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Light on curved backgrounds

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We consider the motion of light on different spacetime manifolds by calculating the deflection angle, lensing properties and by probing into the possibility of bound states. The metrics in which we examine the light motion include, among other items, a general relativistic dark matter metric, a dirty black hole, and a worm hole metric, the last two inspired by noncommutative geometry. The lensing in a holographic screen metric is discussed in detail. We study also the bending of light around naked singularities like, e.g., the Janis-Newman-Winicour metric and include other cases. A generic property of light behavior in these exotic metrics is pointed out. For the standard metric like the Schwarzschild and Schwarzschild-de Sitter cases, we improve the accuracy of the lensing results for the weak and strong regimes.

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I. INTRODUCTION

The year 2015 has been declared by the United Nations (UN) and UNESCO as the "International Year of Light" [1] which commemorates the achievements of light sciences. Simultaneously, the year 2015 is the centenary year of general relativity. Light bending is a genuine effect of general relativity (at least when we confront the theoretical prediction with observations) and was the first experimental confirmation of the newly discovered theory which, looking back, is quite an achievement. It seems therefore timely to revive the subject of light motion on curved backgrounds by including new interesting examples of recently emerged metrics and generalizing or improving/correcting existing analytical results.

To appreciate the development of the subject let us recall that the idea of light bending can be traced back to Newton's *Opticks* [2] which concludes with a number of queries. In query 1 [3] Newton mentions the possibility of light (rays) bending. However, according to [4] the first concrete calculation within the Newtonian framework was done by Henry Cavendish in 1784, but the result remained unpublished. Twenty years later Johann Georg von Soldner did a similar calculation [5] and taking into account that both of these calculations assume a different position of the light source (Cavendish light is emitted at infinity whereas Soldner's comes form a surface of a gravitating body), both results agree in the first order approximation [4]. Moreover, the result, $\Delta\phi_N = 2GM_\odot/R_\odot$ ($M_\odot$ and $R_\odot$ are the mass and radius of the sun), is half the value obtained from general relativity using the Schwarzschild metric. Einstein's first attempt to calculate the effect of gravity on light yielded the same value obtained by Cavendish and Soldner as he also used the Newtonian theory by invoking the energy-mass equivalence. Only in 1916 he obtained what is now considered the correct result (twice the Newtonian value) within the framework of general relativity. Using the Robertson expansion of the metric the deflection angle can be parametrized as $\Delta\phi_N = 4GM_\odot/r_0(1+\gamma)/2$ (r_0 is the closest approach). The most precise experiments on quasars using very long baseline radio interference gives $(1+\gamma)/2 = 0.99992 \pm 0.00023$ [6], an impressive result which excludes the Newtonian value $\gamma = 1$ and confirms general relativity. Of course, the above mentioned theoretical result in general relativity is only the very first approximation in the Schwarzschild metric. More accurate expansions are possible and will be presented in this paper. The result is valid up to $(2M/r_0)^5$ based on expansion of incomplete elliptic integrals of the first kind.

A more pronounced effect of light bending is gravitational lensing. The idea is usually attributed to Einstein [7], but had been already published 12 years earlier by Chwolson [8]. Since then the field has achieved a remarkable level, from the observational as well as from theoretical point of view [9] reaching a high mathematical sophistication [10]. In spite of the seniority of the subject, one can still find some niches where an improvement is possible. As far as the lensing in the Schwarzschild metric is concerned we offer some corrections of expressions existing in the literature and generalization of the formulas. More precisely, we have paid attention to novel perturbation techniques which led us to

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formulas of higher precision. This is to say we presented more accurate results as compared to results found so far in literature. In the case of the Schwarzschild metric we showed that the method used in [11] to derive the deflection formula in the strong regime is mathematically flawed because the regular part of the integral giving the deflection angle is represented by means of an infinite series of definite integrals [see equation (34) in the manuscript] where it can be checked that higher terms in the aforementioned series are all divergent. This contradicts the claim that such a series can be used to compute all coefficients in the expansion for the regular part of a certain integral appearing in the deflection formula. In view of this fact we decided to solve this problem by using a relatively unknown asymptotic formula for the incomplete elliptic integral of the first kind. This in turn allowed us to improve the accuracy of the computation of the deflection angles by introducing Lambert functions.

Another metric, closely related to the Schwarzschild metric, is the Kottler or Schwarzschild-de Sitter metric which includes a positive cosmological constant. The interest in this metric was revived after the accelerated stage of the expansion of the Universe was discovered and a positive cosmological constant could account for the observational data. A natural question arises: does the cosmological constant, given its "measured value" affect the properties of light deflection? This led to some controversy in the literature regarding the observability of the cosmological constant in the lensing. Considering that with the inclusion of the cosmological constant two different scales appear in the theory, it is a priori not excluded that an effect combines these two scales in a way that it becomes, in principle, measurable. This does not seem to happen for the cosmological constant at least as far as lensing is concerned. Our result contains the cosmological constant, but the effect is tiny. In deriving this result we made sure that our expressions reduce to the formulae encountered in the Schwarzschild metric when we put the cosmological constant to zero. Regarding the case of the Schwarzschild-de Sitter metric it is interesting to observe that equation (18) in [12] fails to reproduce correctly the term going together with $(2M/r_0)^2$ in the weak field limit of the Schwarzschild metric $\Lambda \to 0$, while our (55) matches the corresponding formula in the Schwarzschild case even at the order $(2M/r_0)^3$. Concerning strong gravitational lensing in a Schwarzschild-deSitter manifold we solved exactly the integral giving the deflection angle in terms of an incomplete elliptic integral of the first kind and then applied a recent asymptotic formula derived in [13] in the case when the sine of the modular angle and the elliptic modulus both tends to one. Interestingly enough this asymptotic formula seems to be used for the first time in the context of light deflection. The light deflection in the Schwarzschild- de Sitter metric continues to intrigue scientist working in this area [14,15] and we think that we presented here a novel approach.

All this shows that light bending is not a standard exercise and a considerable amount of work in all publications (many of them are recent) goes into the mathematically correct handling of the expressions involved.

The higher precision of theoretical expressions has so far proved fruitful in the development of science as a comparison with observation can lead to new physics. Such an impact on science is well known and we therefore will not quote examples but refer to a possible example below in the case of comparing lensing of a Schwarzschild black hole with the corresponding results in a naked singularity.

In the whole paper we probe into the light motion around naked singularities. One case studied in detail is the Janis-Newman-Winicour (JNW) metric. One interest is to get an insight of the motion of light in naked singularities, in general. Second, in [16] it has been suggested that the center of the galaxy could eventually be modeled by such a singularity. This means that there persists some interest in accurate expressions of light bending and lensing in such a case in order to be able to compare it with accurate expressions obtained for the Schwarzschild black hole. Therefore we focused here also on the improvement of accuracy. But we also embedded the JNW results in a wider context of other naked singularities. With a simple qualitative tool we can show that in many cases of naked singularities (apart from the JNW metric, we examine three other naked singularities) a narrow range of possible parameters leads to bound states of light. This result seems to be a novel footprint of naked singularities.

The relevance of lensing is, of course, manifold. But it is probably its connection to dark matter (DM) [17] which led the area to the avenues of new physics. Based on the nonrelativistic dark matter density of Navarro-Frenk-White, a general relativistic metric had been derived earlier which makes it possible to study the deflection of light in the galactic DM halos from scratch and in the context of general relativity. We offer here the first results of such a study for the weak lensing and leave the strong case for future considerations. Apart from obvious phenomenological impacts, the light bending in this metric is also of pure theoretical interest as the latter is generated by an anisotropic energy-momentum tensor, which by itself is a rare case.

Next we focus on a class of metrics which have been derived having in mind some models of quantum gravity. To be specific, we calculate the light deflection angle for two black hole (BH) metrics: the holographic screen and the dirty BH inspired by noncommutative geometry. The dirty BH contains in a limit the case of a noncommutative BH when the parameters are chosen accordingly. These metrics are based on the fact that noncommutativity, i.e., $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, leads to smeared objects in place of point-like particles [18]. On the other hand, the holographic screen metric has been obtained by "reversed engineering" demanding that the metric has no curvature singularity and "self-implements" the characteristic length scale. It is, of course, of some interest to see how light behaves in exotic metrics like the worm hole metric and mini BH metrics which include aspects of noncommutative geometry. The light bending on a worm hole metric is one of the first attempts to study light behavior in a spacetime where a "shortcut" is possible. Visualizing spacetime as a two-dimensional surface, a worm hole can be pictured as hole in this surface leading to a tube. This is known as the mouth of the worm hole and the mathematics behind this picture requires an embedding analysis which, as a supplement, we have performed for the worm hole and the holographic screen. A similar analysis is possible for the dirty black hole.

The choice of the metrics to study the light deflection covers a wide range: from the standard general relativistic metrics of BHs over a naked singularity and a DM galactic halo metric to metrics inspired by some aspects of quantum gravity out of which one is a worm hole. We hope to have added to the subject of light bending in gravitational fields some new insight by studying these examples.

II. BASIC FORMULAS

We consider a static and radially symmetric gravitational field represented by the line element ($c = G = 1$)

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2C(r)[d\vartheta^2 + \sin^2\vartheta d\varphi^2] \quad (1)$$

with $r \in (0, +\infty)$, $\vartheta \in [-\pi, \pi]$, and $\varphi \in [0, 2\pi)$. We further suppose that the functions A, B, and C are at least k times continuously differentiable on some interval $I \subseteq \mathbb{R}^+$. The corresponding relativistic Kepler problem is defined by the geodesic equation

$$\frac{d^2x^\kappa}{d\lambda^2} = -\Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

for the metric (1) and the relation

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{d\tau}{d\lambda}\right)^2 = \begin{cases} 1 & \text{if } m \neq 0; \\ 0 & \text{if } m = 0. \end{cases} \quad (2)$$

If we compute the Christoffel symbols we find that the geodesic equation gives rise to the following system of ordinary differential equations:

$$\frac{d^2x^0}{d\lambda^2} = -\frac{B'(r)}{B(r)}\frac{dx^0}{d\lambda}\frac{dr}{d\lambda}, \quad (3)$$

$$\frac{d^2r}{d\lambda^2} = -\frac{B'(r)}{2A(r)}\left(\frac{dx^0}{d\lambda}\right)^2 - \frac{A'(r)}{2A(r)}\left(\frac{dr}{d\lambda}\right)^2 + \frac{r^2C'(r) + 2rC(r)}{2A(r)}\left(\frac{d\vartheta}{d\lambda}\right)^2 + \frac{r^2C'(r) + 2rC(r)}{2A(r)}\sin^2\vartheta\left(\frac{d\varphi}{d\lambda}\right)^2, \quad (4)$$

$$\frac{d^2\vartheta}{d\lambda^2} = -\frac{rC'(r) + 2C(r)}{rC(r)}\frac{dr}{d\lambda}\frac{d\vartheta}{d\lambda} + \sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\lambda}\right)^2, \quad (5)$$

$$\frac{d^2\varphi}{d\lambda^2} = -\frac{rC'(r) + 2C(r)}{rC(r)}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} - 2\cot\vartheta\frac{d\vartheta}{d\lambda}\frac{d\varphi}{d\lambda}. \quad (6)$$

Equation (5) can be solved by imposing that $\vartheta = \pi/2$. There is no loss in generality in introducing this condition since at a certain point in time the coordinate system can be rotated in such a way that $\vartheta = \pi/2$ and $d\vartheta/d\lambda = 0$. Then, the coordinate and velocity vectors will belong to the equatorial plane $\vartheta = \pi/2$. This implies that $d^2\vartheta/d\lambda^2 = 0$ and hence $\vartheta(\lambda) = \pi/2$. As a consequence the whole trajectory will belong to the equatorial plane and Eq. (6) becomes

$$\frac{1}{r^2C(r)}\frac{d}{d\lambda}\left(r^2C(r)\frac{d\varphi}{d\lambda}\right) = 0. \quad (7)$$

The above equation can be immediately integrated and we obtain

$$r^2C(r)\frac{d\varphi}{d\lambda} = \ell = \text{const.} \quad (8)$$

We can also motivate the choice $\vartheta = \pi/2$ and Eq. (8) with the help of the isotropy of the problem at hand. This approach is usually adopted in the treatment of the non-relativistic Kepler problem where the angular momentum ℓ is conserved because of the isotropy of the problem. Since the direction of ℓ is constant, we can choose the coordinate system in such a way that $\mathbf{e}_z \| \ell$. This is equivalent to require that $\vartheta = \pi/2$. Since the magnitude of ℓ is constant, Eq. (8) will hold. Hence, the integration constant can be interpreted as the angular momentum per unit mass, i.e. $\ell = L/m$. Let us write (3) in the form

$$\frac{d}{d\lambda}\left[\ln\frac{dx^0}{d\lambda} + \ln B(r)\right] = 0. \quad (9)$$

Integrating the above equation we obtain

$$B(r)\frac{dx^0}{d\lambda} = F = \text{const.} \quad (10)$$

If we set $\vartheta = \pi/2$ and use (8) and (10) in (4), as a result we get the equation

$$\frac{d^2r}{d\lambda^2} + \frac{F^2B'(r)}{2A(r)B^2(r)} + \frac{A'(r)}{2A(r)}\left(\frac{dr}{d\lambda}\right)^2 - \frac{\ell^2[rC'(r) + 2C(r)]}{2r^3A(r)C^2(r)} = 0. \quad (11)$$

Multiplication by $2A(r)(dr/d\lambda)$ yields

$$\frac{d}{d\lambda}\left[A(r)\left(\frac{dr}{d\lambda}\right)^2+\frac{\ell^2}{r^2C(r)}-\frac{F^2}{B(r)}\right]=0. \tag{12}$$

One more integration finally gives

$$A(r)\left(\frac{dr}{d\lambda}\right)^2+\frac{\ell^2}{r^2C(r)}-\frac{F^2}{B(r)}=-\epsilon=\text{const}. \tag{13}$$

This radial equation can be seen as the most important equation of motion since the angular motion is completely specified by (8) and the condition $\vartheta=\pi/2$ whereas the connection between t and λ is fixed by (10). If we integrate (13) once more we obtain $r=r(\lambda)$ and if we substitute this function into (8) and (10), we get after integration $\varphi=\varphi(\lambda)$ and $t=t(\lambda)$. Elimination of the parameter λ yields $r=r(t)$ and $\varphi=\varphi(t)$. Together with $\vartheta=\pi/2$ they represent the full solution of the problem. The involved integrals cannot be in general solved in terms of elementary functions. To determine ϵ, we rewrite (2) as

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=B(r)\left(\frac{dx^0}{d\lambda}\right)^2-A(r)\left(\frac{dr}{d\lambda}\right)^2-r^2C(r)\left(\frac{d\vartheta}{d\lambda}\right)^2 - r^2\sin^2\vartheta C(r)\left(\frac{d\varphi}{d\lambda}\right)^2=\epsilon, \tag{14}$$

where in the last inequality we used the condition $\vartheta=\pi/2$ together with (8), (10) and (13). From (14) and (2) it follows that

$$\epsilon=\begin{cases}1 & \text{if } m\neq 0;\\ 0 & \text{if } m=0.\end{cases} \tag{15}$$

We want to determine the trajectory $\varphi=\varphi(r)$ in the equatorial plane $\vartheta=\pi/2$. First of all, we observe that (13) gives

$$\left(\frac{dr}{d\lambda}\right)^2=\frac{1}{A(r)}\left[\frac{F^2}{B(r)}-\frac{\ell^2}{r^2C(r)}-\epsilon\right]. \tag{16}$$

Taking into account that $d\varphi/d\lambda=(d\varphi/dr)(dr/d\lambda)$ and using (8) together with (16), we obtain

$$\left(\frac{d\varphi}{dr}\right)^2=\frac{A(r)B(r)}{r^4C(r)}\left[\frac{F^2}{\ell^2}C(r)-\frac{B(r)}{r^2}-\frac{\epsilon}{\ell^2}B(r)C(r)\right]^{-1} \tag{17}$$

and integration yields

$$\varphi(r)=\pm\int\frac{dr}{r^2}\sqrt{\frac{A(r)B(r)}{C(r)}}\left[\frac{F^2}{\ell^2}C(r)-\frac{B(r)}{r^2}-\frac{\epsilon}{\ell^2}B(r)C(r)\right]^{-1/2}+\tilde{C} \tag{18}$$

where $\tilde{C}$ is an arbitrary integration constant. The plus and minus sign must be chosen for particles approaching the gravitational object on the equatorial plane and having trajectories exhibiting an anticlockwise and clockwise direction, respectively. This integral determines the trajectory $\varphi=\varphi(r)$ in the plane where the motion takes place. In the case of a massive particle ($\epsilon=1$) the trajectory depends on two integration constants (F and ℓ). In the case of a scattering problem these constants can be expressed in terms of an impact parameter and the initial velocity $r'(r_0)$ where r_0 is the distance of closest approach to the gravitational object attained for some value λ_0 of the affine parameter. For massless particles ($\epsilon=0$) the trajectory depends only on the integration constant F/ℓ that can be interpreted as an impact parameter as follows [19]:

$$\frac{1}{b^2}=\frac{F^2}{\ell^2}=\frac{B(r_0)}{r_0^2C(r_0)}.$$

This in turn permits to express (18) as

$$\varphi(r)=\pm\int\frac{dr}{r}\sqrt{\frac{A(r)}{C(r)}}\left[\left(\frac{r}{r_0}\right)^2\frac{C(r)B(r_0)}{C(r_0)B(r)}-1\right]^{-1/2}+\tilde{C}.$$

It is useful to derive an inequality which gives us information for the qualitative orbits of the particle. It is straightforward to see that from (18) we obtain the condition

$$F^2>\frac{\ell^2B(r)}{r^2C(r)}+\epsilon B(r)\equiv V(r). \tag{19}$$

For massless particles we can write

$$\frac{F^2}{\ell^2}>\frac{B(r)}{r^2C(r)}\equiv\tilde{V}(r). \tag{20}$$

As a first application of Eq. (20), consider the Reissner-Nordström metric with

$$B(r)=1-\frac{1}{r_s}+\frac{r_Q^2}{r^2}$$

where r_s is the standard Schwarzschild radius and r_Q is proportional to the electric charge. We have a naked singularity if $r_s<2r_Q$. It is easy to show that $\tilde{V}$ has a local minimum and maximum if $\alpha\equiv r_s^2/r_Q^2>32/9$. Such a minimum in which the photons would be trapped has a physical significance only if it occurs at a value bigger than the horizon r_+ (the horizons are $r_\pm=1/2/(r_s\pm\sqrt{r_s^2-4r_Q^2})$ or in the case of a naked singularity. In Fig. 1 we demonstrate that this minimum is of relevance only if we have a naked singularity. Hence the

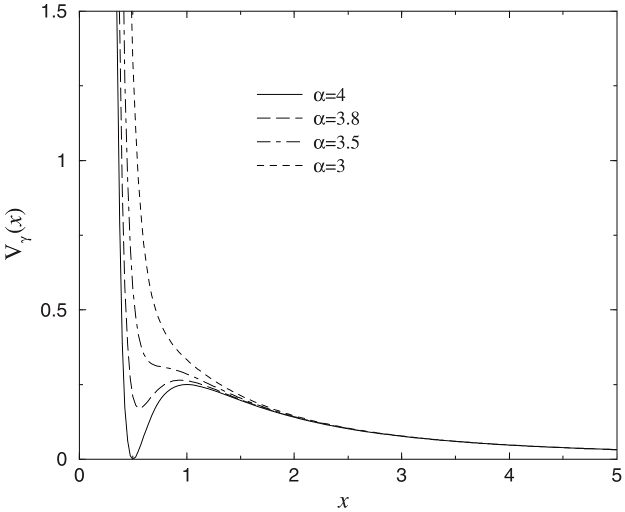

FIG. 1. Behavior of the "photon potential" proportional to $\tilde{V}$ for the Reissner-Nordström naked singularity. As explained in the text in the narrow range of α there is a possibility to trap photons inside a well.

range for such meaningful minimum is very narrow, namely

$$4 > \alpha > \frac{32}{9}.$$

If a particle in classical mechanics has a bound orbit (like in the minimum) we usually talk about an attractive potential. By the same token using similar nomenclature we could claim that the existence of photon's bound orbits around a naked singularity is an indication for its attractive nature for light in contrast to some other results for massive particles [20]. We will see later that this statement is generic for other naked singularities as well.

A. Lens equation for compact gravitational objects

We give a short derivation of the lens equation in the presence of compact gravitational objects, i.e. any astrophysical object whose size is comparable to the event horizon of a black hole. The corresponding lens equation may be also applied to black holes and other compact objects that did not fully undergo the gravitational collapse. Let S, L, O, and I denote the light source, the lens, the observer, and the image of the source seen by the observer, respectively. By OL we denote the optical axis. Furthermore, we introduce angles β and θ giving the position of S with respect to OL and the position of I as seen by O, respectively. In general, the closest approach distance r_0 does not need to be identified with the impact parameter b. By $\Delta\varphi$ we denote the deflection angle. Then, simple trigonometric arguments lead to the full lens equation [11,19,21,22]

$$\tan\beta = \tan\theta - \frac{D_{LS}}{D_{OS}}[\tan\theta + \tan(\Delta\varphi - \theta)], \tag{21}$$

where D_{LS} is the distance between L and S, and $D_{OS} = D_{OL} + D_{LS}$ with D_{OL} the distance between O and L. Given β and $\Delta\varphi$, (21) allows one to compute the positions θ of the images I of S seen by O. The magnification of an image for circularly symmetric gravitational lenses is

$$\mu = \left(\frac{\sin\beta}{\sin\theta}\frac{d\beta}{d\theta}\right)^{-1}, \tag{22}$$

where the sign of μ controls the parity of the image. Critical curves are singularities of μ in the lens plane and the corresponding values in the source plane are called caustics. By images of 0-parity we mean critical images. The tangential and radial magnifications are given by

$$\mu_t = \left(\frac{\sin\beta}{\sin\theta}\right)^{-1}, \qquad \mu_r = \left(\frac{d\beta}{d\theta}\right)^{-1}. \tag{23}$$

Tangential and radial critical curves are simply singularities of μ_t and μ_r, respectively. Their values on the source plane are called tangential and radial caustics. If β, θ, and $\Delta\varphi$ are small (weak gravitational field), the tangent functions appearing in (21) can be Taylor expanded and the deflection angle becomes $\Delta\varphi = 4M/r_0$ so that (21) can be solved producing two images whose separation from OL is quantified by the so-called Einstein angle

$$\theta_E = \sqrt{\frac{4MD_{LS}}{D_{OS}D_{OL}}}. \tag{24}$$

III. BENDING OF LIGHT IN THE SCHWARZSCHILD METRIC

Light rays experience a bending effect due to the presence of a gravitational field. We quantify this effect in the case of a lens represented by a Schwarzschild black hole with

$$B(r) = 1 - \frac{2M}{r},$$

where M denotes the mass of the black hole. We will also suppose that the source, lens, and observer lie along a straight line. Since the Schwarzschild spacetime is asymptotically flat we will assume that the source and observer are located in the flat spacetime region. Setting $\epsilon = 0$ in (18) and taking into account that $AB = 1$, we find that

$$\varphi(r) = \varphi(r_0) + \int_{r_0}^{\infty} \frac{dr}{r^2}\left[\frac{F^2}{\ell^2} - \frac{B(r)}{r^2}\right]^{-1/2}, \tag{25}$$

where the source has been placed in the asymptotically flat region and as a starting point for the integration the minimal distance r_0 of the light ray from the surface of the gravitational object has been chosen. Note that the choice of a plus sign in front of the above integral corresponds to the fact that we are considering light rays moving on trajectories having an anticlockwise direction. Without loss of generality we also require that $\varphi(r_0) = 0$. From $r = r_0$ to $r = \infty$ the angle φ changes by a quantity $\varphi(\infty)$. Along the photon trajectory the radial vector undergoes a rotation with angle $2\varphi(\infty)$. If the gravitational object would be absent, we would have a straight line for the photon trajectory implying that $2\varphi(\infty) = \pi$. Hence, the angle by which light is bent by a spherically symmetric gravitational field described by an asymptotically flat metric is given by [19,23]

$$\Delta\varphi = 2\varphi(\infty) - \pi, \qquad \varphi(r_0) = 0. \tag{26}$$

Since in general $A(r) \neq 1$ the underlying three-dimensional space is not Euclidean. For very large distances we have $A \to 1$ and $B \to 1$. This means that asymptotically far away from the gravitational object the light ray can be described as a straight line in the Euclidean space. The distance of closest approach is determined by the condition $r'(r_0) = 0$ with $r_0 = r(\lambda_0)$ and we find that

$$\frac{1}{b^2} = \frac{F^2}{\ell^2} = \frac{B(r_0)}{r_0^2}, \qquad b^2 = \frac{r_0^3}{r_0 - 2M}. \tag{27}$$

This allows us to eliminate the constant F^2/ℓ^2 in (25) so that we can rewrite (26) as

$$\frac{\Delta\varphi + \pi}{2} = \int_{r_0}^{\infty} \frac{dr}{r^2}\left[\frac{B(r_0)}{r_0^2} - \frac{B(r)}{r^2}\right]^{-1/2}. \tag{28}$$

Depending on the values of the impact parameter we have the following scenarios [19,23,24]:

(1) if $b < 3\sqrt{3}M$, the photon is doomed to be absorbed by the black hole;

(2) if $b > 3\sqrt{3}M$, the photon will be deflected and it can reach spatial infinity. Here, we must consider two further cases

   (a) if $b \gg 3\sqrt{3}M$, the orbit is almost a straight line and the deflection angle is approximately given by $4M/r_0$. Weak gravitational lensing deals with this case which corresponds to the situation when the distance of closest approach is much larger than the radius r_γ of the photon sphere. For spherically symmetric and static spacetimes r_γ can be computed by solving the equation [25]

$$\frac{B'(r)}{B(r)} = \frac{2}{r}. \tag{29}$$

   In the case of the Schwarzschild metric we obtain $r_\gamma = 3M$.

   (b) If $0 < b \ll 3\sqrt{3}M$, we are in the regime of strong gravitational lensing corresponding to a distance of closest approach $r_0 \approx r_\gamma$. In this case the photon can orbit several times around the black hole before it flies off.

Let $r_S = 2M$ denote the Schwarzschild radius. As in [11,25,26] we rescale the time and radial coordinates as $\sigma = t/r_S$ and $\rho = r/r_S$. Then, the Schwarzschild radius is at $\rho_S = 1$, the distance of closest approach will be given by $\rho_0 = r_0/r_S$, and the radius of the photon sphere is located at $\rho_1 = 3/2$. Clearly, we must require that $\rho_0 > \rho_1$. In terms of ρ (28) becomes

$$\Delta\varphi(\rho_0) = -\pi + 2\int_{\rho_0}^{\infty} \frac{d\rho}{\rho^2}\left[\frac{B(\rho_0)}{\rho_0^2} - \frac{B(\rho)}{\rho^2}\right]^{-1/2}. \tag{30}$$

Let $y = B(\rho)$ and $z = (y - y_0)/(1 - y_0)$ where $y_0 = B(\rho_0)$. We find that (30) can be written as

$$\Delta\varphi(\rho_0) = -\pi + 2\int_0^1 f(z,\rho_0)dz,$$
$$f(z,\rho_0) = \left[\left(2 - \frac{3}{\rho_0}\right)z + \left(\frac{3}{\rho_0} - 1\right)z^2 - \frac{z^3}{\rho_0}\right]^{-1/2}. \tag{31}$$

The function f has three singularities located at

$$z_1 = 0, \qquad z_2 = \frac{3 - \rho_0 - \sqrt{\rho_0^2 + 2\rho_0 - 3}}{2},$$
$$z_3 = \frac{3 - \rho_0 + \sqrt{\rho_0^2 + 2\rho_0 - 3}}{2}.$$

It is straightforward to verify that $z_2 < 0$, $z_3 > \rho_1 > 0$, and $z_2 < z_1 < z_3$. If we factorize the argument of the square root in the expression for f in terms of its roots and introduce the variable transformation $z = \tilde{y}^2 + z_2$ [27], we obtain

$$\Delta\varphi(\rho_0) = -\pi + 4\sqrt{\rho_0}\int_{\sqrt{-z_2}}^{\sqrt{1-z_2}} \frac{d\tilde{y}}{\sqrt{(\tilde{y}^2 + z_2)(z_3 - z_2 - \tilde{y}^2)}}.$$

Let $\tilde{y}^2 = z_2(1 - k^2\sin^2\phi)$ with $k = \sqrt{z_3/(z_3 - z_2)}$. Then,

$$\Delta\varphi(\rho_0) = -\pi + A(\rho_0)F(\phi_1, k), \qquad A(\rho_0) = \frac{4\sqrt{\rho_0}}{z_3 - z_2},$$
$$\phi_1 = \sin^{-1}\sqrt{\frac{z_3 - z_2}{z_3(1 - z_2)}}, \tag{32}$$

where F denotes the incomplete elliptic integral of the first kind. Using the expansion 902.00 in [28] for the incomplete elliptic integral of the first kind when $k \ll 1$, we find that the angle by which light is bent in a weak gravitational field ($\rho_0 \gg 1$) is given by

$$\Delta\varphi(\rho_0) = 2\rho_0^{-1} + \left(\frac{15}{16}\pi - 1\right)\rho_0^{-2} + \left(\frac{61}{12} - \frac{15}{16}\pi\right)\rho_0^{-3} + \left(\frac{3465}{1024}\pi - \frac{65}{8}\right)\rho_0^{-4} + \left(\frac{7783}{320} - \frac{3465}{512}\pi\right)\rho_0^{-5} + \mathcal{O}(\rho_0^{-6}). \tag{33}$$

Taking into account that $\rho_0 = r_0/(2M)$ the weak field approximation (33) reproduces correctly the first order term $4M/r_0$ derived in [19,23,24] and generalizes the weak field approximation derived in [21,29]. Moreover, it agrees with Eq. (23) in [30]. Concerning the strong deflection limit ($\rho_0 \to \rho_1$) we will first show that the method used by [25] is mathematically flawed and then we will use an asymptotic formula for the incomplete elliptic integral of the first kind derived in [13]. First of all, [25] starts by observing that the integrand f appearing in the integral giving the deflection angle diverges as $z \to 0$. The order of divergence of the integrand can be found by expanding the argument of the square root in f to the second order at $z = 0$, more precisely

$$f(z,\rho_0) \approx f_0(z,\rho) = \frac{1}{\sqrt{\alpha z + \beta z^2}}, \qquad \alpha = 2 - \frac{3}{\rho_0}, \qquad \beta = \frac{3}{\rho_0} - 1.$$

Hence, for $\alpha \neq 0$ the leading order of the divergence of f_0 is $z^{-1/2}$ which can be integrated while for $\alpha = 0$ the function f_0 diverges as z^{-1} thus leading to a logarithmic divergence. The authors of Ref. [25] split the integral in (31) as follows:

$$2\int_0^1 f(z,\rho_0)dz = I_D(\rho_0) + I_R(\rho_0),$$

where

$$I_D(\rho_0) = 2\int_0^1 f_0(z,\rho_0)dz$$

contains the divergence and

$$I_R(\rho_0) = 2\int_0^1 g_0(z,\rho_0)dz, \qquad g_0(z,\rho_0) = f(z,\rho_0) - f_0(z,\rho_0)$$

is the original integral with the divergence subtracted. At this point one solves the above integrals and the sum of their results will give the deflection angle. The integral I_D can be solved exactly and we get

$$I_D(\rho_0) = \frac{4}{\sqrt{\beta}}\ln\frac{\sqrt{\alpha+\beta}+\sqrt{\beta}}{\sqrt{\beta}} = -2\ln\left(\frac{\rho_0}{\rho_1} - 1\right) + 2\ln 2 + \mathcal{O}(\rho_0 - \rho_1)$$

after having expanded α and β around the radius of the photon sphere. To compute the residual integral I_R [25] employs the following expansion:

$$I_R(\rho_0) = \sum_{n=0}^{\infty}\frac{(\rho_0-\rho_1)^n}{n!}\int_0^1 \left.\frac{\partial^n g}{\partial\rho_0^n}\right|_{\rho_0=\rho_1} dz \tag{34}$$

and at the first order we find

$$I_R(\rho_0) = \int_0^1 g(z,\rho_1)dz + \mathcal{O}(\rho_0-\rho_1) = 2\ln 6(2-\sqrt{3}) + \mathcal{O}(\rho_0-\rho_1).$$

Reference [25] claims that (34) can be used to compute all coefficients in the expansion for the regular part of the integral I_R. This is not true since a closer inspection of the partial derivatives $\partial^n g(z,\rho_1)/\partial\rho_0^n$ shows that they have the following behaviors as $z \to 0$:

$$\left.\frac{\partial g}{\partial\rho_0}\right|_{\rho_0=\frac{3}{2}} = -\frac{4}{3z} - \frac{2}{9} + \mathcal{O}(z),$$

$$\left.\frac{\partial^2 g}{\partial\rho_0^2}\right|_{\rho_0=\frac{3}{2}} = \frac{40}{9z^2} - \frac{4}{27z} + \mathcal{O}(1),$$

$$\left.\frac{\partial^3 g}{\partial\rho_0^3}\right|_{\rho_0=\frac{3}{2}} = -\frac{560}{27z^3} + \mathcal{O}(z^{-2}).$$

Hence, the regular part of the integral giving the deflection angle cannot be represented by means of (34) because otherwise each coefficient with $n \geq 1$ in the expansion for I_R would blow up due to the fact that $\partial^n g(z,\rho_1)/\partial\rho_0^n$ is never integrable at $z = 0$ for all $n \geq 1$. We can overcome this problem by observing that $\sin\phi_1$ and k both approach one as $\rho_0 \to \rho_1$ since

$$\sin\phi_1(\rho_0) = 1 - \frac{2}{9}(\rho_0-\rho_1) + \mathcal{O}(\rho_0-\rho_1),$$

$$k(\rho_0) = 1 - \frac{4}{9}(\rho_0-\rho_1) + \mathcal{O}(\rho_0-\rho_1). \tag{35}$$

For $\sin\phi_1$ and k approaching one simultaneously, the following asymptotic formula for the incomplete elliptic integral of the first kind holds [13]:

$$F(\phi_1,k) = \frac{\sin\phi_1}{4}\left\{[6-(1+k^2)\sin^2\phi_1]\ln\frac{4}{\cos\phi_1+\Delta} - 2 + (1+k^2)\sin^2\phi_1 + \Delta\cos\phi_1\right\} + \theta F(\phi_1,k) \tag{36}$$

with $\Delta = \sqrt{1-k^2\sin^2\phi_1}$ and relative error bound

$$\frac{9\Delta^4 \ln \Delta}{64 \ln(\Delta/16)} < \theta < \frac{3}{8}\Delta^4.$$

Taking into account that

$$A(\rho_0) = 4 - \frac{8}{9}(\rho_0 - \rho_1) + \mathcal{O}(\rho_0 - \rho_1)$$

and expanding (36) around $\rho_0 = \rho_1$ we finally obtain

$$\Delta\varphi(\rho_0) = -\pi + \ln 144(7 - 4\sqrt{3}) - 2\ln\left(\frac{\rho_0}{\rho_1} - 1\right) + \frac{16}{9}(\rho_0 - \rho_1) + \mathcal{O}(\rho_0 - \rho_1)^2. \tag{37}$$

In order to express the deflection angle as a function of θ we must first rewrite ρ_0 in terms of the impact parameter. Taking into account that

$$\tilde{b} = \sqrt{\frac{\rho_0^3}{\rho_0 - 1}} = \tilde{b}_{cr} + \sqrt{3}(\rho_0 - \rho_1)^2 + \mathcal{O}(\rho_0 - \rho_1)^3, \quad \tilde{b}_{cr} = \frac{3\sqrt{3}}{2} \tag{38}$$

we obtain the approximated relation

$$\rho_0 - \rho_1 \approx \sqrt{\frac{\tilde{b} - \tilde{b}_{cr}}{\sqrt{3}}}. \tag{39}$$

Substituting $\tilde{b} = \theta\tilde{D}_{OL}$ with $\tilde{D}_{OL} = D_{OL}/r_S$ in (39) and replacing this relation in (37) we finally get

$$\Delta\varphi(\theta) = -\pi + \ln 216(7 - 4\sqrt{3}) - \ln\left(\frac{\theta\tilde{D}_{OL}}{\tilde{b}_{cr}} - 1\right) + \frac{16}{9\sqrt{3}}\sqrt{\theta\tilde{D}_{OL} - \tilde{b}_{cr}} + \mathcal{O}(\theta\tilde{D}_{OL} - \tilde{b}_{cr}). \tag{40}$$

Our formula (37) generalizes the Schwarzschild deflection angle in the strong field limit given by [11,24,25]. At this point a couple of remarks are in order. An expression for the exact deflection angle of photons in terms of elliptic integrals was first given in [24] [see Eq. (29) therein], an equivalent representation is offered by [11], whereas [31] derives the deflection angle by applying formula 3.131.(5) at page 254 in [32] to our (31). Furthermore, Eq. (9) in [11] expressing the modulus of the elliptic function is not correct and should read in the notation therein

$$\lambda = \sqrt{\frac{3 - x_0 - \sqrt{-3 + 2x_0 + x_0^2}}{3 - x_0 + \sqrt{-3 + 2x_0 + x_0^2}}}.$$

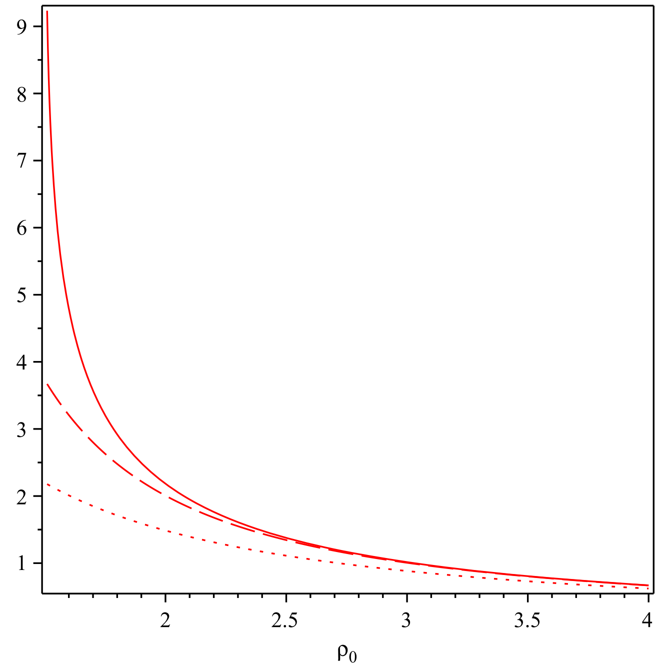

FIG. 2 (color online). Behavior of the exact solution (32) (solid line) versus the weak field approximations (33) (dashed line), and (24) in [29] with $q = 0$ (dotted line).

In Fig. 2 we compare the analytical expression (32) of the deflection angle with (33) and formula (24) with $q = 0$ in [29]. Figure 3 displays the exact solution (32) with the approximated solution (50) offered by [25] and our (37). Figure 4 shows that using our expansion up to $\rho_0 = 1.55$ the error we introduce is about 0.1% whereas the error committed by [11] is 1%. In the strong field approximation, when the light ray gets closer and closer to the photon sphere, $\Delta\varphi$ may become bigger than 2π. This implies that the light ray will wind around the lens one or more times before escaping the gravitational pull. In the extreme case $\tilde{b}_{cr} = 3\sqrt{3}/2$ corresponding to $\rho_0 = 3/2$, $\Delta\varphi$ diverges logarithmically and the photon is captured by the photon sphere. The approximated lens equation in this regime is given by [11]

$$\beta = \theta - \frac{\tilde{D}_{LS}}{\tilde{D}_{OS}}\Delta\varphi_n, \qquad \Delta\varphi_n = \Delta\varphi - 2n\pi, \qquad n \in \mathbb{N}. \tag{41}$$

Let θ_n^0 denote the values of θ such that $\Delta\varphi_n(\theta_n^0) = 0$, or equivalently $\Delta\varphi(\theta_n^0) = 2n\pi$. Using (40) this equation can be solved for θ_n^0 and we obtain

$$\theta_n^0 = \frac{1}{\tilde{D}_{OL}}\left[\frac{3\sqrt{3}}{2} + \frac{243}{64}W_n^2\right], \quad W_n = W\left(-16\sqrt{7 - 4\sqrt{3}}e^{-\pi(n+\frac{1}{2})}\right) \tag{42}$$

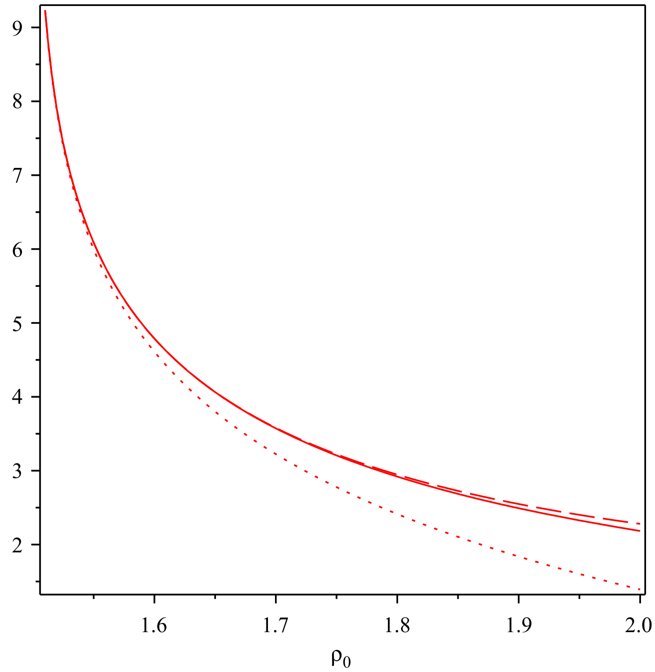

FIG. 3 (color online). Plot of the exact solution (32) (solid line), Bozza's approximation (dotted line), and the approximation (37) (dashed line).

where W_n denotes the Lambert function [33]. Our (42) generalizes formula (16) in [11]. In the limit $n \to \infty$, i.e. when the photon makes an infinite number of loops around the black hole, we correctly obtain $\theta^0_\infty = (3\sqrt{3})/(2\tilde{D}_{OL})$. If we expand $\Delta\varphi$ at the first order around θ^0_n, (40) gives

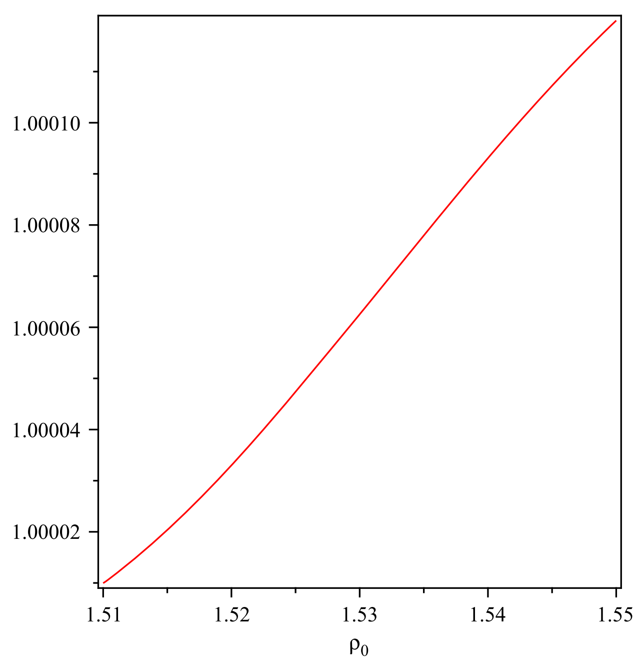

FIG. 4 (color online). Ratio of the exact deflection angle (32) and the approximate one (37) as a function of the closest approach distance.

$$\Delta\varphi_n = \Delta\varphi - \Delta\varphi(\theta^0_n) = \left.\frac{d\Delta\varphi}{d\theta}\right|_{\theta=\theta^0_n} \Delta\theta_n + \mathcal{O}(\Delta\theta^2_n)$$
$$= -\frac{64}{243}\frac{W_n+1}{W^2_n}\tilde{D}_{OL}\Delta\theta_n + \mathcal{O}(\Delta\theta^2_n), \tag{43}$$

where $\Delta\theta_n = \theta_n - \theta^0_n$. The relative error for θ^1_0 is $|\Delta\theta_1|/\theta^0_1$. Using (43) we find that

$$\frac{|\Delta\theta_1|}{\theta^0_1} = \left(\frac{243 W^2_1 \varphi_1}{64\tilde{D}_{OL}\theta^0_1(W_1+1)}\right)\frac{|\Delta\varphi_1|}{\varphi_1}.$$

In the case of the first image we can take $\varphi_1 = 2\pi$. Moreover, we also have $|\Delta\varphi_1|/\varphi_1 \approx 0.001$ for $\rho_0 = 1.55$. By means of (42) with $n = 1$ we find that the corresponding impact parameter is

$$\tilde{b}_1 = \theta^0_1\tilde{D}_{OL} = \frac{3\sqrt{3}}{2} + \frac{243}{64}W^2_1$$

and employing (39) we get that the closest approach distance for the first image is

$$\rho_{0,1} = \frac{3}{2} - \frac{9}{8}W_1 \approx 1.5339$$

instead of 1.545 as given in [11]. In Table I we listed the impact parameters, and the distances of closest approach for $n = 1, \cdots, 4$. Finally, we obtain for the relative error for θ^1_0

$$\frac{|\Delta\theta_1|}{\theta^0_1} = \frac{486\pi W^2_1}{(W_1+1)(96\sqrt{3}+243W^2_1)}\frac{|\Delta\varphi_1|}{\varphi_1} \approx 8.6\times 10^{-6}$$

instead of 8×10^{-5} as given in [11]. Furthermore, Eq. (43) can be replaced into the lens equation (41) to give

$$\beta = \theta^0_n + \left(1 + \frac{64}{243}\frac{\tilde{D}_{LS}\tilde{D}_{OL}}{\tilde{D}_{OS}}\frac{W_n+1}{W^2_n}\right)\Delta\theta_n, \tag{44}$$

which represents the position of the nth image. Since in general $\tilde{D}_{LS}\tilde{D}_{OL}/\tilde{D}_{OS} \gg 1$ and $(64/243)(W_n+1)W_n^{-2} \approx 280.82$, the second term in the bracket in (44) is much bigger than one and therefore, we can approximate (44) as follows:

TABLE I. Impact parameters, distances of closest approach, and positions of the relativistic rings for the black hole believed to be hosted by our galaxy ($M = 2.8\times 10^6 M_\odot$ and $D_{OL} = 8.5$ Kpc [34]).

n	$\tilde{b}_n$	$\rho_{0,n}$	θ^0_n μarcsec
1	2.601529850	1.533929564	16.91272357
2	2.598082299	1.501424471	16.89031076
3	2.598076223	1.500061482	16.89027125
4	2.598076211	1.500002656	16.89027118

$$\beta = \theta_n^0 + \frac{64}{243}\frac{\tilde{D}_{LS}\tilde{D}_{OL}}{\tilde{D}_{OS}}\frac{W_n+1}{W_n^2}\Delta\theta_n. \tag{45}$$

Finally, from (45) we obtain that the position of the nth image is

$$\theta_n = \theta_n^0 + \frac{243}{64}\frac{(\beta-\theta_0^n)\tilde{D}_{OS}}{\tilde{D}_{LS}\tilde{D}_{OL}}\frac{W_n^2}{W_n+1}. \tag{46}$$

From (44) we have $d\beta/d\theta > 0$. This implies that $\mu_r > 0$ in (23) and therefore there are no radial critical curves. On the other hand, μ_t becomes singular when $\beta = 0$. Hence, the only critical curves are of tangential nature and since we already solved the lens equation, it is sufficient to set $\beta = 0$ in (46) to obtain

$$\theta_{n,cr} = \left(1 - \frac{243}{64}\frac{\tilde{D}_{OS}}{\tilde{D}_{LS}\tilde{D}_{OL}}\frac{W_n^2}{W_n+1}\right)\theta_n^0.$$

The magnification of the nth image is given by the formula [11]

$$\mu_n = \frac{\theta_0^n}{\beta\frac{d\beta}{d\theta}|_{\theta=\theta_0^n}}.$$

Since

$$\left.\frac{d\beta}{d\theta}\right|_{\theta=\theta_0^n} = 1 + \frac{64}{243}\frac{\tilde{D}_{LS}\tilde{D}_{OL}}{\tilde{D}_{OS}}\frac{W_n+1}{W_n^2},$$

where the second term is much grater than one, the magnification of the nth image is

$$\mu_n = \frac{\tilde{D}_{OS}B_n}{\beta\tilde{D}_{OL}^2\tilde{D}_{LS}}, \qquad B_n = \frac{243}{4096}\frac{W_n^2(96\sqrt{3}+243W_n^2)}{W_n+1}. \tag{47}$$

and it decreases very quickly because $B_1 = 9.2\times10^{-3}$, $B_2 = 1.5\times10^{-5}$, $B_3 = 2.9\times10^{-8}$. Hence, the luminosity of the first image will dominate over all others. Finally the total magnification is [11]

$$\mu_{\rm tot} = 2\sum_{n=1}^{\infty}\mu_n = \frac{\tilde{D}_{OS}}{\beta\tilde{D}_{OL}^2\tilde{D}_{LS}}\sum_{n=1}^{\infty}B_n. \tag{48}$$

Since B_4 is of the order 10^{-11}, the series above is rapidly convergent and a good approximation for the total magnification is given by

$$\mu_{\rm tot} = \frac{\tilde{D}_{OS}B}{\beta\tilde{D}_{OL}^2\tilde{D}_{LS}}, \qquad B = B_1 + B_2 + B_3.$$

Our formulas (47) and (48) generalize Eqs. (24) and (27) in [11]. The amplification of each of the weak field images is [11]

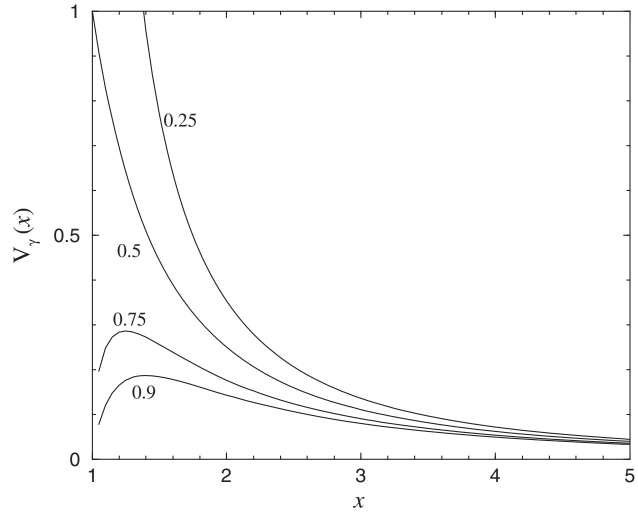

FIG. 5. Behavior of the rescaled "photon potential" proportional $\tilde{V}$ for the JNW metric (naked singularity) for different values of the parameter γ. As long as $\gamma < 1/2$ the metric displays a local maximum at the rescaled position bigger than 1. Since the nonremovable singular point at $x = 1$ cannot be crossed, light can be trapped, in principle, between the singular point and the hump.

$$\mu_{\rm wfi} = \frac{1}{\beta}\sqrt{\frac{2\tilde{D}_{LS}}{\tilde{D}_{OL}\tilde{D}_{OS}}}.$$

Then,

$$\frac{\mu_{\rm wfi}}{\mu_{\rm tot}} = \frac{\sqrt{2}}{B}\left(\frac{\tilde{D}_{LS}\tilde{D}_{OL}}{\tilde{D}_{OS}}\right)^{3/2}$$

with $B \approx 0.0092$ instead of 0.017 as in [11]. This implies that relativistic images are extremely faint in comparison to

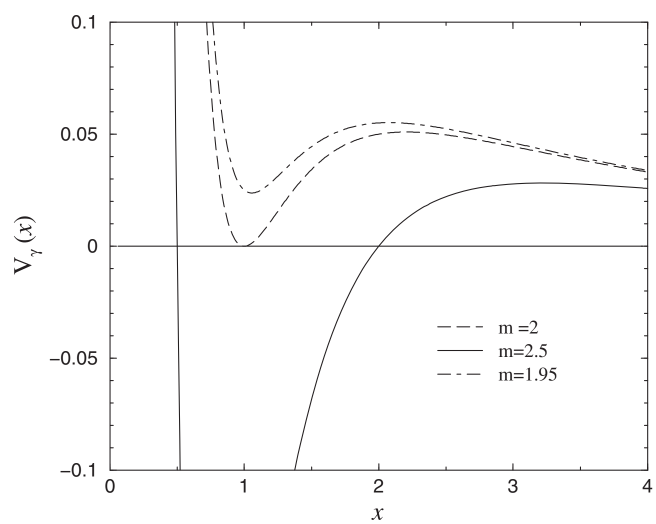

FIG. 6. The rescaled "photon potential" proportional to $\tilde{V}$ for the holographic screen metric. In case of a naked singularity, photons can be trapped in the local minimum.

the weak field images. Moreover, we also find that the observables θ^0_∞ and $s = \theta^0_1 - \theta^0_\infty$ whose numerical values are given in Table I in [25] should read 16.89 μarcsec and 0.02245 μarcsec instead of 16.87 μarcsec and 0.0211 μarcsec in the case of a lens represented by a Schwarzschild black hole with mass $M = 2.8 \times 10^6 M_\odot$ and $D_{OL} = 8.5$ Kpc. Last but not least, the sequence of impact parameters can be computed according to the formula

$$\tilde{b}_n = \tilde{b}_{cr}\left(1 + \frac{27\sqrt{3}}{32} W_n^2\right).$$

The angular size θ^0_n of the relativistic rings generated by photons deflected by $2\pi, 4\pi, 6\pi$, etc. will be given by $\theta^0_n = \tilde{b}_n/\tilde{D}_{OL}$. Note that the above formula generalizes Eq. (15) in [31]. Employing (39) we can compute the corresponding distances of closest approach. In Table I we give the first five values of the impact parameters, the corresponding distances of closest approach, and the size of the relativistic rings. Finally, from (24) we find that the position of the main ring is $\theta_E = 1.157987$ arcsec.

IV. BENDING OF LIGHT IN THE SCHWARZSCHILD-DE SITTER METRIC

The Schwarzschild-de Sitter metric describes a static black hole of mass M in a universe with positive cosmological constant Λ. The corresponding line element is given by (1) with

$$B(r) = 1 - \frac{r_S}{r} - \frac{r^2}{r_\Lambda^2}, \qquad A(r) = \frac{1}{B(r)}, \qquad C(r) = 1,$$

$$r_S = 2M, \qquad r_\Lambda = \sqrt{\frac{3}{\Lambda}}.$$

The numerical value of the cosmological constant is extremely small and can be taken to be $\Lambda \approx 10^{-56}$ cm^{-2} according to [35] even though astrophysical tests such as the perihelion precession of planets in our solar system and other tests based on the large scale geometry of the Universe suggest an upper bound for the cosmological constant given by $\Lambda \lesssim 10^{-42}$ m^{-2} [36–39]. With the inclusion of Λ gravity becomes essentially a two scale theory. It might appear that whereas Λ governs only the cosmological aspects, the only constant entering local gravity effects is the Newtonian constant. However, in certain circumstances a combination of the two scales also appears. For instance, for massive particles the effective potential (19) for the Schwarzschild-de Sitter metric develops a local maximum of the astrophysical order of magnitude $(r_S r_\Lambda)^{1/3}$ signifying the largest radius of a bound state [40]. The valid question is if lensing in the Schwarzschild-de Sitter metric gives us also some surprises.

Following [26] we rescale the time and radial coordinates as $\tilde{t} = t/r_S$ and $\rho = r/r_S$. Then, the metric function B can be rewritten in terms of only one dimensionless parameter y as

$$B(\rho) = 1 - \frac{1}{\rho} - y\rho^2, \qquad y = \frac{r_S^2}{r_\Lambda^2} = \frac{4}{3} M^2 \Lambda.$$

Note that the relation between our parameter y and the corresponding one in [26] reads $y_s = y/4$. Instead of a single event horizon as in the Schwarzschild metric, there are different possibilities for the Schwarzschild-de Sitter metric as follows:

(1) Two distinct horizons for $0 < y < 4/27$ located at

$$\rho_h = \frac{2}{\sqrt{3y}} \cos\frac{\pi + \psi}{3}, \qquad \rho_c = \frac{2}{\sqrt{3y}} \cos\frac{\pi - \psi}{3},$$
$$\psi = \cos^{-1}\left(\frac{3}{2}\sqrt{3y}\right),$$

where ρ_h and ρ_c denote the event and cosmological horizon, respectively. Note that the condition $0 < y < 4/27$ ensures that $\cos\psi \in (0,1)$. Moreover, we also have a negative root located at $\rho_- = -(2/\sqrt{3y})\cos(\psi/3)$. Finally, if we expand the horizons with respect to the parameter y as

$$\rho_h = 1 + y + \mathcal{O}(y^2),$$
$$\rho_c = \frac{1}{\sqrt{y}} - \frac{1}{2} - \frac{3}{8}\sqrt{y} - \frac{y}{2} - \frac{105}{128} y^{3/2} + \mathcal{O}(y^2),$$

we can verify that the formulas for the event and cosmological horizons predict correctly that $\rho_h \to 1$ and $\rho_c \to +\infty$ for $y \to 0^+$ as it should be in the Schwarzschild case.

(2) If $y > 4/27$, there is only one real root of the equation $B(\rho) = 0$ and the space-time describes a naked singularity located at

$$\rho_n = -\frac{2}{\sqrt{3y}} \cosh\frac{\psi}{3}, \qquad \psi = \cosh^{-1}\left(\frac{3}{2}\sqrt{3y}\right).$$

(3) In the case $y = 4/27$ the event and cosmological horizons coincide at $\rho_h = 3/2 = \rho_c$ and there is also a negative root at $\tilde{\rho} = -3$.

We will analyze gravitational lensing for the case $0 < y < 4/27$. By means of Eq. (3) in [25] the photon sphere is found to be again at $\rho_\gamma = 3/2$ as it is the case for the Schwarzschild metric. According to [26] the critical parameter of the photon circular orbit depends on y and is given by

$$\tilde{b}_c(y) = \frac{3\sqrt{3}}{\sqrt{1 - \frac{27}{4} y}}.$$

Note that expanding $\tilde{b}_c$ around y we have $\tilde{b}_c(y) = 3\sqrt{3} + (81\sqrt{3}/8)y + \mathcal{O}(y^2)$ and in the limit $y \to 0$ it reproduces correctly the critical values of the impact parameter in the Schwarzschild case that distinguishes photons which fall into the black hole from those escaping at infinity. Since the Schwarzschild-de Sitter manifold is spherically symmetric, there is no loss in generality if we suppose that the positions of the light source and that of the observer belong to the equatorial plane. Using formula (17) yields

$$\frac{d\varphi}{d\rho} = \pm\frac{1}{\rho^2}\frac{1}{\sqrt{\frac{1}{\tilde{b}^2} - \frac{1}{\rho^2}(1 - \frac{1}{\rho} - y\rho^2)}}$$

and with the help of the transformation $u = 1/\rho$, we find that the photon motion will be governed by the equation

$$\frac{d\varphi}{du} = \mp\frac{1}{\sqrt{u^3 - u^2 + a}}, \qquad a = y + \frac{1}{\tilde{b}^2}. \tag{49}$$

Note that the above differential equation agrees with (10) in [15] and it clearly contains the cosmological constant. The dependence on Λ can be removed if we use (49) to derive a second order nonlinear differential equation for $u = u(\varphi)$ as in [41]. Furthermore, photons with $\tilde{b} < \tilde{b}_c$ are doomed to crash into the central singularity, while those characterized by $\tilde{b} > \tilde{b}_c$ will be able to escape the gravitational pull of the black hole and they will eventually reach the cosmological horizon. We are interested in the latter case. If we look back at the term under the square root in (49), we realize that we need to introduce a motion reality condition represented by $u^3 - u^2 + a \geq 0$. Turning points will be represented by the roots of the associated cubic equation. To study the existence of these points it is more efficient to switch back to the radial variable ρ and consider the cubic equation $a\rho^3 - \rho + 1 = 0$ which is in principle the same equation we would obtain by setting $B(\rho) = 0$ with the parameter y replaced by a. Since we already analyzed the latter equation, we can immediately conclude that we have the following three cases:

(1) Two turning points for $0 < a < 4/27$ at

$$\rho_1 = \frac{2}{\sqrt{3a}}\cos\frac{\pi + \beta}{3}, \qquad \rho_2 = \frac{2}{\sqrt{3a}}\cos\frac{\pi - \beta}{3},$$
$$\beta = \cos^{-1}\left(\frac{3}{2}\sqrt{3a}\right)$$

such that $\rho_h < \rho_1 < \rho_\gamma < \rho_2 < \rho_c$. Moreover, the cubic $a\rho^3 - \rho + 1$ will be negative on the interval (ρ_γ, ρ_1) and positive on (ρ_2, ρ_c).

(2) If $a > 4/27$, there is only one turning point at

$$\rho_t = -\frac{2}{\sqrt{3a}}\cosh\frac{\beta}{3}, \qquad \beta = \cosh^{-1}\left(\frac{3}{2}\sqrt{3a}\right)$$

and since $\rho_t < 0$ the cubic will be positive on the interval (ρ_h, ρ_c).

(3) In the case $a = 4/27$ the turning points found in 1 coalesce into a single turning point at $\rho_t = \rho_\gamma = 3/2$ and the cubic will be positive on the interval (ρ_γ, ρ_c).

It is interesting to observe that the conditions $\tilde{b} > \tilde{b}_c$ and $0 < y < 4/27$ rule out the possibility that $a \geq 4/27$ as it can be seen from the following simple estimate:

$$a = y + \frac{1}{\tilde{b}^2} < y + \frac{1}{\tilde{b}_c^2} = \frac{3}{4}y + \frac{1}{27} < \frac{4}{27}.$$

Hence, we must have $0 < y < a < 4/27$. Let ρ_0 denote the distance of closest approach. According to the above discussion we must take ρ_0 in the interval (ρ_2, ρ_c). By ρ_b we will denote the position of the observer. Then, from (17) we find that

$$\varphi(\rho_0) = \int_{\rho_0}^{\rho_b}\frac{d\rho}{\rho}\sqrt{A(\rho)}\left[\left(\frac{\rho}{\rho_0}\right)^2\frac{B(\rho_0)}{B(\rho)} - 1\right]^{-1/2}. \tag{50}$$

The main difference with the Schwarzschild case is that the observer cannot be positioned asymptotically in a region where we can assume that the space-time is described by the Minkowski metric. For this reason we will suppose that the deflection angle is given by the formula

$$\Delta\varphi(\rho_0) = \kappa_1 I(\rho_0) + \kappa_2 \tag{51}$$

where $I(\rho_0)$ denotes the integral in (50) and κ_1 and κ_2 are two scalars to be determined in such a way that the weak field approximation of (51) reproduces the weak field approximation for the Schwarzschild case in the limit $y \to 0$. Let $x = \rho/\rho_0$. Then, we have

$$I(\rho_0) = \int_1^{x_b}\frac{dx}{x\sqrt{x^2 B(\rho_0) - B(\rho_0 x)}}.$$

Let $\alpha = 1/\rho_0$. The term under the square root in the above expression becomes

$$x^2 B(\rho_0) - B(\rho_0 x) = (1 - \alpha)x^2 + \frac{\alpha}{x} - 1$$

and it does not depend on the cosmological constant Λ. This is not surprising since it is well known that Λ has an influence over the orbits of massive particles but it can be made to disappear from the coordinate orbital equation when photons are considered [41]. At this point the integral I will depend only on the parameter α and will be given by

$$I(\alpha) = \int_1^{x_b}\frac{dx}{x}\left[(1 - \alpha)x^2 + \frac{\alpha}{x} - 1\right]^{-1/2}.$$

Letting $\rho_0 \gg 1$ corresponding to the condition $\alpha \ll 1$ we can expand the integral I in powers of the small parameter α according to

$$I(\alpha)=I(0)+I'(0)\alpha+\frac{I''(0)}{2}\alpha^2+\frac{I'''(0)}{3!}\alpha^3+\mathcal{O}(\alpha^4), \tag{52}$$

where the prime denotes differentiation with respect to α. Note that we can differentiate under the integral since the integrand is continuous on the interval $(1,x_b)$. The coefficients in the above expansion have been computed with the software Maple 14 and they are given by the following formulas:

$$I(0)=\frac{\pi}{2}-f_1(x_b),\qquad f_1(x_b)=\arctan\left(\frac{1}{\sqrt{x_b^2-1}}\right),$$

$$I'(0)=\frac{2x_b^2-x_b-1}{2x_b\sqrt{x_b^2-1}},$$

$$I''(0)=\frac{15x_b^2(x_b+1)^2[\pi-2f_1(x_b)]-2\sqrt{x_b^2-1}f_2(x_b)}{16x_b^2(x_b+1)^2},$$

$$f_2(x_b)=8x_b^3+7x_b^2-6x_b-3,$$

$$I'''(0)=-\frac{45x_b^3(x_b+1)^3[\pi-2f_1(x_b)]-2\sqrt{x_b^2-1}f_3(x_b)}{16x_b^3(x_b+1)^3},$$

$$f_3(x_b)=122x_b^5+306x_b^4+247x_b^3+70x_b^2+15x_b+5.$$

Moreover, for $x_b\gg 1$ the above quantities can be expanded as follows:

$$I(0)=\frac{\pi}{2}-\frac{1}{x_b}+\mathcal{O}(x_b^{-2}),\qquad I'(0)=1-\frac{1}{2x_b}+\mathcal{O}(x_b^{-2}), \tag{53}$$

$$I''(0)=\frac{15}{16}\pi-1-\frac{3}{4x_b}+\mathcal{O}(x_b^{-2}),$$
$$I'''(0)=\frac{61}{4}-\frac{45}{16}\pi-\frac{15}{8x_b}+\mathcal{O}(x_b^{-2}). \tag{54}$$

Hence, in the limit $x_b\to\infty$ corresponding to $y\to 0$ we find that the deflection angle is given by

$$\Delta\varphi(\rho_0)=\frac{\pi}{2}\kappa_1+\kappa_2+\kappa_1\rho_0^{-1}+\frac{\kappa_1}{2}\left(\frac{15}{16}\pi-1\right)\rho_0^{-2}+\frac{\kappa_1}{6}\left(\frac{61}{4}-\frac{45}{16}\pi\right)\rho_0^{-3}+\mathcal{O}(\rho_0^{-4}).$$

Comparison of the above expression with (33) gives $\kappa_1=2$ and $\kappa_2=-\pi$. Letting x_b approach the cosmological horizon at x_c and employing (53) and (54) it is not difficult to verify that the deflection angle can be written at the third order in the parameter α and at the first order in $1/x_c$ as

$$\Delta\varphi(\alpha)=-\frac{2}{x_c}+\mathcal{O}\left(\frac{1}{x_c^2}\right)+\left[2-\frac{1}{x_c}+\mathcal{O}\left(\frac{1}{x_c^2}\right)\right]\alpha+\left[\frac{15}{16}\pi-1-\frac{3}{4x_c}+\mathcal{O}\left(\frac{1}{x_c^2}\right)\right]\alpha^2+\left[\frac{1}{3}\left(\frac{61}{4}-\frac{45}{16}\pi\right)-\frac{5}{8x_c}+\mathcal{O}\left(\frac{1}{x_c^2}\right)\right]\alpha^3+\mathcal{O}(\alpha^4).$$

Finally, expanding the cosmological horizon in powers of Λ we find that the deflection angle as a function of the distance of closest approach for an observer located asymptotically at the cosmological horizon can be approximated by the following formula:

$$\Delta\varphi(r_0)=-\frac{2}{\sqrt{3}}r_0\sqrt{\Lambda}+\left(2-\frac{r_0\sqrt{\Lambda}}{\sqrt{3}}\right)\frac{2M}{r_0}+\left[\left(\frac{15}{16}\pi-1\right)-\frac{\sqrt{3}}{4}r_0\sqrt{\Lambda}\right]\left(\frac{2M}{r_0}\right)^2+\left[\frac{1}{3}\left(\frac{61}{4}\pi-\frac{45}{16}\pi\right)-\frac{5}{8\sqrt{3}}r_0\sqrt{\Lambda}\right]\left(\frac{2M}{r_0}\right)^3+\cdots \tag{55}$$

Hence, we agree with [42] that weak gravitational lensing in the Schwarzschild-de Sitter metric will depend on the cosmological constant Λ. It is interesting to observe that Eq. (18) in [12] in the limit $\Lambda\to 0$ fails to reproduce correctly the term going together with $(2M/r_0)^2$ in the weak field limit of the Schwarzschild metric, while our (55) matches the corresponding formula in the Schwarzschild case even at the order $(2M/r_0)^3$. Reference [43] considered the light orbital equation with the source located at (r_s,φ_s) while the observer is positioned at (r_b,φ_b) with $\varphi_b=0$ and derived an expression for φ_s in powers of $1/b$, $1/r_s$, and $1/r_b$. Concerning strong gravitational lensing in a Schwarzschild-de Sitter manifold, we will first solve exactly the integral in (52) in terms of an incomplete elliptic integral of the first kind and then apply an asymptotic formula derived by [13] in the case when the sine of the modular angle and the elliptic modulus both tends to one. To this purpose we write the deflection angle as $\Delta\varphi(\alpha)=2I(\alpha)-\pi$ where the integral $I(\alpha)$ is given by (52) and introduce the coordinate transformation $u=1/x$ so that we obtain

$$\Delta\varphi(\alpha)=2\int_{1/x_b}^{1}\frac{du}{\sqrt{\alpha u^3-u^2+1-\alpha}}-\pi.$$

The cubic polynomial under the square root in the above expression has zeroes at

$$u_0=1,\qquad u_1=\frac{1-\alpha-\sqrt{1+2\alpha-3\alpha^2}}{2\alpha},$$
$$u_2=\frac{1-\alpha+\sqrt{1+2\alpha-3\alpha^2}}{2\alpha}.$$

Since $\rho_0 > \rho_2 > \rho_\gamma$, then $\alpha < 2/3$ and we can order the roots as $u_2 > u_0 > 0 > u_1$. Using formula 3.131.4 in [32] we can express the deflection angle in terms of an incomplete elliptic integral of the first kind as follows:

$$\Delta\varphi(\alpha) = -\pi + \frac{4F(\phi_1, \kappa)}{\sqrt{\alpha(u_2 - u_1)}},$$
$$\phi_1 = \sin^{-1}\sqrt{\frac{(u_2 - u_1)(1 - 1/x_b)}{(1 - u_1)(u_2 - 1/x_b)}}, \qquad \kappa = \sqrt{\frac{1 - u_1}{u_2 - u_1}}.$$

Note that κ can be expanded around the photon sphere as in (35) while the modular angle admits the Taylor expansion

$$\sin\phi_1 = 1 - \frac{4}{9}\frac{3 + \rho_b}{2\rho_b - 3}(\rho_0 - \rho_\gamma) + \mathcal{O}(\rho_0 - \rho_\gamma),$$

which agrees with the corresponding expansion in (35) when $\rho_b \to \rho_c$ and $\Lambda \to 0$. Since both κ and $\sin\phi_1$ tends to 1 as $\rho_0 \to \rho_\gamma$, we can apply the asymptotic expansion (36) for the incomplete elliptic integral of the first kind developed by [13] and the deflection angle can be approximated as

$$\Delta\varphi(\rho_0) = -\pi + \ln\frac{144(2\rho_b - 3)}{[3 + 4\rho_b + 2\sqrt{3\rho_b(3 + \rho_b)}]^2} - 2\ln\left(\frac{\rho_0}{\rho_\gamma} - 1\right) + \mathcal{O}(\rho_0 - \rho_\gamma).$$

The above formula reduces correctly to the corresponding one derived in the previous section for the Schwarzschild metric when $\rho_b \to \rho_c$ and $\Lambda \to 0$. Finally, if $\rho_b \to \rho_c$ in the above expression and we make an expansion around $\Lambda = 0$ we obtain

$$\Delta\varphi(r_0) = -\pi + \ln 144(7 - 4\sqrt{3}) - \frac{2}{3}(9 - 2\sqrt{3})M\sqrt{\Lambda} + \mathcal{O}(M^2\Lambda) - 2\ln\left(\frac{r_0}{r_\gamma} - 1\right) + \mathcal{O}(r_0 - r_\gamma).$$

V. LIGHT DEFLECTION IN THE JANIS-NEWMAN-WINICOUR METRIC

The Janis-Newman-Winicour metric is the most general spherically symmetric static and asymptotically flat solution of Einstein's field equations coupled to a massless scalar field and is given by [44]

$$ds^2 = \left(1 - \frac{\mu}{r}\right)^\gamma dt^2 - \left(1 - \frac{\mu}{r}\right)^{-\gamma} dr^2 - r^2\left(1 - \frac{\mu}{r}\right)^{1-\gamma}(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{56}$$

with $\gamma = M/\sqrt{M^2 + q^2}$ and $\mu = 2\sqrt{M^2 + q^2}$ where M is the total mass and q is the strength of the scalar field also called the "scalar charge." Note that $\gamma \le 1$. For $\gamma = 1$ the JNW metric reduces to the Schwarzschild solution and the scalar field vanishes. The metric has been rediscovered by Wyman [45] and his solution was shown to be equivalent to the JNW metric in [46]. The metric is not only interesting from the point of view of an example of a naked singularity but also has served as a model for the supermassive galactic center in [16].

In order to study gravitational lensing it is convenient to rescale the time and radial coordinates as $\tilde{t} = t/\mu$ and $\rho = r/\mu$. Then, the above metric can be cast into the form

$$d\tilde{s}^2 = \frac{ds^2}{\mu^2} = \left(1 - \frac{1}{\rho}\right)^\gamma d\tilde{t}^2 - \left(1 - \frac{1}{\rho}\right)^{-\gamma} d\rho^2 - \rho^2\left(1 - \frac{1}{\rho}\right)^{1-\gamma}(d\vartheta^2 + \sin^2\vartheta d\varphi^2),$$
$$\gamma = \frac{r_S}{\mu}.$$

References [29] and [47] constructed weak field approximations of the deflection angle up to the second order. We point out that formula (51) in [47] reproduces correctly the first order term of the weak field limit of the Schwarzschild metric but the second order term fails to do so when $\nu \to 0$ in the aforementioned formula. Before going into the details of light bending in this metric, let us have a closer look at the nature of the naked singularity of this gravitational background for $\rho = 1$. The existence of the latter is indicated by the Kretschmann invariant given by

$$K = R^{abcd}R_{abcd} = \frac{\mu^2(r - \mu)^{2\gamma}}{4r^{2\gamma+4}(r - \mu)^4} f_\gamma(r),$$
$$f_\gamma(r) = 48\gamma^2 r^2 - 16\mu\gamma(\gamma + 1)(2\gamma + 1)r + \mu^2(\gamma + 1)^2(7\gamma^2 + 2\gamma + 3).$$

At the same time it is clear that for certain choices of γ the coordinate system used to write the line element (56) does not represent the full maximal atlas. Indeed, it is possible to find a coordinate system where the metric elements are nonsingular and the naked singularity manifests itself through the noninvertibility of the metric. As an example let us choose $\gamma = 1/3$. The null radial geodesic gives rise to the definition of the tortoise coordinate r_* via the first order differential equation

$$\frac{dr_*}{dr} = \frac{1}{\sqrt[3]{1 - \frac{\mu}{r}}}.$$

Using the integral representation of the hypergeometric function (see 15.3.1 in [48]) one obtains

$$r_* = -\frac{3}{4}r\sqrt[3]{\frac{r}{\mu^2}}F_1\left(\frac{1}{3}, \frac{4}{3}; \frac{7}{3}; \frac{r}{\mu}\right).$$

The analog of the Eddington-Finkelstein coordinates for this metric is now $\tilde{u} = t + r_*$ and $\tilde{v} = t - r_*$. In these coordinates the line element takes now the form

$$ds^2 = \sqrt[3]{1-\frac{\mu}{r}}d\tilde{u}^2 - 2d\tilde{u}dr - C(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

from which one can see that the presence of the naked singularity is not obvious. On the other hand, the determinant of the metric is

$$\det g = -r^4\left(1-\frac{\mu}{r}\right)^{4/3}\sin^2\theta$$

which makes the metric not to be invertible at $r=\mu$. This illustrative example shows the subtle nature of the naked singularity. Here, by means of an alternative method we offer a formula extending the latter expansions up to the fifth order and reducing correctly to the weak field approximation (33) in the limit of a vanishing scalar charge. With the help of (7) in [25] the integral giving the deflection angle can be written as $\Delta\varphi(\rho_0) = I(\rho_0) - \pi$ where

$$I(\rho_0) = 2\int_{\rho_0}^{\infty}\frac{d\rho}{\rho}\sqrt{\frac{A(\rho)}{C(\rho)}}\left[\left(\frac{\rho}{\rho_0}\right)^2\frac{C(\rho)B(\rho_0)}{C(\rho_0)B(\rho)} - 1\right]^{-1/2},$$

$$C(\rho) = \left(1-\frac{1}{\rho}\right)^{1-\gamma}.$$

Letting $\alpha = 1/\rho_0$ and introducing the change of variable $u = (\alpha\rho)^{-1}$ we obtain

$$I(\alpha) = 2\int_0^1 [(1-\alpha u)^{2-2\gamma}(1-\alpha)^{2\gamma-1} - u^2(1-\alpha u)]^{-1/2}.$$

Expanding the integrand in $I(\alpha)$ around $\alpha = 0$ we obtain after a tedious computation

$$I(\alpha) = \pi + 2\gamma\alpha + c_1\alpha^2 + \cdots + c_4\alpha^5 + \mathcal{O}(\alpha^6)$$

with

$$c_1 = (\pi-2)\gamma^2 + \gamma - \frac{\pi}{16},$$

$$c_2 = (7-2\pi)\gamma^3 + (\pi-2)\gamma^2 + \frac{1}{4}\left(\frac{1}{3}+\frac{\pi}{2}\right)\gamma - \frac{\pi}{16},$$

$$c_3 = \left(6\pi - \frac{55}{3}\right)\gamma^4 + \left(\frac{21}{2} - 3\pi\right)\gamma^3 + \frac{1+3\pi}{12}\gamma^2 + \frac{3}{8}\left(\frac{\pi}{2}-1\right)\gamma - \frac{55}{1024}\pi,$$

$$c_4 = \left(\frac{1975}{36} - \frac{52}{3}\pi\right)\gamma^5 + \left(12\pi - \frac{110}{3}\right)\gamma^4 + \left(\frac{323}{72} - \frac{13}{12}\pi\right)\gamma^3 + \left(\frac{13}{6} - \frac{\pi}{2}\right)\gamma^2 + \left(\frac{149}{768}\pi - \frac{1513}{2880}\right)\gamma - \frac{53}{512}\pi.$$

Hence, the weak field limit of the deflection angle reads

$$\Delta\varphi(\rho_0) = \frac{2\gamma}{\rho_0} + \frac{c_1}{\rho_0^2} + \cdots\frac{c_4}{\rho_0^5} + \mathcal{O}\left(\frac{1}{\rho_0^6}\right).$$

The above expansion generalizes the weak field approximations (24) in [29] and (51) in [47]. Moreover, it reproduces the weak field approximation (33) for a Schwarzschild manifold in the limit $\gamma \to 1$. For $\gamma = 1/2$ the deflection angle can be computed analytically in terms of the complete elliptic integral of the first kind K and the incomplete elliptic integral of the first kind F as

$$\Delta\varphi(\rho_0) = -\pi + 2\sqrt{\frac{2}{\rho_0+1}}\left[K\left(\sqrt{\frac{2}{\rho_0+1}}\right) - F\left(\frac{\sqrt{2}}{2}, \frac{2}{\rho_0+1}\right)\right].$$

The above formula is new since we could not find any similar result in the existing literature concerning gravitational lensing in the JNW metric. Concerning a numerical analysis of gravitational lensing for the case $0 < \gamma < 1/2$ we refer to [21] whereas the study of the strong gravitational lensing can be found in [25]. The $\gamma = 1/2$ is indeed a special case of the JNW metric. We are interested in the region $\rho > 1$ assuming that the geodesics cannot be continued through the naked singularity. Then, the function $\tilde{V}$ defined in (20) (see Fig. 5) displays a maximum (unstable circular orbit) at $\rho_m = \gamma + 1/2$. In the very principle, light can be trapped now between the hump of $\tilde{V}$ and the line $\rho = 1$. For $\gamma < 1/2$ the function $\tilde{V}$ is a smoothly decreasing function between one and infinity whereas for $\gamma = 1/2$ we have $\tilde{V}(\rho) = 1/\rho^2$.

VI. LIGHT DEFLECTION IN A SPACETIME WITH GALACTIC DARK MATTER HALOS

One of the pressing problems of astrophysics is to explain the rotational curves of galaxies. The post popular explanation is to postulate nonbaryonic neutral matter (dark matter) [49]. Another explanation prefers to modify the gravity itself [50]. In both cases, an interesting task is to extend the nonrelativistic theories within the framework of general relativity. In the case of DM this means using existing empirical density profiles to construct a DM halo metric.

The line element associated with a galaxy dark matter halo based on the Navarro-Frenk-White (NFW) [51] density profile is a metric of the form (1) [52]

$$B(x) = \begin{cases} 1 + 2\Phi_c + \tilde{\gamma}C_0x^2 & \text{if } 0 \le x \le x_0 \\ 1 + 2\Phi_c + \tilde{\gamma}A_0 - \frac{\tilde{\gamma}}{x}(B_0 + \ln\frac{1+x}{1+x_0}) & \text{if } x > x_0 \end{cases},$$

$$A(x) = \begin{cases} 1 + 2\tilde{\gamma}C_0x^2 & \text{if } 0 \le x \le x_0 \\ 1 + \frac{\tilde{\gamma}}{x}(D_0 + \ln\frac{1+x}{1+x_0} + \frac{1}{1+x}) & \text{if } x > x_0 \end{cases}$$

with

$$D_0 = \frac{x_0^2 - 3x_0 - 3}{3(1+x_0)^2}, \qquad C_0 = \frac{1}{6x_0(1+x_0)^2},$$
$$B_0 = \frac{x_0(4x_0+3)}{3(1+x_0)^2}, \qquad A_0 = \frac{3x_0+2}{2(1+x_0)^2}$$

and

$$\tilde{\gamma} = 2v_0^2, \qquad x = \frac{r}{r_g}$$

where the two free parameters v_0 and r_g are the characteristic speed and radius of the galaxy, respectively. In what follows it will be assumed that the characteristic speed of the galaxy is small so that $\tilde{\gamma} \ll 1$. Moreover,

$$\Phi_c \approx -\epsilon\left[1 + \frac{2\ln(1+c_0)}{c_0} - \frac{1}{1+c_0}\right], \qquad \epsilon = \tilde{\gamma}/2$$

and

$$c_0 = 62.1 \times \left(\frac{M_{\mathrm{vir}}h}{M_\odot}\right)^{-0.06}(1+\epsilon),$$

where M_{vir} is the virial mass. According to [52] for $h = 0.7$ we have $10^8 \lesssim M_{\mathrm{vir}}/M_\odot \lesssim 10^{15}$ which implies that $6 \lesssim c_0 \lesssim 30$. Furthermore, for $M_{\mathrm{vir}}/M_\odot$ varying in the aforementioned range x_0 is of the order 10^{-4} as it can be evinced from Eq. (29) in [52]. At this point a comment is in order. If we let $x_0 \to 0$, we recover the metric derived in [53]. However, there are two conceptual differences. First of all, in [52] there are two regions to be considered and moreover, the calculation is valid only for small $\tilde{\gamma}$. The case distinction above is necessary because of the singular nature of the NFW density profile at the origin. To get around this problem one replaces the inner region by a regular solution [52]. We have recalculated the matching conditions and they differ slightly with our results of the function A and B from [52].

In what follows we are interested in the case of weak gravitational lensing and therefore we will assume that the distance of closest approach $\tilde{x}_0 \gg 1$. For a generic spherically symmetric spacetime the deflection angle is given by the integral [25]

$$\Delta\varphi(\tilde{x}_0) = -\pi + 2\int_{\tilde{x}_0}^{\infty} \frac{\sqrt{A(x)}dx}{x\sqrt{(\frac{x}{\tilde{x}_0})^2\frac{B(\tilde{x}_0)}{B(x)} - 1}}. \tag{57}$$

To compute the deflection angle in the case of weak gravitational lensing $(x, \tilde{x}_0 \gg 1)$ we follow the method outlined by [19] and make an asymptotic expansion in the small parameters x^{-1} and $\tilde{x}_0^{-1}$. Taking into account that

$$B(x) = 1 + 2\Phi_c + \tilde{\gamma}A_0 - \tilde{\gamma}\frac{B_0 - \ln(1+x_0)}{x} - \tilde{\gamma}\frac{\ln x}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)$$

we find that

$$\left(\frac{x}{\tilde{x}_0}\right)^2\frac{B(\tilde{x}_0)}{B(x)} - 1 = \left(\frac{x}{\tilde{x}_0}\right)^2\left[1 - \tilde{\gamma}\frac{B_0 - \ln(1+x_0)}{1+2\Phi_c}\frac{x-\tilde{x}_0}{x\tilde{x}_0} + \frac{\tilde{\gamma}}{1+2\Phi_c}\left(\frac{\ln x}{x} - \frac{\ln\tilde{x}_0}{\tilde{x}_0}\right)\cdots\right] - 1,$$
$$= \left[\left(\frac{x}{\tilde{x}_0}\right)^2 - 1\right][1 + \tilde{\gamma}\omega(x,\tilde{x}_0) + \cdots]$$

with

$$\omega(x,\tilde{x}_0) = -\frac{B_0 - \ln(1+x_0)}{1+2\Phi_c}\frac{x}{\tilde{x}_0(x+\tilde{x}_0)} + \frac{1}{1+2\Phi_c}\frac{x(\tilde{x}_0\ln x - x\ln\tilde{x}_0)}{\tilde{x}_0(x^2-\tilde{x}_0^2)} \ll 1.$$

Moreover,

$$A(x) = 1 + \tilde{\gamma}\frac{D_0 - \ln(1+x_0)}{x} + \tilde{\gamma}\frac{\ln x}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)$$

and the integrand in (57) can be approximated as follows:

$$\frac{\sqrt{A(x)}}{x\sqrt{(\frac{x}{\tilde{x}_0})^2\frac{B(\tilde{x}_0)}{B(x)} - 1}} = \frac{1}{x\sqrt{(\frac{x}{\tilde{x}_0})^2 - 1}}\left[1 + \tilde{\gamma}\frac{B_0 - \ln(1+x_0)}{2(1+2\Phi_c)} \times \frac{x}{\tilde{x}_0(x+\tilde{x}_0)} + \tilde{\gamma}\frac{D_0 - \ln(1+x_0)}{2x} + \frac{\tilde{\gamma}}{2}\frac{\ln x}{x} - \frac{\tilde{\gamma}}{2(1+2\Phi_c)}\frac{x(\tilde{x}_0\ln x - x\ln\tilde{x}_0)}{\tilde{x}_0(x^2-\tilde{x}_0^2)}\right].$$

Let $h(x) = x^{-1}[(x/\tilde{x}_0)^2 - 1]^{-1/2}$. By means of the formulas

$$\int_{\tilde{x}_0}^{\infty} h(x)dx = \frac{\pi}{2}, \qquad \int_{\tilde{x}_0}^{\infty}\frac{xh(x)}{x+\tilde{x}_0}dx = 1,$$
$$\int_{\tilde{x}_0}^{\infty}\frac{h(x)}{x}dx = \frac{1}{\tilde{x}_0},$$
$$\int_{\tilde{x}_0}^{\infty} h(x)\frac{x(\tilde{x}_0\ln x - x\ln\tilde{x}_0)}{x^2-\tilde{x}_0^2}dx = \ln\frac{2}{\tilde{x}_0},$$
$$\int_{\tilde{x}_0}^{\infty} h(x)\frac{\ln x}{x}dx = \frac{\ln\tilde{x}_0 - \ln 2 + 1}{\tilde{x}_0}$$

we find that the deflection angle can be represented at the order $\tilde{\gamma}$ as follows:

$$\Delta\varphi(\tilde{x}_0) = \tilde{\gamma}\left[1 - \frac{2(1+\Phi_c)}{1+2\Phi_c}\ln 2 + D_0 - \ln(1+x_0) + \frac{B_0 - \ln(1+x_0)}{1+2\Phi_c}\right]\frac{1}{\tilde{x}_0} + \tilde{\gamma}\frac{2(1+\Phi_c)}{1+2\Phi_c}\frac{\ln\tilde{x}_0}{\tilde{x}_0} + \cdots. \quad (58)$$

As $\Phi_c \to 0$ which corresponds to $\epsilon \to 0$ and $x_0 \to 0$ the deflection angle behaves as $2\tilde{\gamma}\ln\tilde{x}_0/\tilde{x}_0$. The above results are the first steps for calculating the deflection angle of a dark matter halo within a general relativistic framework. We leave the lensing and the strong lensing for future projects.

VII. GRAVITATIONAL LENSING IN THE PRESENCE OF A HOLOGRAPHIC SCREEN

We recall that a holographic screen can be seen as the event horizon of a black hole characterized by a mass spectrum bounded from below by a mass of the extremal configuration coinciding with the Planck mass. The metric modeling a holographic screen is [54]

$$ds^2 = \left(1 - \frac{2ML_p^2 r}{r^2 + L_p^2}\right)dt^2 - \left(1 - \frac{2ML_p^2 r}{r^2 + L_p^2}\right)^{-1}dr^2 - r^2 d\Omega^2,$$

where the mass of the holographic screen is

$$M = \frac{r_h^2 + L_p^2}{2L_p^2 r_h}, \quad (59)$$

r_h is the radius of the screen, and L_p denotes the Planck length. The above line element admits a pair of distinct horizons at

$$r_\pm = L_p^2(M \pm \sqrt{M^2 - M_p^2}), \quad (60)$$

whenever $M > M_p$ where M_p is the Planck mass, while for $M = M_p$ the two horizons merge together and we have an extremal black hole. For $M \gg M_p$ we get the usual Schwarzschild metric. Inserting (59) into (60) we find that

$$r_+ = r_h, \qquad r_- = \frac{L_p^2}{r_h}.$$

In the present case the function $\tilde{V}$ defined in (20) reads

$$\tilde{V}(x) = \frac{1}{L_p^2 x^2}\left(1 - \frac{mx}{x^2+1}\right), \qquad x = \frac{r}{L_p}, \qquad m = \frac{2M}{M_p}.$$

It is not difficult to verify that for $0 \le m \le 2$ we have $\tilde{V}(x) > 0$ for any $x > 0$ and monotonically decreasing (see Fig. 6 for instance). However, when $m > 2$ the same function intersects the positive x axis at

$${}_1x_2 = \frac{m \pm \sqrt{m^2-4}}{2}$$

so that $\tilde{V}(x)$ is positive on the intervals $(0, x_1)$ and (x_2, ∞) and negative on (x_1, x_2). It possesses a negative minimum at $x_m \in (x_1, x_2)$ and a positive maximum at $x_M \in (x_2, \infty)$ where we will have an unstable circular orbit corresponding to the photon sphere. Before embarking ourselves into the derivation of the formulas for the deflection angle in the weak and strong regimes, we construct the embedding diagram for the holographic screen. To this purpose we consider a two-dimensional surface $\mathcal{H} = \{(t, x, \vartheta, \varphi) \in \mathbb{R} \times (x_2, \infty) \times S^2 | t = \text{const}, \vartheta = \pi/2\}$ with line element

$$d\sigma^2 = -\left(1 - \frac{mx}{1+x^2}\right) - x^2 d\varphi^2$$

and we introduce an additional coordinate z orthogonal to the (x, φ) plane. In the cylindrical coordinate system (x, φ, z) the embedding can be realized by looking for a surface of rotation having the same line element as $\mathcal{H}$, i.e. we search for a surface $z = z(x)$ with line element $d\tilde{\sigma}^2 = -dx^2 - x^2 d\varphi^2 - dz^2 = d\sigma^2$. From this condition we find that in the extreme case $m = 2$ the function $z = z(x)$ must satisfy the differential equation

$$\frac{dz}{dx} = \pm\frac{\sqrt{2x}}{x-1}$$

for $x > x_2 = 1$. The solutions are found to be

$$z_\pm(x) = \pm\sqrt{2}\left(2\sqrt{x} + \ln\frac{\sqrt{x}-1}{\sqrt{x}+1}\right), \quad (61)$$

where the integration constant has been chosen so that the solutions meet at the point $x_0 = 1.4392$ where the expression in the bracket of (61) vanishes. In the nonextreme case $m > 2$ we end up with the differential equation

$$\frac{dz}{dx} = \pm\sqrt{\frac{mx}{x^2 - mx + 1}}$$

that must be solved for $x > x_2$. By means of the transformation $x = \tau^2$ combined with $8a6$ at page 85 in [55] we find that the general solution can be expressed in terms of incomplete elliptic integrals of the first and second kind as follows:

$$z_\pm(x) = \sqrt{m}(m+2)\left[-\alpha(m)F(\phi(x), k) + \beta(m)\left(E(\phi(x), k) + \sqrt{\frac{x^2 - mx + 1}{m^2 - 4}}\frac{1 - \tau_1^2}{\tau_1\sqrt{x} - 1}\right)\right]$$

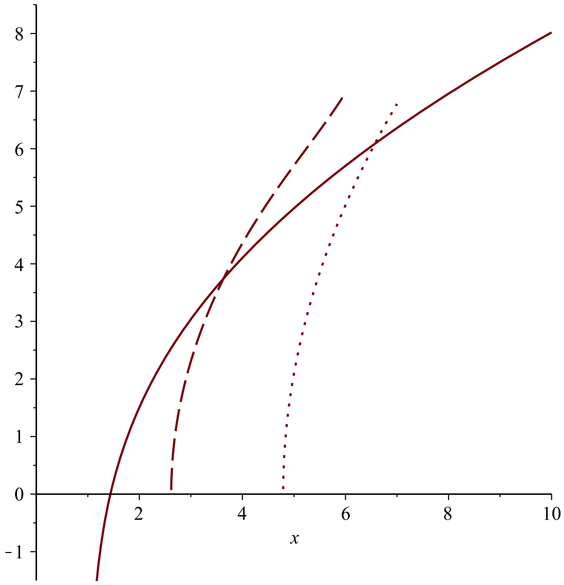

FIG. 7 (color online). Plot of the function $z_+(x)$ in the extreme case (solid line), for $m = 3$ (dashed line), and $m = 6$ (dotted line). Note that the size of the event horizon increases as the value of the rescaled mass parameter m increases. This figure refers to the holographic screen metric.

with

$$\alpha(m) = (m+1)(3m+6+\sqrt{m^2-4}),$$
$$\beta(m) = (m+2)(m+2+\sqrt{m^2-4}),$$

$$\phi(x) = \sin^{-1}\sqrt{\frac{(\tau_1+\tau_2)(\sqrt{x}-\tau_1)}{2\tau_1(\tau_1-\tau_1^{-1})}}, \qquad k = \frac{2\sqrt{\tau_1\tau_2}}{\tau_1+\tau_2},$$

and

$$\tau_1 = \sqrt{\frac{m+\sqrt{m^2-4}}{2}}, \qquad \tau_1 = \sqrt{\frac{m-\sqrt{m^2-4}}{2}}.$$

A plot of the profile function $z_+(x)$ for different values of the rescaled mass parameter can be found in Fig. 7.

A. The nonextreme case

To derive formulas for the deflection angle in the weak and strong regimes we start by considering the nonextreme case $M > M_p$. First of all, we rescale t and r by r_h. Let $\rho = r/r_h > 1$. Then, the original line element can be rewritten as $ds^2 = r_h^2 d\tilde{s}^2$ with

$$d\tilde{s}^2 = B(\rho)d\tilde{t}^2 - \frac{d\rho^2}{B(\rho)} - \rho^2 d\Omega^2,$$
$$B(\rho) = 1 - \frac{(1+\lambda)\rho}{\rho^2+\lambda}, \qquad \lambda = L_p^2/r_h^2.$$

The radius of the photon sphere can be found by solving the equation $B'(\rho)/B(\rho) = 2/\rho$ which leads to the problem of finding the roots of the quartic polynomial equation

$$p(\rho) = 2\rho^4 - 3(1+\lambda)\rho^3 + 4\lambda\rho^2 - \lambda(1+\lambda)\rho + 2\lambda^2 = 0. \tag{62}$$

For $0 < \lambda \ll 1$ we can use a perturbative method to find the position of the photon sphere. In this regard we observe that for $\lambda = 0$ the unperturbed roots of (62) are 0 with algebraic multiplicity 3 and 3/2. Since the latter has algebraic multiplicity 1, we can set up an expansion $\rho_{3/2}(\lambda) = 3/2 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + \mathcal{O}(\lambda^4)$. Substituting $\rho_{3/2}(\lambda)$ into (62), equating powers of λ, and solving the corresponding set of equations, we find

$$\rho_{3/2}(\lambda) = \frac{3}{2} + \frac{7}{18}\lambda + \frac{38}{243}\lambda^2 + \frac{1070}{6561}\lambda^4 + \mathcal{O}(\lambda^4)$$

which is always larger than the event horizon $\rho_h = 1$. The remaining roots of (62) cannot be candidates for the photon sphere since they must approach zero as $\lambda \to 0$ and therefore they have an asymptotic behavior of the form $\rho_j(\lambda) \approx \lambda^{p_j} b_{0,j}$ with $p_j > 0$ and $b_{0,j} \neq 0$ for all $j = 1, 2, 3$. The relation between the impact parameter and the distance of closest approach is

$$\tilde{b} = \rho_0 \sqrt{\frac{\rho_0^2+\lambda}{\rho_0^2-(1+\lambda)\rho_0+\lambda}}. \tag{63}$$

Note that in the limit $\lambda \to 0$ the above relation reproduces correctly (38) in the classic Schwarzschild case. Expanding (63) around $\rho_0 = \rho_{3/2}$ we find $\tilde{b} = \tilde{b}_{cr} + \alpha_1(\rho_0 - \rho_{3/2})^2 + \mathcal{O}(\rho_0 - \rho_{3/2})^3$ with

$$\tilde{b}_{cr} = \frac{3\sqrt{3}}{2} + \frac{5\sqrt{3}}{6}\lambda + \cdots,$$
$$\alpha_1 = \frac{1}{2}\tilde{b}''(\rho_{3/2})$$
$$= \left\{-\frac{B'}{2B\sqrt{B}} + \frac{\rho_0}{2\sqrt{B}}\left[\frac{3}{4}\left(\frac{B'}{B}\right)^2 - \frac{B''}{2B}\right]\right\}\Big|_{\rho_0=\rho_{3/2}}$$
$$= 2\sqrt{3} - \frac{46\sqrt{3}}{27}\lambda + \cdots$$

Note that as in the Schwarzschild case $\tilde{b}'(\rho_{3/2}) = 0$ because $\tilde{b}'(\rho_0) = p(\rho_0)/(2\sqrt{\rho_0^2+\lambda}[\rho_0^2-(1+\lambda)\rho_0+\lambda]^{3/2})$ and $\rho_{3/2}$ is a root of the polynomial $p(\rho_0)$. The deflection angle due to a holographic screen in the weak field limit can be computed from (57) with $\rho, \rho_0 \gg 1$. Taking into account that $B(\rho) = 1 - (1+\lambda)/\rho + \cdots$ we find that

$$\left(\frac{\rho}{\rho_0}\right)^2\frac{B(\rho_0)}{B(\rho)}-1$$
$$=\left(\frac{\rho}{\rho_0}\right)^2\left[1-\frac{1+\lambda}{\rho_0}+\cdots\right]\left[1+\frac{1+\lambda}{\rho}+\cdots\right]$$
$$=\left[\left(\frac{\rho}{\rho_0}\right)^2-1\right]\left[1-\frac{(1+\lambda)\rho}{\rho_0(\rho+\rho_0)}+\cdots\right]$$

and

$$\frac{\sqrt{A(\rho)}}{\rho\sqrt{(\frac{\rho}{\rho_0})^2\frac{B(\rho_0)}{B(\rho)}-1}}=\frac{1}{\rho\sqrt{(\frac{\rho}{\rho_0})^2-1}}\left[1+\frac{1+\lambda}{2\rho}+\frac{(1+\lambda)\rho}{2\rho_0(\rho+\rho_0)}+\cdots\right]$$

where $A=B^{-1}$. Let $h(\rho)=\rho^{-1}[(\rho/\rho_0)^2-1]^{-1/2}$. Since

$$\int_{\rho_0}^{\infty}h(\rho)d\rho=\frac{\pi}{2},\qquad\int_{\rho_0}^{\infty}\frac{h(\rho)}{\rho}d\rho=\frac{1}{\rho_0},$$
$$\int_{\rho_0}^{\infty}\frac{\rho h(\rho)}{\rho+\rho_0}d\rho=1,$$

we find that in the weak field limit and at the first order in $1/\rho_0$ the deflection angle is related to the distance of closest approach through the following relation:

$$\Delta\varphi(\rho_0)=\frac{2(1+\lambda)}{\rho_0}+\cdots.$$

To treat strong gravitational lensing we will adopt the method developed by [25]. First of all, we introduce new variables $y=B(\rho)$ and $z=(1-y)/(1-y_0)$ with $y_0=B(\rho_0)$. Then, ρ can be expressed as a function of z as follows:

$$\rho(z)=\frac{1+\lambda\pm\sqrt{(1+\lambda)^2-4\lambda(1-y_0)^2(1-z)^2}}{2(1-y_0)(1-z)}.$$

Since for $\lambda\to 0$ we should get as in the Schwarzschild case $\rho(z)=1/[(1-y_0)(1-z)]$, we must choose the positive sign in the above expression. The formula for the deflection angle can now be written as

$$\Delta\varphi(\rho_0)=-\pi+\int_0^1 R(z,\rho_0)f(z,\rho_0)dz \qquad (64)$$

with

$$R(z,\rho_0)=\frac{2(1-y_0)\rho_0}{\rho^2(z)B'(\rho)},$$
$$f(z,\rho_0)=\left[y_0-\left[y_0+(1-y_0)z\right]\frac{\rho_0^2}{\rho^2(z)}\right]^{-1/2}, \qquad (65)$$

where the prime denotes differentiation with respect to ρ, Note that the function R does not exhibit singularities for any value of z and ρ_0 while f becomes singular as $z\to 0$ since $\rho(0)=\rho_0$. Expanding the argument of the square root in f at the second order we find $f(z,\rho_0)\approx f_0(z,\rho_0)=[\alpha(\rho_0)z+\beta(\rho_0)z^2]^{-1/2}$ with

$$\alpha(\rho_0)=\frac{1-B(\rho_0)}{\rho_0B'(\rho_0)}[2B(\rho_0)-\rho_0B'(\rho_0)]=\frac{p(\rho_0)}{\rho_0(\rho_0^4-\lambda)},$$
$$\beta(\rho_0)=\frac{[1-B(\rho_0)]^2}{\rho_0^2(B'(\rho_0))^3}[2\rho_0(B'(\rho_0))^2-3B(\rho_0)B'(\rho_0)-\rho_0B(\rho_0)B''(\rho_0)].$$

A closer inspection of α shows that it becomes zero when $\rho=\rho_{3/2}$ and therefore the integral of f will diverge logarithmically. Let us rewrite the integrand in (64) as $R(z,\rho_0)f(z,\rho_0)=R(0,\rho_{3/2})f_0(z,\rho_{3/2})+g(z,\rho_0)$ with $g(z,\rho_0)=R(z,\rho_0)f(z,\rho_0)-R(0,\rho_{3/2})f_0(z,\rho_{3/2})$ where

$$R(0,\rho_{3/2})=\frac{2(\rho_{3/2}^2+\lambda)}{\rho_{3/2}^2-\lambda}=2+\frac{16}{9}\lambda+\cdots$$

tends correctly to the value 2 for $\lambda\to 0$ as one would expect in the classic Schwarzschild case. Then, the integral for the deflection angle can be written as $\Delta\varphi(\rho_0)=-\pi+I_D(\rho_0)+I_R(\rho_0)$ where

$$I_D(\rho_0)=R(0,\rho_{3/2})\int_0^1 f_0(z,\rho_{3/2})dz,$$
$$I_R(\rho_0)=\int_0^1 g(z,\rho_0)dz$$

and the subscripts D and R stay for divergence and regular, respectively. The first integral admits the following exact solution

$$\int_0^1 f_0(z,\rho_{3/2})dz=\frac{2}{\sqrt{\beta}}\ln\frac{\sqrt{\beta}+\sqrt{\alpha+\beta}}{\sqrt{\alpha}}.$$

Expanding α and β around $\rho_{3/2}$ we obtain

$$I_D(\rho_0)=-a\ln\left(\frac{\rho_0}{\rho_{3/2}}-1\right)+b_D+\mathcal{O}(\rho_0-\rho_{3/2}) \qquad (66)$$

with

$$a=\frac{R(0,\rho_{3/2})}{\sqrt{\beta(\rho_{3/2})}}=2+\frac{40}{27}\lambda+\cdots,$$
$$b_D=\frac{R(0,\rho_{3/2})}{\sqrt{\beta(\rho_{3/2})}}\ln\frac{2[1-B(\rho_{3/2})]}{\rho_{3/2}B'(\rho_{3/2})}=2\ln 2+\left(\frac{40}{27}\ln 2+\frac{16}{9}\right)\lambda+\cdots.$$

The regular term in the deflection angle can be found by expanding the integral $I_R(\rho_0)$ in powers of $\rho_0 - \rho_{3/2}$ as follows:

$$I_R(\rho_0) = \sum_{n=0}^{\infty} \frac{(\rho_0 - \rho_{3/2})^n}{n!} \int_0^1 \frac{\partial^n g}{\partial \rho_0^n}\Big|_{\rho_0=\rho_{3/2}} dz = \int_0^1 g(z, \rho_{3/2}) dz + \mathcal{O}(\rho_0 - \rho_{3/2}).$$

Hence, the additional correction to be added to the term $-\pi + b_D$ is represented by $b_R = I_R(\rho_{3/2})$ and in the strong field limit the formula for the deflection angle reads

$$\Delta\varphi(\rho_0) = -a \ln\left(\frac{\rho_0}{\rho_{3/2}} - 1\right) + b_D + b_R \mathcal{O}(\rho_0 - \rho_{3/2}).$$

In this case it is not possible to give an analytical result for the integral representing b_R but we can construct an expansion in the parameter λ. To this purpose note that

$$f_0(z, \rho_{3/2}) = \frac{1}{z} - \frac{4}{27z}\lambda + \cdots,$$
$$R(z, \rho_{3/2}) = 2 + \frac{8}{9}(3z^2 - 6z + 2)\lambda + \cdots,$$
$$f(z, \rho_{3/2}) = \frac{\sqrt{3}}{z\sqrt{3-2z}} - \frac{4\sqrt{3}(3z^2 - 6z + 1)}{27z\sqrt{3-2z}}\lambda + \cdots.$$

Taking into account that $g(z, \rho_{3/2}) = g_1(z) + g_2(z)\lambda + \cdots$ where

$$g_1(z) = \frac{2\sqrt{3} - 2\sqrt{3-2z}}{z\sqrt{3-2z}},$$
$$g_2(z) = \frac{8(6\sqrt{3}z^2 - 12\sqrt{3}z + 5\sqrt{3} - 5\sqrt{3-2z})}{27z\sqrt{3-2z}},$$

and integrating we obtain $b_R = b_{R,\mathrm{Sch}} + \kappa\lambda + \cdots$ where

$$b_{R,\mathrm{Sch}} = \ln 36 - \operatorname{arctanh}(\sqrt{3}/3) = 0.9496,$$
$$\kappa = \frac{40}{27}\ln 6 + \frac{8(2\sqrt{3} - 9)}{27} - \frac{80}{27}\operatorname{arctanh}(\sqrt{3}/3) = -2.5770.$$

Note that $b_{R,\mathrm{Sch}}$ is in agreement with the numerical value found by [25] for the classic Schwarzschild case.

B. The extreme case

Let $M = M_p$. Then, the Cauchy and event horizon coincide at $r_\pm = L_p$. Introducing the rescaling $\rho = r/L_p$ the metric function B is now given by $B(\rho) = (\rho-1)^2/(\rho^2+1)$. The radius of the photon sphere is obtained by solving the equation $B'(\rho)/B(\rho) = 2/\rho$ which gives rise to the cubic equation $q(\rho) = \rho^3 - 2\rho^2 - 1 = 0$. This equation has two imaginary roots and one real root located at

$$\rho_f = \frac{2}{3} + \frac{(172 + 12\sqrt{177})^{2/3} + 16}{(172 + 12\sqrt{177})^{1/3}} = 2.2057.$$

The impact parameter and the distance of closest approach are related to each other through $\tilde{b} = \rho_0\sqrt{\rho_0^2+1}/(\rho_0 - 1)$. An expansion around the point $\rho_0 = \rho_f$ gives $\tilde{b} = \tilde{b}_{cr} + \tilde{\alpha}_1(\rho_0 - \rho_f)^2 + \mathcal{O}(\rho_0 - \rho_f)^2$ with

$$\tilde{b}_{cr} = \frac{\rho_f\sqrt{\rho_f^2+1}}{\rho_f - 1} = 4.4304,$$
$$\tilde{\alpha}_1 = \frac{3\rho_f^3 + 3\rho_f + 2}{2(\rho_f - 1)^3(\rho_f + 1)^{3/2}} = 0.8199.$$

To obtain the weak deflection limit for the angle $\Delta\varphi$ we suppose that $\rho, \rho_0 \gg 1$ and proceeding exactly as we did in the nonextreme case we find at the first order in $1/\rho_0$

$$\Delta\varphi(\rho_0) = \frac{4}{\rho_0} + \cdots.$$

To study the strong gravitational lensing we change variables according to $y = B(\rho)$ and $z = (1-y)/(1-y_0)$. Solving the first equation for ρ we obtain $\rho(y) = (1 \pm \sqrt{2y - y^2})/(1-y)$. Since $\rho \to 1$ as $y \to 0$ and $\rho \to \infty$ for $y \to 1$, we have to pick the solution with the plus sign which can be expressed in terms of the variable z as

$$\rho(z) = \frac{1 + \sqrt{1 - (1-y_0)^2(1-z)^2}}{(1-y_0)(1-z)}.$$

At this point the deflection angle will be given by (64) with the functions R and f formally given by (65). In the present case the coefficients α and β entering in the expansion of f are given by

$$\alpha(\rho_0) = \frac{2q(\rho_0)}{(\rho_0+1)(\rho_0^2+1)},$$
$$\beta(\rho_0) = \frac{\rho_0^4 - 4\rho_0^3 - 2\rho_0^2 - 4\rho_0 - 3}{(1-\rho_0)(\rho_0+1)^3}.$$

Note that α vanishes whenever the distance of closest approach coincides with the radius of the photon sphere. As in the nonextreme case we rewrite the deflection angle as $\Delta\varphi(\rho_0) = -\pi + I_D(\rho_0) + I_R(\rho_0)$. The integral I_D can be expanded about ρ_f and one obtains formally an expression as (66) where $\rho_{3/2}$ is replaced by ρ_f and

$$a = \frac{2(\rho_f^2+1)}{\rho_f - 1}\sqrt{\frac{1-\rho_f^2}{\rho_f^4 - 4\rho_f^3 - 2\rho_f^2 - 4\rho_f - 3}} = 2.9941,$$
$$b_D = a \ln \frac{2(\rho_f^2+1)}{\rho_f - 1} = 6.8120.$$

To compute the coefficient b_R we need the following quantities:

$$R(z,\rho_0) = \frac{4\rho_0^2(\rho_0^2+1)}{\rho_0^4 + (\rho_0^2+1)\sqrt{(\rho_0^2-1)^2 + 4\rho_0^2 z(2-z)} + 2\rho_0^2(4z - 2z^2 - 1) + 1}, \qquad R(0,\rho_0) = \frac{2(\rho_0^2+1)}{\rho_0^2 - 1}.$$

However, the integral giving b_R can be solved only numerically and we find $b_R = -1.0217$. The function in Eq. (20) for the holographic screen is plotted in Figs. 6, 8 and 9.

VIII. GRAVITATIONAL LENSING FOR NONCOMMUTATIVE GEOMETRY INSPIRED WORMHOLES

Noncommutative geometry inspired wormholes are solutions of the Einstein field equations obtained by assuming that the mass/energy distribution is a Gaussian of the form

$$\rho(r) = \frac{M}{(4\pi\theta)^{3/2}} e^{-r^2/(4\theta)},$$

where $\sqrt{\theta}$ is the matter distribution width and it defines the scale where the spacetime coordinates should be replaced by some noncommuting coordinate operators in a suitable Hilbert space [56]. Here, M is the total mass and is given by the integral $M = 4\pi \int_0^\infty r^2 \rho(r) dr$. The gravitational source is then modeled by a fluid-type energy-momentum tensor of the form $T^\mu{}_\nu = \mathrm{diag}(\rho(r), -p_r(r), -p_\perp(r), -p_\perp(r))$ with p_r and $p_\perp$ the radial and tangential pressures, respectively, together with the condition $T^\mu{}_{\nu;\mu} = 0$. If we look for a metric such that it is spherically symmetric, static, and asymptotically flat, then we can write the corresponding line element as

$$ds^2 = e^{2\Phi(r)} dt^2 - \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 - r^2 d\Omega^2,$$

where $\Phi(r)$ and $m(r)$ are the so-called red-shift and shape functions, which must be determined by solving Einstein equations. If we assume that $p_r(r) = -m(r)/(4\pi r^3)$ and $m(r) = 4\pi \int_0^r u^2 \rho(u) du$, then we obtain the following line element:

$$ds^2 = dt^2 - \left(1 - \frac{4M}{\sqrt{\pi} r}\gamma\left(\frac{3}{2}; \frac{r^2}{4\theta}\right)\right)^{-1} dr^2 - r^2 d\Omega^2,$$
$$\gamma\left(\frac{3}{2}; \frac{r^2}{4\theta}\right) = \int_0^{r^2/(4\theta)} \sqrt{s} e^{-s} ds \tag{67}$$

describing a wormhole. Properties of the metric (67) have been investigated in [57]. Here, we extend the results of [57] by studying gravitational lensing in the presence of the noncommutative geometry inspired wormhole given by (67). A throat will exist if $2m(r_t) = r_t$ for some $r_t > \sqrt{\theta}$. As in [58] we will have two distinct throats $r_{t,+} > r_{t,-}$ if $M > M_0 = 1.9042\sqrt{\theta}$, two coinciding throats $r_{t,+} = r_{t,-} = r_e = 3.0226\sqrt{\theta}$ (extreme case) whenever $M = M_0$, and no throats for $M < M_0$. We will consider only weak gravitational lensing for the nonextreme and extreme cases, since in both regimes there is no photon sphere and therefore a light ray approaching the wormhole will be either deflected or disappear into the throat. The absence of a photon sphere is due to the fact that the equation $2/x = B'(x)/B(x)$ can never be satisfied because the metric coefficient B is a constant function. In order to construct the embedding diagram for the extreme and nonextreme noncommutative geometry inspired wormhole, we rescale the time and spatial coordinates and the mass according to $\tilde{t} = t/(2\sqrt{\theta})$, $x = r/(2\sqrt{\theta})$, and $\alpha = r_S/\sqrt{\pi\theta}$ where r_S denotes the classic Schwarzschild radius. Then, the nonextremality condition reads $\alpha > \alpha_0 = 2.1$ and the metric (67) can be brought into the form $ds^2 = 4\theta d\tilde{s}^2$ with

TABLE II. Numerical values of the horizons $x_\pm$, the photon sphere x_p, and the coefficients a, b_D, b_R for $\alpha \geq \alpha_0$. The first line corresponds to the extreme case.

α	x_-	x_+	x_p	a	b_D	b_R
2.1486	1.5113	1.5113	2.8526	2.0339	3.5420	0.3675
2.1800	1.3583	1.6813	2.8952	2.0270	3.5600	0.3721
2.2000	1.3186	1.7321	2.9222	2.0233	3.5724	0.3742
2.3000	1.1972	1.9097	3.0566	2.0108	3.6406	0.3785
2.4000	1.1208	2.0433	3.1901	2.0048	3.7153	0.3787
2.5000	1.0630	2.1596	3.3233	2.0017	3.7914	0.3754
2.6000	1.0161	2.2664	3.4563	2.0007	3.8682	0.3711
2.7000	0.9765	2.3673	3.5892	2.0001	3.9424	0.3622
2.8000	0.9421	2.4643	3.7221	2.0000	4.0152	0.3614
2.9000	0.9118	2.5586	3.8550	2.0000	4.0852	0.3563
3.0000	0.8847	2.6511	3.9880	2.0000	4.1528	0.3507
4.0000	0.7089	3.5448	5.3173	2.0000	4.7284	0.3017
5.0000	0.6111	4.4311	6.6467	2.0000	5.1744	0.2577

$$d\tilde{s}^2 = d\tilde{t}^2 - A(x)dx^2 - x^2 d^2\Omega,$$
$$A(x) = \left[1 - \frac{\alpha}{x}\gamma\left(\frac{3}{2}; x^2\right)\right]^{-1}.$$

The throat condition becomes $1 - (\alpha/x)\gamma(3/2, x^2) = 0$ as for the case of the noncommutative geometry inspired dirty black holes and therefore numerical values of the throat for different choices of the parameter α are listed in the third column of Table II. Proceeding as in the case of the holographic screen we find that the profiles of the surface of rotation are given by

$$z_\pm(x) = \pm \int \sqrt{\frac{\alpha\gamma(3/2, x^2)}{x - \alpha\gamma(3/2, x^2)}}, \qquad x > x_t$$

where the integration constants have been chosen so that both solutions are matched on the x axis at the position of the throat. Observing that x_t is a simple zero of the denominator in the above expression [59], and that the integrand has an integrable singularity there, we find that

$$z_\pm(x) = \pm A\sqrt{x - x_t} + \mathcal{O}(x - x_t)^{3/2},$$
$$A = 2\sqrt{\frac{\alpha\gamma(3/2, x_t^2)}{1 - 2\alpha x_t^2 e^{-x_t^2}}}.$$

A plot of the profile function $z_+(x)$ for different values of the parameter α can be found in Fig. 10.

A. Nonextreme case

Let x_0 denote the distance of closest approach. Further suppose that $x, x_0 \gg 1$. Since the metric coefficient $B(x) = 1$, the integral expressing the deflection angle as a function of x_0 simplifies to

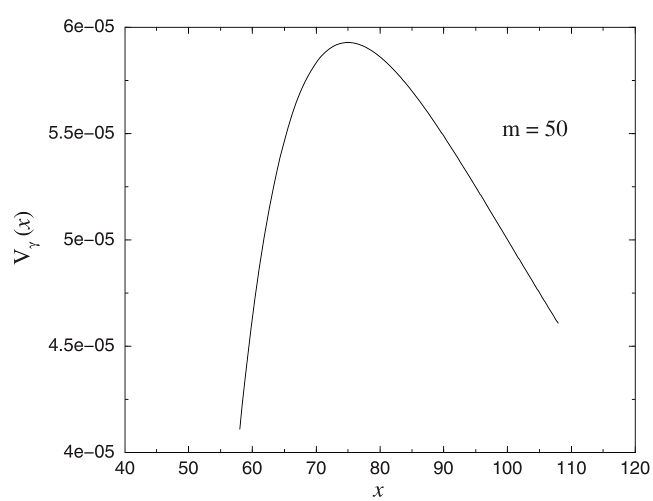

FIG. 8. The rescaled "photon potential" proportional to $\tilde{V}$ for the holographic screen metric showing the photon unstable orbit (maximum) for a large m.

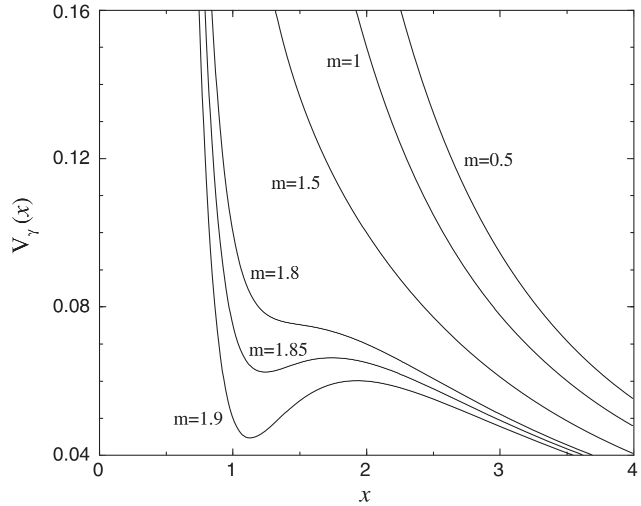

FIG. 9. The rescaled photon potential proportional to $\tilde{V}$ for the holographic screen metric displaying the naked singularity case where a narrow range of the parameter m leads to a local minimum allowing bound states of photons.

$$\Delta\varphi(x_0) = -\pi + 2\int_{x_0}^{\infty} \frac{\sqrt{A(x)}dx}{x\sqrt{(x/x_0)^2 - 1}}.$$

Taking into account that $\gamma(3/2; x^2) = \sqrt{\pi}/2 - \Gamma(3/2; x^2)$ where $\Gamma(\cdot;\cdot)$ denotes the upper incomplete Gamma function and using 6.5.32 in [48] we get the following asymptotic expansion for the lower incomplete Gamma function

$$\gamma\left(\frac{3}{2}; x^2\right) = \frac{\sqrt{\pi}}{2} - xe^{-x^2}[1 + \mathcal{O}(x^{-2})],$$

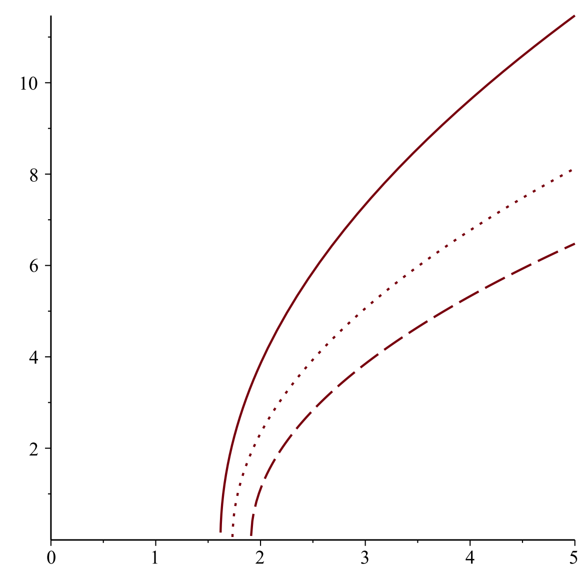

FIG. 10 (color online). Plot of the function $z_+(x)$ for the cases $\alpha = 2.18$ (solid line), $\alpha = 2.20$ (dotted line), and $\alpha = 2.3$ (dashed line). Note that the size of the size of the throat increases as the value of the parameter α increases. This figure refers to the wormhole metric.

which in turn allows one to construct an asymptotic expansion for $\sqrt{A(x)}$ represented by

$$\sqrt{A(x)} = \left[1 - \frac{\alpha\sqrt{\pi}}{2x} + \alpha e^{-x^2} + \mathcal{O}\left(\frac{e^{-x^2}}{x^2}\right)\right]^{-1/2}. \tag{68}$$

To further expand the above expression we make the substitution $x^2 = \ln u$ and we obtain

$$\sqrt{A(u)} = \left[1 - \frac{\alpha\sqrt{\pi}}{2\sqrt{\ln u}} + \frac{\alpha}{u} + \mathcal{O}\left(\frac{1}{u\ln u}\right)\right]^{-1/2}$$
$$= f_0(u) + \frac{f_1(u)}{u} + \frac{f_2(u)}{u^2} + \mathcal{O}\left(\frac{1}{u^3}\right)$$

with

$$f_0(u) = \sqrt{\frac{2\sqrt{\ln u}}{2\sqrt{\ln u} - \alpha\sqrt{\pi}}},$$
$$f_1(u) = -\frac{\alpha f_0(u)\sqrt{\ln u}}{2\sqrt{\ln u} - \alpha\sqrt{\pi}},$$
$$f_2(u) = \frac{3\alpha^2 f_0(u)\ln u}{2(2\sqrt{\ln u} - \alpha\sqrt{\pi})^2}.$$

A further expansion of the functions f_0, f_1, f_2 gives

$$f_0(u) = 1 + \frac{\alpha\sqrt{\pi}}{4\sqrt{\ln u}} + \frac{3\alpha^2\pi}{32\ln u} + \frac{5\alpha^3\pi\sqrt{\pi}}{128\ln u\sqrt{\ln u}} + \mathcal{O}\left(\frac{1}{u\ln u}\right),$$
$$\frac{f_1(u)}{u} = -\frac{\alpha}{2u} - \frac{3\alpha^2\sqrt{\pi}}{8u\sqrt{\ln u}} + \mathcal{O}\left(\frac{1}{u\ln u}\right)$$

whereas f_2/u^2 is of order $\mathcal{O}(1/(u\ln u))$. Finally, going back to the variable x yields, the following asymptotic expansion

$$\sqrt{A(x)} = 1 + \frac{\alpha\sqrt{\pi}}{4x} + \frac{3\alpha^2\pi}{32x^2} + \frac{5\alpha^3\pi\sqrt{\pi}}{128x^3} - \frac{\alpha}{2}e^{-x^2} - \frac{3\alpha^2\sqrt{\pi}}{8}\frac{e^{-x^2}}{x} + \mathcal{O}\left(\frac{e^{-x^2}}{x^2}\right).$$

Let $h(x) = 2x^{-1}[(x/x_0)^2 - 1]^{-1/2}$ so that the integral giving the deflection angle can be written in the more compact form $\Delta\varphi(x_0) = -\pi + \int_{x_0}^{\infty} h(x)\sqrt{A(x)}dx$. Then, we get

$$\int_{x_0}^{\infty} h(x)dx = \pi, \qquad \int_{x_0}^{\infty}\frac{h(x)}{x}dx = \frac{2}{x_0},$$
$$\int_{x_0}^{\infty}\frac{h(x)}{x^2}dx = \frac{\pi}{2x_0^2}, \qquad \int_{x_0}^{\infty}\frac{h(x)}{x^3}dx = \frac{4}{3x_0^3}.$$

Moreover, $\int_{x_0}^{\infty} h(x)e^{-x^2}dx = \pi[1 - \mathrm{erf}(x_0)]$ where $\mathrm{erf}(\cdot)$ is the error function for which we can construct the asymptotic expansion

$$\mathrm{erf}(x_0) = 1 - \frac{e^{-x_0^2}}{\sqrt{\pi}x_0} + \mathcal{O}\left(\frac{e^{-x_0^2}}{x_0^2}\right),$$

by means of relations 7.1.2 and 7.1.23 in [48] and

$$\int_{x_0}^{\infty} h(x)\frac{e^{-x^2}}{x}dx = x_0 e^{-x_0^2/2}\left[K_1\left(\frac{x_0^2}{2}\right) - K_1\left(\frac{x_0^2}{2}\right)\right] = \mathcal{O}\left(\frac{e^{-x_0^2}}{x_0^2}\right)$$

where we used the asymptotic expansion 9.7.2 for the modified Bessel functions given in [48]. Putting things together we find that

$$\Delta\varphi(x_0) = \frac{\alpha\sqrt{\pi}}{2x_0}(1 - e^{-x_0^2}) + \frac{3\alpha^2\pi^2}{64x_0^2} + \frac{5\alpha^3\pi\sqrt{\pi}}{96x_0^3} + \mathcal{O}\left(\frac{e^{-x_0^2}}{x_0^2}\right). \tag{69}$$

We plotted the behavior of (69) in Fig. 11. Going back to the unscaled distance of closest approach, the deflection angle in the weak field limit reads

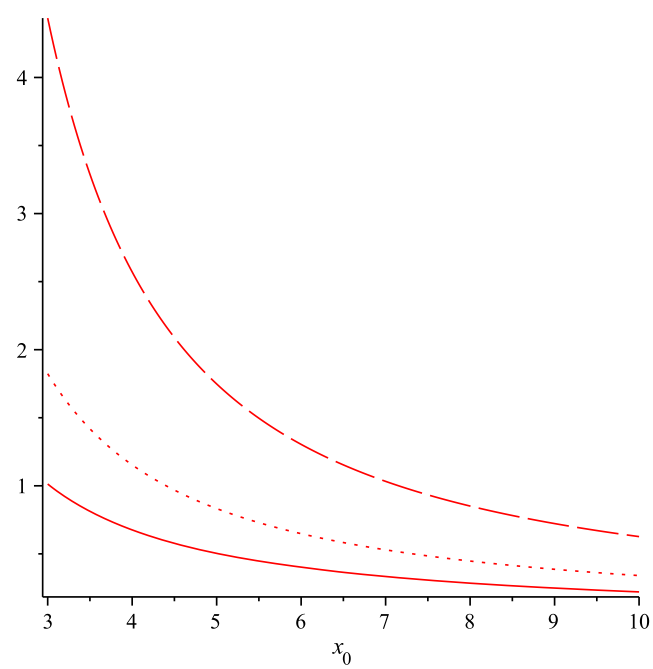

FIG. 11 (color online). Plot of the weak field approximation (69) as a function of the closest distance of approach for $\alpha = 2.2$ (solid line), $\alpha = 3.2$ (dotted line), $\alpha = 5.2$ (long dashed line).

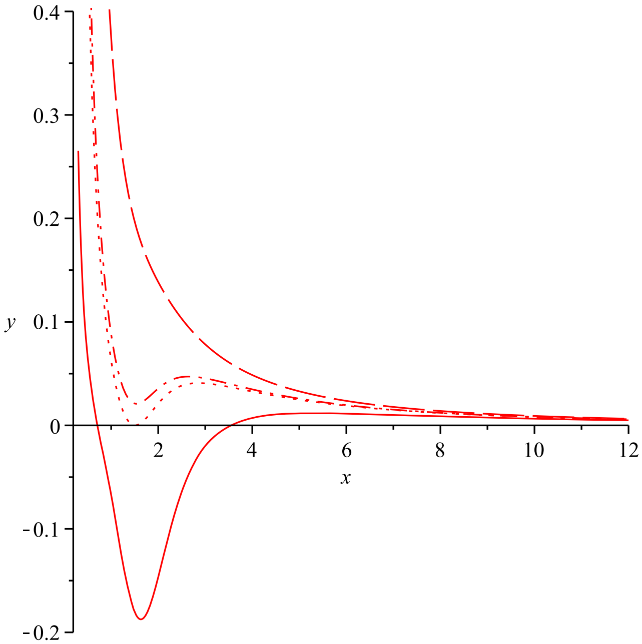

FIG. 12 (color online). Plot of (20) using the dirty black hole metric for the nonxtremal case (solid line, $\alpha = 4$), extremal case (dotted line, $\alpha = \alpha_0$), and the dirty minigravastar (dash dotted and long dashed lines for $\alpha = 2$ and $\alpha = 1$, respectively). It is interesting to observe that in the case of a minigravastar light will form bound states only when the parameter α varies in a certain interval.

$$\Delta\varphi(r_0) = \frac{2M}{r_0}(1 - e^{-r_0^2/(4\theta)}) + \frac{3\pi M^2}{4r_0^2} + \frac{10M^3}{3r_0^3} + \mathcal{O}\left(\frac{e^{-r_0^2/(4\theta)}}{r_0^2/(4\theta)}\right).$$

B. Extreme case

In this case the throats coincide at x_e and we can analyze the metric coefficient $A(x)$ as in [59]. The line element (67) becomes after the usual rescaling $\tilde{t} = t/(2\sqrt{\theta})$, $x = r/(2\sqrt{\theta})$, and $\alpha_0 = 2M_0/(\sqrt{\pi\theta})$

$$ds_E^2 = d\tilde{t}^2 - \frac{dx^2}{(x - x_e)^2\phi(x)} - x^2 d\Omega^2,$$

where ϕ is a differentiable and not vanishing function in the interval $[0, +\infty)$. Moreover,

$$\phi(x_e) = \frac{1}{2}f''(x_e), \phi'(x_e) = \frac{1}{6}f'''(x_e),$$
$$f(x) = 1 - \frac{\alpha}{x}\gamma\left(\frac{3}{2}; x^2\right).$$

Using the software Maple we find the following numerical values $f''(x_1) = 0.5620$ and $f'''(x_1) = -0.3732$. Concerning the weak field limit we can again use formula (69) with α replaced by α_0 since asymptotically at infinity $1/\sqrt{f(x)}$ has the same asymptotic behavior $\sqrt{A(x)}$.

IX. GRAVITATIONAL LENSING FOR NONCOMMUTATIVE GEOMETRY INSPIRED DIRTY BLACK HOLES

Dirty black holes are solutions of Einstein field equations in the presence of various classical matter fields such as electromagnetic fields, dilaton fields, axion fields, Abelian Higgs fields, non-Abelian gauge fields, etc. We will study gravitational lensing for a dirty black hole inspired by noncommutative geometry and described by the line element (1) with [56]

$$A(r) = \left[1 - \frac{4M}{\sqrt{\pi}r}\gamma\left(\frac{3}{2}; \frac{r^2}{4\theta}\right)\right]^{-1},$$
$$B(r) = \frac{1}{A(r)}e^{-\frac{M}{\sqrt{\theta}}[1 - \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}; \frac{r^2}{4\theta})]}$$

with M and $\sqrt{\theta}$ defined as in the previous section. Note that this line element represents a generalization of the noncommutative geometry inspired Schwarzschild metric derived in [58]. After the usual rescaling $\tilde{t} = t/(2\sqrt{\theta})$, $x = r/(2\sqrt{\theta})$, and $\alpha = r_S/\sqrt{\pi\theta}$ the metric functions A and B read

$$A(x) = \left[1 - \frac{\alpha}{x}\gamma\left(\frac{3}{2}; x^2\right)\right]^{-1},$$
$$B(x) = \frac{1}{A(x)}e^{-\frac{\alpha\sqrt{\pi}}{2}[1 - \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}; x^2)]}.$$

The analysis of the existence of horizons x_h can be performed by studying the roots of the equation $A(x) = 0$. Unfortunately their positions can only be given implicitly as

$$x_h = \alpha\gamma\left(\frac{3}{2}; x_h^2\right). \quad (70)$$

In particular, we will have the following scenarios:

(i) two distinct horizons $x_+ > x_-$ for $\alpha > \alpha_0 = 2.1486$ (nonextremal dirty black hole);
(ii) one degenerate horizon at $x_e = x_- = x_+ = 1.5113$ for $\alpha = \alpha_0$ (extremal dirty black hole);
(iii) no horizons for $0 < \alpha < \alpha_0$ (dirty minigravastar).

Note that using (70) and 6.5.3 in [48] we can express the event horizon in terms of the incomplete upper Gamma function as

$$x_+ = \frac{\alpha\sqrt{\pi}}{2} - \alpha\Gamma\left(\frac{3}{2}; x_+^2\right).$$

Since $\Gamma(3/2; x^2) \to 0$ as $x \to \infty$, it can be easily checked that x_+ tends to $\alpha\sqrt{\pi}/2$ which corresponds correctly to

$r_+ \to 2M$. Concerning the embedding diagram we refer to the previous section since noncommutative geometry inspired wormholes and dirty black holes are characterized by having the same metric coefficient A. The radius of the photon sphere can be obtained by finding the roots of the equation

$$\frac{B'(x)}{B(x)} = \frac{2}{x}. \tag{71}$$

Using 6.5.25 in [48] yields $d\gamma(3/2;x^2)/dx = 2x^2 e^{-x^2}$ and the radius x_p of the photon sphere can be given implicitly by the following formula:

$$x_p = \frac{3\alpha\sqrt{\pi}}{4} - \left[\alpha x_p^3 e^{-x_p^2} + \frac{3}{2}\alpha\Gamma\left(\frac{3}{2};x_p^2\right) - \alpha x_p^4 e^{-x_p^2} + \alpha^2 x_p^3 e^{-x_p^2}\gamma\left(\frac{3}{2};x_p^2\right)\right], \tag{72}$$

where we also applied 6.5.3 in [48]. In the Schwarzschild limit $x \to \infty$ the above expression correctly reproduces the radius of the photon sphere at $3M$. Moreover, (72) generalizes formula (2.10) obtained by [60] for the photon sphere of a noncommutative geometry inspired Schwarzschild black hole. Let x_0 denote the distance of closest approach. Then, the rescaled impact parameter is given by

$$\tilde{b}(x_0) = \frac{x_0}{\sqrt{B(x_0)}} = x_0 e^{\frac{\alpha\sqrt{\pi}}{4}[1-\frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2};x_0^2)]}\left[1 - \frac{\alpha}{x_0}\gamma\left(\frac{3}{2};x_0^2\right)\right]^{-1/2}.$$

Since $\tilde{b}'(x_0) = [2B(x_0) - x_0 B'(x_0)]/(2B^{3/2}(x_0))$, it will clearly vanish at $x_0 = x_p$ and the impact parameter can be expanded in a neighbourhood of x_p as

$$\tilde{b}(x_0) = \tilde{b}(x_p) + \frac{1}{2}\tilde{b}''(x_p)(x_0 - x_p)^2 + \mathcal{O}(x_0 - x_p)^3,$$
$$\tilde{b}''(x_p) = \frac{B'(x_p) - x_p B''(x_p)}{2B^{3/2}(x_p)}.$$

Using 6.5.32 in [48] it can be verified that $\tilde{b}(x_p) \to x_p$ as $x_p \to \infty$, from which we recover correctly $b(r_0) = 3\sqrt{3}M$ as one would expect in the case of the Schwarzschild metric. In what follows we will treat at the same time both the nonextreme and the extreme cases by letting $\alpha \geq \alpha_0$. Concerning the weak field limit of the deflection angle we will suppose that $x, x_0 \gg 1$ in

$$\Delta\varphi(x_0) = -\pi + 2\int_{x_0}^{\infty} \frac{\sqrt{A(x)}dx}{x\sqrt{(\frac{x}{x_0})^2 \frac{B(x_0)}{B(x)} - 1}}.$$

An asymptotic expansion for $\sqrt{A(x)}$ has been already derived in the previous section and it is represented by (68). Taking into account that

$$B(x_0) = 1 - \frac{\alpha\sqrt{\pi}}{2x_0} + \alpha e^{-x_0^2} + \mathcal{O}\left(\frac{e^{-x_0^2}}{x_0^2}\right),$$
$$\frac{1}{B(x)} = 1 + \frac{\alpha\sqrt{\pi}}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right),$$

we find that

$$\frac{B(x_0)}{B(x)} = 1 + \frac{\alpha\sqrt{\pi}}{2}\left(\frac{1}{x} - \frac{1}{x_0}\right) + \alpha e^{-x_0^2} + \cdots$$

and hence

$$\left(\frac{x}{x_0}\right)^2 \frac{B(x_0)}{B(x)} - 1 = \left(\frac{x}{x_0}\right)^2 - 1 - \frac{\alpha\sqrt{\pi}}{2}\frac{x(x-x_0)}{x_0^3} + \alpha\frac{x^2}{x_0^2}e^{-x_0^2} + \cdots = \left[\left(\frac{x}{x_0}\right)^2 - 1\right]\Psi(x,x_0),$$

where

$$\Psi(x,x_0) = 1 - \frac{\alpha\sqrt{\pi}}{2}\frac{x}{x_0(x+x_0)} + \frac{\alpha x^2}{x^2 - x_0^2}e^{-x_0^2} + \cdots.$$

Finally, by means of (68) we obtain the asymptotic expansion

$$\frac{\sqrt{A(x)dx}}{x\sqrt{(\frac{x}{x_0})^2\frac{B(x_0)}{B(x)} - 1}} = \frac{1}{x\sqrt{(\frac{x}{x_0})^2 - 1}}\left[1 + \frac{\alpha\sqrt{\pi}}{4}\frac{x^2 + x_0(x+x_0)}{x_0 x(x+x_0)} - \frac{\alpha}{2}e^{-x^2} + \cdots\right].$$

Let $h(x) = 2x^{-1}[(x/x_0)^2 - 1]^{-1/2}$. Then,

$$\int_{x_0}^{\infty} h(x)dx = \pi, \qquad \int_{x_0}^{\infty} h(x)\frac{x^2 + x_0(x+x_0)}{x_0 x(x+x_0)}dx = \frac{4}{x_0},$$
$$\int_{x_0}^{\infty} h(x)e^{-x^2}dx = \pi[1 - \mathrm{erf}(x_0)] = \sqrt{\pi}\frac{e^{-x_0^2}}{x_0} + \mathcal{O}\left(\frac{e^{-x_0^2}}{x_0^2}\right)$$

and the deflection angle can be finally written as

$$\Delta\varphi(x_0) = \frac{\alpha\sqrt{\pi}}{x_0}\left(1 - \frac{e^{-x_0^2}}{2}\right) + \cdots$$

Rewriting the above result by means of the distance of closest approach r_0 as

$$\Delta\varphi(r_0) = \frac{4M}{r_0}\left(1 - \frac{e^{-\frac{r_0^2}{4\theta}}}{2}\right) + \cdots \qquad (73)$$

we see that in the limit $r_0/\sqrt{2\theta} \to \infty$ it correctly reproduces the result one would expect for the classic Schwarzschild metric. From the above formula we see that the effect of noncommutative geometry is that of reducing the deflection angle. Last but not least, formula (73) will also apply to the noncommutative geometry inspired Schwarzschild black hole since for large values of x the metric of a noncommutative dirty black hole goes over into the metric of the aforementioned noncommutative black hole. Concerning the strong gravitational limit of the deflection angle, we cannot introduce the transformations $y = B(x)$ and $z = (1-y)/(1-y_0)$ as in [25] since the metric coefficient B contains the lower incomplete gamma function in such a way that it results in the impossibility of solving analytically the equation $y = B(x)$ for x. For this reason we will adopt the same choice as in [60] and introduce a new variable $z = 1 - x_0/x$ in terms of which the integral giving the deflection angle can be expressed as $\Delta\varphi(x_0) = -\pi + I(x_0)$ with

$$I(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz,$$
$$R(z, x_0) = 2\sqrt{A(z, x_0) B(z, x_0)},$$
$$f(z, x_0) = [B(x_0) - (1-z)^2 B(z, x_0)]^{-1/2},$$

and

$$B(z, x_0) = e^{-\frac{\alpha\sqrt{\pi}}{2}[1 - \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2};\frac{x_0^2}{(1-z)^2})]}\left[1 - \frac{\alpha(1-z)}{x_0}\gamma\left(\frac{3}{2};\frac{x_0^2}{(1-z)^2}\right)\right].$$

At this point some comments are in order. First of all, for the noncommutative geometry inspired Schwarzschild black hole [58] we would have $R(z, x_0) = 2$. Moreover, R is a regular function of z and x_0 with

$$R(0, x_0) = 2e^{-\frac{\alpha\sqrt{\pi}}{4}[1 - \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2};x_0^2)]}, \qquad R(1, x_0) = 2,$$

where for $R(1, x_0)$ we used the result $\lim_{z\to 1}\gamma(3/2;x_0^2/(1-z)^2) = \lim_{x\to\infty}\gamma(3/2;x^2) = \sqrt{\pi}/2$. The result $R(1, x_0) = 2$ is not surprising since for $x \to \infty$ the metric under consideration goes over into the classic Schwarzschild solution. A closer inspection of the function f reveals that there is a singularity at $z = 0$. Expanding the argument of the square root in f to the second order in z we get $f(z, x_0) \approx f_0(z, x_0) = [\tilde{\alpha}(x_0) z^2 + \tilde{\beta}(x_0) z]^{-1/2}$ where

$$\tilde{\alpha}(x_0) = 2B(x_0) - x_0 B'(x_0),$$
$$\tilde{\beta}(x_0) = -B(x_0) + x_0 B'(x_0) - \frac{x_0^2}{2} B''(x_0), \qquad ' = \frac{d}{dx}.$$

We immediately see that at the photon sphere $\tilde{\alpha}(x_p) = 0$ and therefore, f diverges as z^{-1} there, whereas for $x_0 > x_p$ the function f will behave as $z^{-1/2}$ which is clearly integrable at $z = 0$. For $x_0 < x_p$ every photon will be captured by the dirty black hole. As in [25] we split the integral for the deflection angle into a regular and divergent part, respectively, that is $I(x_0) = I_D(x_0) + I_R(x_0)$ where $I_D(x_0) = R(0, x_p)\int_0^1 f_0(z, x_0) dz$ contains the divergence and $I_R(x_0) = \int_0^1 g(z, x_0) dz$ with $g(z, x_0) = R(z, x_0) f(z, x_0) - R(0, x_p) f_0(z, x_0)$ is regular since we subtracted the divergence. The integral I_D can be computed analytically to give

$$I_D(x_0) = \frac{2R(0, x_p)}{\sqrt{\tilde{\beta}(x_0)}} \log\frac{\sqrt{\tilde{\beta}(x_0)} + \sqrt{\tilde{\alpha}(x_0) + \tilde{\beta}(x_0)}}{\sqrt{\tilde{\alpha}(x_0)}}.$$

Expanding $\tilde{\alpha}$ and $\tilde{\beta}$ around the radius x_p of the photon sphere we find

$$\tilde{\alpha}(x_0) = \tilde{\alpha}_1(x_p)(x_0 - x_p) + \mathcal{O}(x_0 - x_p)^2,$$
$$\tilde{\beta}(x_0) = \tilde{\beta}_0(x_p) + \tilde{\beta}_1(x_p)(x_0 - x_p) + \mathcal{O}(x_0 - x_p)^2,$$

TABLE III. Metric components of the line elements for which the gravitational lensing has been studied. Here, NGI stands for noncommutative geometry inspired.

Metric coefficients	B	A	C
Schwarzschild	$1 - \frac{r_S}{r}$	B^{-1}	1
Schwarzschild-de Sitter	$1 - \frac{r_S}{r} - \frac{r^2}{r_\Lambda^2}$	B^{-1}	1
Janis-Newman-Winicour	$(1 - \frac{\mu}{r})^{-\gamma}$	B^{-1}	$(1 - \frac{\mu}{r})^{1-\gamma}$
Dark matter halo	$\begin{cases} 1 + 2\Phi_c + \tilde{\gamma} C_0 x^2 & \text{if } 0 \le x \le x_0 \\ 1 + 2\Phi_c + \tilde{\gamma} A_0 - \frac{\tilde{\gamma}}{x}(B_0 + \ln\frac{1+x}{1+x_0}) & \text{if } x > x_0. \end{cases}$	$\begin{cases} 1 + 2\tilde{\gamma} C_0 x^2 & \text{if } 0 \le x \le x_0 \\ 1 + \frac{\tilde{\gamma}}{x}(D_0 + \ln\frac{1+x}{1+x_0} + \frac{1}{1+x}) & \text{if } x > x_0. \end{cases}$	1
Holographic screen	$1 - \frac{r_S L_p^2 r}{r^2 + L_p^2}$	B^{-1}	1
NGI wormhole	$1 - \frac{2r_S}{\sqrt{\pi} r}\gamma(\frac{3}{2}, \frac{r^2}{4\theta})$	B^{-1}	1
NGI dirty black hole	$\frac{1}{A(r)} e^{-\frac{M}{\sqrt{\theta}}[1 - \frac{2}{\sqrt{\pi}}\gamma(\frac{3}{2}, \frac{r^2}{4\theta})]^{-1}}$	$[1 - \frac{2r_S}{\sqrt{\pi} r}\gamma(\frac{3}{2}, \frac{r^2}{4\theta})]^{-1}$	1

where

$$\tilde{\alpha}_1(x_p) = B'(x_p) - x_p B''(x_p),$$
$$\tilde{\beta}_0(x_p) = -B(x_p) + x_p B'(x_p) - \frac{x_p^2}{2} B''(x_p),$$
$$\tilde{\beta}_1(x_p) = -\frac{x_p^2}{2} B'''(x_p).$$

Hence, the divergent part I_D of the integral can be expanded according to $I_D = -a \log(x_0 - x_p) + b_D + \mathcal{O}(x_0 - x_p)$ where

$$a = \frac{R(0, x_p)}{\sqrt{\tilde{\beta}_0(x_p)}}, \qquad b_D = a \log \frac{4\tilde{\beta}_0(x_p)}{\tilde{\alpha}_1(x_p)}.$$

Following [25] the deflection angle in the strong field limit will be given by $\Delta\varphi(x_0) = -a\log(x_0 - x_p) + b + \mathcal{O}(x_0 - x_p)$ with $b = -\pi + b_D + b_R$ where $b_R = I_R(x_p) = \int_0^1 g(z, x_p)dz + \mathcal{O}(x_0 - x_p)$. Unfortunately, b_R can be only computed numerically. In Table II we present some typical numerical values for the parameters a, b_R, and b_D for the extreme case ($\alpha = \alpha_0$) and the nonextreme case ($\alpha > \alpha_0$). For the treatment of the strong field limit in the presence of the noncommutative geometry inspired Schwarzschild metric, we refer to [60]. In Fig. 12, we have plotted the function defined in Eq. (20) for the dirty black hole under discussion.

X. CONCLUSIONS

In general relativity and cosmology, light becomes an important tool for testing the theories and opening a window to the Universe. The earth- or space-based telescope not only receives the light to give us a picture of the Universe, but the analysis of its red-shift revealed, for instance, the accelerated expansion. Gravitational red-shift and the cosmic microwave background radiation are other examples of the importance of electromagnetic phenomena in gravity.

In this paper we have picked up the classical connection between light and gravity, namely the deflection of light in gravitational field and its lensing. For the "standard" metrics, like Schwarzschild and Schwarzschild-de Sitter we have generalized some formulas and improved upon the numerical accuracy of the final results. With increasing observational accuracy and competing theories of alternative gravity, such precision could prove useful in the future. In the case of a Schwarzschild-de Sitter metric our approach reveals that corrections of the cosmological constant are small given the present value of this constant. This is what one would expect, but other results and approaches do exist in the literature. Therefore, even if we obtained what one might expect, the result is not trivial.

Gravity theory, as many other current physical theories, does not seem to be complete and therefore it is mandatory to venture into new fields of possible future interest (see e.g. [61,62]). Both, on a macroscopic level, where one of the pressing problems is dark matter [49] whose matter content is not known, as well as on the Planck scale where a quantum gravity theory awaits its global acceptance or discovery. We have studied the light bending in a variety of models which correspond to one of the above problems. We have chosen a general relativistic DM metric, a wormhole and a dirty black hole, both inspired by noncommutative geometry to derive the formulas of the deflection angle for light. To this we added a holographic screen metric which is motivated by considerations of physics at the Planck scale. Furthermore, we have given the formulas for the extreme and nonextreme cases making our study quite exhaustive. For an overview we summarize in Table III all the metric elements (A, B, and C entering the general metric (1) of the metrics in which we have studied in detail the motion of light. We paid attention to a global phenomenon emerging in connection with light motion around naked singularities. In all the cases we have examined, we found that there is a narrow range of possible parameters which allow the light to be bound to the source of a naked singularity.

The details of any local metric, indeed its sole existence, can only be revealed by a test particle where light, light scattering, and light bending are ideal tools to do so.

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