the same ADM mass m and charge Q, but different radii R, have the same entropy. Let then an observer sit at infinity and measure m and Q (and thus r_+ and r_-). Then, the observer cannot distinguish the entropy of shells with different radii. This is a kind of thermodynamic mimicker, as a shell near its own gravitational radius and another one far from it have the same entropy.

VI. THE THIN SHELL AND THE BLACK HOLE LIMIT

A. The temperature equation of state and the entropy

Let us consider a charged thin shell, for which the differential of the entropy has been deduced to be Eq. (82). We are free to choose an equation of state for the inverse temperature. Let us pick for convenience the following inverse temperature dependence:

$$b(r_+, r_-) = \gamma \frac{r_+^2}{r_+ - r_-}, \tag{86}$$

where γ is some constant with units of inverse mass times inverse radius, i.e., units of angular momentum.

For a charged shell we must also specify the function $c(r_+, r_-)$, whose form can be taken from the differential equation (83) upon substitution of the function (86). There is a family of solutions for $c(r_+, r_-)$ but for our purposes here we choose the following specific solution:

$$c(r_+, r_-) = \frac{1}{r_+}. \tag{87}$$

The rationale for the choices above becomes clear when we discuss the shell's gravitational radius, i.e., black hole, limit. Inserting the choice for $b(r_+, r_-)$, Eq. (86), along with the choice for the function $c(r_+, r_-)$, Eq. (87), in the differential (82) and integrating, we obtain the entropy differential for the shell

$$dS = \frac{\gamma}{2G} r_+ dr_+. \tag{88}$$

Thus, the entropy of the shell is $S = \frac{\gamma}{4G} r_+^2 + S_0$, where S_0 is an integration constant. Imposing that when the shell vanishes (i.e., $M = 0$ and $Q = 0$, and so $r_+ = 0$) the entropy vanishes we have that S_0 is zero, and so $S = \frac{\gamma}{4G} r_+^2$. Thus, we can write the entropy $S(M, R, Q)$ as

$$S = \frac{\gamma}{16\pi G} A_+, \tag{89}$$

where A_+ is the gravitational area of the shell, as given in Eq. (32). This result shows that the entropy of this thin charged shell depends on (M, R, Q) through r_+^2 only, which itself is a specific function of (M, R, Q).

Now, what is the constant γ? It should be determined by the properties of the matter in the shell, and cannot be decided *a priori*.

B. The stability conditions for the specific temperature ansatz

The thermodynamic stability of the uncharged case ($Q = 0$, i.e., $r_- = 0$) can be worked out [7] and elucidates the issue. In the uncharged case the nontrivial stability conditions are given by Eqs. (58) and (61). Equation (58) gives immediately $R \leq \frac{3}{2} r_+$, i.e., $R \leq 3Gm$. On the other hand, Eq. (61) yields $R \geq r_+$, i.e., $R \geq 2Gm$. Thus, the stability conditions yield the following range for R, $r_+ \leq R \leq \frac{3}{2} r_+$, or in terms of m, $2Gm \leq R \leq 3Gm$. This is precisely the range for stability found by York [8] for a black hole in a canonical ensemble in which a spherical massless thin wall at radius R is maintained at fixed temperature T. In Ref. [8] the criterion used for stability is that the heat capacity of the system should be positive, and physically such a tight range for R means that only when the shell, at a given temperature T, is sufficiently close to the horizon can it smother the black hole enough to make it thermodynamically stable. The positivity of the heat capacity is equivalent to our stability conditions, Eqs. (58) and (61) in the uncharged case.

The stability conditions, Eqs. (58)–(64), for the general charged case cannot be solved analytically in this instance;

they require numerical work, which will shadow what we want to determine. Nevertheless, the approach followed in Refs. [12,13] for the heat capacity of a charged black hole in a grand canonical ensemble gives a hint of the procedure that should be followed.

C. The black hole limit

1. The black hole limit properly stated

Although γ should be determined by the properties of the matter in the shell, there is a case in which the properties of the shell have to adjust to the environmental properties of the spacetime. This is the case when $R \to r_+$. In this case, one must note that, as the shell approaches its own gravitational radius, quantum fields are inevitably present and their backreaction will diverge unless we choose the black hole Hawking temperature $T_{\rm bh}$ for the temperature of the shell. In this case, $R \to r_+$, we must take the temperature of the shell as $T_{\rm bh} = \frac{\hbar}{4\pi}\frac{r_+ - r_-}{r_+^2}$, where $\hbar$ is Planck's constant. So we must choose

$$\gamma = \frac{4\pi}{\hbar}, \tag{90}$$

i.e., γ depends on fundamental constants. Then,

$$b(r_+, r_-) = \frac{1}{T_{\rm bh}} = \frac{4\pi}{\hbar}\frac{r_+^2}{r_+ - r_-}. \tag{91}$$

In this case the entropy of the shell is $S = \frac{1}{4}\frac{A_+}{G\hbar}$, i.e.,

$$S = \frac{1}{4}\frac{A_+}{A_p}, \tag{92}$$

where $l_p = \sqrt{G\hbar}$ is the Planck length, and $A_p = l_p^2$ the Planck area. Note now that the entropy given in Eq. (92) is the black hole Bekenstein-Hawking entropy $S_{\rm bh}$ of a charged black hole since

$$S_{\rm bh} = \frac{1}{4}\frac{A_+}{A_p}, \tag{93}$$

where A_+ is here the horizon area. Thus, when we take the shell to its own gravitational radius the entropy is the Bekenstein-Hawking entropy. The limit also implies that the pressure and the thermodynamic electric potential go to infinity as $1/k$, according to Eqs. (73) and (81), respectively. Note, however, that the local inverse temperature goes to zero as k [see Eq. (75)], and so the local temperature of the shell also goes to infinity as $1/k$. These well-controlled infinities cancel out precisely to give the Bekenstein-Hawking entropy (92).

Note that, since $A = A_+$ when the shell is at its own gravitational radius, at this point the entropy of the shell is proportional to its own area A, indicating that all the shell's fundamental degrees of freedom have been excited.

Note also that the shell at its own gravitational radius, at least in the uncharged case, is thermodynamically stable, since in this case stability requires $r_+ \le R \le \frac{3}{2} r_+$, as mentioned above.

Note yet that our approach and the approach followed in Ref. [10] to find the black hole entropy have some similarities. The two approaches use matter fields, i.e., shells, to find the black hole entropy. Here we use a static shell that decreases its own radius R by steps, maintaining its staticity at each step. In Ref. [10] a reversible contraction of a thin spherical shell down to its own gravitational radius was examined, and it was found that the black hole entropy can be defined as the thermodynamic entropy stored in the matter in the situation that the matter is compressed into a thin layer at its own gravitational radius.

Finally we note that the extremal limit $\sqrt{G}m = Q$ or $r_+ = r_-$ is well defined from above. Indeed, when one takes the limit $r_+ \to r_-$ one finds that $1/b(r_+, r_-) = 0$ (i.e., the Hawking temperature is zero) and the entropy of the extremal black hole is still given by $S_{\text{extremal bh}} = \frac{1}{4}\frac{A_+}{A_p}$. It is well known that extremal black holes and in particular their entropy have to be dealt with care. If, *ab initio*, one starts with the analysis for an extremal black hole one finds that the entropy of the extremal black hole has a more general expression than simply being equal to one quarter of the area [10,11]. This extremal shell is an example of a Majumdar-Papapetrou matter system. Its pressure p is zero, and it remains zero, and thus finite, even when $R \to r_+$. This limit of $R \to r_+$ is called a quasiblack hole, which in the extremal case is a well-behaved one.

2. The rationale for the choice of $b(r_+, r_-)$ and $c(r_+, r_-)$

We have started with a thin shell and imposed a temperature equation of state of the Hawking type [see Eq. (86) as well as Eqs. (90), and (91)], and a specific thermodynamic electric potential [see Eq. (87)]. This set of equations of state gives an entropy for the shell proportional to its gravitational radius area A_+. One can moreover set the temperature of the shell at any R precisely equal to the Hawking temperature $T_{\rm bh}$ [see Eq. (91)]. Remarkably, we have then shown that a self-gravitating electric thin shell at the Hawking temperature and with a specific electric potential has a Bekenstein-Hawking entropy.

A priori, Hawking-type choices for the temperature [Eqs. (86), (90), and (91)], and black hole-type choices for the electric potential [Eq. (87)], are simply choices, and many other choices for the set of equations of state can be taken. However, this set is really imposed on the shell when it approaches its gravitational radius, where it takes the precise forms given in Eqs. (86) and (90) [or, Eq. (91)], and Eq. (87) as the spacetime quantum effects get a hold on the shell.

We would like to stress that the requirement $b = T_{\rm bh}^{-1}$ [see Eqs. (86), (90), and (91)] is compulsory only for shells that approach their own gravitational radius. Otherwise, if we consider the radius of the shell within some constrained region outside the gravitational radius, the shell temperature can be arbitrary since away from the horizon, quantum backreaction remains modest and does not destroy the thermodynamic state. One can discuss whole classes of functions $b(r_+, r_-) \neq T_{\rm bh}^{-1}$.

In addition we stress that the choice for c, Eq. (87), is necessary only for shells at the gravitational radius limit. According to Eq. (80), this gives us $\phi = \frac{\sqrt{r_-}}{\sqrt{G r_+}}$, i.e.,

$$\phi = \frac{Q}{r_+} \tag{94}$$

that coincides with the standard expression for the electric potential for the Reissner-Nordström black hole. In addition, Eq. (81) acquires the form

$$\Phi = \frac{Q}{k}\left(\frac{1}{r_+} - \frac{1}{R}\right) \tag{95}$$

that coincides entirely with the corresponding formula for the Reissner-Nordström black hole in a grand canonical ensemble [12]. Meanwhile, in our case there is no black hole. Moreover, if we go to the uncharged case, $Q \to 0$ or $r_- \to 0$, and thus the outer space is described by the Schwarzschild metric, then it is seen from Eq. (82) that the quantity c drops out from the entropy, so the choice of c is relevant for the charged case only, of course.

3. Similarities between the thin-shell approach and the black hole mechanics approach

There are similarities between the thin-shell approach and the black hole mechanics approach [2]. These are evident if we express the differential of the entropy of the charged shell (82) in terms of the black hole ADM mass m and charge Q, given in terms of the variables (r_+, r_-) by Eqs. (30)–(31). The differential for the entropy of the shell reads in these variables

$$T_0 dS = dm - cQdQ \tag{96}$$

where we have defined $T_0 \equiv 1/b(r_+, r_-)$ which is the temperature the shell would possess if located at infinity. Here, $T_0 = 1/b(r_+, r_-)$ and $c = c(r_+, r_-)$ should be seen as $T_0(m, Q) = 1/b(m, Q)$ and $c(m, Q)$, respectively, since r_+ and r_- are functions of m and Q. As we have seen, if we take the shell to its gravitational radius, we must fix $T_0 = T_{\rm bh}$ and $c = 1/r_+$. This suggests that Q/r_+ should play the role of the black hole electric potential $\Phi_{\rm bh}$, which in fact is true, as shown in Eq. (94) (see Ref. [2]; see also Refs. [12,13]). So the conservation of energy of the shell is expressed as

$$T_{\rm bh} dS_{\rm bh} = dm - \Phi_{\rm bh} dQ. \tag{97}$$

We thus see that the first law of thermodynamics for the shell at its own gravitational radius is equal to the energy conservation for the Reissner-Nordström black hole.

VII. THE THIN SHELL WITH ANOTHER SPECIFIC EQUATION OF STATE FOR THE TEMPERATURE

A. The temperature equation of state and the entropy

The previous equation of state is not prone to a simple stability analysis. Here we give another equation of state that permits finding both an expression for the shell's entropy and performing a simple stability analysis.

We must first specify an adequate thermal equation of state for $b(r_+, r_-)$. A possible simple choice is a power law in the ADM mass m, i.e., $b(r_+, r_-)$ has the form

$$b(r_+, r_-) = 2Ga(r_+ + r_-)^\alpha \tag{98}$$

where a and α are free coefficients related to the properties of the shell. Power laws occur frequently in thermodynamic systems, and so this is a natural choice as well. The simple choice above allows one to find the form of the function c. Indeed, the integrability equation (83) gives that the function c can be put in the form $c(r_+, r_-) = 2G\frac{f(r_+ r_-)}{(r_+ + r_-)^\alpha}$, where $f(r_+ r_-)$ is an arbitrary function of the product $r_+ r_-$ and supposedly also depends on the intrinsic constants of the matter that makes up the shell. For convenience we choose $f(r_+ r_-) = d(r_+ r_-)^\delta$, where d and δ are parameters that reflect the shell's properties, so that

$$c(r_+, r_-) = 2Gd\frac{(r_+ r_-)^\delta}{(r_+ + r_-)^\alpha}. \tag{99}$$

The gravitational constant G was introduced in Eqs. (98) and (99) for convenience. Inserting Eqs. (98)–(99) into Eq. (82) and integrating, gives the entropy

$$S(r_+, r_-) = a\left[\frac{(r_+ + r_-)^{\alpha+1}}{\alpha + 1} - d\frac{(r_+ r_-)^{\delta+1}}{\delta + 1}\right], \tag{100}$$

where the constant of integration S_0 has been put to zero, as expected in the limit $r_+ \to 0$ and $r_- \to 0$. Again, the entropy of this thin charged shell depends on (M, R, Q) through r_+ and r_- only, which in turn are specific functions of (M, R, Q).

We consider positive temperatures and positive electric potentials, so

$$a > 0, \qquad d > 0. \tag{101}$$

We consider only

$$\alpha > 0, \tag{102}$$

for the simplicity of the upcoming stability analysis. Although this choice somewhat narrows down the range of cases to which the analysis is applicable, it only rules out the cases where $-1 < \alpha < 0$, since for values $\alpha \leq -1$ it would give a diverging entropy in the limit $r_+ \to 0$ and $r_- \to 0$, something which is not physically acceptable. Indeed, in such a limit we would expect the entropy to be zero which requires $\alpha > -1$.

B. The stability conditions for the specific temperature ansatz

Proceeding to the thermodynamic stability treatment, we start with Eq. (58), which can be shown to be equivalent to

$$r_+ r_- - 2R^2k^2\alpha + (1-k^2)R^2 \geq 0. \tag{103}$$

Solving for k, this leads to the restriction

$$k \leq \sqrt{\frac{1}{2\alpha+1}\left(1 + \frac{r_+ r_-}{R^2}\right)}. \tag{104}$$

Going now to Eq. (59), it gives

$$[r_+r_- - (1-k)^2R^2][\alpha(r_+r_- - (1-k)^2R^2) + 3(r_+r_- + (1-k^2)R^2)] \leq 0. \tag{105}$$

Since the second multiplicative term on the left must be positive, one can solve for k and obtain the set of values which satisfy the inequality,

$$\frac{\alpha}{\alpha+3} - \sqrt{\frac{9}{(\alpha+3)^2} + \frac{r_+r_-}{R^2}} \leq k \leq \frac{\alpha}{\alpha+3} + \sqrt{\frac{9}{(\alpha+3)^2} + \frac{r_+r_-}{R^2}}. \tag{106}$$

As for Eq. (60), it reduces to

$$\frac{dR(2\delta+1)(r_+r_-)^\delta}{(\frac{r_+r_-}{R} + (1-k^2)R)^\alpha} \geq \frac{R^2(1-k^2) + (2\alpha+1)r_+r_-}{R^2(1-k^2) + r_+r_-}. \tag{107}$$

Although one cannot conclude anything directly from the above inequality, it is nonetheless worth noting that the right-hand side is greater than zero, and so δ must obey the condition

$$\delta \geq -\frac{1}{2}. \tag{108}$$

Regarding Eq. (61), it is possible to show that it implies the condition

$$r_+^2r_-^2(\alpha+3) - 2r_+r_-R^2(2k^2\alpha + 2k^2 - k + \alpha - 1) + (1-k)^2R^4(3k^2\alpha + k^2 + 2\alpha k + \alpha - 1) \leq 0, \tag{109}$$

which does not provide any information on its own since it is a polynomial of order four in the variable k. Nonetheless, it does need to be satisfied once a region of allowed values for k is known, which will be ascertained in the following.

Concerning Eq. (62), we are led to

$$\frac{dR(2\delta+1)(r_+r_-)^\delta}{(\frac{r_+r_-}{R} + (1-k^2)R)^\alpha} \leq \frac{r_+^2r_-^2(3\alpha+1) + 2(1-k)r_+r_-R^2(2\alpha(k-1) + 2k - 1) - (1-k)^3R^4(k(\alpha+3) - \alpha + 3)}{[(1-k)^2R^2 - r_+r_-][(k-1)R^2(k(\alpha+3) - \alpha + 3)]}, \tag{110}$$

which does not contain any new information.

On the other hand, when Eq. (63) is simplified to

$$\frac{dR(2\delta+1)(r_+r_-)^\delta}{(\frac{r_+r_-}{R} + (1-k^2)R)^\alpha} \geq \frac{R^2(1-k^2) + (2\alpha+1)r_+r_- - 2R^2k^2\alpha}{R^2(1-k^2) + r_+r_- - 2R^2k^2\alpha}, \tag{111}$$

and one notices that the numerator on the right side must be positive, another constraint on k naturally appears, namely

$$k \leq \sqrt{\frac{1}{2\alpha+1} + \frac{r_+r_-}{R^2}}. \tag{112}$$

Finally, the last condition (64) gives the inequality

$$r_+r_-(\alpha+1) - R^2[(\alpha+1)k^2 + \alpha - 1] \geq 0 \tag{113}$$

which constricts the values of k to be within the interval

$$k \leq \sqrt{-\frac{\alpha-1}{\alpha+1}+\frac{r_+r_-}{R^2}}. \tag{114}$$

The definitive region of permitted values for k is the intersection of the conditions (104), (106), (112) and (114). It is possible to show that such an intersection gives the range

$$\frac{\alpha}{\alpha+3}-\sqrt{\frac{9}{(\alpha+3)^2}+\frac{r_+r_-}{R^2}} \leq k \leq \sqrt{-\frac{\alpha-1}{\alpha+1}+\frac{r_+r_-}{R^2}} \tag{115}$$

where α must be restricted to

$$\alpha \geq \frac{1+\frac{r_+r_-}{R^2}}{1-\frac{r_+r_-}{R^2}}. \tag{116}$$

Returning to Eq. (109), it is now possible to verify if the interval (115) satisfies said condition, which indeed it does.

C. The black hole limit

If one takes the shell to its own gravitational radius, the chosen temperature equation of state (98) is wiped out, and a new equation of state sets in to adapt to the quantum spacetime properties. The new equation of state is then given by Eq. (91) and the black hole entropy (92) follows.

VIII. OTHER EQUATIONS OF STATE

Naturally, other equations of state can be sought. We give four examples, one fixing $b(r_+,r_-)$ and three others fixing $c(r_+,r_-)$.

If we fix the inverse temperature

$$b(r_+,r_-) = \gamma\frac{r_+^2}{r_+-r_-}, \tag{117}$$

for some γ, as we did before, then generically, from Eq. (83), we find

$$c(r_+,r_-) = \frac{a(r_+r_-)(r_+-r_-)+r_-}{r_+^2}, \tag{118}$$

where $a(r_+r_-)$ is an arbitrary function of integration of the product r_+r_- and presumably also depends on the intrinsic constants of the matter that makes up the shell. Then, from Eq. (82), the entropy is

$$S(r_+,r_-) = \frac{\gamma}{4G}\left(r_+^2+\int_0^{r_+r_-}(1-a(x))dx\right), \tag{119}$$

where it is implied that the function $h(x)$ vanishes at $x=0$ rapidly enough so that the entropy goes to zero when $r_+=0$. In the example we gave previously we have put $a(r_+r_-)=1$, so that $c(r_+,r_-)=\frac{1}{r_+}$. This case $a(r_+r_-)=1$ gives precisely that the entropy of the shell is proportional to the area of its gravitational radius and for $\gamma=\frac{4\pi}{\hbar}$ gives that the entropy of the shell is equal to the corresponding black hole entropy as we have discussed previously. Of course, many other choices can be given for $a(r_+r_-)$ and quite generally the entropy will be a function of r_+ and r_-, $S=S(r_+,r_-)$.

Inversely, instead of $b(r_+,r_-)$ one can give $c(r_+,r_-)$. One equation for $c(r_+,r_-)$ could be

$$c(r_+,r_-) = \frac{1}{r_+}, \tag{120}$$

as for the black hole case. The integrability condition, Eq. (83), for the temperature then gives

$$b(r_+,r_-) = \frac{h(r_+)}{r_+-r_-}, \tag{121}$$

where $h(r_+)$ is a function that can be fixed in accord with the matter properties of the shell. Then, from Eq. (82), the entropy is

$$S(r_+) = \frac{1}{2G}\int_0^{r_+}\frac{h(x)}{x}dx, \tag{122}$$

where we are assuming zero entropy when $r_+=0$. If we choose $h(r_+)=\frac{4\pi}{\hbar}r_+^2$, then one recovers the black hole temperature and the black hole entropy for the shell.

Another equation of state one can choose for $c(r_+,r_-)$ is

$$c(r_+,r_-) = \frac{1}{r_-}. \tag{123}$$

The integrability condition, Eq. (83), similarly gives

$$b(r_+,r_-) = \frac{h(r_-)}{r_+-r_-}, \tag{124}$$

where $h(r_-)$ is a function that can be fixed in accord with the matter properties of the shell. In this case, from Eq. (82), the entropy of the shell depends on r_- only, and is given by

$$S(r_-) = \frac{1}{2G}\int_0^{r_-}\frac{h(x)}{x}dx, \tag{125}$$

where we are assuming zero entropy when $r_-=0$.

Yet another example can be obtained if one puts

$$c(r_+,r_-) = c(r_+r_-), \tag{126}$$

i.e., c is a function of the product r_+r_- and may also depend on the intrinsic constants of the matter that makes up the shell. The integrability condition then gives

$$b = b_0, \tag{127}$$

where b_0 is a constant, and so in this case, the temperature measured at infinity does not depend on r_+ or r_-. The entropy is then

$$S(r_+, r_-) = \frac{b_0}{2G}\left(r_+ + r_- - \int_0^{r_+ r_-} c(x)dx\right), \qquad (128)$$

where we are assuming zero entropy when $r_+ = 0$ and $r_- = 0$.

One could study in detail these four cases for the thermodynamics of a shell performing in addition a stability analysis for each one. We refrain here to do so. Certainly other interesting cases can be thought of.

IX. CONCLUSIONS

We have considered the thermodynamics of a self-gravitating electrically charged thin shell thus generalizing previous works on the thermodynamics of self-gravitating thin-shell systems. Relatively to the simplest shell where there are two independent thermodynamic state variables, namely, the rest mass M and the size R of the shell, we have now a new independent state variable in the thermodynamic system, the electric charge Q, out of which, using the first law of thermodynamics and the equations of state one can construct the entropy of the shell $S(M, R, Q)$. Due to the additional variable, the charge Q, the calculations are somewhat more complex. Concomitantly, the richness in physical results increases in the same proportion.

The equations of state one has to give are the pressure $p(M, R, Q)$, the temperature $T(M, R, Q)$, and the electric potential $\Phi(M, R, Q)$. The pressure can be obtained from dynamics alone, using the thin-shell formalism and the junction conditions for a flat interior and a Reissner-Nordström exterior. The form of the temperature and of the thermodynamic electric potential are obtained using the integrability conditions that follow from the first law of thermodynamics.

The differential for the entropy in its final form shows remarkably that the entropy must be a function of r_+ and r_- alone, i.e., a function of the intrinsic properties of the shell spacetime. Thus, shells with the same r_+ and r_- (i.e., the same ADM mass m and charge Q) but different radii R, have the same entropy. From the thermodynamics properties alone of the shell one cannot distinguish a shell near its own gravitational radius from a shell far from it. In a sense, the shell can mimic a black hole.

The differential for the entropy in its final form gives that T and Φ are related through an integrability condition. One has then to specify either T or Φ and the form of the other function is somewhat constrained. We gave two example cases and mentioned other possibilities.

First, we gave the equations of state where the temperature has the form of the Hawking temperature, apart from a constant factor, and the electric potential has a simple precise form Q/r_+, and found the entropy. When the factor is the Hawking factor it was shown that the resulting entropy was equal to the Bekenstein-Hawking entropy of a non-extremal charged black hole. The need to set the temperature of the shell equal to the Hawking temperature is justified when the shell is taken to its own gravitational radius. At this radius the backreaction of the nearby quantum fields diverges unless the shell has precisely the Hawking temperature. Conversely, one should note that if instead, the function for the electric potential Q/r_+ was given, the integrability equation would then fix the function T apart from an arbitrary function. A simple choice for this arbitrary function is the Hawking temperature.

Second, the other set of equations of state were given as a simple ansatz. For the thermal equation of state, we set the temperature as proportional to some power in the ADM mass m, and the thermodynamic electric potential was set to be a power in the electric charge and an inverse power in m. This choice also allows one to find an expression for the entropy of the shell and, furthermore, allows for an analytic stability analysis. Indeed, despite the increase in complexity in the thermodynamic stability analysis due to the existence of four new stability equations, it was possible to obtain a unique range for the redshift parameter k, as well as the regions of allowed values for the parameters α and δ.

Many other interesting equations of state can be chosen and some of them were indeed given. However at the gravitational radius all turn into the Hawking equation of state, i.e., the Hawking temperature. Since the area of the shell A is equal to the gravitational radius area A_+, $A = A_+$, when the shell is at its own gravitational radius, and $S = \frac{1}{4}\frac{A_+}{A_p}$ in this limit, we conclude that the entropy of the shell is proportional to its own area A. This indicates that all its fundamental degrees of freedom have been excited. Matter systems at their own gravitational radius are called quasi-black holes and have thermodynamic properties similar to black holes.

ACKNOWLEDGMENTS

We thank FCT-Portugal for financial support through Project No. PEst-OE/FIS/UI0099/2013. G. Q. also acknowledges the Grant No. SFRH/BD/92583/2013 from FCT. O. B. Z. has been partially supported by the Kazan Federal University through a state grant for scientific activities.

APPENDIX A: THE DOMINANT ENERGY CONDITION

With the expressions for the mass density and pressure, Eqs. (23)–(24), we can consider some mechanical constraints which the shell should naturally obey. One can impose that the shell satisfies the weak energy condition. It requires that σ and p be positive, which is always verified. One can also insist that the shell satisfies the dominant energy condition, i.e.,

$$p \le \sigma. \tag{A1}$$

It is then possible to show that the dominant energy condition imposes the constraint $k \in [k_1, k_2]$, where

$$k_1 = \frac{3}{5}\left(1 - \sqrt{1 - \frac{5}{9}\left(1 - \frac{r_+ r_-}{R^2}\right)}\right), \tag{A2}$$

and $k_2 = \frac{3}{5}(1 + \sqrt{1 - \frac{5}{9}(1 - \frac{r_+ r_-}{R^2})})$. Since $k_2 > 1$, and k trivially obeys $k \le 1$, we conclude that the dominant energy condition restricts the values of k to obey

$$k_1 \le k. \tag{A3}$$

In the case where there is no charge, i.e., $Q = 0$ or $r_- = 0$, one gets $k_1 = 1/5$, thus regaining the result obtained in Ref. [7]. When expressed in terms of the variables R/m and R/Q, the relation (A3) can be written as

$$\frac{R}{m} \ge \frac{25}{6 + 10\frac{GQ^2}{R^2} + 3\sqrt{4 + 5\frac{GQ^2}{R^2}}} \tag{A4}$$

or in terms of R/r_+ and r_-/R,

$$\frac{R}{r_+} \ge \frac{12\sqrt{1 - \frac{r_-}{R} - (\frac{r_-}{R})^2 + (\frac{r_-}{R})^3} + 31\frac{r_-}{R} - 20(\frac{r_-}{R})^2 - 12}{24\frac{r_-}{R} - 25(\frac{r_-}{R})^2}. \tag{A5}$$

This is a mechanical constraint. A fundamental constraint, the no-trapped-surface condition for the shell, is $R \ge r_+$, as was given in Eq. (48).

APPENDIX B: DERIVATION OF THE EQUATIONS OF THERMODYNAMIC STABILITY FOR A SYSTEM WITH THREE INDEPENDENT VARIABLES

In this appendix we shall show the derivation of the equations of thermodynamic stability for an electrically charged system, i.e., Eqs. (58)–(64). Thus the approach used for two independent variables in Ref. [20] is extended here by us to three independent variables. We name these independent variables M, A, and Q.

We start by considering two identical subsystems, each with an entropy $S = S(M, A, Q)$, where M is the internal energy of the system (equivalent to the rest mass), A is its area and Q its electric charge. The usual state variables of a thermodynamic system are the internal energy U, volume V and the number of particles, N, say. However, the system we wish to study is an electrically charged thin shell, and thus it is natural to use the variables (M, A, Q). Thermodynamic stability is guaranteed if $dS = 0$ and $d^2S < 0$ are both satisfied, or in other words, if the entropy is an extremum and a maximum respectively.

Now suppose we keep A and Q constant and remove a positive amount of internal energy ΔM from one subsystem to the other. The total entropy of the two subsystems goes from the value $2S(M, A, Q)$ to $S(M + \Delta M, A, Q) +$ $S(M - \Delta M, A, Q)$. If the initial entropy $S(M, A, Q)$ is a maximum, then the sum of initial entropies must be greater or equal to the sum of final entropies, i.e.

$$S(M + \Delta M, A, Q) + S(M - \Delta M, A, Q) \le 2S(M, A, Q). \tag{B1}$$

Expanding $S(M + \Delta M, A, Q)$ and $S(M - \Delta M, A, Q)$ in a Taylor series to second order in ΔM, we see that Eq. (B1) becomes

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_{A,Q} \le 0 \tag{B2}$$

in the limit $\Delta M \to 0$. The same reasoning applies if we fix M and Q instead and apply a positive change of area ΔA, so we must have

$$S(M, A + \Delta A, Q) + S(M, A - \Delta A, Q) \le 2S(M, A, Q) \tag{B3}$$

which in the limit $\Delta A \to 0$ gives

$$\left(\frac{\partial^2 S}{\partial A^2}\right)_{M,Q} \le 0. \tag{B4}$$

If we fix M and A and make a positive change ΔQ on the charge, we have

$$S(M, A, Q + \Delta Q) + S(M, A, Q - \Delta Q) \le 2S(M, A, Q) \tag{B5}$$

and so it follows that

$$\left(\frac{\partial^2 S}{\partial Q^2}\right)_{M,A} \le 0. \tag{B6}$$

However, if we keep only one quantity fixed, like Q for example, we must also have a final sum of entropies smaller than the initial sum if we apply a simultaneous change of area and internal energy rather than separately, i.e.

$$S(M + \Delta M, A + \Delta A, Q) + S(M - \Delta M, A - \Delta A, Q) \le 2S(M, A, Q). \tag{B7}$$

This inequality is satisfied by Eq. (B2) and Eq. (B4), but it also implies a new requirement. If we expand the left side in a Taylor series to second order in ΔM and ΔA, and use the abbreviated notation $S_{ij} = \partial^2 S/\partial x_i \partial x_j$, we get

$$S_{MM}(\Delta M)^2 + 2S_{MA}\Delta M\Delta A + S_{AA}(\Delta A)^2 \leq 0. \quad \text{(B8)}$$

Multiplying Eq. (B8) by S_{MM} and adding and subtracting $S_{MA}^2(\Delta A)^2$ to and from the left side, allows the last inequality to be written in the form

$$(S_{MM}\Delta M + S_{MA}\Delta A)^2 + (S_{MM}S_{AA} - S_{MA}^2)(\Delta A)^2 \geq 0. \quad \text{(B9)}$$

Since the first term on the left side is always greater than zero, we see that it is sufficient to have

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial A^2}\right) - \left(\frac{\partial^2 S}{\partial M\partial A}\right)^2 \geq 0. \quad \text{(B10)}$$

This concludes the derivation of Eqs. (58), (59) and (61).

To derive the other stability equations, namely, Eqs. (60), (62), (63), and (64), we note that we can repeat the same calculations but now we fix M and A in turns. It is now straightforward to see that, when fixing M, we must have

$$S_{AA}(\Delta A)^2 + 2S_{AQ}\Delta A\Delta Q + S_{QQ}(\Delta Q)^2 \leq 0, \quad \text{(B11)}$$

which is satisfied by

$$\left(\frac{\partial^2 S}{\partial A^2}\right)\left(\frac{\partial^2 S}{\partial Q^2}\right) - \left(\frac{\partial^2 S}{\partial A\partial Q}\right)^2 \geq 0. \quad \text{(B12)}$$

Finally, by fixing A we get the inequality

$$S_{MM}(\Delta M)^2 + 2S_{MQ}\Delta M\Delta Q + S_{QQ}(\Delta Q)^2 \leq 0 \quad \text{(B13)}$$

which implies the sufficient condition

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial Q^2}\right) - \left(\frac{\partial^2 S}{\partial M\partial Q}\right)^2 \geq 0. \quad \text{(B14)}$$

The last case left consists of doing a simultaneous change in all the state variables of the system, i.e.,

$$S(M + \Delta M, A + \Delta A, Q + \Delta Q) + S(M - \Delta M, A - \Delta A, Q - \Delta Q) \leq 2S(M, A, Q). \quad \text{(B15)}$$

To investigate the sufficient differential condition that this inequality implies, one must first expand $S(M + \Delta M, A + \Delta A, Q + \Delta Q)$ and $S(M - \Delta M, A - \Delta A, Q - \Delta Q)$ in a Taylor series to second order in ΔM, ΔA and ΔQ, which can be shown to lead to

$$S_{MM}(\Delta M)^2 + S_{AA}(\Delta A)^2 + S_{QQ}(\Delta Q)^2 + 2S_{MA}\Delta M\Delta A + 2S_{MQ}\Delta M\Delta Q + 2S_{QA}\Delta A\Delta Q \leq 0. \quad \text{(B16)}$$

Multiplying the above relation by S_{MM}, noting that

$$(S_{MM}\Delta M + S_{MA}\Delta A + S_{MQ}\Delta Q)^2 = S_{MM}^2(\Delta M)^2 + S_{MA}^2(\Delta A)^2 + S_{MQ}^2(\Delta Q)^2 + 2S_{MM}S_{MA}\Delta M\Delta A + 2S_{MM}S_{MQ}\Delta M\Delta Q + 2S_{MA}S_{MQ}\Delta A\Delta Q, \quad \text{(B17)}$$

and inserting this into Eq. (B16), gives

$$(S_{MM}\Delta M + S_{MA}\Delta A + S_{MQ}\Delta Q)^2 + (S_{MM}S_{AA} - S_{MA}^2)(\Delta A)^2 + (S_{MM}S_{QQ} - S_{MQ}^2)(\Delta Q)^2 + 2(S_{MM}S_{QA} - S_{MA}S_{MQ})\Delta A\Delta Q \geq 0. \quad \text{(B18)}$$

Recalling Eq. (B10) and Eq. (B14), and noting that the first term in the above inequality is always positive, we conclude that the condition

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial Q\partial A}\right) - \left(\frac{\partial^2 S}{\partial M\partial A}\right)\left(\frac{\partial^2 S}{\partial M\partial Q}\right) \geq 0 \quad \text{(B19)}$$

is sufficient to satisfy Eq. (B15). This concludes the derivation of Eqs. (60), (62), (63) and (64). Thus all stability equations, Eqs. (58)–(64), have been derived.

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