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New definition of a wormhole throat

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We present a new definition of the wormhole throat including the flare-out condition and the feature corresponding to the traversability for general dynamical spacetimes in terms of null geodesic congruences. We will examine our definition for some examples and see advantages compared to the others.

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I. INTRODUCTION

A wormhole is one of the interesting objects in general relativity [1–6]. However, there is no universal definition that can work for general situations. To discuss a wormhole, we often specify the throat, the flare-out condition, and so on. There are some proposals for those [3,7–9]. As far as we know, for static and spherical symmetric cases, the definition of the throat was given in Ref. [3]. Therein, the static slices are embedded to the Euclidean space to see the throat structure. Since certain spacetime symmetries are used for the definition of wormhole, this proposal is not applicable for dynamical or nonspherical symmetric cases.

The issue about the extension of the concept of a wormhole to general spacetimes has been already addressed in Refs. [7,8] (see also Refs. [9–13]). In Refs. [7,8], using null geodesic congruences, the wormhole throat is defined as the minimal surface on the null hypersurfaces, i.e., the trapping horizon [14]. We know that, in their definition, some exotic matters are required to maintain static wormholes [3,4,10,13,15] and dynamical ones [7,11,12] without any singularities. Nevertheless, our Universe may have the initial singularity, and the cosmological wormhole solutions with the initial singularity, in which the two Friedmann–Lemaître–Robertson–Walker (FLRW) universes are connected, were constructed without any exotic matters [9]. Since these solutions do not meet the definition of Refs. [7,8], the authors proposed an alternative definition focusing on spherical symmetric cases; the wormhole throat is the minimal surface on spacelike hypersurfaces [9]. However, this definition strongly depends on which spacelike hypersurfaces we take, and, because of this dependence, even de Sitter and FLRW spacetimes are categorized into the wormhole.

In this paper, we propose a new definition of the wormhole throat that is better suited for the intuitive image of a wormhole. Our definition seems to be a hybrid one of a null hypersurface-based definition [7,8] and a spacelike hypersurface one [9]. We describe the throat in terms of the expansion rate of null geodesic congruences on a kind of spacelike hypersurface.

The remaining part of this paper is organized as follows. In Sec. II, we introduce a new definition of the wormhole throat and discuss some general features. In Sec. III, we look at several examples to see if our definition can work well. Finally, we give summary and discussion in Sec. IV.

II. NEW DEFINITION

In this section, we propose a new definition of the wormhole throat with the flare-out condition and the feature corresponding to the traversability. We also discuss some general features.

We consider a codimension-2 spacelike compact surface S and the future directed outgoing/ingoing null geodesic congruences with the affine parameter $\lambda_\pm$ emanating from S. Then, we define the null expansion rate $\theta_\pm$, and we introduce the following quantities:

$$k \coloneqq \theta_+ - \theta_- \tag{1}$$

and

$$\bar{k} \coloneqq \theta_+ + \theta_-. \tag{2}$$

Defining the two vectors

$$r^a \coloneqq (\partial_+ - \partial_-)^a \tag{3}$$

and

$$t^a \coloneqq (\partial_+ + \partial_-)^a, \tag{4}$$

where $\partial_\pm \coloneqq \partial_{\lambda_\pm}$, k and $\bar{k}$ are rewritten as

$$k = r^a \nabla_a \ln\sqrt{h}, \qquad \bar{k} = t^a \nabla_a \ln\sqrt{h}. \tag{5}$$

In the above, h is the determinant of the induced metric of the codimension-2 surface S. Although we cannot assume that the affine parameter $\lambda_\pm$ emanating from S provides us the global coordinate for spacetimes in general, we can have a quasilocal null coordinate system $\tilde{\lambda}_\pm$ such that it coincides with $\lambda_\pm$ when it crosses S.

Now, we define the throat as the codimension-2 surface such that

$$k|_S = 0 \tag{6}$$

holds and the following flare-out condition

$$r^a \nabla_a k|_S > 0 \tag{7}$$

is satisfied. We emphasize that, by fixing the coordinate locally with $\tilde{\lambda}_\pm$, there is no ambiguity of the spatial derivative $r^a\nabla_a$. To introduce the feature corresponding to the traversability for the wormhole, we consider the time sequence of the throat. We require that the tangent vector of the sequence of the throat, which is normal to S, is timelike. However, as we will see in Sec. III B 1, the above conditions hold in a black hole. Therefore, we will not consider the 2-surface S satisfying Eq. (6) and inequality (7) as the wormhole throat if there is the event horizon such that it encloses S. This is because travelers cannot come back to the same region once they enter into the black hole.

Let us look at the general properties of our definition for the wormhole. From the condition of Eq. (6), we have

$$\theta_+|_S = \theta_-|_S. \tag{8}$$

When $\theta_+|_S = \theta_-|_S < 0 (> 0)$, it means the existence of the future (past) trapped surface. Then, if the null energy condition holds, the singularity theorem implies the presence of singularity in the future (past) [16]. With the energy condition, assuming the cosmic censorship conjecture to be held, the future trapped region is always inside the event horizon [16], and thus it is not identified as the wormhole throat. Meanwhile, with $\theta_+|_S = \theta_-|_S > 0$, the singularity theorem predicts the existence of the past singularity, but we consider the past trapped region as the place where the wormhole throat exists. The realization of $\theta_-|_S > 0$ will be easy in the expanding Universe, and the past singularity may be unified to the initial one. That is, there is a room to construct a dynamical wormhole in the cosmological context satisfying the energy condition.

Let z^a be the tangent vector of the time sequence of the throat that is normal to S. Since z^a is timelike, we can write it as $z^a = \alpha(\partial_+)^a + \beta(\partial_-)^a$ with $\alpha, \beta > 0$. Along the time sequence,

$$z^a \nabla_a k|_S = 0 \tag{9}$$

holds, and this gives

$$\partial_- k|_S = -\frac{\alpha}{\beta}\partial_+ k|_S. \tag{10}$$

Then, $r^a\nabla_a k|_S$ becomes

$$\begin{aligned} r^a\nabla_a k|_S &= \left(1+\frac{\alpha}{\beta}\right)\partial_+ k|_S \\ &= \left(1+\frac{\alpha}{\beta}\right)(\partial_+\theta_+ - \partial_+\theta_-)|_S > 0. \end{aligned} \tag{11}$$

If the null energy condition is satisfied in D-dimensional spacetimes, the Raychaudhuri equation tells us

$$\partial_+\theta_+ = -\frac{1}{D-2}\theta_+^2 - \sigma_{+ab}\sigma_+^{ab} - R_{ab}n^a n^b \le 0, \tag{12}$$

where σ_{+ab} is the shear and n^a is the tangent vector of null geodesics. Here, we used the fact that the null geodesic congruences are normal to the throat; that is, the rotation of the congruence vanishes. Therefore, Eq. (11) requires

$$\partial_+\theta_-|_S < 0. \tag{13}$$

We can also confirm the well-known fact for static cases. Since

$$r^a\nabla_a k + t^a\nabla_a \bar{k} = 2(\partial_+\theta_+ + \partial_-\theta_-), \tag{14}$$

the Raychaudhuri equation with the null energy condition shows us

$$r^a\nabla_a k + t^a\nabla_a \bar{k} \le 0. \tag{15}$$

In particular, the flare-out condition is not satisfied when $t^a\nabla_a\bar{k} = 0$ holds.

III. EXAMPLES

Let us examine our definition in four-dimensional spacetimes with symmetries including the spherical symmetry.

A. General scheme

In the null coordinate, the metric of spherically symmetric spacetime is generically written as

$$ds^2 = -a^2(u,v)dudv + R^2(u,v)d\Omega_2^2, \tag{16}$$

where $d\Omega_2^2$ is the metric of the unit 2-sphere. The throat is supposed to be a 2-surface located at $u = u_0$, $v = v_0$.

The radial null geodesic will be on u or $v =$ constant lines. Let us consider the geodesic on $v = v_0$ that follows the geodesic equation

$$\frac{d^2u}{d\lambda_u^2} + 2\frac{\partial_u a}{a}\left(\frac{du}{d\lambda_u}\right)^2 = 0. \tag{17}$$

In a formal way, we can solve the above as

$$\lambda_u = C_u^{-1} \int^u a^2(u')du' =: U, \tag{18}$$

where $a(u) \coloneqq a(u, v_0)$. C_u is the positive integration constant, and we choose λ_u such that $du/d\lambda_u > 0$. In the same way, for the geodesic on $u = u_0$, we have

$$\lambda_v = C_v^{-1} \int^v a^2(v')dv' =: V, \tag{19}$$

where $a(v) \coloneqq a(u_0, v)$. C_v is the positive integration constant, and we choose λ_v such that $dv/d\lambda_v > 0$. Employing U, V as the new coordinates, the metric (16) is rewritten as

$$ds^2 = -C_u C_v \frac{a^2(u,v)}{a^2(u)a^2(v)} dU dV + R^2(u,v) d\Omega_2^2. \tag{20}$$

We should stress that U (V) is the affine parameter on the $v_0(u_0) =$ constant geodesic.

The null expansion rates θ_U, θ_V are calculated to be

$$\theta_U = \theta_- = \frac{2}{R}\partial_U R, \qquad \theta_V = \theta_+ = \frac{2}{R}\partial_V R. \tag{21}$$

So, k defined by Eq. (1) becomes

$$\begin{aligned} k &= \frac{2}{R}(\partial_V - \partial_U)R \\ &= \frac{2}{R}(C_v a^{-2}(v)\partial_v - C_u a^{-2}(u)\partial_u)R. \end{aligned} \tag{22}$$

On the throat S, k vanishes, and then we have

$$C_u \partial_u R|_S = C_v \partial_v R|_S. \tag{23}$$

In the above, we used the fact of $a(u_0) = a(v_0) =: a_0$.

We also need to check the flare-out condition (7) and the timelike condition (9), which are written with the coordinate (20) as

$$\begin{aligned} r^a \nabla_a k|_S &= (\partial_V - \partial_U)k|_S \\ &= \frac{2}{a_0^4 R}[-2C_v^2 \partial_v \ln a(v) \partial_v R - 2C_u^2 \partial_u \ln a(u) \partial_u R \\ &\quad + C_v^2 \partial_v^2 R - 2C_u C_v \partial_u \partial_v R + C_u^2 \partial_u^2 R]|_S \\ &> 0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} z^a \nabla_a k|_S &= (\alpha\partial_V + \beta\partial_U)k|_S \\ &= \frac{2}{a^4 R}[-2\alpha C_v^2 \partial_v \ln a(v) \partial_v R + 2\beta C_u^2 \partial_u \ln a(u) \partial_u R \\ &\quad + \alpha C_v^2 \partial_v^2 R - (\alpha - \beta) C_u C_v \partial_u \partial_v R - \beta C_u^2 \partial_u^2 R]|_S \\ &= 0. \end{aligned} \tag{25}$$

Equalities (23) and (25) and inequality (24) with the positivities of C_u, C_v, α, and β are the conditions that the wormhole should satisfy in spherically symmetric spacetimes.

B. Examples

In this subsection, we look at concrete examples which include nonwormhole spacetimes.

1. Schwarzschild spacetime

It is well known that the throat of the Schwarzschild spacetime is not that of the wormhole due to the presence of the event horizon. Nevertheless, it is nice to see the feature bearing our definition of the throat in mind. To see this, we adopt the Kruskal coordinate

$$ds^2 = \frac{4r_g^3 e^{-r/r_g}}{r}(-dT^2 + dX^2) + r^2 d\Omega_2^2, \tag{26}$$

where $r_g = 2M$ and M is the Arnowitt–Deser–Misner mass. The coordinate transformation from the Kruskal to the ordinal one is given by

$$(r/r_g - 1)e^{r/r_g} = X^2 - T^2 \tag{27}$$

and

$$T/X = \tanh(t/2r_g) \tag{28}$$

for $r > r_g$, or

$$X/T = \tanh(t/2r_g) \tag{29}$$

for $0 < r < r_g$.

In this case, choosing u, v as $u = T - X$, $v = T + X$, $k|_S = 0$ [Eq. (23)] gives us

$$C_u(T + X)|_S = C_v(T - X)|_S. \tag{30}$$

This implies that the candidate of a throat is in the region $0 < r \le r_g$ because of C_u, $C_v > 0$. In addition, inequality (24) and Eq. (25) become

$$r^a \nabla_a k|_S = \frac{4r_g^4}{a_0^4 r^4} e^{-r/r_g} C_u C_v|_S > 0, \tag{31}$$

$$z^a \nabla_a k|_S = \frac{2r_g^4}{a_0^4 r^4} e^{-r/r_g} (\alpha - \beta) C_u C_v|_S = 0, \tag{32}$$

where $a_0^2 = 4(r_g^3/r)e^{-r/r_g}$. The flare-out condition is satisfied as expected, and the tangent vector of the throat orbit of $\alpha = \beta$ is timelike. However, there is the event horizon at $r = r_g$, which is the boundary of the region satisfying Eq. (23) and inequality (24). Therefore, as we have commented in Sec. II, the Schwarzschild spacetime does not have the wormhole throat.

2. De Sitter spacetime

Next, we examine the de Sitter spacetime. If one elaborates the selection of a spacelike hypersurface and follows the definition of Maeda *et al.* [9] for the wormhole throat, there is a case in which the wormhole exists. This is because their definition is not precise. Meanwhile, our definition excludes this case.

In the flat chart, the metric of the de Sitter spacetime is given by

$$\begin{aligned} ds^2 &= a^2(\eta)(-d\eta^2 + dr^2 + r^2 d\Omega_2^2) \\ &= a^2(\eta)(-du dv + r^2 d\Omega_2^2), \end{aligned} \tag{33}$$

where $a(\eta) = -1/(H\eta)$, H is the Hubble constant, and $u = \eta - r$, $v = \eta + r$. Then, Eq. (23) implies

$$C_u(Har - 1)|_S = C_v(Har + 1)|_S. \tag{34}$$

This has a solution

$$Har = \frac{C_u + C_v}{C_u - C_v} > 1, \tag{35}$$

if one chooses C_u, C_v satisfying $C_u > C_v$. This means that the throat candidate is in outside of the cosmological horizon.

Let us check the flare-out condition (24). With the metric (33), we have

$$r^a \nabla_a k|_S = -\frac{2H^2}{a^2} C_u C_v|_S < 0. \tag{36}$$

This disagrees with the flare-out condition (24). Therefore, there is no throat in the de Sitter spacetime as expected.

3. FLRW spacetime

Now, we consider the FLRW spacetime. The metric is given by

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)[(1 - kr^2)^{-1} dr^2 + r^2 d\Omega_2^2] \\ &= a^2(\eta)[-d\eta^2 + d\zeta^2 + r^2 d\Omega_2^2] \\ &= a^2(\eta)[-du dv + r^2 d\Omega_2^2], \end{aligned} \tag{37}$$

where $k = -1, 0, 1$ depending on the spatial topology, η is the conformal time defined by $d\eta = a^{-1}(t)dt$, $d\zeta = dr/\sqrt{1 - kr^2}$, and $u = \eta - \zeta$, $v = \eta + \zeta$.

For the FLRW spacetime, Eq. (23) becomes

$$C_u(\dot{a}r - \sqrt{1 - kr^2})|_S = C_v(\dot{a}r + \sqrt{1 - kr^2})|_S, \tag{38}$$

where $\dot{a} = da(t)/dt$. If one chooses C_u, C_v satisfying $C_u > C_v$, the above has the solution as

$$H(t)\frac{a(t)r}{\sqrt{1 - kr^2}} = \frac{C_u + C_v}{C_u - C_v} > 1, \tag{39}$$

where $H(t) \coloneqq \dot{a}(t)/a(t)$. Roughly speaking, as in the de Sitter spacetime, this means that the throat candidate is outside of the cosmological horizon.

For the current case, the flare-out condition (24) becomes

$$\begin{aligned} r^a \nabla_a k|_S &= \frac{2C_u C_v[a\ddot{a}(1 - kr^2) - \dot{a}^2 r^2(\dot{a}^2 + k)]}{a^4[\dot{a}^2 r^2 - (1 - kr^2)]}\bigg|_S \\ &> 0. \end{aligned} \tag{40}$$

This requires

$$\dot{a}^2 r^2(\dot{a}^2 + k) < a\ddot{a}(1 - kr^2). \tag{41}$$

Together with Eq. (39), the above implies

$$\dot{a}^2 r^2(\dot{a}^2 + k) < a\ddot{a}(1 - kr^2) < a\ddot{a}\dot{a}^2 r^2. \tag{42}$$

Using the Friedmann equation, it is easy to see that the inequality $\dot{a}^2 + k < a\ddot{a}$ obtained from inequality (42) is equivalent with the violation of the null energy condition,

$$\rho + p < 0, \tag{43}$$

where ρ and p are the energy density and the pressure of the perfect fluid, respectively. This is compatible with common sense.

4. Morris–Thorne wormhole

The Morris–Thorne wormhole, which is static and spherically symmetric, is often investigated [3–6,13,15]. The metric is given by

$$\begin{aligned} ds^2 &= -e^{2\Phi(r)}dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \\ &= e^{2\Phi}(-dt^2 + d\zeta^2) + r^2 d\Omega_2^2 \\ &= -e^{2\Phi} du dv + r^2 d\Omega_2^2, \end{aligned} \tag{44}$$

where $\Phi(r), b(r)$ are functions of r, ζ is defined by $d\zeta = e^{-\Phi}dr/\sqrt{1 - b/r}$, and $u = t - \zeta$, $v = t + \zeta$. Here, we suppose that $g_{tt} = -e^{2\Phi}$ is negative and regular. Note that this metric is not obtained as a solution of the Einstein equation with a given matter field action.

For this spacetime, Eq. (23) becomes

$$-C_u\sqrt{1-b/r}e^{\Phi}|_S = C_v\sqrt{1-b/r}e^{\Phi}|_S. \tag{45}$$

This implies that the throat candidate is the surface that satisfies $b(r) = r$ because of $C_u, C_v > 0$. The flare-out condition (24) becomes

$$r^a\nabla_a k|_S = \frac{(C_u + C_v)^2(1-b')}{4r^2}e^{-2\Phi}\bigg|_S > 0, \tag{46}$$

where $b' = db(r)/dr$. This flare-out condition is satisfied if

$$b' < 1 \tag{47}$$

on S. Equation (25) becomes

$$z^a\nabla_a k|_S = \frac{(\alpha C_v - \beta C_u)(C_v + C_u)(1-b')}{4r^2}e^{-2\Phi}\bigg|_S = 0. \tag{48}$$

From this, we see that the tangent vector of the throat orbit of $\alpha C_v = \beta C_u$ is timelike.

To sum up, the conditions for the wormhole are $b(r) = r$ and $b' < 1$, which are the same as the well-known results of Ref. [3].

5. *Dynamical Ellis wormhole*

The dynamical Ellis wormhole is the typical example with dynamics [9,17]. The metric is given by

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)[dl^2 + (l^2+b^2)d\Omega_2^2] \\ &= a^2(\eta)[-d\eta^2 + dl^2 + (l^2+b^2)d\Omega_2^2] \\ &= a^2(\eta)[-dudv + (l^2+b^2)d\Omega_2^2], \end{aligned} \tag{49}$$

where $a(\eta)$ is a function of the conformal time η, b is a constant, and $u = \eta - l$, $v = \eta + l$. Note that the metric (49) is not obtained as a solution of the Einstein equation with a given matter field action.

For this spacetime, Eq. (23) becomes

$$C_u(\dot{a}(l^2+b^2) - l)|_S = C_v(\dot{a}(l^2+b^2) + l)|_S, \tag{50}$$

where $\dot{a} = da/dt$. Since $C_u, C_v > 0$, this gives us the rather trivial condition

$$l^2 < \dot{a}^2(l^2+b^2)^2 \tag{51}$$

at the throat candidate. The flare-out condition (24) becomes

$$r^a\nabla_a k|_S = \frac{2C_uC_v[a\ddot{a}l^2 - \dot{a}^4(l^2+b^2)^2 + \dot{a}^2b^2]}{a^4[\dot{a}^2(l^2+b^2)^2 - l^2]}\bigg|_S > 0. \tag{52}$$

Inequality (52) gives

$$f(l) \coloneqq -\dot{a}^4(l^2+b^2)^2 + \dot{a}^2b^2 + a\ddot{a}l^2 > 0. \tag{53}$$

Using the Einstein equation $R_{\mu\nu} - Rg_{\mu\nu}/2 = T_{\mu\nu}$ with the given metric (49), we compute the energy-momentum tensor $T_{\mu\nu}$. Then, the dominant energy condition requires

$$-T^t_t - T^l_l = \frac{2}{a^2}(a\ddot{a} + 2\dot{a}^2) \geq 0, \tag{54}$$

$$-T^t_t + T^\theta_\theta = \frac{2}{a^2}(-a\ddot{a} + \dot{a}^2) \geq 0, \tag{55}$$

$$-T^t_t + T^l_l = \frac{2}{a^2}\left(-a\ddot{a} + \dot{a}^2 - \frac{b^2}{(l^2+b^2)^2}\right) \geq 0, \tag{56}$$

$$-T^t_t - T^\theta_\theta = \frac{2}{a^2}\left(a\ddot{a} + 2\dot{a}^2 - \frac{b^2}{(l^2+b^2)^2}\right) \geq 0, \tag{57}$$

where θ is the angular coordinate appearing as $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$. Inequality (57) is stronger than inequality (54) and gives

$$a\ddot{a} + 2\dot{a}^2 \geq \frac{b^2}{(l^2+b^2)^2}. \tag{58}$$

The tightest constraint is given at $l = 0$ as

$$a\ddot{a} + 2\dot{a}^2 \geq b^{-2}. \tag{59}$$

In a similar way, from inequality (56), we have

$$\dot{a}^2 - b^{-2} \geq a\ddot{a}. \tag{60}$$

The above two inequalities imply

$$-2\dot{a}^2 + b^{-2} \leq a\ddot{a} \leq \dot{a}^2 - b^{-2}, \tag{61}$$

and then we see

$$\dot{a}^2b^2 \geq \frac{2}{3}. \tag{62}$$

Under the energy condition (61), we can see that $f(l)$ has the maximum value at $l = 0$. Therefore, the condition for the existence of the region satisfying inequality (53) is $f(0) > 0$, which becomes

$$\dot{a}^2 < b^{-2}. \tag{63}$$

Using this, inequality (61) tells us

$$-\dot{a}^2 < -2\dot{a}^2 + b^{-2} \le a\ddot{a} \le \dot{a}^2 - b^{-2} < 0. \qquad (64)$$

Here, let us suppose $a(t)$ to be proportional to $t^{\frac{2}{3(1+w)}}$, where w is a constant. In this case, the loosest inequality in (64), $-\dot{a}^2 < a\ddot{a} < 0$, implies that w satisfies

$$-\frac{1}{3} \le w < \frac{1}{3}. \qquad (65)$$

We can see from inequalities (62) and (63) that the wormhole satisfying the dominant energy condition is realized in a certain time interval of the Universe and the size is about the Hubble radius.

Note that we did not give the equation of state like $p = w\rho$ here. In the current case, w will be determined through the Einstein equation. Moreover, the energy-momentum tensor derived through the Einstein equation does not have isotropic pressure. Therefore, w in inequality (65) is not directly related to the equation of state.

Setting $C_u = C_v = 1$, we see that the throat is located at $l = 0$ and the tangent of the throat orbit is obviously timelike.

6. Dynamical Schwarzschild wormhole

One may be interested in a special case with the metric given by [17]

$$\begin{aligned} ds^2 &= -e^{2\Phi(r)}dt^2 + a^2(t)\left[\left(1-\frac{r_g}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2\right] \\ &= a^2 e^{2\Phi}(-d\eta^2 + d\zeta^2) + r^2 d\Omega_2^2 \\ &= -a^2 e^{2\Phi} du dv + r^2 d\Omega_2^2, \end{aligned} \qquad (66)$$

where $\Phi(r)$ is the function of r, $a(t)$ is the function of t, η is defined by $d\eta = a^{-1}dt$, ζ is defined by $d\zeta = e^{-\Phi}dr/\sqrt{1-r_g/r}$, and $u = \eta - \zeta$, $v = \eta + \zeta$. Here, we suppose that $g_{tt} = -e^{2\Phi}$ is negative and regular. We call this the dynamical Schwarzschild wormhole because the spatial metric on the $t =$ constant hypersurface is the same as the Schwarzschild one.

Here, we focus on the case of $\Phi(r) = 0$ to compare with the dynamical Ellis wormhole. The argument below is almost similar to the previous subsubsection. For this case, Eq. (23) becomes

$$C_u a(\dot{a}r - \sqrt{1-r_g/r})|_S = C_v a(\dot{a}r + \sqrt{1-r_g/r})|_S, \qquad (67)$$

where $\dot{a} = da/dt$. Since C_u, $C_v > 0$, this gives us

$$1 - r_g/r < \dot{a}^2 r^2 \qquad (68)$$

at the throat candidate. The flare-out condition (24) becomes

$$r^a \nabla_a k|_S = \frac{2C_u C_v[a\ddot{a} - \dot{a}^4 r^2 + \frac{r_g}{2r}(\dot{a}^2 - 2a\ddot{a})]}{a^4(\dot{a}^2 r^2 - 1 + \frac{r_g}{r})}\bigg|_S > 0. \qquad (69)$$

Inequality (69) gives

$$f(r) \coloneqq a\ddot{a}r - \dot{a}^4 r^3 + \frac{r_g}{2}(\dot{a}^2 - 2a\ddot{a}) > 0. \qquad (70)$$

Using the Einstein equation, we compute the energy-momentum tensor $T_{\mu\nu}$. Then, the dominant energy condition requires

$$-T^t_t - T^r_r = \frac{1}{a^2}\left(2a\ddot{a} + 4\dot{a}^2 + \frac{r_g}{r^3}\right) \ge 0, \qquad (71)$$

$$-T^t_t + T^\theta_\theta = \frac{1}{a^2}\left(-2a\ddot{a} + 2\dot{a}^2 + \frac{r_g}{2r^3}\right) \ge 0, \qquad (72)$$

$$-T^t_t + T^r_r = \frac{1}{a^2}\left(-2a\ddot{a} + 2\dot{a}^2 - \frac{r_g}{r^3}\right) \ge 0, \qquad (73)$$

$$-T^t_t - T^\theta_\theta = \frac{1}{a^2}\left(2a\ddot{a} + 4\dot{a}^2 - \frac{r_g}{2r^3}\right) \ge 0. \qquad (74)$$

From these, we have the inequality

$$-2\dot{a}^2 + \frac{r_g}{4r^3} \le a\ddot{a} \le \dot{a}^2 - \frac{r_g}{2r^3}. \qquad (75)$$

Since we see from the above that $f(r)$ has the maximum value at $r = r_g$, and then inequality (70) at $r = r_g$ gives us

$$\dot{a}^2 < \frac{1}{2r_g^2}. \qquad (76)$$

The tightest condition comes from inequality (75) at $r = r_g$ as

$$-2\dot{a}^2 + \frac{1}{4r_g^2} \le a\ddot{a} \le \dot{a}^2 - \frac{1}{2r_g^2}. \qquad (77)$$

Then, with (76), inequality (77) tells us

$$-\frac{3}{2}\dot{a}^2 < -2\dot{a}^2 + \frac{1}{4r_g^2} \le a\ddot{a} \le \dot{a}^2 - \frac{1}{2r_g^2} < 0. \qquad (78)$$

Here, let us suppose $a(t) \propto t^{\frac{2}{3(1+w)}}$, where w is a constant. In this case, the loosest condition of inequality (78), $-\frac{3}{2}\dot{a}^2 < a\ddot{a} < 0$, implies the constraint for w as

$$-\frac{1}{3} \le w < \frac{2}{3}. \qquad (79)$$

We have the similar constraint to the dynamical Ellis wormhole one. Together with $1/(4r_g^2) \le \dot{a}^2$ derived from (75), Eq. (76) tells us that the wormhole satisfying the dominant energy condition is realized in a certain time interval of the Universe.

Setting $C_u = C_v = 1$, we see that the throat is located at $r = r_g$ and the tangent of the throat orbit is obviously timelike.

7. DGP wormhole

Finally, we consider the Dvali-Gabadadze-Porrati (DGP) wormhole discussed in Refs. [18,19]. The DGP is one of the braneworld models, and our four-dimensional spacetime is realized as a membrane in five-dimensional spacetime. In Ref. [19], the definition of Maeda *et al.* [9] was employed, and it turned out that the spacetime on the brane has the wormhole throat. Here, we reconsider the brane geometry using our current definition.

The induced metric is

$$ds^2 = \gamma^{-2}(r)dr^2 + r^2(-d\tau^2 + \cosh^2\tau d\Omega_2^2), \quad (80)$$

where

$$\gamma^2(r) = \frac{-(r^2 - 2r_c^2) + \sqrt{r^4 - 4r_0^2 r_c^2}}{2r_c^2} \quad (81)$$

and r_0, r_c are positive constants satisfying $r_0 > r_c$. The range of r is limited as $r \ge r_* \coloneqq \sqrt{r_0^2 + r_c^2}$ so that $\gamma^2(r)$ is positive, and we see $0 \le \gamma^2(r) < 1$.

To investigate the spacetime structure in the current scheme, it is better to introduce new coordinates $(T, \bar{R})$ defined by $T = rh(r)\sinh\tau$ and $\bar{R} = rh(r)\cosh\tau$, where

$$\ln h(r) = \int \frac{1-\gamma}{\gamma r} dr. \quad (82)$$

Then, the metric is written as

$$ds^2 = h^{-2}(r)(-dT^2 + d\bar{R}^2 + \bar{R}^2 d\Omega_2^2). \quad (83)$$

Here, we choose u, v as $u = T - \bar{R}$, $v = T + \bar{R}$, and $a(u,v) = h^{-1}(r)$.

Now, we look at Eq. (23):

$$C_u[(1-\gamma)e^{\tau}\cosh\tau - 1]|_S = -C_v[(1-\gamma)e^{-\tau}\cosh\tau - 1]|_S. \quad (84)$$

Because of C_u, $C_v > 0$, this implies

$$\gamma^2 < \tanh^2\tau. \quad (85)$$

Note that the apparent horizon of the DGP wormhole is located at the surface satisfying $\gamma^2 = \tanh^2\tau$.

Inequality (24) becomes

$$r^a\nabla_a k|_S = \frac{2C_v^2 h^2[\gamma^2(1-\gamma^2)\cosh^2\tau + r\gamma\gamma'\sinh^2\tau]}{r^2\{(1-\gamma)e^{\tau}\cosh\tau - 1\}^2}\Bigg|_S > 0, \quad (86)$$

where $\gamma' = d\gamma(r)/dr$. Using the fact of $0 \le \gamma^2(r) < 1$ and

$$r\gamma\gamma' = \frac{r^2}{\sqrt{r^4 - 4r_0^2 r_c^2}}(1-\gamma^2) > 0 \quad (87)$$

derived from Eq. (81), it is easy to see that the flare-out condition (86) is always satisfied.

Equation (25) is calculated to be

$$\begin{aligned}\alpha[C_v^2 e^{-2\tau}\{r\gamma\gamma' + (1-\gamma^2)\} \\ + C_u C_v\{r\gamma\gamma' - (1-\gamma^2)\}]|_S \\ = \beta[C_u^2 e^{2\tau}\{r\gamma\gamma' + (1-\gamma^2)\} \\ + C_u C_v\{r\gamma\gamma' - (1-\gamma^2)\}]|_S. \quad (88)\end{aligned}$$

From Eq. (87), we see $r\gamma\gamma' - (1-\gamma^2) > 0$. This implies that α and β exist and they must be positive. Therefore, the region satisfying inequality (85) is the wormhole throat. This result is consistent with that in Ref. [19].

Setting $C_u = C_v = 1$, Eq. (84) is solved as

$$r_S^2(\tau) = r_c^2(1 - \tanh^4\tau) + r_0^2(1 - \tanh^4\tau)^{-1}. \quad (89)$$

This is the same with result in Ref. [19].

Although one considers the vacuum brane, as stressed in Ref. [19], the energy conditions are not satisfied for the *effective* energy-momentum tensor computed from the four-dimensional Einstein tensor on the brane.

IV. SUMMARY

In this paper, we proposed a new definition of the wormhole throat with the flare-out condition and the feature corresponding to the traversability for general cases in terms of the null expansion rate. This formulation refines one of the former studies [7–9]. It can appropriately represent not only wormholes without singularities, which are mainly investigated in this field, but also the cosmological wormholes proposed in the recent work [9].

As a demonstration, we applied our formulation to several examples that include nonwormhole spacetimes, too. As a result, we could confirm that our definition can work at least for the concrete examples considered here. All of our examples are spherically symmetric cases, while it is interesting to investigate whether in generic spacetimes our definition coincides with the intuitive image of wormhole. This is left for future study.

Practically interesting objects are wormholes that we can actually pass through. The dynamical Ellis wormhole is that in the FLRW universe without violating any exotic matters, and thus it could exist in our Universe. However, it is too large. Because of the similar size to the Hubble radius, even if it exists, it is not observed as a compact object but rather affects to the cosmological scale physics. For actual use, small wormholes are fascinating, but it seems hard or impossible to construct such wormholes without the violation of the energy condition.

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