PHYSICAL REVIEW D **91**, 104014 (2015)

Weak-field spherically symmetric solutions in $f(T)$ gravity

Matteo Luca Ruggiero*
DISAT, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
INFN, Sezione di Torino, Via Pietro Giuria 1, 10125 Torino, Italy

Ninfa Radicella†
Dipartimento di Fisica E.R. Caianiello, Universita‘ di Salerno, Via Giovanni Paolo II 132, 84084 Fisciano (Sa), Italy
INFN, Sezione di Napoli, Gruppo Collegato di Salerno, via Cintia, 80126 Napoli, Italy
(Received 26 January 2015; published 19 May 2015)

We study weak-field solutions having spherical symmetry in $f(T)$ gravity; to this end, we solve the field equations for a nondiagonal tetrad, starting from Lagrangian in the form $f(T) = T + \alpha T^n$, where α is a small constant, parametrizing the departure of the theory from general relativity. We show that the classical spherically symmetric solutions of general relativity, i.e., the Schwarzschild and Schwarzschild–de Sitter solutions, are perturbed by terms in the form $\propto r^{2-2n}$ and discuss the impact of these perturbations in observational tests.

DOI: 10.1103/PhysRevD.91.104014 PACS numbers: 04.50.Kd, 04.50.-h, 04.80.Cc

I. INTRODUCTION

Since the very discovery of the accelerated cosmic expansion [1,2], and its confirmation due to multiple observations [3–5], it has been customary to investigate theories that extend general relativity (GR), in order to get an agreement with the observations, without requiring the existence of *dark entities*. Hence, motivated by this instance, which recognizes that GR fails in describing gravity at large scales (consider also the old issue of the rotation curves of spiral galaxies [6]), several theories have been proposed to generalize Einstein's theory. Some of these modified models of gravity are geometric extensions of GR; in other words, they are based on a richer geometric structure, which is supposed to give the required ingredients to support the observations.

For a prototype of this approach, one can consider the $f(R)$ theories, in which the gravitational Lagrangian depends on a function f of the curvature scalar R (see Refs. [7,8] and references therein): when $f(R) = R$, the action reduces to the usual Einstein–Hilbert action, and Einstein's theory is obtained. Another example is given by the so-called $f(T)$ theories, which have similarities and differences with respect to $f(R)$. To begin with, they are based on teleparallel gravity (TEGR) [9], in which the gravitational interaction is determined by torsion, and the torsion scalar T appears in the Lagrangian instead of the curvature scalar. Furthermore, the underlying Riemann–Cartan space-time is endowed with the Weitzenböck connection (instead of the Levi-Civitá connection), which is not commutative under the exchange of the lower indices and has non zero curvature but nonzero torsion. Actually, Einstein himself proposed such an alternative point of view on gravitation in terms of torsion and tetrads [10]. In fact, in the TEGR picture, the tetrads field is promoted to be the dynamical field instead of the metric tensor. Despite these differences, TEGR and GR have equivalent dynamics; in other words, every solution of GR is also solution of TEGR. However, when TEGR is generalized to $f(T)$ by considering a gravitational Lagrangian that is a function of the torsion scalar, the equivalence breaks down [11,12]. As a consequence $f(T)$ theories can be considered potential candidates for explaining (on a purely geometric ground) the accelerated expansion of the Universe, without requiring the existence of exotic cosmic fluids (see, e.g., Ref. [13]).

While $f(R)$ theories gives fourth-order equations (at least in the metric formalism, while they are still second order in the Palatini approach; see, e.g., Ref. [8]), the $f(T)$ field equations are second order in the field derivatives since the torsion scalar is a function of the square of the first derivatives of the tetrads field. Furthermore, as for $f(R)$ theories, the generalized TEGR displays additional degrees of freedom (the physical nature of which is still under investigation [14]) related to the fact that the equations of motion are not invariant under local Lorentz transformations [15]. In particular, this implies the existence of a preferential global reference frame defined by the autoparallel curves of the manifold that solve the equations of motion. Consequently, even though the symmetry can help in choosing suitable coordinates to write the metric in a simple way, this does not give any hint on the form of the tetrad. As discussed in Ref. [16], a diagonal tetrad—which could in principle be a good working ansatz for dealing with diagonal metrics—is not a good choice to properly parallelize the spacetime both in the context of nonflat homogenous and isotropic cosmologies (Friedman–Lemaitre–Robertson–Walker universes) and in spherically symmetric space-times (Schwarzschild or Schwarzschild–de Sitter solutions).

*matteo.ruggiero@polito.it
†ninfa.radicella@sa.infn.it

The cases of the Schwarzschild solution and, more in general, of the spherically symmetric solutions in $f(T)$ gravity are particularly important because these solutions, which describe the gravitational field of pointlike sources, allow one to test $f(T)$ theories at scales different from the cosmological ones, e.g., in the Solar System. Such a class of solutions—both with diagonal and nondiagonal tetrads—have been receiving much attention during the last few years; see, for instance, Refs. [17–23]. Indeed, $f(T)$ theories can be used to explain the cosmic acceleration and observations on large scales (e.g., via galaxy clustering and cosmic shear measurements [24]), but we must remember that, since GR is in excellent agreement with Solar System and binary pulsar observations [25], every theory that aims at explaining the large-scale dynamics of the Universe should reproduce GR in a suitable weak-field limit; the same holds true for $f(T)$ theories. Recently, Solar System data [26,27] have been used to constrain $f(T)$ theories; these results are based on the spherical symmetry solution found by Iorio and Saridakis [26], who used a diagonal tetrad. In this paper, we follow the approach described in Ref. [16] to define a "good tetrad" in $f(T)$ gravity—which is consistent with the equations of motion without constraining the functional form of the Lagrangian—and solve the field equations to obtain weak-field solutions with a power-law ansatz for an additive term to the TEGR Lagrangian, $f(T) = T + \alpha T^n$.

This paper is organized as follows. In Sec. II, we review the theoretical framework of $f(T)$ gravity and write the field equations, of which the solutions for spherically symmetric space-times, in the weak-field approximation, are given in Sec. III. Eventually, a discussion and conclusions are in Secs. IV and V.

II. $f(T)$ GRAVITY FIELD EQUATIONS

We start by briefly discussing the $f(T)$ gravity framework that leads to the field equations. To begin with, we point out that, in this scenario, the metric tensor can be viewed as a subsidiary field, and the vierbein field is the dynamical object of which the components in a given coordinate basis e^a_μ are related to the metric tensor by

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x), \tag{1}$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$. Notice that Latin indices refer to the tangent space while Greek indices label coordinates on the manifold. Hence, the dynamics is obtained by the action[1]

$$\mathcal{S} = \frac{1}{16\pi G}\int f(T) e d^4x + \mathcal{S}_M, \tag{2}$$

[1]We use units such as $c = 1$.

where $e = \det\ e^a_\mu = \sqrt{-\det(g_{\mu\nu})}$ and $\mathcal{S}_M$ is the action for the matter fields.[2] Here, $f(T)$ is a differentiable function of the *torsion scalar* T, which is defined as

$$T = S^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu}, \tag{3}$$

where the *contorsion tensor* $S^\rho{}_{\mu\nu}$ is defined by

$$S^\rho{}_{\mu\nu} = \frac{1}{4}(T^\rho{}_{\mu\nu} - T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho) + \frac{1}{2}\delta^\rho_\mu T_{\sigma\nu}{}^\sigma - \frac{1}{2}\delta^\rho_\nu T_{\sigma\mu}{}^\sigma \tag{4}$$

and the *torsion tensor* $T^\lambda{}_{\mu\nu}$ is

$$T^\lambda{}_{\mu\nu} = e^\lambda_a(\partial_\nu e^a_\mu - \partial_\mu e^a_\nu). \tag{5}$$

Varying the action with respect to the vierbein $e^a_\mu(x)$, one gets the field equations

$$e^{-1}\partial_\mu(e e_a{}^\rho S_\rho{}^{\mu\nu}) f_T + e_a{}^\lambda S_\rho{}^{\nu\mu} T^\rho{}_{\mu\lambda} f_T + e_a{}^\rho S_\rho{}^{\mu\nu}\partial_\mu(T) f_{TT} + \frac{1}{4} e^\nu_a f = 4\pi G e_a{}^\mu \mathcal{T}^\nu_\mu, \tag{6}$$

where $\mathcal{T}^\nu_\mu$ is the matter energy-momentum tensor and subscripts T denote differentiation with respect to T.

We look for spherically symmetric solutions of the field equations, so we start from the metric

$$ds^2 = e^{A(r)}dt^2 - e^{B(r)}dr^2 - r^2 d\Omega^2, \tag{7}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Because of the lack of local Lorentz invariance, tetrads connected by local Lorentz transformations lead to the same metric—i.e., the same causal structure—but different equations of motions, thus physically inequivalent solutions. This means that, even in the spherically symmetric case, for which symmetry helps us in choosing the coordinates and the metric tensor in a simple form, it is quite complicated to do an ansatz for the tetrad field. In particular, for the symmetry and coordinates with which we are dealing, it turns out to be a mistake to choose a diagonal form for $e^a{}_\mu$; it does not properly parallelize the static spherically symmetric geometry in the context of $f(T)$ gravity.

Then, with this *caveat* in mind, it is possible to derive the field equations for the nondiagonal tetrad

$$e^a_\mu = \begin{pmatrix} e^{A/2} & 0 & 0 & 0 \\ 0 & e^{B/2}\sin\theta\cos\phi & e^{B/2}\sin\theta\sin\phi & e^{B/2}\cos\theta \\ 0 & -r\cos\theta\cos\phi & -r\cos\theta\sin\phi & r\sin\theta \\ 0 & r\sin\theta\sin\phi & -r\sin\theta\cos\phi & 0 \end{pmatrix},$$

[2]Notice that many authors write the gravitational Lagrangian in the form $T + f(T)$, thus denoting the deviation from GR by means of the function $f(T)$; on the contrary, here $f(T)$ is the *whole* Lagrangian.

following the approach described in Ref. [16]; in doing so, the functional form of the Lagrangian and the specific form of the torsion scalar are not constrained *a priori*. We remark once more that different choices of tetrads, while giving back the same metric, represent different physical theories. In this work, we are interested in a specific tetrads that does not lead to a constant torsion scalar. In such a theory, the Birkhoff theorem does hold, as shown in Ref. [16], while the most general vacuum solution is not of Scwharzschild–de Sitter kind, as it happens with tetrads for which $T = \mathrm{const}$. Then, it is worthwhile to investigate the features of the spherically symmetric solutions in this case for a generic Lagrangian, which nevertheless should admit the Schwarzschild solution when it reduces to the teleparallel equivalent of GR. The field equations are

$$\frac{f(T)}{4} - f_T \frac{e^{-B(r)}}{4r^2}(2 - 2e^{B(r)} + r^2 e^{B(r)} T - 2rB'(r)) - f_{TT} \frac{T'(r)e^{-B(r)}}{r}(1 + e^{B(r)/2}) = 4\pi\rho \tag{8}$$

$$-\frac{f(T)}{4} + f_T \frac{e^{-B(r)}}{4r^2}(2 - 2e^{B(r)} + r^2 e^{B(r)} T - 2rA'(r)) = 4\pi p \tag{9}$$

$$f_T[-4 + 4e^{B(r)} - 2rA'(r) - 2rB'(r) + r^2 A'(r)^2 - r^2 A'(r) B'(r) + 2r^2 A''(r)] + 2r f_{TT} T'(2 + 2e^{B(r)/2} + rA'(r)) = 0, \tag{10}$$

where ρ and p are the energy density and pressure of the matter energy-momentum tensor and the prime denotes differentiation with respect to the radial coordinate r. Moreover, the torsion scalar is

$$T = \frac{2e^{-B(r)}(1 + e^{B(r)/2})}{r^2}[1 + e^{B(r)/2} + rA'(r)]. \tag{11}$$

III. WEAK-FIELD SOLUTIONS

Exact solutions in vacuum ($\rho = p = 0$) and in the presence of a cosmological constant ($\rho = -p$) of the above field equations are thoroughly discussed in Ref. [16]; here, we are interested in weak-field solutions with a nonconstant torsion scalar, i.e., $T' = dT/dr \neq 0$.

Indeed, for actual physical situations such as in the Solar System, the gravitational field is expected to be just a small perturbation of a flat background Minkowski spacetime. As a consequence, we write

$$e^{A(r)} = 1 + A(r), \qquad e^{B(r)} = 1 + B(r) \tag{12}$$

and confine ourselves to linear perturbations. Moreover, we consider Lagrangians of sufficient generality, which we write in the form $f(T) = T + \alpha T^n$, where α is a small constant, parameterizing the departure of these theories from GR, and $|n| \neq 1$.

To begin with, we consider the case $n = 2$, which has been already analyzed in Ref. [26]. From Eqs. (8)–(10), we obtain the solutions

$$A(r) = -32\frac{\alpha}{r^2} - \frac{C_1}{r} \tag{13}$$

$$B(r) = 96\frac{\alpha}{r^2} + \frac{C_1}{r}, \tag{14}$$

where C_1 is an integration constant. Then, on setting $C_1 = 2M$, we get the weak-field limit of the Schwarzschild solution plus a correction due to α:

$$ds^2 = \left(1 - \frac{2M}{r} - 32\frac{\alpha}{r^2}\right)dt^2 - \left(1 + \frac{2M}{r} + 96\frac{\alpha}{r^2}\right)dr^2 - r^2 d\Omega^2. \tag{15}$$

Eventually, the torsion scalar turns out to be

$$T(r) = \frac{8}{r^2} - 128\frac{\alpha}{r^4}. \tag{16}$$

These results can be compared to those obtained in Ref. [26], in which a Lagrangian in the form $f(T) = T + \alpha T^2$ was considered. While to lowest-order approximation in both cases the perturbations are proportional to $1/r^2$, the numerical coefficients are different; this is not surprising, since the authors in Ref. [26] solve different field equations. In particular, they use a diagonal tetrad, which constrains the torsion scalar to be constant (see, e.g., Ref. [16] and references therein); however, the solution given in Ref. [26] does not seem to have a constant torsion scalar, which makes it inconsistent.

Likewise, if we look for solutions of the equations (8)–(10) with $\rho = k$, $p = -k$, which corresponds to a cosmological constant, we obtain

$$ds^2 = \left(1 - \frac{2M}{r} - 32\frac{\alpha}{r^2} - \frac{1}{3}\Lambda r^2\right)dt^2 - \left(1 + \frac{2M}{r} + 96\frac{\alpha}{r^2} + \frac{1}{3}\Lambda r^2\right)dr^2 - r^2 d\Omega^2, \tag{17}$$

where we set $k = \frac{\Lambda}{8\pi}$ and Λ is the cosmological constant. The torsion scalar is the same as Eq. (16). So, the weak-field limit of the Schwarzschild–de Sitter solution is perturbed by terms that are proportional to α.

The previous results can be generalized to the case of a Lagrangian in the form $f(T) = T + \alpha T^n$, and we get

$$A(r) = -\frac{C_1}{r} - \alpha\frac{r^{2-2n}}{2n-3}2^{3n-1} - \frac{1}{3}\Lambda r^2 \tag{18}$$

$$B(r) = \frac{C_1}{r} + \alpha \frac{r^{2-2n}}{2n-3} 2^{3n-1}(-3n+1+2n^2) + \frac{1}{3}\Lambda r^2. \tag{19}$$

In particular, if $\Lambda = 0$, we obtain vacuum solutions. Notice that, on setting $C_1 = 2M$, we obtain a weak-field Schwarzschild–de Sitter solution perturbed by terms that are proportional to α and decay with a power of the radial coordinate, the specific value depending on the power-law chosen in the Lagrangian. The torsion scalar is

$$T(r) = \frac{8}{r^2} + 2\alpha r^{-2n} 2^{3n}(n+1), \tag{20}$$

while the perturbation terms due to the deviation from GR are in the form

$$A_\alpha(r) = \alpha a_n r^{2-2n}, \qquad B_\alpha(r) = \alpha b_n r^{2-2n}, \tag{21}$$

where $a_n = \frac{2^{3n-1}}{2n-3}$, $b_n = \frac{2^{3n-1}}{2n-3}(2n^2 - 3n + 1)$. A close inspection of the perturbation terms reveals that they go to zero both when $r \to \infty$ with $n > 1$ and when $r \to 0$ with $n < 1$. In the latter case, to keep the perturbative approach self-consistent, a maximum value of r must be defined to consider these terms as perturbations of the flat space-time background.

We remark here that our linearized approach can be applied to arbitrary polynomial corrections to the torsion scalar; as a consequence, by writing an arbitrary function as a suitable power series, it is possible to evaluate its impact as a perturbation of the weak-field spherically symmetric solution in GR, and the nth term of the series gives a contribution proportional to r^{2-2n}.

It could be interesting to test the impact of the perturbations (21). To this end, we remember that it is possible to obtain the secular variations of the Keplerian orbital elements due to general spherically symmetric perturbations of the GR solution, describing the gravitational field around a pointlike mass, as one of us showed in Ref. [28]. For instance, the average over one orbital period of the secular precession of the pericenter turns out to be

$$\langle\dot{\omega}\rangle = \frac{1}{4}\alpha \frac{2^{3n-1}(2n-2)(1-e^2)^{3-2n}}{n_b a^{2n}} \mathrm{F}\left(2-n, \frac{5}{2}-n, 2, e^2\right),$$
for $n > \frac{3}{2}$ (22)

$$\langle\dot{\omega}\rangle = \frac{1}{4}\alpha \frac{2^{3n-1}(2-2n)(3-2n)\sqrt{1-e^2}}{(2n-3)n_b a^{2n}} \mathrm{F}\left(n, n-\frac{1}{2}, 2, e^2\right),$$
for $n \leq \frac{1}{2}$. (23)

In the above equations, n_b, a, and e are, respectively, the mean motion, the semimajor axis, and the eccentricity of the unperturbed orbit, while F is the hypergeometric function. These relations can be used to constrain the parameters α and n, on the bases of the ephemerides data.

IV. DISCUSSION

It is useful to comment on the constraints one can infer for the parameters of our model from Solar System data. But before proceeding, it is important to emphasize a point about the tests of $f(T)$ gravity. In theories with torsion, there is a sharp distinction between the test particles trajectories: *autoparallels,* or affine geodesics, are curves along which the velocity vector is transported parallel to itself, by the space-time connection; *extremals*, or metric geodesics, are curves of the extremal space-time interval with respect to the space-time metric [29]. While in GR autoparallels and extremals curves do coincide and we can simply speak of geodesics, the same is not true when torsion is present. So, it is not trivial to define the actual trajectories of test particles. The results obtained by Refs. [26] and [27], together with the expressions (22) and (23) of the secular precession of the pericenter, strictly apply to the case of metric geodesics. According to us, this is a very important issue, which is often neglected in the literature pertaining to theories alternative to GR based to torsion; we will focus on this issue in a forthcoming publication [30]. In the same publication, we will constrain the parameters α and n, taking into account the recent data of the ephemerides of the Solar System provided by INPOP10a [31,32] and EPM2011 [33–35]. Actually, perturbations in the form of the power law are present in different models of modified gravity, and their impact on the Solar System dynamics has been analyzed, for instance, in Refs. [36–38].

Bearing this in mind, it is possible to comment on our results and compare them to those already available in the literature pertaining to $f(T)$ theories. In particular, because of the different choice of the tetrad, our solution, even in the case of a quadratic deformation of the TEGR Lagrangian, differs from the one found by Iorio and Saridakis. Both corrections are proportional to $1/r^2$, but they have different numerical coefficients.

In particular, on substituting $n = 2$ in Eq. (22), we obtain

$$\langle\dot{\omega}\rangle = \frac{16\alpha}{a^4 n_b(1-e^2)}. \tag{24}$$

On the contrary, the corresponding expression obtained by Iorio and Saridakis [26] is

$$\langle\dot{\omega}\rangle_{\mathrm{IS}} = \frac{3\alpha}{a^4 n_b(1-e^2)}. \tag{25}$$

We see that they differ for a factor $16/3$; the same happens to the constraints that can be obtained from our solution, by applying the approach described in Refs. [26] and [27].

In particular, Iorio and Saridakis [26] derive constraints from the rate of change of perihelia of the first four inner planets, obtaining

$$|\Lambda| \leq 6.1 \times 10^{-42}\ \mathrm{m}^{-2}$$
$$|\alpha| \leq 1.8 \times 10^{4}\ \mathrm{m}^{2}.$$

Tighter results have been obtained in a subsequent paper [27], in which the authors consider upper bounds deriving from different phenomena: perihelion advance, light bending, and gravitational time delay [39–42]. But the strongest constraints come from the perihelion advance, in particular, from some supplementary advances constructed by considering that the effects due to the Sun's quadrupole mass moment might represent possible unexplained parts of perihelion advance in GR [31]. This gives

$$|\Lambda| \leq 1.8 \times 10^{-43}\ \mathrm{m}^{-2}$$
$$|\alpha| \leq 1.2 \times 10^{2}\ \mathrm{m}^{2}.$$

The upper bound for the solution in Eqs. (17) would be $3/16$ smaller; that is $|\alpha| \leq 2.3 \times 10\ \mathrm{m}^2$.

Eventually, we comment on the issue of the parametrized post-Newtonian formalism (PPN), in the framework of $f(T)$ gravity. To test theories of gravity that give rise to detectable torsion effects in the Solar System, a theory-independent formalism that generalizes the PPN formalism when torsion is present was developed in Ref. [29] (see also Ref. [43]). Starting from symmetry arguments, the metric and the connection around a massive body are perturbatively expressed in terms of dimensionless parameters related to the matter-energy content of the source, namely, its mass and its angular momentum per unit mass. In doing so, the new parameters, which add to the original PPN ones, can be constrained by the experiments. Our results, however, cannot be directly described in this framework: an inspection of our solutions (21) clearly shows that the perturbations are *not related* to the matter-energy content of the source, but rather they depend on α, which parametrizes the departure of the $f(T)$ theory from GR (see, e.g., Eq. (4.2) in Ref. [43]). So, a new formalism is required to test the content of the Lagrangian by means of observations; in a sense, α can be considered a *new* post-Newtonian parameter of this formalism.

V. CONCLUSIONS

We studied spherically symmetric solutions in the weak approximation of $f(T)$ gravity. In particular, we started from a Lagrangian in the form $f(T) = T + \alpha T^n$, with $|n| \neq 1$, where α is a small constant that parametrizes the departure of these theories from GR, and solved the field equations using a nondiagonal tetrad, showing that, to lowest approximation order, the perturbations of the corresponding GR solutions (Schwarzchild or Schwarzchild–de Sitter) are in the form $\propto \alpha r^{2-2n}$. These results can be used to evaluate the impact of the nonlinearity of the Lagrangian, for instance, in the Solar System.

The case $n = 2$, corresponding to the Lagrangian $f(T) = T + \alpha T^2$, has been already analyzed by Refs. [26] and [27], in which the authors used Solar System observations to set constraints on the parameter α. It is important to point out that the latter results are based on the solution obtained by Iorio and Saridakis [26], in which a diagonal tetrad was used, which forces the torsion scalar to be constant; however, that solution does not seem to have a constant torsion scalar, which makes the consequent constraints not reliable.

On the other hand, since because of the invariance properties of $f(T)$ the choice of the tetrad field is crucial, we performed our calculations by using a more general nondiagonal tetrad, according to the prescriptions given in Ref. [16], and obtained a new solution of the $f(T)$ quadratic model, for which the torsion scalar is not forced to be zero.

We used the results already available in the literature to obtain the correct constraints from Solar System data on the α parameter for the Lagrangian $f(T) = T + \alpha T^2$, even if we pointed out that the distinction between autoparallels, or affine geodesics, and extremals, or metric geodesics, is crucial in $f(T)$ gravity and deserves further investigation that we are going to carry out in forthcoming publications.

ACKNOWLEDGMENTS

The authors acknowledge support by Istituto Nazionale di Fisica Nucleare and by the Italian Ministero dell'Istruzione dell'Università e della Ricerca.

[1] A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116**, 1009 (1998).
[2] S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **517**, 565 (1999).
[3] G. Hinshaw *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **208**, 19 (2013).
[4] D. J. Eisenstein *et al.* (SDSS Collaboration), Astrophys. J. **633**, 560 (2005).

[5] Y. Wang, Phys. Rev. D **78**, 123532 (2008).
[6] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University, Princeton, 1987).
[7] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).
[8] S. Capozziello and M. Francaviglia, Gen. Relativ. Gravit. **40**, 357 (2008).
[9] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction* (Springer Dordrecth, Germany, 2012).
[10] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. **22**, 414 (1925); 217 (1928); A. Einstein, Math. Ann. **102**, 685 (1930).
[11] R. Ferraro and F. Fiorini, Phys. Rev. D **78**, 124019 (2008).
[12] F. Fiorini and R. Ferraro, Int. J. Mod. Phys. A **24**, 1686 (2009).
[13] V. F. Cardone, N. Radicella, and S. Camera, Phys. Rev. D **85**, 124007 (2012).
[14] M. Li, R. X. Miao, and Y. G. Miao, J. High Energy Phys. 07 (2011) 108.
[15] B. Li, T. P. Sotiriou, and J. D. Barrow, Phys. Rev. D **83**, 064035 (2011).
[16] N. Tamanini and C. G. Boehmer, Phys. Rev. D **86**, 044009 (2012).
[17] R. Ferraro and F. Fiorini, Phys. Rev. D **84**, 083518 (2011).
[18] T. Wang, Phys. Rev. D **84**, 024042 (2011).
[19] C. G. Bohmer, A. Mussa, and N. Tamanini, Classical Quantum Gravity **28**, 245020 (2011).
[20] H. Dong, Y.-b. Wang, and X.-h. Meng, Eur. Phys. J. C **72**, 2002 (2012).
[21] M. E. Rodrigues, M. J. Houndjo, J. Tassa, D. Moment, and R. Myrzakulov, J. Cosmol. Astropart. Phys. 11 (2013) 024.
[22] G. G. L. Nashed, Phys. Rev. D **88**, 104034 (2013).
[23] A. Paliathanasis, S. Basilakos, E. N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabo, and M. Tsamparlis, Phys. Rev. D **89**, 104042 (2014).
[24] S. Camera, V. F. Cardone, and N. Radicella, Phys. Rev. D **89**, 083520 (2014).
[25] C. M. Will, Living Rev. Relativity **9**, 3 (2006).
[26] L. Iorio and E. N. Saridakis, Mon. Not. R. Astron. Soc. **427**, 1555 (2012).
[27] Y. Xie and X. M. Deng, Mon. Not. R. Astron. Soc. **433**, 3584 (2013).
[28] M. L. Ruggiero, Int. J. Mod. Phys. D **23**, 1450049 (2014).
[29] Y. Mao, M. Tegmark, A. H. Guth, and S. Cabi, Phys. Rev. D **76**, 104029 (2007).
[30] N. Radicella, M. L. Ruggiero, and L. Iorio (unpublished).
[31] A. Fienga *et al.*, *IAU Symposium, Gravity Tests in INPOP Planetary Ephemerides, USA, 2009*, edited by S. A. Klioner, P. K. Seidelmann, and M. H. Soffel (Cambridge University Press, Cambridge, 2009), Vol. 261, p. 159.
[32] A. Fienga, J. Laskar, P. Kuchynka, H. Manche, G. Desvignes, M. Gastineau, I. Cognard, and G. Theureau, Celestial Mechanics and Dynamical Astronomy **111**, 363 (2011).
[33] N. P. Pitjev and E. V. Pitjeva, Astron. Lett. **39**, 141 (2013).
[34] E. V. Pitjeva, Solar System Research **47**, 386 (2013).
[35] E. V. Pitjeva and N. P. Pitjev, Mon. Not. R. Astron. Soc. **432**, 3431 (2013).
[36] L. Iorio, Adv. High Energy Phys. **2007**, 90731 (2007).
[37] G. S. Adkins and J. McDonnell, Phys. Rev. D **75**, 082001 (2007).
[38] M.-Y. Liu, Z.-H. Zhong, Y.-C. Han, X.-Y. Wang, Z.-S. Yang, and Y. Xie, Res. Astron. Astrophys. **14**, 1019 (2014).
[39] K. Lake, Phys. Rev. D **65**, 087301 (2002).
[40] V. Kagramanova, J. Kunz, and C. Lammerzhal, Phys. Lett. B **634**, 465 (2006).
[41] R. A. Nelson, Metrologia **48**, S171 (2011).
[42] E. B. Fomalont and S. M. Kopeikin, Astrophys. J. **598**, 704 (2003).
[43] R. March, G. Bellettini, R. Tauraso, and S. Dell'Agnello, Phys. Rev. D **83**, 104008 (2011).