Consistent massive graviton on arbitrary backgrounds

Laura Bernard,[1] Cédric Deffayet,[1,2] and Mikael von Strauss[1]

[1]*UPMC-CNRS, UMR7095, Institut d'Astrophysique de Paris,*
GReCO, 98bis boulevard Arago, F-75014 Paris, France
[2]*IHES, Le Bois-Marie, 35 route de Chartres, F-91440 Bures-sur-Yvette, France*

We obtain the fully covariant linearized field equations for the metric perturbation in the de Rham-Gabadadze-Tolley (dRGT) ghost-free massive gravities. For a subset of these theories, we show that the nondynamical metric that appears in the dRGT setup can be completely eliminated leading to the theory of a massive graviton moving in a single metric. This has a mass term which contains nontrivial contributions of the space-time curvature. We show further how five covariant constraints can be obtained including one which leads to the tracelessness of the graviton on flat space-time and removes the Boulware-Deser ghost. The five constraints are obtained for a background metric which is arbitrary, i.e. which does not have to obey the background field equations.

Lately there has been a renewal of interest in massive gravity with interesting applications to cosmology (see e.g. [1] for reviews). The only consistent linear theory for a massive graviton on flat space-time has been known for a long time since the work of Fierz and Pauli [2]. It propagates 5 degrees of freedom of positive energy, those of a transverse, traceless, symmetric, 2 times covariant tensor $h_{\mu\nu}$. It can easily be extended to an Einstein space-time background keeping the same number of propagating polarizations [3,4]. However, a similar theory for an arbitrary background metric has not been written so far. A starting point to do so is the set of fully nonlinear theories formulated by de Rham, Gabadadze and Tolley (dRGT in the following) [5]. Such a theory was shown to contain only 5 dynamical degrees of freedom [5,6] and hence is devoid from a pathology long thought unavoidable: the presence of an extra ghostlike sixth degree of freedom in a generic nonlinear extension of Fierz-Pauli theory first discussed by Boulware and Deser in the seminal paper [7]. Hence, *a priori* one should be able to extract from the dRGT family a consistent linear theory for a massive graviton moving on a large class of metrics [8]. However, the dRGT theory is formulated using two metrics, a dynamical one called $g_{\mu\nu}$ in the following, and a nondynamical one, usually taken to be flat, called here $f_{\mu\nu}$. Hence, the expectation is that when one linearizes the dRGT field equations one will obtain a theory for a massive graviton moving in a space-time endowed with two background metrics which has various drawbacks (see e.g. Ref. [9]). Moreover, this linearization is not easy, in part because dRGT theories involve a matrix square root S of the tensor $\mathfrak{F}$ both defined by

$$S^{\mu}{}_{\sigma}S^{\sigma}{}_{\nu} = g^{\mu\sigma}f_{\sigma\nu} = \mathfrak{F}^{\mu}{}_{\nu}. \quad (1)$$

Lastly, it may not be easy to show that the obtained theory contains the correct number of degrees of freedom without

using the elegant but involved proof obtained for the nonlinear theories [6]. Here, we will overcome these various difficulties and show how to obtain from dRGT models a fully covariant theory for a massive graviton moving in a single, totally arbitrary, metric (hence eliminating the need for the nondynamical metric). We will also show how, for such a theory, one can obtain five covariant constraints, including one which leads to the tracelessness of the graviton on flat space-time and removes the Boulware-Deser ghost [10]. This last constraint involves combinations of the curvature of the background metric which become trivial when this metric describes a flat space-time or a more general Einstein space-time.

Our starting point is the set of massive gravity theories defined by the following action in *four* dimensions [5,6]

$$S_{g,m} = M_g^2 \int \mathrm{d}^4x \sqrt{|g|} \left[R(g) - 2m^2 \sum_{n=0}^{3} \beta_n e_n(S) \right], \quad (2)$$

the β_n being dimensionless parameters, and $e_n(S)$ the nth order elementary symmetric polynomial of the eigenvalues of its matrix argument S. One has in particular $e_0 = 1$ and $e_1 = \mathrm{Tr}[S]$, where here and henceforth $\mathrm{Tr}[X] = X^{\rho}{}_{\rho}$ indicates a matrix trace operation and we do not write out anymore the functional dependence of the e_n when they depend only on S [i.e. it is to be understood that $e_n \equiv e_n(S)$]. The e_n can be constructed iteratively (with $e_0 = 1$) from the relation

$$e_n = -\frac{1}{n}\sum_{k=1}^{n}(-1)^k \mathrm{Tr}[S^k]e_{n-k}, \qquad n \geq 1, \quad (3)$$

where S^k is the kth power of the tensor $S^{\mu}{}_{\nu}$ (considered as a matrix), and S^0 is just the identity. The field equations deriving from the action (2) for the dynamical metric $g_{\mu\nu}$ are

$$E_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + m^2 V_{\mu\nu} = 0, \qquad (4)$$

where $\mathcal{G}_{\mu\nu}$ is the Einstein tensor built from the metric $g_{\mu\nu}$, and $V_{\mu\nu}$ is given by

$$V_{\mu\nu} = g_{\mu\rho} \sum_{n=0}^{3} \sum_{k=0}^{n} (-1)^{n+k} \beta_n [S^{n-k}]^\rho{}_\nu e_k. \qquad (5)$$

The next step in our derivation consists of linearizing these field equations around a background solution for the dynamical metric $g_{\mu\nu}$, calling $h_{\mu\nu}$ the small perturbation of this metric. This may seem at first sight an easy task; however, it is not because it involves in general computing the variation at first order in $h_{\mu\nu}$ of the matrix square root S.

This variation δS obeys (with obvious notations), as seen from (1),

$$S^\mu{}_\nu (\delta S)^\nu{}_\sigma + (\delta S)^\mu{}_\nu S^\nu{}_\sigma = \delta \mathfrak{F}^\mu{}_\sigma, \qquad (6)$$

which is a special kind of Sylvester matrix equation, where the right-hand side is easy to get in terms of $h_{\mu\nu}$ from the definition of $\mathfrak{F}$. It is known that this equation has a unique solution for δS if and only if the spectra of $S^\mu{}_\nu$ and $-S^\mu{}_\nu$ do no intersect (which is generically the case here). In this case, one can express the solution for δS linearly in terms of $\delta \mathfrak{F}$ [11]. Using our own derivation [12] which is more convenient for our purpose here, we obtain

$$
\begin{aligned}
\frac{\delta S^\lambda{}_\mu}{\delta g_{\rho\sigma}} = \frac{1}{2} g^{\nu\lambda} & [e_4 c_1 (\delta^\rho_\nu \delta^\sigma_\mu + \delta^\sigma_\nu \delta^\rho_\mu - g_{\mu\nu} g^{\rho\sigma}) + e_4 c_2 (S^\rho_\nu \delta^\sigma_\mu + S^\sigma_\nu \delta^\rho_\mu - S_{\mu\nu} g^{\rho\sigma} - g_{\mu\nu} S^{\rho\sigma}) - e_3 c_1 (\delta^\rho_\nu S^\sigma_\mu + \delta^\sigma_\nu S^\rho_\mu) \\
& + e_4 c_3 [\delta^\sigma_\mu [S^2]^\rho_\nu + \delta^\rho_\mu [S^2]^\sigma_\nu - g^{\rho\sigma} [S^2]_{\mu\nu} + \delta^\rho_\nu [S^2]^\sigma_\mu + \delta^\sigma_\nu [S^2]^\rho_\mu - g_{\mu\nu} [S^2]^{\rho\sigma}] + (e_2 c_1 - e_4 c_3 + e_3 c_2) S_{\mu\nu} S^{\rho\sigma} \\
& - e_3 c_2 (S^\rho_\nu S^\sigma_\mu + S^\sigma_\nu S^\rho_\mu) - e_3 c_3 (S^\sigma_\mu [S^2]^\rho_\nu + S^\rho_\mu [S^2]^\sigma_\nu + S^\rho_\nu [S^2]^\sigma_\mu + S^\sigma_\nu [S^2]^\rho_\mu) \\
& + (e_3 c_3 - e_1 c_1)(S^{\rho\sigma} [S^2]_{\mu\nu} + S_{\mu\nu} [S^2]^{\rho\sigma}) - (c_1 - e_2 c_3)([S^2]^\rho_\nu [S^2]^\sigma_\mu + [S^2]^\sigma_\nu [S^2]^\rho_\mu) \\
& + c_4 [S^2]_{\mu\nu} [S^2]^{\rho\sigma} + c_1 ([S^3]_{\mu\nu} S^{\rho\sigma} + S_{\mu\nu} [S^3]^{\rho\sigma}) + c_2 ([S^3]_{\mu\nu} [S^2]^{\rho\sigma} + [S^2]_{\mu\nu} [S^3]^{\rho\sigma}) + c_3 [S^3]_{\mu\nu} [S^3]^{\rho\sigma}], \qquad (7)
\end{aligned}
$$

where the coefficients c_i are given by

$$c_1 = \frac{e_3 - e_1 e_2}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}, \qquad c_2 = \frac{e_1^2}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}, \qquad c_3 = \frac{-e_1}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}, \qquad c_4 = \frac{e_3 - e_1^3}{-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4}, \qquad (8)$$

and here and henceforth all indices are moved with the metric $g_{\mu\nu}$. Obviously, expression (7) makes sense only if $-e_1 e_2 e_3 + e_3^2 + e_1^2 e_4$ does not vanish, which in turn can be shown to be equivalent to the nonintersection of the spectra of $S^\mu{}_\nu$ and $-S^\mu{}_\nu$ mentioned above.

This result was checked to agree with the one we know from the mathematical literature [11] using in particular nontrivial identities—syzygies—which also play a fundamental role for the derivation of the covariant constraints below. These identities, as a consequence of the "second fundamental theorem" of invariant theory [13], can be derived using the Cayley-Hamilton theorem stating that for an arbitrary 4×4 matrix M, one has

$$M^4 = e_1(M) M^3 - e_2(M) M^2 + e_3(M) M - e_4(M) \mathbb{1}. \qquad (9)$$

One can then apply this to a matrix M built out of four arbitrary matrices A, B, C, D and four arbitrary real numbers $\{x_i\}$ in the form, $M = x_0 A + x_1 B + x_2 C + x_3 D$. Now, because the $\{x_i\}$ as well as the matrices A, B, C, D are arbitrary, it means that in Eq. (9) the terms which have the same degree of homogeneity in the $\{x_i\}$ must each yield

separate identities between the matrices A, B, C, D. Once these identities are obtained, one can replace in them A by h, B by S, C by S^2 and D by S^3 to get nontrivial matrix syzygies denoted here as $[\mathcal{I}_k]^\mu{}_\nu = 0$, between the tensors of interest here. Notice that the above Cayley-Hamilton equation (9) can also be used iteratively, when applied to the matrix S, to replace any power of S, S^k, with $k \geq 4$ by a linear combination of powers of S, S^i with $i \leq 3$. This was done systematically in order to reach the expression (7).

Using (7), one can get the linearization of field equations (4) around an arbitrary metric $g_{\mu\nu}$ reading [14]

$$\delta E_{\mu\nu} \equiv \delta \mathcal{G}_{\mu\nu} + m^2 \mathcal{M}_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} = 0, \qquad (10)$$

where $\delta \mathcal{G}_{\mu\nu}$ is the linearization of the Einstein tensor

$$
\begin{aligned}
\delta \mathcal{G}_{\mu\nu} = -\frac{1}{2} & [\delta^\rho_\mu \delta^\sigma_\nu \nabla^2 + g^{\rho\sigma} \nabla_\mu \nabla_\nu - \delta^\rho_\mu \nabla^\sigma \nabla_\nu \\
& - \delta^\rho_\nu \nabla^\sigma \nabla_\mu - g_{\mu\nu} g^{\rho\sigma} \nabla^2 + g_{\mu\nu} \nabla^\rho \nabla^\sigma] h_{\rho\sigma} \\
& + \frac{1}{2} [g_{\mu\nu} R^{\rho\sigma} - \delta^\rho_\mu \delta^\sigma_\nu R] h_{\rho\sigma} \qquad (11)
\end{aligned}
$$

and the "mass matrix" $\mathcal{M}_{\mu\nu}{}^{\rho\sigma}$ is defined through

$$\mathcal{M}_{\mu\nu}{}^{\rho\sigma} \equiv \frac{\partial V_{\mu\nu}}{\partial g_{\rho\sigma}}, \tag{12}$$

and is given by the following expression,

$$\mathcal{M}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2} V_{\mu}{}^{\sigma} \delta_{\nu}^{\rho} - \frac{1}{2} (\beta_2 \delta_{\mu}^{\lambda} + \beta_3 (e_1 \delta_{\mu}^{\lambda} - S_{\mu}^{\lambda}))[S^2]_{\lambda}^{\sigma} \delta_{\nu}^{\rho} + \frac{1}{4} \sum_{n=1}^{3} \sum_{k=1}^{n} \sum_{m=1}^{k} (-1)^{n+k+m} \beta_n e_{k-m} [S^{n-k}]^{\lambda}{}_{\mu} g_{\nu\lambda} g^{\tau(\rho} [S^m]^{\sigma)}{}_{\tau}$$

$$- \frac{1}{2} (\beta_1 \delta^{\tau}{}_{\lambda} + \beta_2 e_1 \delta^{\tau}{}_{\lambda} + \beta_3 (e_2 \delta^{\tau}{}_{\lambda} + [S^2]^{\tau}{}_{\lambda})) \frac{\delta S^{\lambda}{}_{\mu}}{\delta g_{\rho\sigma}} g_{\nu\tau} + (\mu \leftrightarrow \nu). \tag{13}$$

Note that here, for the sake of simplicity, we have only considered the case of a massive graviton in vacuum, but it can easily be coupled to some matter source by introducing an energy-momentum tensor $\delta T_{\mu\nu}$ on the right-hand side of Eq. (10). Provided that this tensor is conserved with respect to the background metric $g_{\mu\nu}$ (as it is usually assumed in massive gravity, and follows from dRGT action if one just couples matter minimally to the dynamical metric), the results presented in the following on the presence of the extra scalar constraint will hold, since this follows from the linearized Bianchi identity.

In the most general case, Eq. (10) still contains two different metrics, the background dynamical metric $g_{\mu\nu}$ and the nondynamical metric $f_{\mu\nu}$ which can be traded for the tensor S^2. However, there is a subclass of dRGT models where one can explicitly get rid of this second metric and obtain field equations for a massive graviton on the background of just one metric $g_{\mu\nu}$. This subclass of models (called here β_1 models) is defined by setting to zero the parameters β_2 and β_3. In this case, indeed, $V_{\mu\nu}$ is linear in S and the background field equations can just be reexpressed as [15]

$$S^{\rho}{}_{\nu} = \frac{1}{\beta_1 m^2} \left[R^{\rho}{}_{\nu} - \frac{1}{6} \delta^{\rho}_{\nu} R - \frac{m^2 \beta_0}{3} \delta^{\rho}_{\nu} \right], \tag{14}$$

where $R_{\rho\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$, and R the corresponding Ricci scalar. Using (1), this can equivalently be seen (by squaring the above equation) as expressing $f_{\mu\nu}$ in terms of $g_{\mu\nu}$ and its Ricci curvatures. This remarkable feature means that in the linearized equations of motion, we can eliminate any and all occurrences of the auxiliary metric $f_{\mu\nu}$ in favor of $g_{\mu\nu}$ and its curvature. This feature only requires a nonvanishing β_1 [16].

Having carried out this elimination, we take the obtained linearized field equation (10) as a new starting point, and ask if one can show from these equations that the graviton $h_{\mu\nu}$ propagates five polarizations (or less) for a completely generic metric $g_{\mu\nu}$ (i.e. without assuming it obeys the background equations). The idea here is to try to parallel

what can be done for a massive graviton on flat space-time with metric $\eta_{\mu\nu}$ (and easily extended to a massive graviton on an Einstein space-time [3,4]). In this case, the linearized Bianchi identities lead, by taking one derivative of the field equations, to four constraints reading

$$\partial^{\mu} h_{\mu\nu} - \partial_{\nu} h = 0. \tag{15}$$

Taking another derivative of this equation and subtracting this from the trace (tracing with $\eta_{\mu\nu}$) of the field equations, one then concludes that h defined as $h = \eta^{\mu\nu} h_{\mu\nu}$ vanishes in vacuum. Together with (15) this gives five Lagrangian constraints, which eliminate as many degrees of freedom out of the *a priori* 10 dynamical degrees of freedom of $h_{\mu\nu}$.

Let us then try to follow a similar path from the field equations (10). First, it is easy to find four vector constraints similar to (15). Indeed, as a consequence of the Bianchi identities, one has

$$\nabla^{\mu} \delta\mathcal{G}_{\mu\nu} \sim 0, \tag{16}$$

where here and henceforth ∇ denotes the covariant derivative taken with respect to the background metric and two expressions separated by the symbol "$\sim$" are by definition equal *off shell* (i.e. without using the field equations) up to terms containing no second or higher order derivatives acting on $h_{\mu\nu}$. Hence, the field equations yield the four vector constraints (being first order in derivatives)

$$\nabla^{\mu} \delta E_{\mu\nu} = 0. \tag{17}$$

In analogy with the flat space case we are interested in finding a fifth scalar constraint which generalizes the constraint $h = 0$. This fifth constraint should be the one which eliminates the Boulware-Deser ghost and reduces the number of degrees of freedom from 6 to 5. Accordingly, we look for a linear combination of scalars made by tracing over the field equations (10), and its second derivatives, which would not contain any derivatives of $h_{\mu\nu}$ of order strictly higher than one. However, we have now at hand two (symmetric) tensors which can be used to take traces,

namely the metric $g_{\mu\nu}$ and its Ricci curvature $R_{\mu\nu}$. Equivalently we can also use the metric and the tensor $S_{\mu\nu}$ trading $R_{\mu\nu}$ for $S_{\mu\nu}$ via Eq. (14). Choosing the second solution turns out to be more convenient for technical reasons. We stress however that the two possibilities are strictly equivalent and do not impose any restriction on $g_{\mu\nu}$, since (14) can just be considered as a definition of $S_{\mu\nu}$ in terms of $R_{\mu\nu}$ and $g_{\mu\nu}$ (as opposed to a background field equation). Hence we define the scalars Φ_i obtained by tracing the equations of motion with powers of S,

$$\Phi_i \equiv [S^i]^{\mu\nu} \delta E_{\mu\nu}, \qquad (18)$$

together with the scalars Ψ_i obtained by tracing the derivative of the divergence of the equations of motion in various ways,

$$\Psi_i \equiv \frac{1}{2} [S^i]^{\mu\nu} \nabla_\nu \nabla^\lambda \delta E_{\lambda\mu}. \qquad (19)$$

An exhaustive set of linearly independent scalars is obtained by restricting i, $0 \leq i \leq 3$, due the Cayley-Hamilton identity. To summarize we look for a specific linear combination of the scalars Φ_i and Ψ_i, $i = 0, \dots, 3$ with scalar coefficients $\{u_i, v_i\}$ to be determined, such that

$$\sum_{i=0}^{3} (u_i \Phi_i + v_i \Psi_i) \sim 0, \qquad (20)$$

i.e. which contains no second (or higher) derivatives of $h_{\mu\nu}$. Computing explicitly the scalars Φ_i and Ψ_i [12], one obtains that, in these scalars, the second derivatives of $h_{\mu\nu}$ appear in the form of linear combinations (with S-dependent coefficients) of 26 different scalars $\aleph_i$ made by contracting $\nabla_\mu \nabla_\nu h_{\rho\sigma}$ with powers of S (including zeroth power which is simply the metric) in various ways. Two of these scalars are e.g. $\nabla_\rho \nabla_\sigma h^{\rho\sigma}$ and $[S^3]^{\rho\sigma}[S^3]^{\mu\nu} \nabla_\rho \nabla_\sigma h_{\mu\nu}$.

We get *a priori* 26 equations for the seven unknowns $\{u_i, v_i\}$ [17] by setting to zero each coefficient of the $\aleph_i$ which appears in (20). However, one can show that not all the scalars $\aleph_i$ are independent, thanks to the syzygies $\mathcal{I}_k$. Indeed, the equation $\nabla_\mu \nabla^\nu [\mathcal{I}_k]^\mu_{\ \nu} = 0$, together with the use of the Cayley-Hamilton theorem for S yields four independent identities between the scalars $\aleph_i$ which vanish up to terms ~ 0. These identities are just enough to reduce to seven the number of equations to be solved to fulfill (20). This yields a unique solution for the coefficients $\{u_i, v_i\}$ which translates into the identity

$$\frac{m^2 \beta_1 e_4}{4} \Phi_0 \sim -e_3 \Psi_0 + e_2 \Psi_1 - e_1 \Psi_2 + \Psi_3. \qquad (21)$$

Hence, using the field equations, we get the scalar constraint

$$-\frac{m^2 \beta_1 e_4}{4} \Phi_0 - e_3 \Psi_0 + e_2 \Psi_1 - e_1 \Psi_2 + \Psi_3 = 0, \qquad (22)$$

valid now for an arbitrary metric $g_{\mu\nu}$. Notice that the curvature of this metric enters this constraint in a nontrivial way via the tensor S. One can check in fact that to the lowest nontrivial order (i.e. at linear order) in curvature, the analysis done here agrees with the one of [4] which investigates at this order the consistent coupling of a massive graviton to a curved background.

A nontrivial check is obtained considering the case where the background metric is covering an Einstein space-time with a cosmological constant Λ. In this case one has $R_{\mu\nu} = \Lambda g_{\mu\nu}$ and the constraint (22) has the usual form found in Einstein space [3,4], i.e. it reads

$$h\left(1 - \frac{2\Lambda}{3m_{FP}^2}\right) = 0, \qquad (23)$$

where m_{FP} is the graviton mass as appearing in the Fierz-Pauli action extended to Einstein space-time backgrounds. This mass is given here by the relation $3m_{FP}^2 = \Lambda - m^2 \beta_0$. For generic Einstein space-times, including flat ones, this constraint reduces to $h = 0$ which also shows that (22) is independent of the vector constraints (17), as it should be. One also recovers the partially massless case (for $3m_{FP}^2 = 2\Lambda$) where this constraint vanishes identically (i.e. does not impose any constraint on the field $h_{\mu\nu}$) [18]. In fact the current analysis can be used to further investigate the issue of partial masslessness on more general backgrounds.

Another interesting direction to go would be to investigate whether an action for a single dynamical metric could be obtained leading to the linearized equations (10). It appears however unlikely that such an action exists with a simple form given what we know from the dRGT theory.

[1] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012); C. de Rham, Living Rev. Relativity **17**, 7 (2014).

[2] M. Fierz, Helv. Phys. Acta **12**, 3 (1939); M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939).

[3] A. Higuchi, Nucl. Phys. **B282**, 397 (1987); I. Bengtsson, J. Math. Phys. (N.Y.) **36**, 5805 (1995); A. Cucchieri, M. Porrati, and S. Deser, Phys. Rev. D **51**, 4543 (1995); M. Porrati, Phys. Lett. B **498**, 92 (2001); C. Deffayet and S. Randjbar-Daemi, Phys. Rev. D **84**, 044053 (2011).

[4] I. L. Buchbinder, D. M. Gitman, V. A. Krykhtin, and V. D. Pershin, Nucl. Phys. **B584**, 615 (2000); I. L. Buchbinder, D. M. Gitman, and V. D. Pershin, Phys. Lett. B **492**, 161 (2000).

[5] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011); C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010); C. de Rham, G. Gabadadze, and A. Tolley, Phys. Lett. B **711**, 190 (2012).

[6] S. F. Hassan and R. A. Rosen, J. High Energy Phys. 04 (2012) 123; S. F. Hassan and R. A. Rosen, J. High Energy Phys. 02 (2012) 126; S. F. Hassan, R. A. Rosen, and A. Schmidt-May, J. High Energy Phys. 02 (2012) 026; S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012); S. F. Hassan, A. Schmidt-May, and M. von Strauss, Phys. Lett. B **715**, 335 (2012).

[7] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).

[8] Note that several potential problems of massive gravity have been, and still are, debated in the literature. We refer the reader to Ref. [1] for discussions of these issues, some of which (such as the meaning and consequences of super-luminality which was shown to arise for some specific backgrounds) are equally relevant for the linear theory extracted here.

[9] C. Deffayet and T. Jacobson, Classical Quantum Gravity **29**, 065009 (2012).

[10] Note that a covariant constraint counting can be obtained in the vierbein and nonperturbative formulation [20] of dRGT theories [21,22]. However this formulation is not fully equivalent to the metric formulation [21,23].

[11] Q. Hu and D. Cheng, Appl. Math. Lett. **19**, 859 (2006).

[12] L. Bernard, C. Deffayet, and M. von Strauss, arXiv:1504.04382.

[13] C. Procesi, Adv. Math. **19**, 306 (1976); G. E. Sneddon, J. Math. Phys. (N.Y.) **39**, 1659 (1998).

[14] Note that Guarato and Durrer [24] also obtained the quadratic action of dRGT theories around an arbitrary metric using different techniques from ours. Their analysis relied on being able to put S in triangular form, which is only possible provided $S^2 = g^{-1}f$ has no negative eigenvalues and if one consistently treats $f_{\mu\nu}$ as a tensor under general coordinate transformations. This however does not cover all possible real matrices $g^{-1}f$ with a well-defined real square root (see e.g. [25]). Moreover, Ref. [24] then specialized to FLRW backgrounds using metrics which are not actual solutions of the background equations (see [12]).

[15] S. F. Hassan, A. Schmidt-May, and M. von Strauss, arXiv:1303.6940.

[16] We keep otherwise β_0 and β_1 arbitrary which means that these models, if one sets there $f_{\mu\nu}$ equals to the flat space-time metric $\eta_{\mu\nu}$, will not always have $g_{\mu\nu} = \eta_{\mu\nu}$ as a background solution, and hence will not always be non-linear extensions of Fierz-Pauli theories *stricto sensu*. It is however always possible to get a flat space-time solution for $g_{\mu\nu}$ by adding a constant conformal factor in $f_{\mu\nu}$.

[17] There are eight scalars $\{u_i, v_i\}$, but they only need to be determined up to an overall factor.

[18] S. Deser and R. I. Nepomechie, Ann. Phys. (N.Y.) **154**, 396 (1984).

[19] J.-M. Martín-García, Comput. Phys. Commun. **179**, 597 (2008); http://metric.iem.csic.es/Martin-Garcia/xAct/.

[20] K. Hinterbichler and R. A. Rosen, J. High Energy Phys. 07 (2012) 047.

[21] C. Deffayet, J. Mourad, and G. Zahariade, J. Cosmol. Astropart. Phys. 01 (2013) 032.

[22] S. Deser, M. Sandora, A. Waldron, and G. Zahariade, arXiv:1408.0561.

[23] M. Banados, C. Deffayet, and M. Pino, Phys. Rev. D **88**, 124016 (2013).

[24] P. Guarato and R. Durrer, Phys. Rev. D **89**, 084016 (2014).

[25] C. Deffayet, J. Mourad and G. Zahariade, J. High Energy Phys. 03 (2013) 086.