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Quantum inequality in spacetimes with small curvature

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Quantum inequalities bound the extent to which weighted time averages of the renormalized energy density of a quantum field can be negative. Such inequalities in curved spacetime can be used to disprove the existence of exotic phenomena, such as closed timelike curves. Starting with a general result of Fewster and Smith, we derive a quantum inequality for a minimally coupled scalar field on a geodesic in a spacetime with small curvature, working to first order in the Ricci tensor and its derivatives. Since only the Ricci tensor enters, there are no first-order corrections to the flat-space quantum inequality on paths which do not encounter any matter or energy.

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I. INTRODUCTION

In the context of general relativity all kinds of exotic spacetimes are allowed. With the appropriate stress-energy tensor $T_{\mu\nu}$, following Einstein's equations, the spacetime can contain wormholes and allow superluminal travel and the construction of "time machines." However, in quantum field theory, there are restrictions on $T_{\mu\nu}$. Two examples of these are the energy conditions and the quantum inequalities. Pointwise energy conditions bound the stress-energy tensor at each spacetime point, but they are easily violated, since quantum field theory allows arbitrary negative energies (e.g., in the Casimir effect). On the other hand, averaged energy conditions bound the stress-energy tensor integrated along a complete geodesic, and quantum inequalities bound a weighted time average of the total energy. These have been proven to hold in a variety of spacetimes.

Energy conditions have been used to address the possibility of exotic spacetimes. Specifically, Ref. [1] showed that the achronal averaged null energy condition (achronal ANEC) is sufficient to rule out most known spacetimes with exotic curvature. In previous work [2], we proved achronal ANEC for spacetimes with a classical source. However, to do that we assumed that with a timescale small compared to any curvature radius the quantum inequality for flat spacetime still holds with small corrections. Ford, Pfenning and Roman [3,4] also have suggested that the flat-space quantum inequalities can be used in spacetimes with small curvature. However none of these results have been explicitly proven.

Quantum inequalities were first introduced by Ford [5] to prevent the violation of the second law of thermodynamics. Since then, they have been derived for a wide range of spacetimes, fields, and weighting functions. We concentrate here on quantum inequalities for minimally coupled scalar fields in curved spacetime. Some of these, for example Ref. [6], are difference quantum inequalities, which bound the difference of $T_{\mu\nu}$ between some state and a reference state. But difference inequalities cannot be used to rule out exotic spacetimes arising from vacuum energies.

Fewster and Smith [7] proved an absolute quantum inequality (i.e., one without dependence on a reference state) that applies to spacetimes with curvature. Their bound involves the Fourier transform of differentiated terms of the Hadamard series up to fifth order. In recent work [8], we used their result to provide a bound for flat spacetimes with a background potential. In the same paper we also showed that is sufficient to consider only terms up to first order, which makes Fewster and Smith's result more practical. Using this result we will now show, in accordance with our past conjecture and previous work, that in spacetimes with small curvature, the quantum inequality is the flat-space one with small corrections that depend on the curvature.

The present paper closely follows Ref. [8]. We begin by stating the general absolute quantum inequality of Fewster and Smith [7] in Sec. II. The inequality bounds the time-averaged, renormalized energy density using the Fourier transform of a point-split energy operator applied to $\tilde{H}$, which is a combination of the Hadamard series and the advanced-minus-retarded Green's function. In Sec. III we discuss and simplify this operator. In Sec. IV we compute the Green's function to first order for a spacetime with curvature, and in Sec. V we use that result to calculate $\tilde{H}$. In Sec. VI we apply the point-split energy operator and compute $\tilde{H}$. Finally we perform the Fourier transform, and find the resulting quantum inequality in Sec. VII. We conclude in Sec. VIII.

We use the sign convention $(-,-,-)$ in the classification of Misner, Thorne and Wheeler [9]. Indices $a, b, c, \ldots$ denote all spacetime coordinates while $i, j, k\ldots$ only spatial coordinates.

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II. ABSOLUTE QUANTUM ENERGY INEQUALITY

We consider a massless, minimally coupled scalar field with the usual classical stress-energy tensor,

$$T_{ab} = \nabla_a\Phi\nabla_b\Phi - \frac{1}{2}g_{ab}g^{cd}\nabla_c\Phi\nabla_d\Phi. \tag{1}$$

Let γ be any timelike geodesic parametrized by proper time t, and let $g(t)$ be any smooth, positive, compactly supported sampling function. In flat spacetime, Fewster and Eveson [10] showed that

$$\int_{-\infty}^{\infty} dt T_{tt}(\gamma(t))g(t)^2 \geq -\frac{1}{16\pi^2}\int_{-\infty}^{\infty} dt g''(t)^2. \tag{2}$$

We will generalize Eq. (2) to geodesics in curved spacetime.

First we construct Fermi normal coordinates [11] in the usual way: We let the vector $e_0(t)$ be the unit tangent to the geodesic γ, and construct a tetrad by choosing arbitrary normalized vectors $e_i(0), i = 1, 2, 3$, orthogonal to $e_0(0)$ and to each other, and define $\{e_i(t)\}$ by parallel transport along γ. The point with coordinates (x^0, x^1, x^2, x^3) is found by traveling unit distance along the geodesic given by $x^i e_i(0)$ from the point $\gamma(0)$.

We work only in first order in the curvature and its derivatives, but do not otherwise assume that it is small. We assume that the components of the Ricci tensor in any Fermi coordinate system, and their derivatives, are bounded,

$$|R_{ab}| \leq R_{\max} \quad |R_{ab,cd}| \leq R''_{\max} \quad |R_{ab,cde}| \leq R'''_{\max}. \tag{3a}$$

These lead to bounds on the Ricci scalar and its derivatives,

$$|R| \leq 4R_{\max} \quad |R_{,cd}| \leq 4R''_{\max} \quad |R_{,cde}| \leq 4R'''_{\max}, \tag{3b}$$

since we are working in four dimensions.

Equations (3) are intended as universal bounds which hold without regard to the specific choice of Fermi coordinate system above. We will not need a bound on the first derivative. The reason that we bound the Ricci tensor and not the Riemann tensor is that, as we will prove, the additional terms of the quantum inequality do not depend on any other components of the Riemann tensor. We will discuss this result further in the conclusions.

Following Ref. [7], we define the renormalized energy density,

$$\langle T^{\mathrm{ren}}_{tt}\rangle \equiv \lim_{x\to x'} T^{\mathrm{split}}(\langle\phi(x)\phi(x')\rangle - H(x,x')) - Q + C_{tt}, \tag{4}$$

with quantities appearing in Eq. (4) defined as follows. T^{split} is the point-split energy density operator,

$$T^{\mathrm{split}} = \frac{1}{2}\sum_{a=0}^{3} e^\alpha_a\nabla_\alpha \otimes e^{\beta'}_a\nabla_{\beta'} = \frac{1}{2}\sum_{a=0}^{3}\partial_a\partial_{a'}, \tag{5}$$

where $\partial_a f$ or $f_{,a}$ denotes the gradient of a function f with respect to x in the direction of $e_a(x)$, and $\partial_{a'} f$ or $f_{,a'}$ the same with x' in place of x.

We renormalize the energy density according to the procedure of Wald [12], by taking the difference between the two-point function, $\langle\phi(x)\phi(x')\rangle$, and the Hadamard series,

$$\begin{aligned} H(x,x') = \frac{1}{4\pi^2}\Bigg[&\frac{\Delta^{1/2}}{\sigma_+(x,x')} \\ &+ \sum_{j=0}^{\infty} v_j(x,x')\sigma^j_+(x,x')\ln(\sigma_+(x,x')/\ell^2) \\ &+ \sum_{j=0}^{\infty} w_j(x,x')\sigma^j(x,x')\Bigg], \end{aligned} \tag{6}$$

where v_j and w_j are smooth, real functions, and σ is the squared invariant length of the geodesic between x and x', negative for timelike separation. In flat space,

$$\sigma(x,x') = -\eta_{ab}(x-x')^a(x-x')^b. \tag{7}$$

By $F(\sigma_+)$, for some function F, we mean the distributional limit

$$F(\sigma_+) = \lim_{\epsilon\to 0^+} F(\sigma_\epsilon), \tag{8}$$

where

$$\sigma_\epsilon(x,x') = \sigma(x,x') + 2i\epsilon(t(x) - t(x')) + \epsilon^2. \tag{9}$$

In some parts of the calculation it is possible to assume that both points lie on the geodesic, so we define

$$\tau = t - t' \tag{10}$$

and write

$$F(\sigma_+) = F(\tau_-) = \lim_{\epsilon\to 0} F(\tau_\epsilon), \tag{11}$$

where

$$\tau_\epsilon = \tau - i\epsilon. \tag{12}$$

We have introduced a length ℓ so that the argument of the logarithm in Eq. (6) is dimensionless. The possibility of changing this scale creates an ambiguity in the definition of H, but this ambiguity can be absorbed into the ambiguity involving local curvature terms discussed below. For simplicity of notation, we will assume we are working in units where $\ell = 1$.

The function Δ is the Van Vleck-Morette determinant biscalar, given by

$$\Delta(x,x') = -\frac{\det(\nabla_a \otimes \nabla_{b'}\sigma(x,x')/2)}{\sqrt{-g(x)}\sqrt{-g(x')}}. \tag{13}$$

All the v_j, and the w_j for $j \geq 1$ are determined by differential equations discussed below, but w_0 is undetermined. Here we will follow Wald [12] and choose $w_0 = 0$.

The term Q is the one introduced by Wald to preserve the conservation of the stress-energy tensor. Wald [13] calculated this term in the coincidence limit,

$$Q = \frac{1}{12\pi^2} w_1(x, x). \tag{14}$$

The term C_{tt} handles the ambiguities in the definition of the stress-energy tensor T in curved spacetime. We will adopt the axiomatic definition given by Wald [12], but there remains the ambiguity of adding local curvature terms with arbitrary coefficients. From Ref. [14] we find that these terms include

$$^{(1)}H_{ab} = 2R_{;ab} - 2g_{ab}\Box R - g_{ab}R^2/2 + 2RR_{ab} \tag{15a}$$

$$^{(2)}H_{ab} = R_{;ab} - \Box R_{ab} - g_{ab}\Box R/2 - g_{ab}R^{cd}R_{cd}/2 + 2R^{cd}R_{acbd}. \tag{15b}$$

Thus in Eq. (19) we must include a term given by a linear combination of Eqs. (15a) and (15b) to first order in R,

$$\begin{aligned} C_{tt} &= a{}^{(1)}H_{tt} + b{}^{(2)}H_{tt} \\ &= 2aR_{,ii} - \frac{b}{2}(R_{tt,tt} + R_{ii,tt} - 3R_{tt,ii} + R_{ii,jj}), \end{aligned} \tag{16}$$

where a and b are undetermined constants.[1]

From Ref. [7] we have the definition

$$\tilde{H}(x, x') = \frac{1}{2}[H(x, x') + H(x', x) + iE(x, x')], \tag{17}$$

where iE is the antisymmetric part of the two-point function, which we calculate in Sec. IV. We will use the Fourier transform convention

$$\hat{f}(k) \quad \text{or} \quad f^\wedge[k] = \int_{-\infty}^{\infty} dx f(x) e^{ixk}. \tag{18}$$

We can now state the quantum inequality of Ref. [7],

$$\begin{aligned} &\int_{-\infty}^{\infty} dt g(t)^2 \langle T_{tt}^{\text{ren}} \rangle_\omega (t, 0) \\ &\quad \geq - \int_0^\infty \frac{d\xi}{\pi} [(g \otimes g)(\theta^* T^{\text{split}} \tilde{H}_{(5)})(t, t')]^\wedge(-\xi, \xi) \\ &\qquad + \int_{-\infty}^{\infty} dt g^2(t)(-Q + C_{tt}), \end{aligned} \tag{19}$$

where the operator θ^* denotes the pullback of the function to the geodesic,

$$(\theta^* T^{\text{split}} \tilde{H}_{(5)})(t, t') \equiv (T^{\text{split}} \tilde{H}_{(5)})(\gamma(t), \gamma(t')), \tag{20}$$

and the subscript (5) means that we include only terms through $j = 5$ in the sums of Eq. (6). However, as we proved in Ref. [8], terms of order $j > 1$ make no contribution to Eq. (19).

Thus we can write Eq. (19) in our case as

$$\int_{-\infty}^{\infty} dt g(t)^2 \langle T_{tt}^{\text{ren}} \rangle (t, 0) \geq -B, \tag{21}$$

where

$$\begin{aligned} B = &\int_0^\infty \frac{d\xi}{\pi} \hat{F}(-\xi, \xi) + \int_{-\infty}^{\infty} dt g^2(t) \bigg(Q - 2aR_{,ii} \\ &- \frac{b}{2}(R_{tt,tt} + R_{ii,tt} - 3R_{tt,ii} + R_{ii,jj}) \bigg), \end{aligned} \tag{22}$$

$$F(t, t') = g(t) g(t') T^{\text{split}} \tilde{H}_{(1)}((t, 0), (t', 0)), \tag{23}$$

and $\hat{F}$ denotes the Fourier transform in both arguments according to Eq. (18).

III. SIMPLIFICATION OF T^{split}

The T^{split} operator, Eq. (5), can be written

$$T^{\text{split}} = \frac{1}{2}\left[\partial_t \partial_{t'} + \sum_{i=1}^{3} \partial_i \partial_{i'}\right]. \tag{24}$$

To simplify it, we will define the following operator:

$$\nabla_{\bar{x}}^2 = \nabla_x^2 + 2\sum_{i=1}^{3} \partial_i \partial_{i'} + \nabla_{x'}^2, \tag{25}$$

which in flat space would be the derivative with respect to the center point. Then Eqs. (24) and (25) give

[1] There are also ambiguities corresponding to adding multiples of the metric and the Einstein tensor to the stress tensor. The first can be considered renormalization of the cosmological constant and the second renormalization of Newton's constant. We will assume that these renormalizations have been performed, and that the cosmological constant is considered part of the gravitational sector, so neither of these affects T_{ab}.

$$T^{\rm split} = \frac{1}{2}\left[\partial_t\partial_{t'} + \frac{1}{2}(\nabla^2_{\bar{x}} - \nabla^2_x - \nabla^2_{x'})\right]$$
$$= \frac{1}{4}[\nabla^2_{\bar{x}} + \Box_x - \partial^2_t + \Box_{x'} - \partial^2_{t'} + 2\partial_t\partial_{t'}], \tag{26}$$

where $\Box_x$ and $\Box_{x'}$ denote the D'Alembertian operator with respect to x and x'. Because we are using Fermi coordinates and are on the generating geodesic, the D'Alembertian and Laplacian operators have the same form with respect to Fermi coordinates as they do in flat space. Then using

$$\partial^2_\tau = \frac{1}{4}[\partial^2_t - 2\partial_t\partial_{t'} + \partial^2_{t'}], \tag{27}$$

we can write

$$T^{\rm split}\tilde{H} = \frac{1}{4}[\Box_x\tilde{H} + \Box_{x'}\tilde{H} + \nabla^2_{\bar{x}}\tilde{H}] - \partial^2_\tau\tilde{H}. \tag{28}$$

Consider the first term. The function $H(x,x')$ obeys the equation of motion[2] in x and so does $E(x,x')$. Thus

$$\Box_x\tilde{H} = \frac{1}{2}\Box_x H(x',x). \tag{29}$$

The quantities $\Delta(x,x')$ and $v_j(x,x')$ are symmetric, so the only sources of asymmetry in H are the functions w_j, and the fact that the imaginary part of $\sigma_+(x,x')$ is antisymmetric. The functions w_j, when constructed by the procedure above, are real but may not be symmetric. Thus

$$H(x',x) = H(x,x')^* + \frac{1}{4\pi^2}\sum_j (w_j(x',x) - w_j(x,x'))\sigma^j(x,x'), \tag{30}$$

where $*$ denotes complex conjugation. Since $\Box_x$ is real, $\Box_x H(x,x')^* = 0$, and we have

$$\Box_x\tilde{H} = \frac{1}{8\pi^2}\Box_x\sum_j (w_j(x',x) - w_j(x,x'))\sigma^j(x,x'). \tag{31}$$

In the coincidence limit Eq. (31) vanishes. There is no $j = 0$ term because $w_0 = 0$. In the $j = 1$ term, we have σ, which vanishes at coincidence unless both derivatives of the $\Box$ are applied to it, in which case w_1 cancel each other, and for $j > 1$, even $\Box_x\sigma^j$ vanishes.

The second term in brackets in Eq. (28) gives

$$\Box_{x'}\tilde{H} = \frac{1}{8\pi^2}\Box_{x'}\sum_j (w_j(x,x') - w_j(x',x))\sigma^j(x,x'). \tag{32}$$

Adding together Eqs. (31) and (32), we get something which is smooth, symmetric in x and x', and vanishes in the coincidence limit. Following the analysis of Sec. 3 A of Ref. [8], such a term makes no contribution to Eq. (22) and for our purposes we can take

$$T^{\rm split}\tilde{H} = \left[\frac{1}{4}\nabla^2_{\bar{x}} - \partial^2_\tau\right]\tilde{H}. \tag{33}$$

IV. GENERAL COMPUTATION OF E

The function E is the advanced minus the retarded Green's function,

$$E(x,x') = G_A(x,x') - G_R(x,x'), \tag{34}$$

and iE is the imaginary, antisymmetric part of the two-point function. The Green's functions satisfy

$$\Box G(x,x') = \frac{\delta^{(4)}(x-x')}{\sqrt{-g}}. \tag{35}$$

Following Poisson *et al.* [15] and adjusting for different sign and normalization conventions,

$$G(x,x') = \frac{1}{4\pi}(2U(x,x')\delta(\sigma) + V(x,x')\Theta(-\sigma)), \tag{36}$$

where $U(x,x') = \Delta^{1/2}(x,x')$ and $V(x,x')$ are smooth biscalars.

For points y null separated from x', V is called $\check{V}$ [15] and satisfies

$$\check{V}_{,a}\sigma^{\cdot a} + \left[\frac{1}{2}\Box\sigma + 2\right]\check{V} = -\Box U, \tag{37}$$

with all derivatives with respect to y. Now $\check{V}$ is first order in the curvature, so we will do the rest of the calculation as though we were in flat space. Under this approximation, we will neglect coefficients which depend on the curvature, and also evaluate curvature components at locations that would be relevant if we were in flat space. The distance between these locations and the proper locations is first order in the curvature, so the overall inaccuracy will always be second order in the curvature and its derivatives.

Thus we use $\sigma^{\cdot a} = -2(y-x')^a$ and $\Box\sigma = -8$ in Eq. (37) to get

$$(y-x')^a\check{V}_{,a}(y) + \check{V}(y) = \frac{1}{2}\Box U(y). \tag{38}$$

Now suppose we want to compute $\check{V}$ at some point x''. We need to integrate along the geodesic going from x' to x''. So let $y = x' + \lambda(x''-x')$ and observe that

$$\frac{d(\lambda\check{V}(y))}{d\lambda} = \lambda\frac{d\check{V}(y)}{d\lambda} + \check{V}(y) = \lambda(x''-x')^a\check{V}_{,a} + \check{V}(y)$$
$$= (y-x')^a\check{V}_{,a} + \check{V}(y) = \frac{1}{2}\Box U(y), \tag{39}$$

so

[2]In general the sums in Eq. (6) do not converge and we should work only to some finite order in σ. In that case $H(x,x')$ obeys the equation of motion to that order.

$$\check{V}(x'', x') = \frac{1}{2}\int_0^1 d\lambda \Box U(y). \tag{40}$$

The function V obeys [15]

$$\Box_x V(x, x') = 0. \tag{41}$$

Let us consider points x and x' on the geodesic γ, which in the flat-space approximation means they are separated only in time, let $\bar{x} = (x + x')/2$, and establish a spherical coordinate system (r, θ, ϕ) with origin at the common spatial position of x and x'. Now $V(x, x')$ can be found in terms of V and its derivatives evaluated at the time $\bar{t}$ (the time component of $\bar{x}$) using Kirchhoff's formula,

$$V(x, x') = \frac{1}{4\pi}\int d\Omega \left[\check{V}(x'', x') + \frac{\tau}{2}\frac{\partial}{\partial r}\check{V}(x'', x') + \frac{\tau}{2}\frac{\partial}{\partial t}\check{V}(x'', x')\right], \tag{42}$$

where the derivatives act on the first argument of $\check{V}$, $\int d\Omega$ means to integrate over all spatial unit vectors $\hat{\Omega}$, and we now set

$$x'' = \bar{x} + (\tau/2)\Omega, \tag{43}$$

with the 4-vector Ω given by $\hat{\Omega}$ with zero time component.

Now define null coordinates $u = t + r$ and $v = t - r$. Then x'' has $u = \tau$, $v = 0$. The derivative $\partial/\partial u$ can be written $(\partial/\partial t + \partial/\partial r)/2$ and so

$$V(x, x') = \frac{1}{4\pi}\int d\Omega \frac{d}{du}[u\check{V}((u/2)\Omega_1, x')]_{u=\tau}, \tag{44}$$

where Ω_1 is $\hat{\Omega}$ with unit time component. From Eq. (40),

$$u\check{V}\left(\frac{u}{2}\Omega, x'\right) = \frac{1}{2}\int_0^u du'(\Box U)((u'/2)\Omega_1, x'), \tag{45}$$

with the D'Alembertian applied to the first argument, and so

$$V(x, x') = \frac{1}{8\pi}\int d\Omega \Box_{x''} U(x'', x'). \tag{46}$$

We are only interested in the first order of curvature, so we can expand U, which is just the square root of the Van Vleck determinant, to first order. From Ref. [16],

$$\Delta^{1/2}(x, x') = 1 - \frac{1}{2}\int_0^1 ds(1-s)sR_{ab}(sx + (1-s)x') \times (x - x')^a (x - x')^b + O(R^2), \tag{47}$$

so in the case at hand we can use

$$U(x'', x') = \Delta^{1/2}(x'', x') = 1 - \frac{1}{2}\int_0^1 ds(1-s)sR_{ab}(y)X^a X^b \tag{48}$$

where $y = sx'' = (su'', sv'', \theta'', \phi'')$ is a point between 0 and x'', and the tangent vector $X = dy/ds$. We are interested in $\Box_{x''} U(x'', 0)$. To bring the $\Box$ inside the integral, we define $Y = sX = (su'', sv'', 0, 0)$, and

$$\Box U(x'', 0) = -\frac{1}{2}\int_0^1 ds(1-s)s\Box_{x''}[R_{ab}(y)X^a X^b] = -\frac{1}{2}\int_0^1 ds(1-s)s\Box_y [R_{ab}(y)Y^a Y^b]. \tag{49}$$

For the rest of this section, all occurrences of u, v, θ, ϕ, and derivatives with respect to these variables will refer to these components of y or Y.

Now we expand the D'Alembertian in Eq. (46), in terms of an angular part,

$$\nabla^2_\Omega = \frac{4}{(v-u)^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{4}{(v-u)^2\sin\theta^2}\frac{\partial^2}{\partial\phi^2}, \tag{50}$$

and a radial and temporal part, which we can write in terms of derivatives in u and v,

$$4\frac{\partial^2}{\partial v\partial u} - \frac{4}{u-v}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right). \tag{51}$$

The angular part vanishes on integration, leaving

$$V(x, x') = -\frac{1}{4\pi}\int d\Omega \int_0^1 dss(1-s) \times \left[\partial_u\partial_v - \frac{1}{u-v}(\partial_u - \partial_v)\right](R_{ab}(y)Y^a Y^b). \tag{52}$$

Outside the derivatives, we can take $v = 0$ and change variables to $u = s\tau$, giving

$$\begin{aligned} V(x, x') &= -\frac{1}{4\pi\tau^3}\int d\Omega\int_0^\tau du(\tau - u) \times [u\partial_u\partial_v - \partial_u + \partial_v](R_{ab}(y)Y^a Y^b) \\ &= -\frac{1}{4\pi\tau^3}\int d\Omega\int_0^\tau du(\tau - u) \times \partial_u[(u\partial_v - 1)(R_{ab}(y)Y^a Y^b)]. \end{aligned} \tag{53}$$

We can integrate by parts with no surface contribution, giving

$$V(x,x') = \frac{1}{4\pi\tau^3}\int d\Omega \int_0^\tau du(1-u\partial_v)(R_{ab}(y)Y^aY^b)$$
$$= \frac{1}{4\pi\tau^3}\int d\Omega \int_0^\tau duu^2[-uR_{uu,v}(y) - 2R_{uv}(y) + R_{uu}(y)]. \tag{54}$$

Now

$$R_{ab} = G_{ab} - (1/2)g_{ab}G, \tag{55}$$

where G_{ab} is the Einstein tensor and G its trace. Thus

$$V(x,x') = \frac{1}{4\pi\tau^3}\int d\Omega \int_0^\tau duu^2[-uG_{uu,v}(y) - 2G_{uv}(y) + (1/2)G(y) + G_{uu}(y)]. \tag{56}$$

Now define a vector field $Q_a(y) = G_{ab}(y)Y^b$. Then

$$Q_{a;c} = G_{ab;c}(y)Y^b + G_{ab}(y)Y^b{}_{;c}. \tag{57}$$

We write the covariant derivative only because we are working in null-spherical coordinates, rather than because of spacetime curvature, which we are ignoring because we already have first-order quantities.

Since the covariant divergence of G vanishes,

$$g^{ac}Q_{a;c} = g^{ac}G_{ab}(y)Y^b{}_{;c}. \tag{58}$$

In Cartesian coordinates, $Y^b = y^b$, and $y^b{}_{;c} = \delta^b_c$, which means that (in any coordinate system)

$$g^{ac}Q_{a;c} = G. \tag{59}$$

Explicit expansion gives

$$g^{ac}Q_{a;c} = 2(Q_{v,u} + Q_{u,v}) - \frac{4}{u-v}(Q_u - Q_v) - \frac{4}{(v-u)^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta Q_\theta) + \frac{1}{\sin\theta^2}Q_{\phi,\phi}\right], \tag{60}$$

but the angular terms vanish on integration. Now we expand the derivatives in u and v and set $v = 0$, giving

$$Q_{v,u} = uG_{uv,u} + G_{uv} \tag{61a}$$

$$Q_{u,v} = uG_{uu,v} + G_{uv}, \tag{61b}$$

so

$$\int d\Omega(2uG_{uv,u} + 2uG_{uu,v} + 8G_{uv} - 4G_{uu}) = \int d\Omega G. \tag{62}$$

Substituting Eq. (62) into Eq. (56), we find

$$V(x,x') = \frac{1}{4\pi\tau^3}\int d\Omega \int_0^\tau duu^2[uG_{uv,u}(y) + 2G_{uv}(y) - G_{uu}(y)] \tag{63}$$

and integration by parts yields

$$V(x,x') = \frac{1}{4\pi}\int d\Omega\left[G_{uv}(x'') - \frac{1}{\tau^3}\int_0^\tau duu^2(G_{uv}(y) + G_{uu}(y))\right]. \tag{64}$$

Now

$$\int d\Omega \int_0^\tau duu^2(G_{uv}(y) + G_{uu}(y)) = \frac{1}{2}\int d\Omega \int_0^\tau duu^2(G_{tt}(y) + G_{tr}(y)) = \frac{1}{2}\int d\Omega \int_0^\tau duu^2(G^{tt}(y) - G^{tr}(y)), \tag{65}$$

which is 4 times the total flux of G^{ta} crossing inward through the light cone. Since this quantity is conserved, $G^{ta}{}_{;a} = 0$, we can integrate instead over a ball at constant time $\bar{t}$, giving

$$4\int d\Omega \int_0^{\tau/2} drr^2 G^{tt}(\bar{x} + r\Omega) = \frac{\tau^3}{2}\int d\Omega \int_0^1 dss^2 G^{tt}(\bar{x} + s(\tau/2)\Omega) \tag{66}$$

so

$$V(x,x') = \frac{1}{8\pi}\int d\Omega\left[\frac{1}{2}[G_{tt}(x'') - G_{rr}(x'')] - \int_0^1 dss^2 G_{tt}(x''_s)\right], \tag{67}$$

where $x''_s = \bar{x} + s(\tau/2)\Omega$, and

$$G_R(x,x') = \Delta^{1/2}(x,x')\frac{\delta(\sigma)}{2\pi} + \frac{1}{32\pi^2}\int d\Omega\left\{\frac{1}{2}[G_{tt}(x'') - G_{rr}(x'')] - \int_0^1 dss^2 G_{tt}(x''_s)\right\}, \quad (68)$$

$$E(x,x') = \Delta^{1/2}(x,x')\frac{\delta(\tau - |\mathbf{x}-\mathbf{x}'|) - \delta(\tau + |\mathbf{x}-\mathbf{x}'|)}{4\pi|\mathbf{x}-\mathbf{x}'|} + \frac{1}{32\pi^2}\int d\Omega\left\{\frac{1}{2}[G_{tt}(x'') - G_{rr}(x'')] - \int_0^1 dss^2 G_{tt}(x''_s)\right\}\mathrm{sgn}\tau. \quad (69)$$

V. COMPUTATION OF $\tilde{H}$

We now need to compute $\tilde{H}(x,x')$ and apply T^{split}. First we consider the term in $\tilde{H}(x,x')$ that has no dependence on the curvature. It has the same form as it would in flat space [7,8],

$$\tilde{H}_{-1}(x,x') = H_{-1}(x,x') = \frac{1}{4\pi^2\sigma_+(x,x')}. \quad (70)$$

In Sec. VI, we will apply the fully general T^{split} from Eq. (33) with $\nabla_{\bar{x}}$ defined in Eq. (25) to $\tilde{H}_{-1}(x,x')$.

All the remaining terms that we need are first order in the curvature, so for these it is sufficient to take $\nabla_{\bar{x}}$ as the flat-space Laplacian with respect to the center point, $\bar{x}$. For this we only need to compute $\tilde{H}$ at positions given by time coordinates t and t' but the same spatial position.

As we discussed, we only need to keep terms in $\tilde{H}$ with powers of τ up to τ^2, but we need E exactly. The terms from H alone give a function whose Fourier transform does not decline fast enough for positive ξ for the integral in Eq. (22) to converge. Thus we extract the leading order terms from iE and combine these with the terms from H. This combination gives a result that has the appropriate behavior after the Fourier transform.

Following the notation of Ref [8], we let $H_j(t,t')$, $j = 0,1,\ldots$, denote the term in H involving τ^{2j} (with or without $\ln\tau$), and $H_{(j)}$ denote the sum of all terms from H_{-1} through H_j. We will split up $E(x,x')$ in similar fashion, define a "remainder term",

$$R_j = E - \sum_{k=-1}^{j} E_k, \quad (71)$$

and let

$$\tilde{H}_j(x,x') = \frac{1}{2}[H_j(x,x') + H_j(x',x) + iE_j(x,x')] \quad (72a)$$

$$\tilde{H}_{(j)}(x,x') = \frac{1}{2}[H_{(j)}(x,x') + H_{(j)}(x',x) + iE(x,x')]. \quad (72b)$$

A. Terms with no powers of τ

First we want to calculate the zeroth order of the Hadamard series. The Hadamard coefficients are given by the Hadamard recursion relations, which are the solutions to

$$\Box H(x,x') = 0. \quad (73)$$

The recursion relations for the massless field in a curved background are [7]

$$\Box\Delta^{1/2} + 2v_{0,a}\sigma^{;a} + 4v_0 + v_0\Box\sigma = 0, \quad (74)$$

$$\Box v_j + 2(j+1)v_{j+1,a}\sigma^{;a} - 4j(j+1)v_{j+1} + (j+1)v_{j+1}\Box\sigma = 0. \quad (75)$$

To find the zeroth order of the Hadamard series we need only $v_0(x,x')$, which we find by integrating Eq. (74) along the geodesic from x' to x. Since we are computing a first-order quantity, we can work in flat space by letting $y' = x' + \lambda(x - x')$ and using the first-order formulas $\Box\sigma = -8$ and $\sigma^{;a} = -2(y'-x')^a$. From Eq. (74), we have

$$(y'-x')^a v_{0,a} + v_0 = \frac{1}{4}\Box\Delta^{1/2}(y',x'), \quad (76)$$

and thus

$$v_0(x,x') = \frac{1}{4}\int_0^1 d\lambda(\Box\Delta^{1/2})(x' + \lambda(x-x'),x') \quad (77)$$

by the same analysis as Eq. (40).

Using the expansion for $\Delta^{1/2}$ from Eq. (47) gives

$$\begin{aligned} v_0(x,x') &= -\frac{1}{8}\int_0^1 d\lambda\int_0^1 ds(1-s)s\Box_{y'}[R_{ab}(sy' + (1-s)x')(y'-x')^a(y'-x')^b] \\ &= -\frac{1}{8}\int_0^1 d\lambda\int_0^1 ds(1-s)s[(\lambda s)^2(\Box R_{ab})(x' + s\lambda(x-x'))(x-x')^a(x-x')^b \\ &\quad + 2\lambda s R_{,b}(x' + s\lambda(x-x'))(x-x')^b + 2R(x' + s\lambda(x-x'))]. \end{aligned} \quad (78)$$

We can combine the s and λ integrals by defining a new variable $\sigma = s\lambda$,

$$\int_0^1 d\lambda \int_0^1 ds(1-s)sf(\lambda s) = \int_0^1 d\lambda \int_0^\lambda d\sigma\left(\frac{\sigma}{\lambda^2}-\frac{\sigma^2}{\lambda^3}\right)f(\sigma) = \int_0^1 d\sigma f(\sigma)\int_\sigma^1 d\lambda\left(\frac{\sigma}{\lambda^2}-\frac{\sigma^2}{\lambda^3}\right)$$
$$= \int_0^1 d\sigma f(\sigma)\left[-\frac{\sigma}{\lambda}+\frac{\sigma^2}{2\lambda^2}\right]_\sigma^1 = \frac{1}{2}\int_0^1 d\sigma f(\sigma)(1-\sigma)^2. \tag{79}$$

Then, changing σ to s, we find

$$v_0(x,x') = -\frac{1}{16}\int_0^1 ds(1-s)^2[s^2(\Box R_{ab})(x'+s(x-x'))(x-x')^a(x-x')^b$$
$$+ 2sR_{,b}(x'+s(x-x'))(x-x')^b + 2R(x'+s(x-x'))] \tag{80}$$

or when the two points are on the geodesic,

$$v_0(t,t') = -\frac{1}{16}\int_0^1 ds(1-s)^2[s^2(\Box R_{tt})(x'+s\tau)\tau^2 + 4s\eta^{cd}R_{ct,d}(x'+s\tau)\tau + 2R(x'+s\tau)]. \tag{81}$$

In the second term we use the contracted Bianchi identity, $\eta^{cd}R_{ct,d} = R_{,t}/2$, giving

$$2\int_0^1 ds(1-s)^2 s\tau R_{,t}(x'+s\tau) = 2\int_0^1 ds(1-s)^2 s\frac{d}{ds}R(x'+s\tau) = -2\int_0^1 ds(1-s)(1-3s)R(x'+s\tau), \tag{82}$$

so the final expression for v_0 is

$$v_0(t,t') = -\frac{1}{16}\int_0^1 ds(1-s)[s^2(1-s)\Box R_{tt}(\bar{x}+(s-1/2)\tau)\tau^2 + 4sR(\bar{x}+(s-1/2)\tau)]. \tag{83}$$

To calculate H_0 we only need the zeroth order in τ from v_0, so the first term does not contribute. In the second term, we make a Taylor series expansion,

$$R(\bar{x}+(s-1/2)\tau) = R(\bar{x}) + R_{,t}(\bar{x})\tau(s-1/2) + \frac{1}{2}R_{,tt}(\bar{x})\tau^2(s-1/2)^2 + O(\tau^3), \tag{84}$$

but only the first term is relevant here. Thus

$$v_0(t,t') = -\frac{1}{4}\int_0^1 ds(1-s)sR(\bar{x}) = -\frac{1}{24}R(\bar{x}). \tag{85}$$

We also need to expand the Van Vleck determinant appearing in the Hadamard series. From Eq. (47),

$$\Delta^{1/2}(t,t') = 1 - \frac{1}{12}R_{tt}(\bar{x})\tau^2 - \frac{1}{480}R_{tt,tt}(\bar{x})\tau^4 + O(\tau^6). \tag{86}$$

Keeping the first-order term from Eq. (86) and using Eq. (85), we have

$$H_0(x,x') = \frac{1}{48\pi^2}\left[R_{tt}(\bar{x}) - \frac{1}{2}R(\bar{x})\ln(-\tau_-^2)\right]. \tag{87}$$

Now we can add the $H_0(x',x)$ which is the same except that t and t' interchange,

$$H_0(x,x') + H_0(x',x) = \frac{1}{24\pi^2}[R_{tt}(\bar{x}) - R(\bar{x})\ln|\tau|]. \tag{88}$$

Next we must include E from Eq. (69). We can expand the components of the Einstein tensor around $\bar{x}$,

$$G_{ab}(x'') = G_{ab}(\bar{x}) + G^{(1)}_{ab}(x''), \tag{89}$$

where $G^{(1)}_{ab}$ is the remainder of the Taylor series,

$$G^{(1)}_{ab}(x'') = G_{ab}(x'') - G_{ab}(\bar{x}) = \int_0^{\tau/2} drG_{ab,i}(\bar{x}+r\Omega)\Omega^i. \tag{90}$$

To find E_0, we put the first term of Eq. (89) into the second term of Eq. (69). We use $G_{rr} = G_{ij}\Omega^i\Omega^j$ and $\int d\Omega\Omega^i\Omega^j = (4\pi/3)\delta^{ij}$ and find

$$E_0(x,x') = \frac{1}{8\pi}\left\{\frac{1}{2}G_{tt}(\bar{x}) - \frac{1}{6}G_{ii}(\bar{x}) - \int_0^1 ds s^2 G_{tt}(\bar{x})\right\}\mathrm{sgn}\tau$$
$$= \frac{1}{48\pi}G(\bar{x})\mathrm{sgn}\tau = -\frac{1}{48\pi}R(\bar{x})\mathrm{sgn}\tau. \tag{91}$$

For the remainder term R_0, we put the second term of Eq. (89) into the second term of Eq. (69),

$$R_0(x,x') = \frac{1}{32\pi^2}\int d\Omega\left\{\frac{1}{2}[G^{(1)}_{tt}(x'') - G^{(1)}_{rr}(x'')] - \int_0^1 ds s^2 G^{(1)}_{tt}(x''_s)\right\}\mathrm{sgn}\tau. \tag{92}$$

Using

$$2\ln|\tau| + \pi i\,\mathrm{sgn}\tau = \ln(-\tau_-^2), \tag{93}$$

we combine Eqs. (88) and (91) to find

$$\tilde{H}_0(t,t') = \frac{1}{48\pi^2}\left[R_{tt}(\bar{x}) - \frac{1}{2}R(\bar{x})\ln(-\tau_-^2)\right]. \tag{94}$$

Combining all terms through order 0 gives

$$\tilde{H}_{(0)}(t,t') = \tilde{H}_{-1}(t,t') + \tilde{H}_0(t,t') + \frac{1}{2}iR_0(t,t'). \tag{95}$$

B. Terms of order τ^2

Now we compute the terms of order τ^2 in H and E. To find v_0 at this order we take Eqs. (83) and (84) and include terms through second order in τ. The first-order term vanishes, leaving

$$v_0(x,x') = -\frac{1}{24}R(\bar{x}) - \frac{1}{16}\int_0^1 ds(1-s)[s^2(1-s)\Box R_{tt}(\bar{x}) + 2s(s-1/2)^2 R_{,tt}(\bar{x})]\tau^2 + \cdots$$
$$= -\frac{1}{24}R(\bar{x}) - \frac{1}{480}\left(\Box R_{tt}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\tau^2 + \cdots. \tag{96}$$

Next we need v_1 but since it is multiplied by τ^2 in H we need only the τ-independent term. From Eq. (75),

$$\Box v_0 + 2v_{1,a}\sigma^{;a} + v_1\Box\sigma = 0. \tag{97}$$

At $x = x'$, $\sigma^{;a} = 0$ so

$$v_1(x,x) = \frac{1}{8}\lim_{x\to x'}\Box_x v_0(x,x'). \tag{98}$$

Using Eq. (80) in Eq. (98), the only terms that survive in the coincidence limit are those that have no powers of $x - x'$ after differentiation, so

$$v_1(x,x) = -\frac{1}{16}\int_0^1 ds(1-s)^2 s^2\Box R(\bar{x}) = -\frac{1}{480}\Box R(\bar{x}). \tag{99}$$

Equations (83), (96) and (99) agree with Ref. [17] if we note that their expansions are around x instead of $\bar{x}$.

The w_1 at coincidence is given by Ref. [13],

$$w_1(x,x) = -\frac{3}{2}v_1(x,x) = \frac{1}{320}\Box R(\bar{x}). \tag{100}$$

Combining Eqs. (96), (99), and (100), and the fourth order term from the Van Vleck determinant of Eq. (86), and keeping in mind that $\sigma = -\tau^2$ when both points are on the geodesic, we find

$$H_1(x,x') = \frac{1}{640\pi^2}\left[\frac{1}{3}R_{tt,tt}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\ln(-\tau_-^2)\right]\tau^2. \tag{101}$$

Then $H_1(x',x)$ is given by symmetry, so

$$H_1(x,x') + H_1(x',x) = \frac{1}{160\pi^2}\left[\frac{1}{6}R_{tt,tt}(\bar{x}) - \frac{1}{4}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\ln|\tau|\right]\tau^2. \tag{102}$$

The calculation of E_1 is similar to E_0, but now we have to include more terms to the Taylor expansion,

$$G_{ab}(x'') = G_{ab}(\bar{x}) + \frac{\tau}{2}G_{ab,i}(\bar{x})\Omega^i + \frac{\tau^2}{8}G_{ab,ij}\Omega^i\Omega^j(\bar{x}) + G^{(3)}_{ab}(x''), \tag{103}$$

where the remainder of the Taylor series is

$$G^{(3)}_{ab}(x'') = \frac{1}{2}\int_0^{\tau/2} dr G_{ab,ijk}(\bar{x} + r\Omega)\left(\frac{\tau}{2} - r\right)^2\Omega^i\Omega^j\Omega^k. \tag{104}$$

Now we put Eq. (103) into Eq. (69), and again use $G_{rr} = G_{ij}\Omega^i\Omega^j$. The first term of Eq. (103) gives E_0, which we computed before, and the second term gives nothing, because $\int d\Omega\Omega^i = \int d\Omega\Omega^i\Omega^j\Omega^k = 0$. Using $\int d\Omega\Omega^i\Omega^j = 4\pi/3\delta^{ij}$ and $\int d\Omega\Omega^i\Omega^j\Omega^k\Omega^l = (4\pi/15)(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$, the third term gives

$$E_1(x,x') = -\frac{1}{192\pi}\left[\frac{1}{10}G_{ii,jj}(\bar{x}) + \frac{1}{5}G_{ij,ij}(\bar{x}) - \frac{1}{2}G_{tt,ii}(\bar{x}) + \int_0^1 ds s^4 G_{tt,ii}(\bar{x})\right]\tau^2 \mathrm{sgn}\tau$$
$$= -\frac{1}{320\pi}\left[\frac{1}{6}G_{ii,jj}(\bar{x}) + \frac{1}{3}G_{ij,ij}(\bar{x}) - \frac{1}{2}G_{tt,ii}(\bar{x})\right] \times \tau^2 \mathrm{sgn}\tau. \tag{105}$$

Using the conservation of the Einstein tensor, $0 = \eta^{ab}G_{ia,b} = G_{it,t} - G_{ij,j}$ and $0 = \eta^{ab}G_{ta,b} = G_{tt,t} - G_{it,i}$ we can write

$$G_{ij,ij}(\bar{x}) = G_{tt,tt}(\bar{x}). \tag{106}$$

So

$$E_1(x,x') = -\frac{1}{960\pi}\left(\frac{1}{2}G_{ii,jj}(\bar{x}) + G_{tt,tt}(\bar{x}) - \frac{3}{2}G_{tt,ii}(\bar{x})\right) \times \tau^2 \mathrm{sgn}\tau. \tag{107}$$

Now $G_{ab} = R_{ab} - (1/2)R$, so

$$G_{ii} = (3/2)R_{tt} - (1/2)R_{ii} \tag{108a}$$

$$G_{tt} = (1/2)R_{tt} + (1/2)R_{ii}. \tag{108b}$$

Putting these in Eq. (107) gives

$$E_1(x,x') = -\frac{1}{960\pi}\left(R_{ii,jj}(\bar{x}) + \frac{1}{2}R_{tt,tt}(\bar{x}) + \frac{1}{2}R_{ii,tt}(\bar{x})\right) \times \tau^2 \mathrm{sgn}\tau$$
$$= -\frac{1}{960\pi}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\tau^2 \mathrm{sgn}\tau. \tag{109}$$

The fourth term of Eq. (103) gives the remainder,

$$R_1(x,x') = \frac{1}{32\pi^2}\int d\Omega\left\{\frac{1}{2}[G^{(3)}_{tt}(x'') - G^{(3)}_{rr}(x'')] - \int_0^1 ds s^2 G^{(3)}_{tt}(x''_s)\right\}\mathrm{sgn}\tau. \tag{110}$$

To calculate $\tilde{H}_1$, we combine Eqs. (102) and (109) and use Eq. (93) to get

$$\tilde{H}_1(x,x') = \frac{\tau^2}{640\pi^2}\left[\frac{1}{3}R_{tt,tt}(\bar{x}) - \frac{1}{2}\Box R(\bar{x}) - \frac{1}{3}\left(\Box R_{ii}(\bar{x}) + \frac{1}{2}R_{,tt}(\bar{x})\right)\ln(-\tau_-^2)\right]. \tag{111}$$

All terms through order 1 are then given by

$$\tilde{H}_{(1)}(t,t') = \tilde{H}_{-1}(t,t') + \tilde{H}_0(t,t') + \tilde{H}_1(t,t') + \frac{1}{2}iR_1(t,t'). \tag{112}$$

VI. THE $T^{\mathrm{split}}\tilde{H}$

We can easily take the derivatives of $\tilde{H}_0$ and $\tilde{H}_1$ using Eq. (33), because they are already first order in R. However in the case of the term $\nabla^2_{\bar{x}}\tilde{H}_{-1}$ we have to proceed more carefully. From Eqs. (25) and (70) we have

$$\nabla^2_{\bar{x}}\tilde{H}_{-1} = \frac{1}{4\pi^2}\sum_{i=1}^3\left(\frac{\partial^2}{\partial(x^i)^2} + 2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x'^i} + \frac{\partial^2}{\partial(x'^i)^2}\right)\left(\frac{1}{\sigma_+}\right)$$
$$= -\frac{1}{4\pi^2\sigma_+^2}\sum_{i=1}^3\left(\frac{\partial^2\sigma}{\partial(x^i)^2} + 2\frac{\partial^2\sigma}{\partial x^i\partial x'^i} + \frac{\partial^2\sigma}{\partial(x'^i)^2}\right), \tag{113}$$

where we used $\partial\sigma/\partial x^i = \partial\sigma/\partial x'^i = 0$ when the two points are on the geodesic. From [17], after we shift the Taylor series so that the Riemann tensor is evaluated at $\bar{x}$, we have

$$\frac{\partial^2\sigma}{\partial(x^i)^2} = -2\eta_{ii} - \frac{2}{3}R_{itit}(\bar{x})\tau^2 - \frac{1}{2}R_{itit,t}(\bar{x})\tau^3 - \frac{1}{5}R_{itit,tt}\tau^4 + O(\tau^5) \tag{114a}$$

$$\frac{\partial^2\sigma}{\partial(x'^i)^2} = -2\eta_{ii} - \frac{2}{3}R_{itit}(\bar{x})\tau^2 + \frac{1}{2}R_{itit,t}(\bar{x})\tau^3 - \frac{1}{5}R_{itit,tt}\tau^4 + O(\tau^5) \tag{114b}$$

$$\frac{\partial^2\sigma}{\partial x^i\partial x'^i} = 2\eta_{ii} - \frac{1}{3}R_{itit}(\bar{x})\tau^2 - \frac{7}{40}R_{itit,tt}\tau^4 + O(\tau^5). \tag{114c}$$

From Eqs. (113) and (115), and using $R_{itit} = -R_{tt}$ we have

$$\nabla^2_{\bar{x}}\tilde{H}_{-1} = -\frac{1}{4\pi^2}\left[\frac{2}{\tau_-^2}R_{tt}(\bar{x}) + \frac{3}{4}R_{tt,tt}(\bar{x})\right]. \tag{115}$$

From Eqs. (23) and (33), we need to compute

$$\int_0^\infty \frac{d\xi}{\pi}\hat{F}(-\xi,\xi'), \tag{116}$$

where

$$F(t,t') = g(t)g(t')\left[\frac{1}{4}\nabla^2_{\bar{x}}\tilde{H}_{(0)}(t,t') - \partial^2_\tau\tilde{H}_{(1)}(t,t')\right]. \tag{117}$$

In the first term in brackets it is sufficient to use $\tilde{H}_{(0)}(t,t')$, because higher order terms in H are smooth, even in τ, and vanish at coincidence, and so they do not contribute, as discussed in Sec. 3 A of Ref. [8]. In the second term, two powers of τ are removed by differentiation, so we need $\tilde{H}_{(1)}(t,t')$.

Using Eqs. (70), (92), (94), (95), (110), (111), (112) and (115) we can combine all terms in F to write

$$F(t,t') = g(t)g(t')\sum_{i=1}^{6} f_i(t,t'), \tag{118}$$

with

$$f_1 = \frac{3}{2\pi^2\tau_-^4} \tag{119a}$$

$$f_2 = \frac{1}{48\pi^2\tau_-^2}[R_{ii}(\bar{x}) - 7R_{tt}(\bar{x})] \tag{119b}$$

$$f_3 = \frac{1}{384\pi^2}\left[\frac{1}{5}R_{tt,tt}(\bar{x}) + \frac{1}{5}R_{ii,tt}(\bar{x}) - R_{tt,ii}(\bar{x}) + \frac{3}{5}R_{ii,jj}(\bar{x})\right]\ln(-\tau_-^2) \tag{119c}$$

$$f_4 = \frac{1}{320\pi^2}\left[-\frac{43}{3}R_{tt,tt}(\bar{x}) + \frac{7}{6}R_{tt,ii}(\bar{x}) - \frac{1}{2}R_{ii,jj}(\bar{x})\right] \tag{119d}$$

$$f_5 = \frac{1}{256\pi^2}\int d\Omega\nabla_{\bar{x}}^2\left\{\frac{1}{2}[G_{tt}^{(1)}(x'') - G_{rr}^{(1)}(x'')] - \int_0^1 dss^2 G_{tt}^{(1)}(x_s'')\right\} i\,\mathrm{sgn}\tau \tag{119e}$$

$$f_6 = -\frac{1}{64\pi^2}\int d\Omega\partial_\tau^2\left\{\frac{1}{2}[G_{tt}^{(3)}(x'') - G_{rr}^{(3)}(x'')] - \int_0^1 dss^2 G_{tt}^{(3)}(x_s'')\right\} i\,\mathrm{sgn}\tau. \tag{119f}$$

VII. THE QUANTUM INEQUALITY

We want to calculate the quantum inequality bound B, given by Eq. (22). We can write it

$$B = \sum_{i=1}^{8} B_i, \tag{120}$$

where

$$B_i = \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' g(t)g(t')f_i(t,t')e^{i\xi(t'-t)} = \int_0^\infty \frac{d\xi}{\pi}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t} g\left(\bar{t}-\frac{\tau}{2}\right)g\left(\bar{t}+\frac{\tau}{2}\right)f_i(\bar{t},\tau)e^{-i\xi\tau} \qquad i = 1\ldots6 \tag{121a}$$

$$B_7 = \int_{-\infty}^{\infty} dt g^2(t)Q(t) = \frac{1}{3840\pi^2}\int_{-\infty}^{\infty} dt g^2(t)\Box R(\bar{t}) \tag{121b}$$

$$B_8 = -\int_{-\infty}^{\infty} dt g^2(t)\left[2aR_{,ii}(\bar{x}) - \frac{b}{2}(R_{tt,tt}(\bar{x}) + R_{ii,tt}(\bar{x}) - 3R_{tt,ii}(\bar{x}) + R_{ii,jj}(\bar{x}))\right] \tag{121c}$$

using Eqs. (14), (16), (22) and (100). The first 6 terms have exactly the same τ dependence as the corresponding terms in Ref. [8]. So the Fourier transform proceeds in the same way, except that instead of dependence on the potential and its derivatives, we have dependence on the Ricci tensor and its derivatives. After the Fourier transform, we see that B_4 and B_7 have exactly the same form so we merge them in one term. Thus

$$B = \frac{1}{16\pi^2}\left[I_1 + \frac{1}{12}I_2 - \frac{1}{12}I_3 + \frac{1}{240}I_4 + \frac{1}{16\pi}I_5 - \frac{1}{4\pi}I_6\right] - I_7, \tag{122}$$

where

$$I_1 = \int_{-\infty}^{\infty} dt g''(t)^2 \tag{123a}$$

$$I_2 = \int_{-\infty}^{\infty} d\bar{t}[R_{ii}(\bar{x}) - 7R_{tt}(\bar{x})](g(\bar{t})g''(\bar{t}) - g'(\bar{t})g'(\bar{t})) \tag{123b}$$

$$I_3 = \int_{-\infty}^{\infty} d\tau \ln|\tau| \mathrm{sgn}\tau \int_{-\infty}^{\infty} d\bar{t}\left[\frac{1}{5}R_{tt,tt}(\bar{x}) + \frac{1}{5}R_{ii,tt}(\bar{x}) - R_{tt,ii}(\bar{x}) + \frac{3}{5}R_{ii,jj}(\bar{x})\right] g\left(\bar{t} - \frac{\tau}{2}\right) g'\left(\bar{t} + \frac{\tau}{2}\right) \tag{123c}$$

$$I_4 = \int_{-\infty}^{\infty} d\bar{t} g(\bar{t})^2[-171R_{tt,tt}(\bar{x}) - R_{ii,tt}(\bar{x}) + 13R_{tt,ii}(\bar{x}) - 5R_{ii,jj}(\bar{x})] \tag{123d}$$

$$I_5 = \int_{-\infty}^{\infty} d\tau \frac{1}{\tau} \int_{-\infty}^{\infty} d\bar{t} g(\bar{t} - \tau/2) g(\bar{t} + \tau/2) \int d\Omega \nabla_{\bar{x}}^2 \left\{\frac{1}{2}[G_{tt}^{(1)}(x'') - G_{rr}^{(1)}(x'')] - \int_0^1 ds s^2 [G_{tt}^{(1)}(x_s'')]\right\} \mathrm{sgn}\tau \tag{123e}$$

$$I_6 = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\bar{t} \partial_\tau^2 \left[\frac{1}{\tau} g(\bar{t} - \tau/2) g(\bar{t} + \tau/2)\right] \int d\Omega \left\{\frac{1}{2}[G_{tt}^{(3)}(x'') - G_{rr}^{(3)}(x'')] - \int_0^1 ds s^2 G_{tt}^{(3)}(x_s'')\right\} \mathrm{sgn}\tau \tag{123f}$$

$$I_7 = \int_{-\infty}^{\infty} dt g^2(t) \left[2aR_{,ii}(\bar{x}) - \frac{b}{2}(R_{tt,tt}(\bar{x}) + R_{ii,tt}(\bar{x}) - 3R_{tt,ii}(\bar{x}) + R_{ii,jj}(\bar{x}))\right]. \tag{123g}$$

If we only know that the Ricci tensor and its derivatives are bounded, as in Eqs. (3), we can restrict the magnitude of each term of Eq. (122). We start with the second term

$$|I_2| \leq \int_{-\infty}^{\infty} d\bar{t} |R_{ii}(\bar{x}) - 7R_{tt}(\bar{x})||g(\bar{t})g''(\bar{t}) - g'(\bar{t})g'(\bar{t})| \leq 10R_{\max} \int_{-\infty}^{\infty} d\bar{t}[g(\bar{t})|g''(\bar{t})| + g'(\bar{t})^2]. \tag{124}$$

Terms I_3, I_4 and I_7 are similar. For I_5 and I_6, we need bounds on the components of G. From Eq. (108b), $|G_{tt}| < 2R_{\max}$. Since Eq. (3) holds regardless of rotation, we can bound G_{rr} at any given point by taking the x-axis to point in the radial direction. Then $G_{rr} = G_{xx} = (1/2)[R_{xx} - R_{yy} - R_{zz} + R_{tt}]$ and $|G_{rr}| < 2R_{\max}$. Taking derivatives of G just differentiates the corresponding components of R, which are also bounded. In particular, since there are 3 terms in $\nabla_{\bar{x}}^2$, we have $|\nabla_{\bar{x}}^2 G_{tt}|, |\nabla_{\bar{x}}^2 G_{rr}| < 6R''_{\max}$. Using these results and Eq. (90) for the remainder we have

$$\begin{aligned} &\left| \int d\Omega \nabla_{\bar{x}}^2 \left\{\frac{1}{2}[G_{rr}^{(1)}(x'') - G_{tt}^{(1)}(x'')] + \int_0^1 ds s^2 G_{tt}^{(1)}(x_s'')\right\}\right| \\ &\quad \leq \frac{|\tau|}{2} \int d\Omega \left\{\frac{1}{2}[|\nabla^2 G_{rr,i}(\bar{x})| + |\nabla^2 G_{tt,i}(\bar{x})|] + \int_0^1 ds s^3 |\nabla^2 G_{tt,i}(\bar{x})|\right\} |\Omega^i| \leq R'''_{\max} \frac{15|\tau|}{4} \sum_i \int d\Omega |\Omega^i| = \frac{45\pi}{2} |\tau| R'''_{\max}. \end{aligned} \tag{125}$$

For I_6 we use Eq. (104) for the remainder,

$$\begin{aligned} &\left| \int d\Omega \left\{\frac{1}{2}[G_{rr}^{(3)}(x'') - G_{tt}^{(3)}(x'')] + \int_0^1 ds s^2 G_{tt}^{(3)}(x_s'')\right\}\right| \\ &\quad \leq \frac{|\tau|^3}{48} \int d\Omega \left\{\frac{1}{2}[|G_{rr,ijk}(\bar{x})| + |G_{tt,ijk}(\bar{x})|] + \int_0^1 ds s^5 |G_{tt,ijk}(\bar{x})|\right\} |\Omega^i||\Omega^j||\Omega^k| \\ &\quad \leq R'''_{\max} \frac{7|\tau|^3}{144} \sum_{i,j,k} \int d\Omega |\Omega^i||\Omega^j||\Omega^k| = \frac{7(2\pi + 1)}{24} |\tau|^3 R'''_{\max}. \end{aligned} \tag{126}$$

After we bound all the terms and calculate the derivatives in I_6 we can define

$$J_2 = \int_{-\infty}^{\infty} dt[g(t)|g''(t)| + g'(t)^2] \tag{127a}$$

$$J_3 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' |g'(t')||g(t)| \ln|t'-t|| \tag{127b}$$

$$J_4 = \int_{-\infty}^{\infty} dt g(t)^2 \tag{127c}$$

$$J_5 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' g(t)g(t') \tag{127d}$$

$$J_6 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' |g'(t')||g(t)||t'-t| \tag{127e}$$

$$J_7 = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' [g(t)|g''(t')| + g'(t)g'(t')](t'-t)^2 \tag{127f}$$

and find

$$|I_2| \le 10 R_{\max} J_2 \tag{128a}$$

$$|I_3| \le \frac{46}{5} R''_{\max} J_3 \tag{128b}$$

$$|I_4| \le 258 R''_{\max} J_4 \tag{128c}$$

$$|I_5| \le \frac{45\pi}{2} R'''_{\max} J_5 \tag{128d}$$

$$|I_6| \le \frac{7(2\pi+1)}{48} R'''_{\max}(4J_5 + 4J_6 + J_7) \tag{128e}$$

$$|I_7| \le (24|a| + 11|b|) R''_{\max} J_4. \tag{128f}$$

Thus the final form of the inequality is

$$\begin{aligned}
&\int_{\mathbb{R}} d\tau g(t)^2 \langle T^{\text{ren}}_{tt}\rangle_\omega(t,0) \\
&\quad \ge -\frac{1}{16\pi^2}\left\{ I_1 + \frac{5}{6} R_{\max} J_2 \right. \\
&\qquad + R''_{\max}\left[\frac{23}{30} J_3 + \left(\frac{43}{40} + 16\pi^2(24|a| + 11|b|)\right) J_4\right] \\
&\qquad \left. + R'''_{\max}\left[\frac{163\pi + 14}{96\pi} J_5 + \frac{7(2\pi+1)}{192\pi}(4J_6 + J_7)\right]\right\}.
\end{aligned} \tag{129}$$

Once we have a specific sampling function g, we can compute the integrals of Eqs. (129) to get a specific bound. In the case of a Gaussian sampling function,

$$g(t) = e^{-t^2/t_0^2}, \tag{130}$$

we computed these integrals numerically in Ref. [8]. Using those results the right-hand side of Eq. (129) becomes

$$\begin{aligned}
&-\frac{1}{16\pi^2 t_0^3}\{3.76 + 2.63 R_{\max} t_0^2 \\
&\quad + [3.42 + 197.9(24|a| + 11|b|)] R''_{\max} t_0^4 + 6.99 R'''_{\max} t_0^5\}.
\end{aligned} \tag{131}$$

The leading term is just the flat spacetime bound of Ref. [10] for g given by Eq. (130). The possibility of curvature weakens the bound by introducing additional terms, which have the same dependence on t_0 as in Ref. [8], with the Ricci tensor bounds in place of the bounds on the potential.

VIII. CONCLUSION

In this work, using a general quantum inequality of Fewster and Smith [7], we derived an inequality for a minimally coupled quantum scalar field on spacetimes with small curvature. We calculated the necessary Hadamard series terms and the Green's function for this problem to first order in the curvature. Combining these terms gives $\tilde{H}$, and taking the Fourier transform gives a bound in terms of the Ricci tensor and its derivatives.

If we know the spacetime explicitly, Eqs. (21), (122), and (124) give an explicit bound on the weighted average of the energy density along the geodesic. This bound depends on integrals of the Ricci tensor and its derivatives combined with the weighting function g.

If we do not know the spacetime explicitly but we know that the Ricci tensor and its first 3 derivatives are bounded, Eqs. (129) and (129) give a quantum inequality depending on the bounds and the weighting function. If we take a Gaussian weighting function, Eq. (131) gives a bound depending on the Ricci tensor bounds and the width of the Gaussian, t_0.

As expected, the result shows that the corrections due to curvature are small if the quantities $R_{\max} t_0^2$, $R''_{\max} t_0^4$, and $R'''_{\max} t_0^5$ are all much less than 1. That will be true if the curvature is small when we consider its effect over a distance equal to the characteristic sampling time t_0 (or equivalently if t_0 is much smaller than any curvature radius), and if the scale of variation of the curvature is also small compared to t_0.

In all bounds, there is unfortunately an ambiguity resulting from the unknown coefficients of local curvature terms in the gravitational Lagrangian. This ambiguity is parametrized by the quantities a and b.

Ford and Roman [3] have argued that flat-space quantum inequalities can be applied in curved spacetime, so long as the radius of curvature is small as compared to the sampling time. The present paper explicitly confirms this claim and calculates the magnitude of the deviation. The curvature must be small not only on the path where the quantum inequality is to be applied but also at any point that is in both the causal future of some point of this path and the causal past of another. All such points are included in the integrals in Eqs. (123e) and (123f).

Is is interesting to consider vacuum spacetimes, i.e., those whose Ricci tensor vanishes. These include, for example, the Schwarzschild and Kerr spacetimes, and those consisting only of gravitational waves. In such spacetimes, the flat-space quantum inequality will hold to first order without modification. There are, of course, second-order corrections. For the Schwarzschild spacetime, for example, these were calculated explicitly by Visser [18–20].

In Ref. [2] we proved a theorem ruling out achronal ANEC violation, given a conjecture that paths with small acceleration in spacetimes with small curvature obey the same null-projected timelike-averaged quantum inequality as in flat space [21], with corrections of the form discussed here. The present result is a step toward proving that conjecture. In future work we intend to extend the present result to null-projected instead of timelike-projected quantum inequalities and to handle slightly nongeodesic curves.

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