PHYSICAL REVIEW D **91**, 103536 (2015)

Phenomenology of dark matter via a bimetric extension of general relativity

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(Received 5 February 2015; published 27 May 2015)

We propose a relativistic model of dark matter reproducing at once the concordance cosmological model Λ-cold-dark-matter (Λ-CDM) at cosmological scales and the phenomenology of the modified Newtonian dynamics (MOND) at galactic scales. To achieve this we postulate a nonstandard form of dark matter, consisting of two different species of particles coupled to gravity via a bimetric extension of general relativity and linked together through an internal vector field (a "graviphoton") generated by the mass of these particles. We prove that this dark matter behaves like ordinary cold dark matter at the level of first-order cosmological perturbation, while a pure cosmological constant plays the role of dark energy. The MOND equation emerges in the nonrelativistic limit through a mechanism of gravitational polarization of the dark matter medium in the gravitational field of ordinary matter. Finally we show that the model is viable in the Solar System as it predicts the same parametrized post-Newtonian parameters as general relativity.

DOI: 10.1103/PhysRevD.91.103536 PACS numbers: 95.35.+d, 04.50.Kd, 95.36.+x

I. INTRODUCTION

The goal of the present article is to reproduce within a single relativistic framework, consisting of a nonstandard form of dark matter particles coupled to a bimetric extension of general relativity (GR), both

(1) the concordance cosmological model Λ-CDM and its tremendous successes at cosmological scales and notably the cosmic microwave background (CMB) at first-order cosmological perturbations (see [1–4] for reviews), in which cold dark matter (CDM) is a fluid of collisionless particles without interactions, and Λ is a pure cosmological constant added to the Einstein field equations; and
(2) the phenomenology of MOND (i.e. modified Newtonian dynamics or Milgromian dynamics [5–7]), which is a basic set of observational phenomena relevant to galaxy dynamics and dark matter distribution at galactic scales (see [8–10] for reviews), including most importantly the almost flat rotation curves of galaxies, the famous baryonic Tully-Fisher (BTF) relation for spiral galaxies [11–13], and its equivalent for elliptical galaxies, the Faber-Jackson relation [14].

It has long been known, and so far disappointing, that the cosmological model Λ-CDM, when extrapolated down to galactic scales, seems to be fundamentally incompatible with the phenomenology of MOND. Within the Λ-CDM picture one can only take notice of that phenomenology and suppose that it emerges from some (physical or astrophysical) mechanism taking place in the interaction between dark matter and baryons. A lot of work on astrophysical feedbacks (e.g. supernova winds) has been done to reconcile Λ-CDM with observations; see e.g. [15,16]. However, because of the problem of fine-tuning of complicated phenomena to simple empirical laws like the BTF relation, and because of the baffling presence of the MOND acceleration scale a_0 in the data, it appears to be practically impossible that Λ-CDM could provide a satisfactory explanation of the MOND phenomenology [10]. By contrast, the MOND empirical formula is extremely predictive and successful for galaxy dynamics [8,9] but is antagonistic to anything we would like to call a fundamental theory. Furthermore, it has problems at larger scales where it fails to reproduce about one-half of the dark matter we see in galaxy clusters [17–22] and unfortunately has *a priori* little to say about cosmology at still larger scales.

Most relativistic MOND theories extend GR with appropriate extra fundamental fields, so as to recover MOND in the nonrelativistic limit; see Refs. [23–32]. None of these theories assume dark matter, so they can be called pure modified gravity theories. They have been extensively studied in cosmology, notably the tensor-vector-scalar theory [23–25] and noncanonical Einstein-æther theories [26,27], at first-order perturbation around a cosmological background (see e.g. Refs. [33–38]). However, because they do not assume dark matter, the pure modified gravity theories have difficulties at reproducing the cosmological observations, notably the full spectrum of anisotropies of the CMB.

A different approach, called dipolar dark matter (DDM), is more promising in order to fit cosmological observations. This approach is motivated by the *dielectric analogy* of MOND [39,40], a remarkable property of the MOND formula which may have deep physical implications (but of course, which could also be merely coincidental). The idea is that the phenomenology of MOND could arise from some property of dark matter itself, namely a spacelike

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vector field called the gravitational dipole moment and able to polarize the DDM medium in the gravitational field of ordinary matter. A relativistic version of this idea has been proposed in Refs. [41,42] and correctly reproduces the cosmological Λ-CDM model at the level of first-order cosmological perturbations. The deviations from Λ-CDM at second-order cosmological perturbations in that model have also been investigated [43].

In the model [41,42] the phenomenology of MOND is recovered when the DDM medium is polarized, i.e. when the polarization field is aligned with the local gravitational field. This is obtained at the price of a hypothesis of "weak clustering" of DDM, namely the fact that the DDM medium stays essentially at rest and does not cluster much in galaxies compared to ordinary matter. This hypothesis is made plausible by the fact that the internal force due to the presence of the dipole moment will balance the gravitational force. Furthermore the hypothesis has been explicitly verified in the case of the static gravitational field of a spherical mass distribution (see the Appendix of Ref. [41]). However, in more general situations, either highly dynamical or involving nonspherical gravitational fields, it is likely that the polarization will not be exactly aligned with the gravitational field, and in that case the model [41,42] would deviate from MOND *stricto sensu*. We have in mind situations like the dynamical evolution of galaxies including the formation of bars [44], the collision of spiral galaxies yielding famous antenna structures [45] and the formation of tidal dwarf galaxies [46], and the problem of nonspherical polar ring galaxies [47].

In the present paper we propose a new relativistic model for DDM, which is free of the weak clustering hypothesis of the DDM and thus permits one to recover the phenomenology of MOND in all situations, either spherical or nonspherical and/or highly dynamical. Furthermore we shall show that this model also recovers the essential features of the standard cosmological model Λ-CDM and in particular is indistinguishable from it at first-order perturbation around a cosmological background.

The present model is actually closer to the original concept of gravitational polarization and dipolar dark matter [39,40]. Indeed it involves *two* species of dark matter particles, interacting together via some internal force field. The DDM medium appears to be the gravitational analogue of a plasma in electrodynamics, oscillating at the natural "plasma" frequency, and which can be polarized by the gravitational field of ordinary matter, mimicking the presence of dark matter. The nonrelativistic approximation of our model has already been reviewed in Sec. III of Ref. [40] and exhibits all the desirable features we would expect for gravitational polarization and MOND.[1]

[1]Interpreting this polarizable dark matter medium as a sea of virtual pairs of particles and antiparticles, we gave in Ref. [40] a few numerical estimates that such a hypothetical medium would have [48,49].

To achieve these results we assume that the two species of dark matter particles are coupled to two different metrics, reducing to two different Newtonian potentials in the nonrelativistic limit [40]. Therefore, in this new model and contrary to the previous one [41,42], we do consider a modification of gravity, in the form of a bimetric extension of GR. Furthermore, the internal field is chosen to be a vector field whose associated charge is the mass of particles—i.e. a "graviphoton." Thus our model is a compromise between particle dark matter and modified gravity, which can be seen as the result of the antinomic phenomenologies of dark matter when it is seen either in cosmology or in galaxies. Note that the model is very different from BIMOND [28,29], a bimetric theory that has been proposed for MOND and which is a pure modified gravity theory without dark matter.

As our model uses a bimetric extension of general relativity, it is necessary to check the consistency of its gravitational sector. In particular the number of propagating gravitational degrees of freedom should be investigated, together with their possible ghostlike behavior. This work would be along the lines of studies of ghost-free bimetric theory motivated by a nontrivial generalization of de Rham-Gabadadze-Tolley (dRGT) massive gravity [50–54]. In the spirit of the search for relativistic MOND theories [23–32,41,42], we focus here on the physical consequences of the model. The counting of gravitational propagating modes is treated in Ref. [55]. However we shall indicate in Appendix C that some aspects of the gravitational sector of the model are safe at linear order around a Minkowski background.

Finally since the present theory involves a modification of gravity it is very important to check its viability in the Solar System (SS). We compute the first post-Newtonian (1PN) limit of the model in the regime of the SS, i.e. when typical accelerations are much above the MOND scale a_0, and find the same parametrized post-Newtonian (PPN) parameters as in GR [56], which allows us to conclude that the theory is viable in this regime.

The plan of this paper is as follows. In Sec. II we describe the model using a relativistic action for the ordinary matter, the two types of dark matter coupled to two different metrics, and an internal vector field. We also look at the perturbative solution of the field and matter equations. In Sec. III we investigate the cosmology of the model up to first order in perturbations. In Sec. IV we investigate the nonrelativistic limit of the model, describe the mechanism of polarization that yields the MOND phenomenology at galactic scales (see [40] for a review), and check the 1PN limit of the model. The paper ends with a short conclusion in Sec. V and with appendixes presenting technical details, notably Appendix C which investigates the gravitational sector of the model at linear order.

II. DIPOLAR DARK MATTER AND MODIFIED GRAVITY

A. Dynamical action and field equations

Let us consider a model involving, in addition to the ordinary matter simply described by baryons, two species of dark matter particles. The gravitational sector is composed of two Lorentzian metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ and one vector field K_μ sourced by the dark matter masses and which will be called a *graviphoton*. The baryons are coupled in the usual way to the metric $g_{\mu\nu}$. Though we model the ordinary matter only by baryons, we have in mind that all ordinary matter fields (fermions, neutrinos, electromagnetic radiation, etc.) are coupled in the standard way to the ordinary metric $g_{\mu\nu}$. As a way to recover the dipolar behavior of dark matter we assume that one species of dark matter particles is, like the baryons, minimally coupled to the ordinary metric $g_{\mu\nu}$, while the other one is minimally coupled to the second metric $\underline{g}_{\mu\nu}$. The vector field K_μ links together the two species of dark matter particles and is crucial in order to ensure the stability of the dipolar medium.

The gravitational-plus-matter action of our model reads[2]

$$S = \int \mathrm{d}^4x \left\{ \sqrt{-g}\left(\frac{R-2\lambda}{32\pi} - \rho_\mathrm{b} - \rho\right) + \sqrt{-\underline{g}}\left(\frac{\underline{R}-2\underline{\lambda}}{32\pi} - \underline{\rho}\right) + \sqrt{-f}\left[\frac{\mathcal{R}-2\lambda_f}{16\pi\varepsilon} + (j^\mu - \underline{j}^\mu)K_\mu + \frac{a_0^2}{8\pi}W(X)\right]\right\}. \tag{2.1}$$

We describe baryons and dark matter particles in their respective sectors by their conserved scalar densities ρ_b, ρ and $\underline{\rho}$, without pressure, and define their four-velocities u_b^μ, u^μ and $\underline{u}^\mu$, normalized with their respective metrics, i.e. $g_{\mu\nu}u_\mathrm{b}^\mu u_\mathrm{b}^\nu = g_{\mu\nu}u^\mu u^\nu = \underline{g}_{\mu\nu}\underline{u}^\mu\underline{u}^\nu = -1$. We denote by $R \equiv R[g]$ and $\underline{R} \equiv R[\underline{g}]$ the Ricci scalars associated with the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$. These metrics interact with each other through an interaction term involving the Ricci scalar $\mathcal{R} \equiv R[f]$ associated with an additional Lorentzian metric $f_{\mu\nu}$, defined nonperturbatively from $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ by the implicit relations

$$f_{\mu\nu} = f^{\rho\sigma}g_{\rho\mu}\underline{g}_{\nu\sigma} = f^{\rho\sigma}g_{\rho\nu}\underline{g}_{\mu\sigma}, \tag{2.2}$$

where $f^{\rho\sigma}$ is the inverse metric, i.e. $f^{\rho\sigma}f_{\sigma\tau} = \delta^\rho_\tau$. Note that (2.2) implies $f^2 = g\underline{g}$ for the determinants [e.g. $f = \det(f_{\mu\nu})$]. In applications the relations (2.2) will be solved perturbatively and the solution in the form of a full perturbative series is analyzed in Appendix A. The action (2.1) is thus composed of an ordinary sector coupled to $g_{\mu\nu}$, first term in (2.1), a dark sector coupled to $\underline{g}_{\mu\nu}$ in the second term, and an interacting sector with metric $f_{\mu\nu}$ in the third term, which also entirely contains the contribution of the internal field K_μ. The ordinary and dark sectors are not symmetrical due to the baryons in the ordinary sector, and we may imagine that this is somewhat similar to the matter-antimatter asymmetry.

The model is specified by several constants: the MOND acceleration scale a_0 [5–7] which has been introduced only in the interacting sector, some cosmological constants λ, $\underline{\lambda}$ and λ_f that have been inserted in the three sectors and will be related to the true cosmological constant Λ of the model Λ-CDM, and a dimensionless coupling constant ε ruling the strength of the interaction between the two metrics and which will be assumed to be very small, $\varepsilon \ll 1$, in Sec. IV. We can write the latter coupling constant as $\varepsilon = (m_\mathrm{P}/M)^2$, where m_P is the Planck mass and M represents a new mass scale that we shall not need to specify here except in Sec. IV where $M \gg m_\mathrm{P}$ will be assumed.

Like in the previous model [41,42] we have in mind that the MOND scale a_0 is a fundamental constant and that the observed cosmological constant Λ (to which we shall relate the constants λ, $\underline{\lambda}$, λ_f in the action) would be derived from it in a more fundamental theory and so naturally satisfies the appropriate scaling relation $\Lambda \sim a_0^2$ which is in very good agreement with observations [9]. It would be interesting to investigate whether the coupling constant ε (or mass M) could also be related to the acceleration scale a_0.

The internal vector field K_μ obeys a noncanonical kinetic term $W(X)$, where

$$X \equiv -\frac{H^{\mu\nu}H_{\mu\nu}}{2a_0^2}. \tag{2.3}$$

We pose $H_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu$ and $H^{\mu\nu} = f^{\mu\rho}f^{\nu\sigma}H_{\rho\sigma}$ since the metric in this sector is $f_{\mu\nu}$. Note that the vector field strength in (2.3) has been rescaled by the MOND acceleration a_0. We refer to [57] for discussions on the stability and Cauchy problem for vector field theories involving noncanonical kinetic terms.

The function W is determined phenomenologically in order to recover MOND from the nonrelativistic limit of the model studied in Sec. IV A and to be in agreement with the usual Solar-System tests as investigated in Sec. IV B. This function, which is related to the MOND interpolating function, should in principle be interpreted within some more fundamental theory. However this task is not addressed in this work. In the limit $X \ll 1$, which corresponds to the MOND weak-acceleration regime below the scale a_0, we impose

$$W(X) = X - \frac{2}{3}X^{3/2} + \mathcal{O}(X^2). \tag{2.4}$$

[2] Greek indices $\mu, \nu, \ldots$ take space-time values 0,1,2,3 and Latin ones space values 1,2,3. The signature of the three Lorentzian metrics $g_{\mu\nu}$, $\underline{g}_{\mu\nu}$ and $f_{\mu\nu}$ is $(-,+,+,+)$. In most of the paper we use geometrical units with $G = c = 1$. Symmetrization of indices is defined by $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$.

On the other hand we also impose the following behavior of W when $X \gg 1$ so as to recover the usual 1PN limit of GR in an acceleration regime much above a_0 [see Sec. IV B]:

$$W(X) = A + \frac{B}{X^b} + o\left(\frac{1}{X^b}\right), \tag{2.5}$$

where A and B are constants and where the power b can be any strictly positive real number, $b > 0$. The limit of the action (2.1) in the strong-field regime $X \gg 1$ is given in Eq. (4.24) of Sec. IV.

The graviphoton K_μ is sourced by the dark matter currents j^μ and $\underline{j}^\mu$ in the interacting sector of the action, defined as follows. First we define the baryons and dark matter currents in their respective sector by $J_\mathrm{b}^\mu = \rho_\mathrm{b} u_\mathrm{b}^\mu$, $J^\mu = \rho u^\mu$ and $\underline{J}^\mu = \underline{\rho}\,\underline{u}^\mu$. These currents are conserved in the sense that $\nabla_\mu J_\mathrm{b}^\mu = 0$ and $\nabla_\mu J^\mu = 0$, where ∇_μ is the covariant derivative associated with $g_{\mu\nu}$, and $\underline{\nabla}_\mu \underline{J}^\mu = 0$, where $\underline{\nabla}_\mu$ is the covariant derivative of $\underline{g}_{\mu\nu}$. Then both dark matter currents j^μ and $\underline{j}^\mu$ in the action (2.1) are defined with respect to the metric $f_{\mu\nu}$, solution of Eq. (2.2). They are thus given by

$$j^\mu = \beta J^\mu, \qquad \underline{j}^\mu = \underline{\beta}\,\underline{J}^\mu, \tag{2.6}$$

where we pose $\beta \equiv \sqrt{-g}/\sqrt{-f}$ and $\underline{\beta} \equiv \sqrt{-\underline{g}}/\sqrt{-f}$,[3] and obey the conservation laws $\mathcal{D}_\mu j^\mu = 0$ and $\mathcal{D}_\mu \underline{j}^\mu = 0$, where $\mathcal{D}_\mu$ is the covariant derivative associated with $f_{\mu\nu}$.

As a preliminary check of the consistency of our model, we investigate in Appendix C the gravitational part of the action (2.1) at quadratic order around a Minkowski background and show that it reduces to the sum of the actions for two noninteracting massless spin-2 fields. We conclude that the model is consistent (i.e. ghost-free) at that order.

First we vary the action with respect to the metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$. For the moment we write the linear variation of $f_{\mu\nu}$ as

$$\delta f_{\mu\nu} = \frac{1}{2}\mathcal{A}^{\rho\sigma}_{\mu\nu}\delta g_{\rho\sigma} + \frac{1}{2}\underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu}\delta \underline{g}_{\rho\sigma}, \tag{2.7}$$

where $\mathcal{A}^{\rho\sigma}_{\mu\nu}$ and $\underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu}$ denote some tensorial coefficients obeying two implicit equations given in Appendix A and which will be computed perturbatively in applications; see for instance (A11) in Appendix A. We then obtain the two Einstein field equations

$$\begin{aligned}\beta(E^{\mu\nu} + \lambda g^{\mu\nu}) &+ \frac{1}{\varepsilon}\mathcal{A}^{\mu\nu}_{\rho\sigma}(\mathcal{E}^{\rho\sigma} + \lambda_f f^{\rho\sigma}) \\ &= 16\pi[\beta(T_\mathrm{b}^{\mu\nu} + T^{\mu\nu}) + \mathcal{A}^{\mu\nu}_{\rho\sigma}\tau^{\rho\sigma}],\end{aligned} \tag{2.8a}$$

$$\begin{aligned}\underline{\beta}(\underline{E}^{\mu\nu} + \underline{\lambda}\,\underline{g}^{\mu\nu}) &+ \frac{1}{\varepsilon}\underline{\mathcal{A}}^{\mu\nu}_{\rho\sigma}(\mathcal{E}^{\rho\sigma} + \lambda_f f^{\rho\sigma}) \\ &= 16\pi[\underline{\beta}\,\underline{T}^{\mu\nu} + \underline{\mathcal{A}}^{\mu\nu}_{\rho\sigma}\tau^{\rho\sigma}],\end{aligned} \tag{2.8b}$$

where the Einstein tensors associated with their respective metrics are $E^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$, $\underline{E}^{\mu\nu} = \underline{R}^{\mu\nu} - \frac{1}{2}\underline{g}^{\mu\nu}\underline{R}$ and $\mathcal{E}^{\mu\nu} = \mathcal{R}^{\mu\nu} - \frac{1}{2}f^{\mu\nu}\mathcal{R}$. The stress-energy tensors of the matter particles are given by $T_\mathrm{b}^{\mu\nu} = \rho_\mathrm{b} u_\mathrm{b}^\mu u_\mathrm{b}^\nu$, $T^{\mu\nu} = \rho u^\mu u^\nu$ and $\underline{T}^{\mu\nu} = \underline{\rho}\,\underline{u}^\mu \underline{u}^\nu$, each one being defined with its respective metric. In addition, the stress-energy tensor of the internal graviphoton field K_μ, living in the sector $f_{\mu\nu}$, reads

$$\tau^{\mu\nu} = \frac{1}{8\pi}\left[W' H^{\mu\rho}H^\nu_\rho + \frac{a_0^2}{2}W f^{\mu\nu}\right], \tag{2.9}$$

where $W' \equiv \mathrm{d}W/\mathrm{d}X$. Next, varying the action with respect to K_μ we obtain

$$\mathcal{D}_\nu[W' H^{\mu\nu}] = 4\pi(j^\mu - \underline{j}^\mu), \tag{2.10}$$

which is obviously compatible with the conservation laws $\mathcal{D}_\mu j^\mu = \mathcal{D}_\mu \underline{j}^\mu = 0$.

Finally, we vary the action with respect to the particles. Since the baryons are minimally coupled to the metric $g_{\mu\nu}$, their equation of motion is simply the geodesic equation $a_\mathrm{b}^\mu = 0$, where $a_\mathrm{b}^\mu \equiv u_\mathrm{b}^\nu \nabla_\nu u_\mathrm{b}^\mu$. On the contrary, because of the presence of the internal field K_μ, the motion of dark matter particles is nongeodesic:

$$a_\mu = u^\nu H_{\mu\nu}, \tag{2.11a}$$

$$\underline{a}_\mu = -\underline{u}^\nu H_{\mu\nu}, \tag{2.11b}$$

where $a^\mu \equiv u^\nu \nabla_\nu u^\mu$ and $a_\mu = g_{\mu\nu}a^\nu$, and similarly $\underline{a}^\mu \equiv \underline{u}^\nu \underline{\nabla}_\nu \underline{u}^\mu$ and $\underline{a}_\mu = \underline{g}_{\mu\nu}\underline{a}^\nu$. Note that the forces acting on the two species of dark matter particles are spacelike and are completely analogous to the usual Lorentz force acting on charged particles.

The stress-energy tensors of the dark matter particles and of the internal field are not conserved separately, but we can derive a "global" conservation law. Indeed Eq. (2.10) can be equivalently written by means of the stress-energy tensor (2.9) as

$$\mathcal{D}_\nu \tau^\nu_\mu = -\frac{1}{2}(j^\nu - \underline{j}^\nu)H_{\mu\nu}, \tag{2.12}$$

where we pose $\tau^\nu_\mu = f_{\mu\rho}\tau^{\nu\rho}$. As a result of Eq. (2.12) the two dark matter equations of motion (2.11) can be combined to give

$$\mathcal{D}_\nu \tau^\nu_\mu + \frac{1}{2}(\beta \nabla_\nu T^\nu_\mu + \underline{\beta}\,\underline{\nabla}_\nu \underline{T}^\nu_\mu) = 0, \tag{2.13}$$

[3]Note that with our choice (2.2) for the metric $f_{\mu\nu}$ we have $\beta\underline{\beta} = 1$.

where $T_\mu^\nu = g_{\mu\rho} T^{\nu\rho}$ and $\underline{T}_\mu^\nu = \underline{g}_{\mu\rho} \underline{T}^{\nu\rho}$. This conservation law describes the exchanges of stress-energy between the dark matter particles and the internal field.

B. First-order perturbation of the matter and gravitational fields

We now make a crucial assumption regarding the two fluids of dark matter particles, namely that they differ by some small displacement vectors y^μ and $\underline{y}^\mu$ from a common equilibrium configuration where they superpose on top of each other. This assumption permits one to obtain a solution of the field equations, which is at the basis of the cosmological, MOND and Solar-System solutions, respectively investigated in Secs. III, IV A and IV B. Such a solution suggests a description of the dark matter medium as the analogue of a relativistic plasma in electromagnetism, polarizable in the gravitational field of ordinary matter and oscillating at its natural plasma frequency [39,40].

Looking for such a solution we make a perturbative assumption regarding the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$. We note that if they are related by a conformal transformation, $g_{\mu\nu} = \alpha^2 \underline{g}_{\mu\nu}$, then there is a simple, "conformal" solution of (2.2) given by $f_{\mu\nu} = \alpha^{-1} g_{\mu\nu} = \alpha \underline{g}_{\mu\nu}$. Here we assume that our solution differs from the latter conformal solution by a small metric perturbation $h_{\mu\nu} = \frac{1}{2}(\alpha^{-1} g_{\mu\nu} - \alpha \underline{g}_{\mu\nu})$. Then we can solve Eq. (2.2) at first order in $h_{\mu\nu}$ as

$$g_{\mu\nu} = \alpha(f_{\mu\nu} + h_{\mu\nu}) + \mathcal{O}(2), \tag{2.14a}$$

$$\underline{g}_{\mu\nu} = \frac{1}{\alpha}(f_{\mu\nu} - h_{\mu\nu}) + \mathcal{O}(2), \tag{2.14b}$$

where second-order terms in $h_{\mu\nu}$ are systematically neglected in this section and we define $\mathcal{O}(n) \equiv \mathcal{O}(h^n)$. Our introduction of the factor α is motivated by the application to cosmology in Sec. III in order to allow for two different cosmological backgrounds for the metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$. For this application it will be sufficient to assume that α is constant.

As we have seen the two dark matter fluids are described by the conserved currents j^μ and $\underline{j}^\mu$ defined by Eqs. (2.6). We now suppose that they slightly differ from an equilibrium configuration described by the equilibrium current $j_0^\mu = \rho_0 u_0^\mu$, conveniently defined with respect to the metric $f_{\mu\nu}$, so that $f_{\mu\nu} u_0^\mu u_0^\nu = -1$ and $\mathcal{D}_\mu j_0^\mu = 0$. In Appendix B we give details of the plasmalike hypothesis. In particular we obtain that

$$j^\mu = j_0^\mu + \mathcal{D}_\nu(j_0^\nu y_\perp^\mu - j_0^\mu y_\perp^\nu) + \mathcal{O}(2), \tag{2.15a}$$

$$\underline{j}^\mu = j_0^\mu + \mathcal{D}_\nu(j_0^\nu \underline{y}_\perp^\mu - j_0^\mu \underline{y}_\perp^\nu) + \mathcal{O}(2). \tag{2.15b}$$

Our explicit plasmalike solution is now obtained when we insert the ansatz (2.15) into the graviphoton field equation (2.10). Indeed, posing for the two displacements $y^\mu = y_0^\mu + \frac{1}{2}\xi^\mu$ and $\underline{y}^\mu = y_0^\mu - \frac{1}{2}\xi^\mu$, where $\xi^\mu = y^\mu - \underline{y}^\mu$ is the relative displacement, we can straightforwardly integrate this equation with the result

$$W' H^{\mu\nu} = -4\pi(j_0^\mu \xi_\perp^\nu - j_0^\nu \xi_\perp^\mu) + \mathcal{O}(2). \tag{2.16}$$

This is valid for any function $W(X)$ in the action, where X is defined by (2.3), showing that $W' H^{\mu\nu} = \mathcal{O}(1)$. In the MOND weak-field regime and also for first-order cosmological perturbations where $X \ll 1$, the function W behave as $W' = 1 + \mathcal{O}(1)$; see Eq. (2.4). Thus Eq. (2.16) tells us that $H^{\mu\nu}$ itself is a perturbative quantity and reduces at first order to

$$H^{\mu\nu} = -4\pi(j_0^\mu \xi_\perp^\nu - j_0^\nu \xi_\perp^\mu) + \mathcal{O}(2). \tag{2.17}$$

This solution is analogous to a classic one in relativistic plasma physics and is at the basis of our model of dipolar dark matter. It implies that the stress-energy tensor (2.9) of the internal field is of second order in the MOND regime and in cosmology:

$$\tau^{\mu\nu} = \mathcal{O}(2). \tag{2.18}$$

On the other hand, in the limiting case $X \gg 1$ appropriate to the Solar System where we have the postulated behavior (2.5) hence $W' \sim X^{-b-1}$, Eq. (2.16) tells us that the dipole moment scales as $\xi_\perp \sim X^{-b-1/2}$ and can be neglected since $b > 0$. We shall use this result in Sec. IV B for the study of the post-Newtonian limit of the theory in the Solar System.

We shall now investigate the matter equations and Einstein field equations at first perturbative order in the weak-field limit $X \ll 1$, for which we have already derived the solutions (2.17) and (2.18). Inserting (2.17) into the equations of motion (2.11) of the dark matter particles, and using (B4), we obtain

$$a^\mu = -4\pi\alpha^{-3/2} \rho_0 \xi_\perp^\mu + \mathcal{O}(2), \tag{2.19a}$$

$$\underline{a}^\mu = 4\pi\alpha^{3/2} \rho_0 \xi_\perp^\mu + \mathcal{O}(2). \tag{2.19b}$$

Thus a^μ and $\underline{a}^\mu$ are perturbative quantities of order $\mathcal{O}(1)$.

From now on we shall often view the dark matter, instead of being composed of the two fluids j^μ and $\underline{j}^\mu$, as composed of a single fluid with current j_0^μ but endowed with the vector field $\xi_\perp^\mu$. In analogy with the previous model of dipolar dark matter [41,42] we can call the vector field $\xi_\perp^\mu$ a dipole moment. Note that $\xi_\perp^\mu$ is necessarily spacelike, because of the projection orthogonal to the timelike four-velocity u_0^μ of the equilibrium configuration.

We next make use of the relations (2.15), or equivalently (B3) and (B4), to transform the two equations of motion

(2.19) into two equivalent equations. First, we obtain the equation of evolution for the dipole moment:

$$\ddot{\xi}^\mu_\perp + \xi^\rho_\perp \mathcal{R}^\mu{}_{\nu\rho\sigma} u_0^\nu u_0^\sigma = -\perp^\mu_\sigma (2\mathcal{D}_\nu h^\sigma_\rho - \mathcal{D}^\sigma h_{\nu\rho}) u_0^\nu u_0^\rho - 4\pi(\alpha^{-1/2} + \alpha^{1/2})\rho_0 \xi^\mu_\perp + \mathcal{O}(2), \tag{2.20}$$

where we denote $\ddot{\xi}^\mu_\perp \equiv u_0^\rho \mathcal{D}_\rho (u_0^\sigma \mathcal{D}_\sigma \xi^\mu_\perp)$, and the Riemann curvature tensor $\mathcal{R}^\mu{}_{\nu\rho\sigma} \equiv R^\mu{}_{\nu\rho\sigma}[f]$ of the metric $f_{\mu\nu}$ arises from the commutator of covariant derivatives. Second, we get

$$a_0^\mu + \ddot{y}^\mu_{0\perp} + y^\rho_{0\perp} \mathcal{R}^\mu{}_{\nu\rho\sigma} u_0^\nu u_0^\sigma = -2\pi(\alpha^{-1/2} - \alpha^{1/2})\rho_0 \xi^\mu_\perp + \mathcal{O}(2), \tag{2.21}$$

where we pose $\ddot{y}^\mu_{0\perp} \equiv u_0^\rho \mathcal{D}_\rho (u_0^\sigma \mathcal{D}_\sigma y^\mu_{0\perp})$ and recall that $y_0^\mu = \frac{1}{2}(y^\mu + \underline{y}^\mu)$. The evolution of the vector y_0^μ, which is the "center of position" of y^μ and $\underline{y}^\mu$, is thus governed by (2.21). We now specify the equilibrium configuration by choosing $y_0^\mu = 0$, which implies that the fluid at equilibrium obeys

$$a_0^\mu = -2\pi(\alpha^{-1/2} - \alpha^{1/2})\rho_0 \xi^\mu_\perp + \mathcal{O}(2). \tag{2.22}$$

The equilibrium fluid is geodesic with respect to the metric $f_{\mu\nu}$ in the special case where the two metrics have the same background, i.e. $\alpha = 1$. We shall see that when the coupling constant ε is very small (as will be assumed in Sec. IV to reproduce MOND and to study the 1PN limit), α is indeed very close to one so that the equilibrium fluid is almost geodesic.

For the choice $y_0^\mu = 0$ adopted here, we can easily relate the dark matter stress-energy tensors $T^{\mu\nu}$ and $\underline{T}^{\mu\nu}$ to the one of the equilibrium fluid, $T_0^{\mu\nu} = \rho_0 u_0^\mu u_0^\nu$, and to the dipole moment $\xi^\mu_\perp$ and its time derivative $\dot{\xi}^\mu_\perp \equiv u_0^\rho \mathcal{D}_\rho \xi^\mu_\perp$:

$$\beta T^{\mu\nu} = \alpha^{-1/2}\left[T_0^{\mu\nu}\left(1 + \frac{1}{2} h_{\rho\sigma} u_0^\rho u_0^\sigma\right) + j_0^{(\mu} \dot{\xi}^{\nu)}_\perp - \frac{1}{2}\mathcal{D}_\rho(\xi^\rho_\perp T_0^{\mu\nu})\right] + \mathcal{O}(2), \tag{2.23a}$$

$$\underline{\beta T}^{\mu\nu} = \alpha^{1/2}\left[T_0^{\mu\nu}\left(1 - \frac{1}{2} h_{\rho\sigma} u_0^\rho u_0^\sigma\right) - j_0^{(\mu} \dot{\xi}^{\nu)}_\perp + \frac{1}{2}\mathcal{D}_\rho(\xi^\rho_\perp T_0^{\mu\nu})\right] + \mathcal{O}(2). \tag{2.23b}$$

Concerning the baryons (defined with respect to the metric $g_{\mu\nu}$) we get the simpler relation

$$\beta T_{\rm b}^{\mu\nu} = \alpha^{-1/2} T_{0\rm b}^{\mu\nu}\left(1 + \frac{1}{2} h_{\rho\sigma} u_{0\rm b}^\rho u_{0\rm b}^\sigma\right) + \mathcal{O}(2). \tag{2.24}$$

Finally we provide the two Einstein field equations (2.8) at first order in both the metric perturbation and the dipole moment and in the weak-field regime for which we have $H^{\mu\nu} = \mathcal{O}(1)$ and $\tau^{\mu\nu} = \mathcal{O}(2)$, according to (2.17) and (2.18). We apply a standard perturbation analysis to relate both Einstein tensors $E^{\mu\nu}$ and $\underline{E}^{\mu\nu}$ to the Einstein tensor $\mathcal{E}^{\mu\nu}$ of the metric $f_{\mu\nu}$ at first order in the metric perturbation $h_{\mu\nu}$. At zeroth order $\alpha^{-1} g_{\mu\nu}$ and $\alpha \underline{g}_{\mu\nu}$ reduce to the same background $f_{\mu\nu}$ and we get a consistency condition on the matter tensors $T_0^{\mu\nu}$ and $T_{0\rm b}^{\mu\nu}$ in Eqs. (2.23) and (2.24) so that the two corresponding Einstein field equations for the background are the same:

$$T_{0\rm b}^{\mu\nu} = \frac{(\alpha - 1)(\varepsilon - 1)}{\alpha + \varepsilon} T_0^{\mu\nu}. \tag{2.25}$$

We thus see that when the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ have the same background (i.e. $\alpha = 1$) the baryons must be perturbative. In the application to cosmology in Sec. III we shall adjust the parameter α so that (2.25) reflects the correct baryonic and dark matter content of the cosmological background. In addition we find some constraint relating the constants λ, $\underline{\lambda}$, λ_f in the original action (2.1), for the two backgrounds to be consistent. We shall further restrict this constraint by requiring that the observed cosmological constant Λ be a true constant even at the level of cosmological perturbations (see Sec. III). This entails

$$\lambda = \Lambda, \qquad \underline{\lambda} = \alpha^2 \Lambda, \qquad \lambda_f = \alpha \Lambda. \tag{2.26}$$

To work out the field equations to first order in perturbations, we need to control to first order the tensorial coefficients $\mathcal{A}^{\rho\sigma}_{\mu\nu}$ and $\underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu}$ defined in Eq. (2.7). The results are derived in Appendix A where we obtain

$$\mathcal{A}^{\rho\sigma}_{\mu\nu} = \frac{1}{\alpha}[\delta^\rho_{(\mu}\delta^\sigma_{\nu)} - h^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)}] + \mathcal{O}(2),$$
$$\underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu} = \alpha[\delta^\rho_{(\mu}\delta^\sigma_{\nu)} + h^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)}] + \mathcal{O}(2). \tag{2.27}$$

As the last ingredient we need also to find the link between the two Einstein tensors $E^{\mu\nu}$ and $\underline{E}^{\mu\nu}$ and the one $\mathcal{E}^{\mu\nu}$ of the metric $f_{\mu\nu}$. This is provided by

$$\beta E^{\mu\nu} = \mathcal{E}^{\mu\nu} - \frac{1}{2}\Box_{\rm L} h^{\mu\nu} + \mathcal{O}(2), \tag{2.28a}$$

$$\underline{\beta E}^{\mu\nu} = \mathcal{E}^{\mu\nu} + \frac{1}{2}\Box_{\rm L} h^{\mu\nu} + \mathcal{O}(2), \tag{2.28b}$$

where $\Box_{\rm L}$ denotes a standard linear operator acting on the metric perturbation for any background metric $f_{\mu\nu}$.[4] Finally, we find that both Einstein field equations can be written into the ordinary forms

$$E^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{16\pi}{1+\alpha^2+2\alpha\varepsilon}\left[\alpha(\alpha+2\varepsilon)(T_{\rm b}^{\mu\nu}+T^{\mu\nu}) - \frac{1}{\alpha^4}(1-h)\underline{T}^{\mu\nu} + \frac{2}{\alpha^{3/2}}h_\rho^{(\mu}T_0^{\nu)\rho}\right] + \mathcal{O}(2), \tag{2.29a}$$

$$\underline{E}^{\mu\nu} + \alpha^2\Lambda\underline{g}^{\mu\nu} = -\frac{16\pi\alpha^2}{1+\alpha^2+2\alpha\varepsilon}\left[\alpha^4(1+h)(T_{\rm b}^{\mu\nu}+T^{\mu\nu}) - \frac{1+2\alpha\varepsilon}{\alpha^2}\underline{T}^{\mu\nu} + 2\alpha^{3/2}\frac{1+\alpha\varepsilon}{\alpha+\varepsilon}h_\rho^{(\mu}T_0^{\nu)\rho}\right] + \mathcal{O}(2). \tag{2.29b}$$

When deriving Eqs. (2.29) we have used the consistency relation (2.25) and explicitly assumed that α is constant (if not, further terms have to be added to these equations).

III. FIRST-ORDER COSMOLOGICAL PERTURBATIONS

We expand the model around a homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, writing both metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ (and therefore also $f_{\mu\nu}$) as first-order perturbations around some FLRW background metrics and solving Eqs. (2.29) by applying cosmological perturbation techniques to the three metrics. In the end we shall compare the results with those of the Λ-CDM model by looking at the ordinary sector with metric $g_{\mu\nu}$. The other sector with metric $\underline{g}_{\mu\nu}$ will in principle be unobservable directly, but since the two sectors are coupled together in the action (2.1) via terms involving the metric $f_{\mu\nu}$, our solution for the perturbations of the ordinary sector $g_{\mu\nu}$ will be strongly affected by our solution for the dark sector $\underline{g}_{\mu\nu}$ and vice versa.

[4]Its explicit expression will not be used, because we only need that $\mathcal{E}^{\mu\nu} = \frac{1}{2}(\beta E^{\mu\nu} + \underline{\beta}\underline{E}^{\mu\nu}) + \mathcal{O}(2)$, but is given here for completeness:

$$\Box_{\rm L}h^{\mu\nu} = \Box\hat{h}^{\mu\nu} - 2\mathcal{D}^{(\mu}\hat{H}^{\nu)} + f^{\mu\nu}\mathcal{D}_\rho\hat{H}^\rho - 2\mathcal{C}^{\mu\rho\sigma\nu}\hat{h}_{\rho\sigma} - \frac{2}{3}\left(\hat{h}^{\mu\nu} - \frac{1}{4}\hat{h}f^{\mu\nu}\right)\mathcal{R},$$

where $\Box = \mathcal{D}_\rho\mathcal{D}^\rho$, $\hat{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}f^{\mu\nu}h$, $\hat{h} = f^{\mu\nu}\hat{h}_{\mu\nu} = -h$, $\hat{H}^\mu = \mathcal{D}_\nu\hat{h}^{\mu\nu}$, and $\mathcal{C}^{\mu\rho\sigma\nu}$ and $\mathcal{R}$ denote the Weyl curvature and scalar curvature of the background, respectively.

A. Background cosmology

The two background FLRW metric intervals for the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ read (with the symbol $^\circ$ referring to quantities defined in the background):

$$\mathrm{d}\overset{\circ}{s}{}^2 = a^2[-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j], \tag{3.1a}$$

$$\mathrm{d}\overset{\circ}{\underline{s}}{}^2 = \underline{a}^2[-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j], \tag{3.1b}$$

where η denotes the conformal time and $a(\eta)$ and $\underline{a}(\eta)$ are the scale factors, such that $\mathrm{d}t = a\mathrm{d}\eta$ and $\mathrm{d}\underline{t} = \underline{a}\mathrm{d}\eta$ are the cosmic time intervals, and x^i are the spatial coordinates. The spatial metric γ_{ij}, assumed to be the same for the two backgrounds, is the metric of maximally symmetric spatial hypersurfaces of constant curvature $K = 0$ or $K = \pm 1$. The covariant derivative associated with the spatial metric γ_{ij} will be denoted D_i. The prime will stand for the derivative with respect to the conformal time η, and $\mathcal{H} \equiv a'/a$ and $\underline{\mathcal{H}} \equiv \underline{a}'/\underline{a}$ denote the conformal Hubble parameters. Solving (2.2) we obtain the FLRW background for the metric $f_{\mu\nu}$:

$$\mathrm{d}\overset{\circ}{s}{}_f^2 = a\underline{a}[-\mathrm{d}\eta^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j], \tag{3.2}$$

whose scale factor is $\sqrt{a\underline{a}}$. Recall that we introduced the parameter α in our perturbation assumptions (2.14) to account for the fact that the baryons have been inserted in the ordinary sector with metric $g_{\mu\nu}$ but not in the dark sector with metric $\underline{g}_{\mu\nu}$. We thus see that, in cosmology,

$$\alpha = \frac{a}{\underline{a}}. \tag{3.3}$$

Since α has been assumed from the start in Sec. II B to be constant we are thus looking for two background cosmologies with identical Hubble parameters[5]:

$$\mathcal{H} = \underline{\mathcal{H}}. \tag{3.4}$$

We also assume that the three matter fluids are comoving in their respective backgrounds; hence their background velocities read

$$\overset{\circ}{u}{}_{\rm b}^\mu = \overset{\circ}{u}{}^\mu = \left(\frac{1}{a}, \mathbf{0}\right), \qquad \overset{\circ}{\underline{u}}{}^\mu = \left(\frac{1}{\underline{a}}, \mathbf{0}\right). \tag{3.5}$$

The background matter densities obey the standard evolution laws

$$\overset{\circ}{\rho}{}_{\rm b}' + 3\mathcal{H}\overset{\circ}{\rho}_{\rm b} = 0, \quad \overset{\circ}{\rho}{}' + 3\mathcal{H}\overset{\circ}{\rho} = 0, \quad \overset{\circ}{\underline{\rho}}{}' + 3\mathcal{H}\overset{\circ}{\underline{\rho}} = 0. \tag{3.6}$$

[5]Note that this also agrees with $\mathcal{H}_f = \frac{1}{2}(\mathcal{H} + \underline{\mathcal{H}})$.

In Sec. II B we have shown how the two dark matter fluids ρ, u^μ and $\underline{\rho}$, $\underline{u}^\mu$ are related together through the equilibrium fluid configuration ρ_0, u_0^μ; see Eqs. (2.15) or equivalently (B3) and (B4). In particular, such relations imply that in the background the two dark matter fluid densities obey

$$\overset{\circ}{\rho} = \alpha^{-3/2}\overset{\circ}{\rho}_0, \qquad \overset{\circ}{\underline{\rho}} = \alpha^{3/2}\overset{\circ}{\rho}_0, \tag{3.7}$$

which implies that $\overset{\circ}{\underline{\rho}} = \alpha^3\overset{\circ}{\rho}$. Hence Eqs. (3.6) can be solved as

$$\overset{\circ}{\rho}_{\rm b} = \frac{k_{\rm b}}{a^3}, \qquad \overset{\circ}{\rho} = \frac{k}{a^3}, \qquad \overset{\circ}{\underline{\rho}} = \frac{k}{\underline{a}^3}, \tag{3.8}$$

with $k_{\rm b}$ and k denoting two constants. Note also that the equilibrium fluid ρ_0, u_0^μ is obviously given in the background by

$$\overset{\circ}{u}{}_0^\mu = \left(\frac{1}{(a\underline{a})^{1/2}}, \mathbf{0}\right), \qquad \overset{\circ}{\rho}_0 = \frac{k}{(a\underline{a})^{3/2}}. \tag{3.9}$$

The Friedmann equations of the two backgrounds are now obtained from Eqs. (2.29) or, alternatively, directly from Eqs. (2.8) as

$$3(\mathcal{H}^2 + K) - \Lambda a^2 = \frac{16\pi}{1+\alpha^2+2\alpha\varepsilon}\left[\alpha(\alpha+2\varepsilon)(\overset{\circ}{\rho}_{\rm b} + \overset{\circ}{\rho}) - \frac{1}{\alpha^2}\overset{\circ}{\underline{\rho}}\right]a^2, \tag{3.10a}$$

$$3(\underline{\mathcal{H}}^2 + K) - \alpha^2\Lambda\underline{a}^2 = \frac{16\pi}{1+\alpha^2+2\alpha\varepsilon}[-\alpha^4(\overset{\circ}{\rho}_{\rm b} + \overset{\circ}{\rho}) + (1+2\alpha\varepsilon)\overset{\circ}{\underline{\rho}}]\underline{a}^2. \tag{3.10b}$$

Finally we must impose the equivalence between the two Friedmann equations (3.10). The left-hand sides of these equations are obviously consistent because $\mathcal{H} = \underline{\mathcal{H}}$ and $\alpha = a/\underline{a}$. Now the consistency of the right-hand sides is ensured by the condition

$$k_{\rm b} = \frac{(\alpha-1)(\varepsilon-1)}{\alpha+\varepsilon}k, \tag{3.11}$$

which is nothing but the general relation (2.25) when translated to the case of comoving fluids in a FLRW background. Physically it states how the ratio between the two scale factors $\alpha = a/\underline{a}$ is to be related to the relative proportion of baryonic and dark matter in the two cosmological backgrounds, given that the baryons have been included into the ordinary sector of the action (2.1) but not into the dark sector (nor in the interacting sector). Thus, with this condition, the total matter density seen in the background of the ordinary sector (and thus directly measurable in cosmology) reads

$$\overset{\circ}{\rho}_{\rm M} = \frac{2\alpha\varepsilon}{\alpha+\varepsilon}\overset{\circ}{\rho}. \tag{3.12}$$

When studying cosmological perturbations it will be convenient to define *separately* the effective baryonic and dark matter densities as seen in the ordinary sector:

$$\overset{\circ}{\rho}_{\rm B} = \frac{2\alpha(\alpha+2\varepsilon)}{1+\alpha^2+2\alpha\varepsilon}\overset{\circ}{\rho}_{\rm b}, \qquad \overset{\circ}{\rho}_{\rm DM} = \frac{2\alpha(\alpha-1+2\varepsilon)}{1+\alpha^2+2\alpha\varepsilon}\overset{\circ}{\rho}. \tag{3.13}$$

These definitions come directly from the right-hand side of the Friedmann equation (3.10a) in the ordinary sector and satisfy $\overset{\circ}{\rho}_{\rm M} = \overset{\circ}{\rho}_{\rm B} + \overset{\circ}{\rho}_{\rm DM}$. Let us then suppose that there is a fraction p of baryons with respect to the total matter, so that

$$\frac{\overset{\circ}{\rho}_{\rm B}}{\overset{\circ}{\rho}_{\rm M}} = \frac{1}{p}. \tag{3.14}$$

According to the latest results from Planck we have $p \simeq 6.4$ [58]. Computing the ratio (3.14) from Eqs. (3.12) and (3.13) and solving for α, we obtain an analytic expression in terms of the baryonic fraction p and the coupling constant ε.[6]

In Sec. IV we shall recover the MOND phenomenology for dark matter in galaxies and the correct post-Newtonian limit in the Solar System when $\varepsilon \ll 1$. The interesting application of the present model will therefore be the limit where $\varepsilon \to 0$, in which case we get

$$\alpha = 1 - \frac{2\varepsilon}{p} + \mathcal{O}(\varepsilon^3). \tag{3.15}$$

Our conclusion is that, although we shall work out the cosmology of the model for an arbitrary parameter α and a general coupling constant ε, we can always have in mind that α is very close to one; hence the two backgrounds of $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ are very close to each other. This means in particular that the equilibrium dark matter fluid is almost geodesic with respect to the metric $f_{\mu\nu}$. Indeed $a_0^\mu = \mathcal{O}(\varepsilon)$ from Eq. (2.22), which constitutes a useful fact further

[6] It reads explicitly

$$\alpha(p,\varepsilon) = \frac{p(1-3\varepsilon+2\varepsilon^2) - 2\varepsilon^2 + \sqrt{(1-\varepsilon)[p^2(1+3\varepsilon-4\varepsilon^3) + 4p\varepsilon(-1+\varepsilon+2\varepsilon^2) - 4\varepsilon^2(1+\varepsilon)]}}{2[p(1-\varepsilon)+\varepsilon]}.$$

discussed in Sec. IV B. Note also that Eq. (3.12) tells us that in the limit $\varepsilon \to 0$, the measured matter density at cosmological scales is

$$\overset{\circ}{\rho}_{\mathrm{M}} \sim 2\varepsilon\overset{\circ}{\rho} \sim 10^{-29}\ \mathrm{g\,cm^{-3}}, \tag{3.16}$$

which is much smaller than the "bare" dark matter density $\overset{\circ}{\rho}$ which has been introduced into the action (2.1) and could take a huge value. By extension we see that in the limit $\varepsilon \to 0$, the density of baryons ρ_{b} should be much smaller than the bare density of dark matter ρ in the initial action (2.1). The baryons could be seen as resulting from a small "symmetry breaking" between the ordinary and dark sectors of the model.

B. First-order perturbations in the ordinary sector

We already concluded in Sec. III A that the background evolution is standard, driven by a cosmological constant and by the matter density defined by (3.12). We shall now show that the perturbation equations for the metric $g_{\mu\nu}$, which in our model represents the metric felt by the baryons and ordinary matter fields (including ordinary electromagnetic radiation), take the same form as those for the Λ-CDM model.[7] For the sake of clarity we relegate the definition of standard gravitational and matter perturbations to Appendix D.

We now introduce new effective variables describing the dark matter seen in first-order cosmological perturbations. In terms of these variables the perturbation equations for the ordinary metric $g_{\mu\nu}$ in our model take the standard form. The effective density contrast and scalar-vector-tensor (SVT) velocity of dark matter are defined by

$$\delta^{\mathrm{F}}_{\mathrm{DM}} = \delta^{\mathrm{F}} - \frac{\Delta z - dA}{\alpha - 1 + 2\varepsilon}, \tag{3.17a}$$

$$V_{\mathrm{DM}} = V + \frac{z' + \frac{1}{2}dB}{\alpha - 1 + 2\varepsilon}, \tag{3.17b}$$

$$V^i_{\mathrm{DM}} = V^i + \frac{z'^i + \frac{1}{2}dB^i}{\alpha - 1 + 2\varepsilon}, \tag{3.17c}$$

together with the usual variables $\delta^{\mathrm{F}}_{\mathrm{b}}$, V_{b} and V^i_{b} for the baryons. All relevant quantities are introduced in Appendix D, notably the dipole moment variables z and z^i defined in (D19). Furthermore we shall use the effective background baryonic and dark matter densities defined by Eqs. (3.13). With those definitions we find the following gravitational perturbation equations for the scalar, vectorial and tensorial modes in the *ordinary* sector $g_{\mu\nu}$:

$$\Delta\Psi - 3\mathcal{H}^2 X = 4\pi a^2(\overset{\circ}{\rho}_{\mathrm{B}}\delta^{\mathrm{F}}_{\mathrm{b}} + \overset{\circ}{\rho}_{\mathrm{DM}}\delta^{\mathrm{F}}_{\mathrm{DM}}), \tag{3.18a}$$

$$\Psi - \Phi = 0, \tag{3.18b}$$

$$\Psi' + \mathcal{H}\Phi = -4\pi a^2(\overset{\circ}{\rho}_{\mathrm{B}}V_{\mathrm{b}} + \overset{\circ}{\rho}_{\mathrm{DM}}V_{\mathrm{DM}}), \tag{3.18c}$$

$$\mathcal{H}X' + (\mathcal{H}^2 + 2\mathcal{H}')X = 0, \tag{3.18d}$$

$$(\Delta + 2K)\Phi^i = -16\pi a^2(\overset{\circ}{\rho}_{\mathrm{B}}V^i_{\mathrm{b}} + \overset{\circ}{\rho}_{\mathrm{DM}}V^i_{\mathrm{DM}}), \tag{3.18e}$$

$$\Phi'^i + 2\mathcal{H}\Phi^i = 0, \tag{3.18f}$$

$$E''^{ij} + 2\mathcal{H}E'^{ij} + (2K - \Delta)E^{ij} = 0, \tag{3.18g}$$

where the unknowns are the five gravitational variables Ψ, Φ, X, Φ^i, E^{ij} and the six matter variables $\delta^{\mathrm{F}}_{\mathrm{DM}}$, V_{DM}, V^i_{DM} and $\delta^{\mathrm{F}}_{\mathrm{b}}$, V_{b}, V^i_{b}. Recall that according to Eq. (D7), X is not independent from the other variables.

As Eqs. (3.18) are exactly the same as the perturbation equations of the standard cosmological model [59], we conclude that the present model is indistinguishable from standard Λ-CDM at the level of first-order perturbations and therefore should reproduce the observed anisotropies of the CMB. Indeed, these equations can be evolved without any reference to the dipole moment, which is unobservable in cosmology (but which will play a crucial role at galactic scales; see Sec. IV A). Note also that this result is obtained for any value of the coupling constant ε, as this coupling constant has been absorbed into the definition of the effective matter densities (3.12) and (3.13), and that the MOND acceleration scale a_0 does not appear at this level in cosmology.

To be consistent with the field equations (3.18) and with the equations of motion for the baryons which are standard, the effective dark matter variables introduced in Eqs. (3.17) must obey the continuity equation

$$\delta'^{\mathrm{F}}_{\mathrm{DM}} + \Delta V_{\mathrm{DM}} = 0, \tag{3.19}$$

together with the Euler equations

$$V'_{\mathrm{DM}} + \mathcal{H}V_{\mathrm{DM}} + \Psi = 0, \tag{3.20a}$$

$$V'^i_{\mathrm{DM}} + \mathcal{H}V^i_{\mathrm{DM}} = 0. \tag{3.20b}$$

The standard form of Eqs. (3.19) and (3.20) means that the effective dark matter described by the effective variables (3.17) obeys the ordinary geodesic equation with respect to the metric $g_{\mu\nu}$. In principle, all other variables in the model are unobservable using current cosmological observations performed in the ordinary sector.

[7]We have imposed the relations (2.26) in order to have a true cosmological constant in the background and at the level of perturbations. We could have imposed weaker conditions such that it would be constant only in the background but would deviate from a pure cosmological constant at the first order in perturbations.

Besides the ordinary sector we have similar equations for the dark sector $\underline{g}_{\mu\nu}$. It is very important to check that the latter equations are consistent with Eqs. (3.18) and permit one to determine all the variables of the model, even those that are unobservable in the ordinary sector. The full investigation of the dark sector is relegated to Appendix E where we shall see that Eqs. (3.19) and (3.20) can equivalently be obtained from the perturbation equations in the dark sector $\underline{g}_{\mu\nu}$. In particular the continuity and Euler equations (3.19) and (3.20) are consistent with the equations of motion (D15) and (D22), provided that the equations in the dark sector are satisfied. Finally, we show in Appendix E that all variables in the model can be determined by solving well-defined linear evolution equations.

IV. NONRELATIVISTIC AND POST-NEWTONIAN LIMITS

A. Phenomenology of MOND at galactic scales

In this section, we investigate the nonrelativistic (NR) limit of our model (i.e. formally when the speed of light $c \to +\infty$) and recover the Bekenstein and Milgrom [60] modified Poisson equation for the gravitational field. The MOND function μ that we shall obtain is directly related to the function W introduced into the action (2.1). We have already adjusted this function in Eqs. (2.4) and (2.5) in such a way that the model will be in agreement with the phenomenology of MOND at galactic scales [5–7]. Furthermore, thanks to this adjustment we shall investigate the model in the Solar System in Sec. IV B.

We now work out the NR limit directly at the level of the action (2.1). For convenience we restore for a while the gravitational constant G and the speed of light c such that the action has the dimension of the Planck constant. We insert into the action the standard ansatz for the metric at lowest order, namely

$$g_{00} = -1 + \frac{2U}{c^2} + \mathcal{O}(c^{-4}), \tag{4.1}$$

together with $g_{0i} = \mathcal{O}(c^{-3})$ and $g_{ij} = \delta_{ij} + \mathcal{O}(c^{-2})$, where U represents the ordinary Newtonian potential felt by ordinary baryonic matter and $\mathcal{O}(c^{-n})$ denotes the small post-Newtonian remainder. Similarly we write[8]

$$\underline{g}_{00} = -1 + \frac{2\underline{U}}{c^2} + \mathcal{O}(c^{-4}), \tag{4.2}$$

and $\underline{g}_{0i} = \mathcal{O}(c^{-3})$, $\underline{g}_{ij} = \delta_{ij} + \mathcal{O}(c^{-2})$, where $\underline{U}$ is the Newtonian potential of the dark sector.

We also write a similar ansatz for the vector field K_μ, namely

$$K_0 = \frac{\phi}{c^2} + \mathcal{O}(c^{-4}), \tag{4.3}$$

with $K_i = \mathcal{O}(c^{-3})$, where ϕ denotes an appropriate Coulomb-type potential. For the dipole vector field our ansatz is

$$\xi^i_\perp = \lambda^i + \mathcal{O}(c^{-2}), \tag{4.4}$$

where λ^i is the dipole moment in the NR limit, together with $\xi^0_\perp = \mathcal{O}(c^{-1})$ which is consistent with $u_{0\mu}\xi^\mu_\perp = 0$ in the NR limit.

The baryonic and dark matter particles are described by their Newtonian coordinate densities $\rho^*_{\rm b}$, ρ^* and $\underline{\rho}^*$ and their Newtonian coordinate velocities $\boldsymbol{v}_{\rm b}$, $\boldsymbol{v}$ and $\underline{\boldsymbol{v}}$.[9] These quantities are linked by the usual continuity equations, for instance $\partial_t\rho^* + \boldsymbol{\nabla}\cdot(\rho^*\boldsymbol{v}) = 0$. It is well known that the NR limit has to be performed holding these variables fixed. Furthermore, denoting by $\boldsymbol{v}_0$ the ordinary velocity of the equilibrium configuration we get from Eqs. (B4) $\boldsymbol{v} = \boldsymbol{v}_0 + \frac{1}{2}\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} + \mathcal{O}(c^{-2})$ and $\underline{\boldsymbol{v}} = \boldsymbol{v}_0 - \frac{1}{2}\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} + \mathcal{O}(c^{-2})$, where $\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \boldsymbol{v}_0\cdot\nabla$ is the usual convective time derivative. Thus

$$\boldsymbol{v} - \underline{\boldsymbol{v}} = \frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} + \mathcal{O}(c^{-2}). \tag{4.5}$$

Note also that $\xi^0_\perp = \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{\lambda} + \mathcal{O}(c^{-3})$.

The nonrelativistic action $S_{\rm NR}$ is defined as the limit when $c \to +\infty$ of the action S to which we substract the contributions coming from the rest masses of the particles, for instance $m^* = \int \mathrm{d}^3\mathbf{x}\rho^*$, namely

$$S_{\rm NR} = \lim_{c\to+\infty}\left[S + (m^*_{\rm b} + m^* + \underline{m}^*)c^2\int \mathrm{d}t\right]. \tag{4.6}$$

The NR limit is straightforwardly computed from the action (2.1) using the fact that the Ricci scalar density admits the limit $\sqrt{-g}R = -\frac{2}{c^4}(\boldsymbol{\nabla}U)^2 + \mathrm{div} + \mathcal{O}(c^{-6})$ where we can discard the total divergence which does not contribute to the dynamics. We obtain

$$\begin{aligned} S_{\rm NR} = \int \mathrm{d}t\mathrm{d}^3\mathbf{x}\bigg\{ &-\frac{1}{16\pi G}\left[(\boldsymbol{\nabla}U)^2 + (\boldsymbol{\nabla}\underline{U})^2 + \frac{1}{2\varepsilon}(\boldsymbol{\nabla}[U+\underline{U}])^2 \right.\\ &\left. - 2a_0^2W(X)\right] + \rho^*_{\rm b}\left(U + \frac{\boldsymbol{v}_{\rm b}^2}{2}\right) + \rho^*\left(U + \phi + \frac{\boldsymbol{v}^2}{2}\right)\\ &+ \underline{\rho}^*\left(\underline{U} - \phi + \frac{\underline{\boldsymbol{v}}^2}{2}\right)\bigg\}, \end{aligned} \tag{4.7}$$

[8]Thus the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ (and $f_{\mu\nu}$ as well) differ by small post-Newtonian corrections from the same Minkowskian background, which implies $\alpha = 1$ in the notation of Eqs. (2.14). We adopt $\alpha = 1$ for this application, all over the present section and also in the next one IV B.

[9]We use boldface notation to represent ordinary three-dimensional Euclidean vectors.

where $X = (\nabla\phi)^2/a_0^2$ in the NR limit. Note that when applying the NR limit we assume that the cosmological constant parameters λ, $\underline{\lambda}$, λ_f scale like $\Lambda \sim a_0^2/c^4$ and are therefore negligible when $c \to \infty$ (see Ref. [41] for a discussion). The NR action (4.7) is independent of c and from now on we conveniently redefine $G = 1$.

We then vary the action with respect to all fields and particles. Of course, the results can alternatively be obtained as the NR limit of the relativistic equations derived in Sec. II. The baryons obey the standard Newtonian law of dynamics,

$$\frac{\mathrm{d}\boldsymbol{v}_\mathrm{b}}{\mathrm{d}t} = \nabla U, \tag{4.8}$$

but because of the internal potential ϕ, the dark matter particles receive a supplementary Coulomb-type acceleration:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla(U + \phi), \tag{4.9a}$$

$$\frac{\mathrm{d}\underline{\boldsymbol{v}}}{\mathrm{d}t} = \nabla(\underline{U} - \phi), \tag{4.9b}$$

where the Coulomb potential ϕ obeys the modified Gauss equation

$$\nabla \cdot \left[W'\nabla\phi\right] = 4\pi(\rho^* - \underline{\rho}^*), \tag{4.10}$$

and we recall that $W' = \mathrm{d}W/\mathrm{d}X$. Note that Eqs. (4.9) imply that $\mathrm{d}\boldsymbol{v}_0/\mathrm{d}t = \frac{1}{2}\nabla(U + \underline{U})$ which is consistent with $a_0^\mu = 0$, as we have found in Eq. (2.22) with $\alpha = 1$. Finally, the Newtonian potentials U and $\underline{U}$ obey two equations, which can be rearranged into

$$\Delta U = -\frac{4\pi}{1+\varepsilon}[(1+2\varepsilon)(\rho_\mathrm{b}^* + \rho^*) - \underline{\rho}^*], \tag{4.11a}$$

$$\Delta(U + \underline{U}) = -\frac{8\pi\varepsilon}{1+\varepsilon}(\rho_\mathrm{b}^* + \rho^* + \underline{\rho}^*). \tag{4.11b}$$

With these equations in hands we now look for a plasmalike solution. Namely, the densities ρ^* and $\underline{\rho}^*$ are related to the density ρ_0^* of the equilibrium configuration by

$$\rho^* = \rho_0^* - \frac{1}{2}\nabla\cdot\boldsymbol{P}, \tag{4.12a}$$

$$\underline{\rho}^* = \rho_0^* + \frac{1}{2}\nabla\cdot\boldsymbol{P}. \tag{4.12b}$$

In these relations, which represent the NR limit of Eqs. (B3), we define the polarization field $\boldsymbol{P} = \rho_0^*\boldsymbol{\lambda}$, with $\boldsymbol{\lambda}$ being the NR limit of the dipole moment in Eq. (4.4). Inserting (4.12) into (4.10) and integrating we obtain

$$W'\nabla\phi = -4\pi\boldsymbol{P}, \tag{4.13}$$

which is the NR limit of Eq. (2.16). Thus, quite naturally the internal force field is aligned with the polarization vector.

Let us now show that a mechanism of "gravitational polarization" takes place when the coupling constant ε is very small, $\varepsilon \ll 1$. Indeed, we expect from the form of the coupling term in (4.7) that the latter condition will enforce the two potentials U and $\underline{U}$ to be opposite to each other. In the limit $\varepsilon \ll 1$, Eq. (4.11b) reduces to $\Delta(U + \underline{U}) = 0$; hence we can take $U + \underline{U} = 0$. Then Eq. (4.11) reduces to a simple Poisson equation for the ordinary Newtonian potential felt by baryonic matter,[10]

$$\Delta U = -4\pi(\rho_\mathrm{b}^* + \rho^* - \underline{\rho}^*), \tag{4.14}$$

while the equations of motion of the dark matter particles now read

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nabla(U + \phi), \tag{4.15a}$$

$$\frac{\mathrm{d}\underline{\boldsymbol{v}}}{\mathrm{d}t} = -\nabla(U + \phi). \tag{4.15b}$$

With this mechanism we observe that the "effective" gravitational to inertial mass ratio $m_\mathrm{g}/m_\mathrm{i}$ of the two species of dark matter particles is ±1, and we can interpret the dark matter medium as a "gravitational plasma" composed of particles with masses $(m_\mathrm{i}, m_\mathrm{g}) = (m, \pm m)$ interacting via the gravitoelectric field ϕ generated by the gravitational masses (or charges) $m_\mathrm{g} = \pm m$ (see [39,40] for further discussions). We however note that in the present model no negative masses have been introduced, since each species of dark matter particles in the relativistic action (2.1) has been coupled in a standard way to its respective metric.

In such a gravitational plasma the particles reach equilibrium when the internal force exactly balances the gravitational field, namely

$$\nabla\phi = -\nabla U. \tag{4.16}$$

At equilibrium the dark matter fluid is unaccelerated (in the ordinary three-dimensional sense) while the ordinary matter is accelerated in the standard way. Under this condition the polarization field (4.13) at equilibrium is therefore

$$\boldsymbol{P} = \frac{W'}{4\pi}\nabla U, \tag{4.17}$$

where $W'(X)$ is now a function of the norm of the gravitational field through $X = (\nabla U)^2/a_0^2$. At equilibrium

[10] See the end of Sec. IV B for the discussion of a residual dark matter contribution $\rho_\mathrm{DM}^* = 2\varepsilon\rho^*$ coming from the right side of Eq. (4.11a).

the polarization $\boldsymbol{P}$ is thus aligned with the local value of the gravitational field $\boldsymbol{g} = \boldsymbol{\nabla} U$, which is what we mean by gravitational polarization.

Finally the MOND equation follows immediately from Eq. (4.14), which can be transformed thanks to (4.12) into

$$\boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} U - 4\pi \boldsymbol{P}] = -4\pi \rho^*_{\rm b}. \tag{4.18}$$

Using the constitutive relation (4.17) the latter equation takes exactly the form of the modified Poisson equation [60]:

$$\boldsymbol{\nabla} \cdot \left[\mu\left(\frac{|\boldsymbol{\nabla} U|}{a_0}\right)\boldsymbol{\nabla} U\right] = -4\pi \rho^*_{\rm b}, \tag{4.19}$$

where the MOND interpolating function is given by $\mu = 1 - W'$. It is then easy to see that with the postulated form (2.4) of the function W in the regime $X \to 0$, one recovers the correct MOND regime when $g \ll a_0$, namely

$$\mu = 1 - W' = \frac{g}{a_0} + \mathcal{O}\left(\frac{g^2}{a_0^2}\right). \tag{4.20}$$

On the other hand, we want to recover the ordinary Poisson equation in the Newtonian regime $g \gg a_0$. From Eqs. (4.19) and (4.20) we see that it suffices to impose that $W'(X)$ tends to zero in the formal limit when $X \to +\infty$. However, in order to suppress any residual polarization (4.17) when $g \gg a_0$, we prefer to impose the stronger condition that $\sqrt{X} W' \to 0$ when $X \to \infty$, hence the behavior postulated in Eq. (2.5). The choice $b > 0$ rather than $b > -\frac{1}{2}$ is to ensure that W remains finite in the limit $X \to \infty$. In the next section IV B we shall study the 1PN approximation of the theory in the Solar System under the assumption (2.5).

It remains to show that the equilibrium defined by the condition (4.16) is stable. To prove it we show that the dark matter medium undergoes stable plasmalike oscillations. Indeed, by computing the relative acceleration of the two particle species combining Eqs. (4.15) and (4.5), and using the solution (4.13) for the internal field, we obtain the following harmonic oscillator governing the evolution of the dipole moment $\boldsymbol{\lambda}$[11]:

$$\frac{\mathrm{d}^2 \boldsymbol{\lambda}}{\mathrm{d}t^2} + \omega^2 \boldsymbol{\lambda} = 2\boldsymbol{\nabla} U. \tag{4.21}$$

The derivation is of course analogous to the classic derivation of the plasma oscillations in electrodynamics [61]. The plasma frequency we get in the present context reads

[11]This equation can also be recovered from the more general equation of evolution of the dipole moment (2.20).

$$\omega = \sqrt{\frac{8\pi \rho^*_0}{W'}}. \tag{4.22}$$

In the MOND regime we have $W' \to 1$, and this frequency is simply the one associated with the self-gravitating dynamical time scale $\tau = \frac{2\pi}{\omega} = \sqrt{\frac{\pi}{2\rho^*_0}}$.

B. Post-Newtonian limit in the Solar System

In this section we investigate the theory in the regime of the SS where $g \gg a_0$ hence $X \gg 1$. We have already postulated in Eq. (2.5) the form of the function $W(X)$ in this regime,

$$W(X) = A + \frac{B}{X^b} + o\left(\frac{1}{X^b}\right), \tag{4.23}$$

in which $b > 0$. With this choice we have seen that we recover the usual Poisson equation (4.19) since $W' \to 0$, and we suppress any polarization effect in the NR limit since $\sqrt{X} W' \to 0$; see Eq. (4.17). Furthermore it is clear that the suppression of polarization effects goes beyond the NR limit. Indeed Eq. (2.16) tells us that when $\sqrt{X} W' \to 0$ the dipole moment $\xi^\mu_\perp$ is negligible and therefore the dark matter medium becomes inactive.

In addition we want to impose that $W(X)$ itself tends to zero or a constant in the limit $X \to +\infty$, which is the reason for our choice $b > 0$. The constant A will simply add to the value of the cosmological constant in the regime $g \gg a_0$. Our conclusion is that the action (2.1) in the strong-field regime $g \gg a_0$ reduces to

$$\begin{aligned} S_{\text{strong field}} = \int \mathrm{d}^4 x \Bigg\{ & \frac{\sqrt{-g}}{32\pi}(R - 2\lambda) + \frac{\sqrt{-\underline{g}}}{32\pi}(\underline{R} - 2\underline{\lambda}) \\ & + \frac{\sqrt{-f}}{16\pi\varepsilon}(\mathcal{R} - 2\lambda'_f) - \sqrt{-g}\rho_{\rm b} - 2\sqrt{-f}\rho_0 \Bigg\}, \end{aligned} \tag{4.24}$$

where we have posed $\lambda'_f = \lambda_f - \varepsilon a_0^2 A$. To derive (4.24) we used the fact that when $\xi^\mu_\perp$ is negligible the coupling between the currents j^μ and $\underline{j}^\mu$ and the graviphoton field K_μ disappears because $j^\mu = \underline{j}^\mu$ from Eqs. (2.15). Note the residual contribution of dark matter in this action, and that we shall discuss at the end of this section.[12]

Here we shall explore the consequences of the action (4.24) in a post-Newtonian context, to study the 1PN limit of this theory in the SS. As usual we can neglect all

[12]Here ρ_0 is the density of dark matter in the equilibrium configuration defined with respect to $f_{\mu\nu}$. Its contribution in (4.24) comes from Eqs. (B3) in the case $\alpha = 1$ and is valid only up to second-order terms $\mathcal{O}(2)$, negligible for the present discussion.

cosmological constant terms in the SS. The ordinary metric $g_{\mu\nu}$ at 1PN order is parametrized by two potentials, the "gravitoelectric" scalar potential V and the "gravitomagnetic" vector potential V_i, say $g^{\rm 1PN}_{\mu\nu} = g_{\mu\nu}[V, V_i]$, by which we mean that

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \mathcal{O}(c^{-6}), \qquad (4.25a)$$

$$g_{0i} = -\frac{4V_i}{c^3} + \mathcal{O}(c^{-5}), \qquad (4.25b)$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2V}{c^2}\right) + \mathcal{O}(c^{-4}). \qquad (4.25c)$$

In exactly the same way we parametrize the 1PN metric in the dark sector with two other 1PN potentials $\underline{V}$ and $\underline{V}_i$, namely $\underline{g}^{\rm 1PN}_{\mu\nu} = \underline{g}_{\mu\nu}[\underline{V}, \underline{V}_i]$.[13] The point now is to find the 1PN parametrization of the metric $f_{\mu\nu}$ in the interacting sector of the action (4.24). For this purpose we make use of the result derived in Eq. (A8) of Appendix A for the perturbative expansion of the metric $f_{\mu\nu}$. Keeping only the leading nonlinear correction we obtain (recall that we choose $\alpha = 1$ for this application)

$$f_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} + \underline{g}_{\mu\nu}) - \frac{1}{2} f^{\rho\sigma} h_{\mu\rho} h_{\nu\sigma} + \mathcal{O}(h^4), \qquad (4.26)$$

where we recall that $h_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - \underline{g}_{\mu\nu})$ by definition. The nonlinear correction plays a crucial role for the 1PN limit as it rules the value of the PPN parameter β [56]. Actually it happens that the elegant prescription (2.2) we have adopted for the metric $f_{\mu\nu}$ yields the correct value for the parameter β. Working out Eq. (4.26) at 1PN order we find that the 1PN parametrization of the metric $f_{\mu\nu}$ is simply obtained from the half sum of the 1PN potentials parametrizing the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$, namely

$$f^{\rm 1PN}_{\mu\nu} = f_{\mu\nu}\left[\frac{V + \underline{V}}{2}, \frac{V_i + \underline{V}_i}{2}\right]. \qquad (4.27)$$

The 1PN metrics being properly parametrized, we insert them into the action (4.24) and vary it with respect to V, V_i, $\underline{V}$ and $\underline{V}_i$. We thus obtain two equations for V and $\underline{V}$ valid at order 1PN, which can be rearranged into [extending (4.11) to 1PN order]

$$\Delta V + \frac{1}{c^2}(3\partial_t^2 V + 4\partial_t\partial_i V_i) = -\frac{4\pi}{1+\varepsilon}[(1+2\varepsilon)\sigma_{\rm b} + 2\varepsilon\sigma_0], \qquad (4.28a)$$

$$\Delta(V + \underline{V}) + \frac{1}{c^2}(3\partial_t^2(V + \underline{V}) + 4\partial_t\partial_i(V_i + \underline{V}_i)) = -\frac{8\pi\varepsilon}{1+\varepsilon}(\sigma_{\rm b} + 2\sigma_0). \qquad (4.28b)$$

Similarly we obtain two equations for V_i and $\underline{V}_i$:

$$\Delta V_i - \partial_i(\partial_t V + \partial_j V_j) = -\frac{4\pi}{1+\varepsilon}[(1+2\varepsilon)\sigma^i_{\rm b} + 2\varepsilon\sigma^i_0], \qquad (4.29a)$$

$$\Delta(V_i + \underline{V}_i) - \partial_i(\partial_t(V + \underline{V}) + \partial_j(V_j + \underline{V}_j)) = -\frac{8\pi\varepsilon}{1+\varepsilon}(\sigma^i_{\rm b} + 2\sigma^i_0), \qquad (4.29b)$$

valid only at Newtonian order. The matter sources in these equations are defined from the stress-energy tensor of the baryons as

$$\sigma_{\rm b} = \frac{T^{00}_{\rm b} + T^{ii}_{\rm b}}{c^2}, \qquad \sigma^i_{\rm b} = \frac{T^{0i}_{\rm b}}{c}. \qquad (4.30)$$

These definitions are also valid if one includes some internal energy and pressure into the baryonic part of the action (4.24). A 1PN order we obtain for the matter sources

$$\sigma_{\rm b} = \rho^*_b\left(1 - \frac{V}{c^2} + \frac{3}{2}\frac{\boldsymbol{v}^2_b}{c^2}\right), \qquad \sigma^i_{\rm b} = \rho^*_b v^i_b, \qquad (4.31)$$

which can easily be generalized to the case when adding internal energy and pressure. Similarly we have posed for the dark matter

$$\sigma_0 = \frac{T^{00}_0 + T^{ii}_0}{c^2}, \qquad \sigma^i_0 = \frac{T^{0i}_0}{c}. \qquad (4.32)$$

Like in Sec. IV A the relevant physics of our model is the limiting case where $\varepsilon \ll 1$. Applying this limit on Eqs. (4.28) and (4.29) we obtain the equations for the 1PN potentials parametrizing the ordinary metric $g_{\mu\nu}$ felt by the baryons as

$$\Box V = -4\pi\sigma_{\rm b}, \qquad \Delta V_i = -4\pi\sigma^i_{\rm b}, \qquad (4.33)$$

where we have used the harmonic coordinate condition in the ordinary sector $\partial_t V + \partial_i V^i = \mathcal{O}(c^{-2})$, with the potentials in the dark sector being given by $\underline{V} = -V$ and $\underline{V}_i = -V_i$. As Eqs. (4.33) are the same as the standard equations of the 1PN limit of GR (see e.g. Ref. [62]), we conclude that the model has the same 1PN limit as GR and is therefore viable in the SS. One can check directly from Eqs. (4.33) that all the PPN parameters of the theory agree with their GR values [56]. We emphasize again that the PN limit works thanks to our particular prescription (2.2) for defining the interaction metric $f_{\mu\nu}$ in the original action. Indeed the nonlinear term coming from that prescription

[13]The two forms of the metrics that we postulated above will be justified when we find a consistent solution of the 1PN equations.

[see Eq. (4.26)] turns out to be exactly the one necessary to ensure that $\beta^{\rm PPN} = 1$.

To fully support the latter conclusions, let us look in more detail at the fate of the residual dark matter contributions in Eqs. (4.28) and (4.29). Indeed, when taking the limit $\varepsilon \to 0$ one must be careful with the fact that the effective dark matter observed in cosmology has been found to be ε times the "bare" dark matter; see Eqs. (3.12) or (3.13) with $\alpha = 1$. Posing thus $\sigma_{\rm DM} = 2\varepsilon\sigma_0$ and $\sigma^i_{\rm DM} = 2\varepsilon\sigma^i_0$ we could expect that there should be some remaining dark matter terms $\sigma_{\rm DM}$ and $\sigma^i_{\rm DM}$ in the right-hand sides of (4.33). Similarly, we could expect the presence of a residual dark matter contribution $\rho^*_{\rm DM} = 2\varepsilon\rho^*$ in the right side of the MOND equation; see (4.18) or (4.19).

However we now argue that this dark matter is negligible with respect to baryonic matter, so that we can blindly apply the limit $\varepsilon \to 0$ as we did to obtain (4.33). This is due to a property of "*weak clustering of dipolar dark matter*" which is at work in the present model. According to this property the dark matter medium should not cluster much during the cosmological evolution, so that the dark matter density contrast in a typical galaxy at low redshift after a long cosmological evolution should be smaller than the density contrast of baryonic matter. In the present model this property is the consequence of the fact that the dipolar dark matter particles obey the geodesic equation $a^\mu_0 = 0$ with respect to the metric $f_{\mu\nu}$,[14] while the baryons obey the geodesic equation $a^\mu_{\rm b} = 0$ with respect to the ordinary metric $g_{\mu\nu}$. Therefore the baryons are accelerated relatively to the dark matter medium. Using the result that in the limit $\varepsilon \to 0$ the metric $f_{\mu\nu}$ is almost flat, we see that $a^\mu_0 = 0$ implies that the dark matter fluid is unaccelerated in the ordinary three-dimensional sense with respect to some averaged cosmological matter distribution. In the Newtonian approximation we have indeed seen that $\mathrm{d}\boldsymbol{v}_0/\mathrm{d}t = \frac{1}{2}\boldsymbol{\nabla}(U + \underline{U}) = 0$. We thus expect that $\sigma_{\rm DM}$ and $\sigma^i_{\rm DM}$ (or $\rho^*_{\rm DM}$ in the MOND equation) will be negligible compared to the baryonic contributions in generic galaxies and in the Solar System and may even take very small typical average cosmological values, e.g. $\sigma_{\rm DM} \sim 10^{-29}\ \mathrm{g\,cm^{-3}}$. The property of weak clustering of dark matter in the present model[15] could be checked by implementing numerical N-body cosmological simulations.

V. CONCLUSION

In this paper we have shown how a specific form of dark matter, made of two different species of particles coupled to two different metrics and interacting through a specific internal force field, could permit to interpret in the most natural way the phenomenology of MOND by a mechanism of gravitational polarization. In this approach the dark matter medium appears as a polarizable plasmalike fluid of spacelike dipole moments, aligned with the local gravitational field generated by ordinary baryonic matter. On the other hand, that particular form of dark matter reproduces the cosmological model Λ-CDM at first-order cosmological perturbations and is thus consistent with the observed spectrum of anisotropies of the CMB [2]. Furthermore we have shown that the theory is viable in the Solar System as it predicts the same PPN parameters as GR. Finally the gravitational sector of the model is consistent (ghost-free) at linear order around a Minkowski background.

Improvements with respect to the previous model of dipolar dark matter [41,42] include the hypothesis of weak clustering of dipolar dark matter, which is probably built in the model, and the fact that the dark matter medium is stable, as it undergoes stable plasmalike oscillations when analyzed in perturbations. Another important feature of the present model is that the mechanism of alignment of the polarization with the gravitational field, and consequently the validity of the MOND equation *stricto sensu*, is expected to hold in any nonstatic and nonspherical cases. This is important because it has been shown that MOND works well in describing the highly dynamical evolution and collision of galaxies [44–46] and the nonspherical polar ring structures of galaxies [47].

On the other hand, while Refs. [41,42] investigate a pure model of modified dark matter in standard GR, the present model is less economical in that it postulates both a nonstandard form of dark matter and a modification of gravity in the form of a bimetric extension of GR. Such a compromise between dark matter and modified gravity is perhaps the price to pay for reconciling within a single relativistic framework the conflicting observations of dark matter at large cosmological scales and at small galactic scales. It would be very interesting to test the model by performing N-body cosmological numerical simulations and notably to investigate the intermediate scale of galaxy clusters at which the pure modified gravity theories generally meet problems [9].

ACKNOWLEDGMENTS

It is a pleasure to thank Gilles Esposito-Farèse, Benoit Famaey, Alexandre Le Tiec and Moti Milgrom for interesting discussions and especially for very useful remarks on a preliminary version of this work. We acknowledge partial support from Agence Nationale de la Recherche via the Grant THALES (ANR-10-BLAN-0507-01-02).

[14]Indeed, the acceleration a^μ_0 is given by Eq. (2.22) where we recall that the parameter α is very close to one in the physically relevant case $\varepsilon \to 0$, i.e. $\alpha = 1 + \mathcal{O}(\varepsilon)$ from Eq. (3.15).

[15]Recall that in the previous model of dipolar dark matter [41,42], the weak clustering of dipolar dark matter was used as a *hypothesis* but not as a property logically deduced within that model.

APPENDIX A: PERTURBATIVE SOLUTION FOR THE METRIC $f_{\mu\nu}$

In this Appendix we find the perturbative solution of the implicit definition (2.2) of the metric $f_{\mu\nu}$ given the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$, namely

$$f^{\rho\sigma} g_{\rho\mu} \underline{g}_{\nu\sigma} = f^{\rho\sigma} g_{\rho\nu} \underline{g}_{\mu\sigma} = f_{\mu\nu}. \tag{A1}$$

Let us first gain an insight into the meaning of this prescription by looking at the solution in terms of matrices. For this purpose we pose $G^{\nu}_{\mu} = f^{\nu\rho} g_{\mu\rho}$ and $\underline{G}^{\nu}_{\mu} = f^{\nu\rho} \underline{g}_{\mu\rho}$ and define the associated two matrices $G = (G^{\nu}_{\mu})$ and $\underline{G} = (\underline{G}^{\nu}_{\mu})$. With such a matrix notation the relation (A1) becomes, with $\mathbb{1} = (\delta^{\nu}_{\mu})$ denoting the unit matrix,

$$G\underline{G} = \underline{G}G = \mathbb{1}, \tag{A2}$$

which means that $\underline{G}$ is the inverse of G.

Next we look for the solution of Eqs. (A2) in the form of the perturbative expansion

$$G = \alpha(\mathbb{1} + H + X), \qquad \underline{G} = \frac{1}{\alpha}(\mathbb{1} - H + X), \tag{A3}$$

where α denotes a constant, the matrix H represents the first-order perturbation and is defined by $H = \frac{1}{2}(\alpha^{-1} G - \alpha \underline{G})$, and the matrix X admits an expansion series in powers of H starting at the *second* order in H. The matrix equation to be solved is found to be $X^2 + 2X - H^2 = 0$, whose appropriate solution reads $X = -\mathbb{1} + \sqrt{\mathbb{1} + H^2}$, where we have defined the matrix $\sqrt{\mathbb{1} + H^2}$ by its expansion series in powers of H, that is,

$$\sqrt{\mathbb{1} + H^2} = \sum_{p=0}^{+\infty} \gamma_p H^{2p} \quad \text{with} \quad \gamma_p = \frac{(-)^{p+1}(2p-3)!!}{2^p p!}. \tag{A4}$$

It is interesting to note that the same expansion series plays a crucial role in the definition of the mass term in resummed ghost-free massive gravity theories; see e.g. [63]. Finally our perturbative solution is

$$G = \alpha(H + \sqrt{\mathbb{1} + H^2}), \tag{A5a}$$

$$\underline{G} = \frac{1}{\alpha}(-H + \sqrt{\mathbb{1} + H^2}). \tag{A5b}$$

Notice that such a perturbative solution G obviously commutes with $\underline{G}$ and therefore only one out of the two equations (A2) is sufficient.

Having the above solution in hands we conveniently lower back the contravariant index so as to restore the metrics in a standard form. The expansion variable is $h_{\mu\nu} = H^{\rho}_{\mu} f_{\rho\nu} = \frac{1}{2}(\alpha^{-1} g_{\mu\nu} - \alpha \underline{g}_{\mu\nu})$ which was used as the metric perturbation in Sec. II B. The solution reads

$$g_{\mu\nu} = \alpha(f_{\mu\nu} + h_{\mu\nu} + x_{\mu\nu}), \qquad \underline{g}_{\mu\nu} = \frac{1}{\alpha}(f_{\mu\nu} - h_{\mu\nu} + x_{\mu\nu}), \tag{A6}$$

where $x_{\mu\nu} = X^{\rho}_{\mu} f_{\rho\nu}$ is at least of second order and is given by

$$x_{\mu\nu} = \sum_{p=1}^{+\infty} \gamma_p H^{\rho_1}_{\mu} H^{\rho_2}_{\rho_1} \ldots H^{\rho_{2p-1}}_{\rho_{2p-2}} h_{\nu\rho_{2p-1}}. \tag{A7}$$

In particular $f_{\mu\nu}$ can be determined from the two metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ by the relation

$$f_{\mu\nu} = \frac{1}{2}(\alpha^{-1} g_{\mu\nu} + \alpha \underline{g}_{\mu\nu}) - \sum_{p=1}^{+\infty} \gamma_p H^{\rho_1}_{\mu} H^{\rho_2}_{\rho_1} \ldots H^{\rho_{2p-1}}_{\rho_{2p-2}} h_{\nu\rho_{2p-1}}, \tag{A8}$$

which is nevertheless implicit because $H^{\rho}_{\mu} = f^{\rho\sigma} h_{\mu\sigma} = \frac{1}{2} f^{\rho\sigma}(\alpha^{-1} g_{\mu\sigma} - \alpha \underline{g}_{\mu\sigma})$ still depends on $f^{\rho\sigma}$. The first nonlinear correction term in Eq. (A8) plays an important role when investigating the 1PN limit of the theory in Sec. IV B.

Finally we can vary Eq. (A8) with respect to $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$ to determine perturbatively (i.e. order by order) the tensorial coefficients $\mathcal{A}^{\rho\sigma}_{\mu\nu}$ and $\underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu}$ defined in Eq. (2.7) as

$$\delta f_{\mu\nu} = \frac{1}{2} \mathcal{A}^{\rho\sigma}_{\mu\nu} \delta g_{\rho\sigma} + \frac{1}{2} \underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu} \delta \underline{g}_{\rho\sigma}. \tag{A9}$$

From Eqs. (A1) we find that such coefficients must obey the equations

$$\frac{1}{2}(\mathcal{A}^{\rho\sigma}_{\mu\nu} + G^{\lambda}_{\mu} \underline{G}^{\tau}_{\nu} \mathcal{A}^{\rho\sigma}_{\lambda\tau}) = \delta^{(\sigma}_{\mu} \underline{G}^{\rho)}_{\nu}, \tag{A10a}$$

$$\frac{1}{2}(\underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu} + G^{\lambda}_{\mu} \underline{G}^{\tau}_{\nu} \underline{\mathcal{A}}^{\rho\sigma}_{\lambda\tau}) = G^{(\rho}_{\mu} \delta^{\sigma)}_{\nu}, \tag{A10b}$$

together with the same equations with μ and ν exchanged. These equations can be solved iteratively to any order. For instance we find the solutions up to second order as

$$\mathcal{A}^{\rho\sigma}_{\mu\nu} = \frac{1}{\alpha}\left(\delta^{\rho}_{(\mu}\delta^{\sigma}_{\nu)} - H^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)} + \frac{1}{2} H^{\rho}_{(\mu} H^{\sigma}_{\nu)}\right) + \mathcal{O}(3), \tag{A11a}$$

$$\underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu} = \alpha\left(\delta^{\rho}_{(\mu}\delta^{\sigma}_{\nu)} + H^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)} + \frac{1}{2} H^{\rho}_{(\mu} H^{\sigma}_{\nu)}\right) + \mathcal{O}(3). \tag{A11b}$$

One can check that the relations $f^{\mu\nu} \mathcal{A}^{\rho\sigma}_{\mu\nu} = g^{\rho\sigma}$ and $f^{\mu\nu} \underline{\mathcal{A}}^{\rho\sigma}_{\mu\nu} = \underline{g}^{\rho\sigma}$, which are direct consequences of $f^2 = g\underline{g}$, are satisfied to this order.

APPENDIX B: PLASMALIKE HYPOTHESIS

The dark matter fluids are described by the conserved currents j^μ and $\underline{j}^\mu$ defined by Eqs. (2.6). Here we implement the idea that they perturbatively differ from a single equilibrium fluid described by the current $j_0^\mu = \rho_0 u_0^\mu$, such that $f_{\mu\nu}u_0^\mu u_0^\nu = -1$ and $\mathcal{D}_\mu j_0^\mu = 0$. To do that, suppose for simplicity that the equilibrium fluid is made of particles with coordinate density $\rho_0^*(\mathbf{x},t) = \sum_A m_A \delta[\mathbf{x} - \boldsymbol{x}_A(t)]$ (with δ being the usual three-dimensional Dirac function), satisfying the usual continuity equation $\partial_t \rho_0^* + \partial_i(\rho_0^* v_0^i) = 0$, where $v_0^i(\mathbf{x},t)$ is the Eulerian velocity field. Then the coordinate density of the displaced fluid is defined with respect to that of the equilibrium fluid as $\rho^*(\mathbf{x},t) = \sum_A m_A \delta[\mathbf{x} - \boldsymbol{x}_A(t) - \boldsymbol{y}_A(t)]$, where $y_A^i(t)$ is the displacement of the particles' positions. Introducing the Eulerian displacement field $y^i(\mathbf{x},t)$ associated with $y_A^i(t)$, we find that $\rho^* = \rho_0^* - \partial_i(\rho_0^* y^i)$ to first order in the displacement, while the coordinate velocity reads $v^i = v_0^i + \frac{\mathrm{d}y^i}{\mathrm{d}t} - y^j \partial_j v_0^i$, where $\mathrm{d}/\mathrm{d}t$ is the convective derivative. Introducing the coordinate current $J_*^0 = \rho^*$ and $J_*^i = \rho^* v^i$ such that $\partial_\mu J_*^\mu = 0$, and two displacement vectors y^μ and $\underline{y}^\mu$ for the two fluids, we obtain[16]

$$J_*^\mu = J_{0*}^\mu + \partial_\nu(J_{0*}^\nu y^\mu - J_{0*}^\mu y^\nu) + \mathcal{O}(2), \tag{B1a}$$

$$\underline{J}_*^\mu = J_{0*}^\mu + \partial_\nu(J_{0*}^\nu \underline{y}^\mu - J_{0*}^\mu \underline{y}^\nu) + \mathcal{O}(2). \tag{B1b}$$

In what follows we systematically work at first order in the displacement vectors y^μ and $\underline{y}^\mu$ and assume that their gradients are numerically of the same order as the metric perturbation $h_{\mu\nu}$, namely that $\nabla y \sim \underline{\nabla}\,\underline{y} \sim h = \mathcal{O}(1)$, so that the remainders $\mathcal{O}(2)$ in Eqs. (B1) are of the same order as those in Eqs. (2.14). The expressions (B1) are covariantized in the usual way by defining $j^\mu = J_*^\mu/\sqrt{-f}$, etc., and we obtain (see e.g. [64])

$$j^\mu = j_0^\mu + \mathcal{D}_\nu(j_0^\nu y_\perp^\mu - j_0^\mu y_\perp^\nu) + \mathcal{O}(2), \tag{B2a}$$

$$\underline{j}^\mu = j_0^\mu + \mathcal{D}_\nu(j_0^\nu \underline{y}_\perp^\mu - j_0^\mu \underline{y}_\perp^\nu) + \mathcal{O}(2). \tag{B2b}$$

We have taken advantage of the structure of the terms to replace the displacement vectors by their projections perpendicular to the four-velocity of the equilibrium fluid, namely $y_\perp^\mu = \perp^\mu_\nu y^\nu$ and $\underline{y}_\perp^\mu = \perp^\mu_\nu \underline{y}^\nu$, where $\perp^{\mu\nu} \equiv f^{\mu\nu} + u_0^\mu u_0^\nu$. Coming back to the scalar densities $\rho = \sqrt{-g_{\mu\nu}J^\mu J^\nu}$ and $\underline{\rho} = \sqrt{-\underline{g}_{\mu\nu}\underline{J}^\mu \underline{J}^\nu}$, taking into account the relations (2.6) between currents and using at first order $\beta = \alpha^2[1 + \frac{h}{2} + \mathcal{O}(2)]$ and $\underline{\beta} = \alpha^{-2}[1 - \frac{h}{2} + \mathcal{O}(2)]$, where $h \equiv f^{\mu\nu}h_{\mu\nu}$, we obtain

$$\rho = \alpha^{-3/2}\left[\rho_0\left(1 - \frac{h}{2} - \frac{1}{2}h_{\mu\nu}u_0^\mu u_0^\nu + a_{0\mu}y_\perp^\mu\right) - \mathcal{D}_\mu(\rho_0 y_\perp^\mu)\right] + \mathcal{O}(2), \tag{B3a}$$

$$\underline{\rho} = \alpha^{3/2}\left[\rho_0\left(1 + \frac{h}{2} + \frac{1}{2}h_{\mu\nu}u_0^\mu u_0^\nu + a_{0\mu}\underline{y}_\perp^\mu\right) - \mathcal{D}_\mu(\rho_0 \underline{y}_\perp^\mu)\right] + \mathcal{O}(2), \tag{B3b}$$

where $a_0^\mu \equiv u_0^\nu \mathcal{D}_\nu u_0^\mu$ is the acceleration of the equilibrium configuration and $a_{0\mu} = f_{\mu\nu}a_0^\nu$. For the four-velocities we get

$$u^\mu = \alpha^{-1/2}\left[u_0^\mu\left(1 + \frac{1}{2}h_{\rho\sigma}u_0^\rho u_0^\sigma - a_{0\mu}y_\perp^\mu\right) + \mathcal{L}_{u_0}y_\perp^\mu\right] + \mathcal{O}(2), \tag{B4a}$$

$$\underline{u}^\mu = \alpha^{1/2}\left[u_0^\mu\left(1 - \frac{1}{2}h_{\rho\sigma}u_0^\rho u_0^\sigma - a_{0\mu}\underline{y}_\perp^\mu\right) + \mathcal{L}_{u_0}\underline{y}_\perp^\mu\right] + \mathcal{O}(2), \tag{B4b}$$

in which we made use of the Lie derivative, e.g. $\mathcal{L}_{u_0}y_\perp^\mu = u_0^\nu \mathcal{D}_\nu y_\perp^\mu - y_\perp^\nu \mathcal{D}_\nu u_0^\mu$.

APPENDIX C: LINEARIZATION AROUND A MINKOWSKI BACKGROUND

In this Appendix we derive the gravitational part S_g of the action (2.1) at quadratic order in perturbation around a Minkowski background. Ignoring for simplicity the cosmological constants, we thus start from

$$S_g = \frac{1}{32\pi}\int \mathrm{d}^4x\left\{\sqrt{-g}R + \sqrt{-\underline{g}}\underline{R} + \frac{2}{\varepsilon}\sqrt{-f}\mathcal{R}\right\}, \tag{C1}$$

where the interaction metric $f_{\mu\nu}$ is defined from the two metrics $g_{\mu\nu}$, $\underline{g}_{\mu\nu}$ by the prescription (2.2). To linear order we have $g_{\mu\nu} = \eta_{\mu\nu} + k_{\mu\nu} + \mathcal{O}(2)$, $\underline{g}_{\mu\nu} = \eta_{\mu\nu} + \underline{k}_{\mu\nu} + \mathcal{O}(2)$ and $f_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu} + \mathcal{O}(2)$, where $\eta_{\mu\nu}$ is the Minkowski metric and

$$s_{\mu\nu} = \frac{1}{2}(k_{\mu\nu} + \underline{k}_{\mu\nu}). \tag{C2}$$

With these notations the variable $h_{\mu\nu}$ defined by Eq. (2.14) (with $\alpha = 1$) reads

$$h_{\mu\nu} = \frac{1}{2}(k_{\mu\nu} - \underline{k}_{\mu\nu}). \tag{C3}$$

It is now straightforward to derive the quadratic part of the action in terms of the two variables (C2) and (C3). We find that the two sectors associated with those variables decouple from each other, namely

[16]Note that one can always choose $y^0 = \underline{y}^0 = 0$ to define the two displacement four-vectors y^μ and $\underline{y}^\mu$.

$$S_g = \frac{1}{32\pi}\int \mathrm{d}^4x\left\{-\frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\mu \hat{h}^{\nu\rho} + \hat{H}_\mu \hat{H}^\mu + \frac{1+\varepsilon}{\varepsilon}\left(-\frac{1}{2}\partial_\mu s_{\nu\rho}\partial^\mu \hat{s}^{\nu\rho} + \hat{S}_\mu \hat{S}^\mu\right)\right\} + \mathcal{O}(3), \quad \text{(C4)}$$

where we define $\hat{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$, $\hat{H}^\mu = \partial_\nu \hat{h}^{\mu\nu}$ and similarly for $\hat{s}^{\mu\nu}$ and $\hat{S}^\mu$. Thus the action appears at that order as the sum of two massless noninteracting spin-2 fields, with positive sign in the case where $\varepsilon > 0$. Since this action enjoys two reparametrization invariances $\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}$ and $\delta s_{\mu\nu} = 2\partial_{(\mu}\chi_{\nu)}$, where ξ_ν and χ_ν are two independent functions, each spin-2 field propagates only two degrees of freedom as expected for massless gravitons [65]. However the full action of the model should still be investigated at the nonlinear level for which the number of propagating gravitational modes should be investigated. This question is addressed in Ref. [55].

APPENDIX D: COSMOLOGICAL PERTURBATIONS

1. Gravitational perturbations

We assume that both metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$, which as we have seen in Sec. II B differ perturbatively from each other, take the form of a linear perturbation around the FLRW background (3.1). The metric intervals read then[17]

$$\mathrm{d}s^2 = a^2[-(1+2A)\mathrm{d}\eta^2 + 2\tilde{h}_i \mathrm{d}\eta \mathrm{d}x^i + (\gamma_{ij} + \tilde{h}_{ij})\mathrm{d}x^i \mathrm{d}x^j], \quad \text{(D1)}$$

and similarly for the other metric interval $\mathrm{d}\underline{s}^2$,

$$\mathrm{d}\underline{s}^2 = \underline{a}^2[-(1+2\underline{A})\mathrm{d}\eta^2 + 2\underline{\tilde{h}}_i \mathrm{d}\eta \mathrm{d}x^i + (\gamma_{ij} + \underline{\tilde{h}}_{ij})\mathrm{d}x^i \mathrm{d}x^j]. \quad \text{(D2)}$$

The variables A, $\tilde{h}_i$, $\tilde{h}_{ij}$ and $\underline{A}$, $\underline{\tilde{h}}_i$, $\underline{\tilde{h}}_{ij}$, respectively, denote the metric perturbations for the metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$. An easy computation yields the perturbation of the metric $f_{\mu\nu}$ as

$$\mathrm{d}s_f^2 = a\underline{a}\left[-(1+A+\underline{A})\mathrm{d}\eta^2 + (\tilde{h}_i + \underline{\tilde{h}}_i)\mathrm{d}\eta \mathrm{d}x^i + \left(\gamma_{ij} + \frac{1}{2}\tilde{h}_{ij} + \frac{1}{2}\underline{\tilde{h}}_{ij}\right)\mathrm{d}x^i \mathrm{d}x^j\right]. \quad \text{(D3)}$$

Next we perform the standard SVT decomposition of the metric perturbations (see [59] for a review). For the ordinary sector associated to $g_{\mu\nu}$ we pose

$$\tilde{h}_i = \mathrm{D}_i B + B_i, \quad \text{(D4a)}$$

$$\tilde{h}_{ij} = 2C\gamma_{ij} + 2\mathrm{D}_i \mathrm{D}_j E + 2\mathrm{D}_{(i}E_{j)} + 2E_{ij}, \quad \text{(D4b)}$$

and identically for the dark sector $\underline{g}_{\mu\nu}$. All spatial indices are raised and lowered with γ_{ij} and its inverse γ^{ij}. The vectors B^i, E^i, $\underline{B}^i$, $\underline{E}^i$ defined in this way are divergenceless, while the second-rank tensors E^{ij}, $\underline{E}^{ij}$ are divergenceless and traceless:

$$\mathrm{D}_i B^i = \mathrm{D}_i E^i = \mathrm{D}_i \underline{B}^i = \mathrm{D}_i \underline{E}^i = 0, \quad \text{(D5a)}$$

$$\mathrm{D}_j E^{ij} = E^i_i = \mathrm{D}_j \underline{E}^{ij} = \underline{E}^i_i = 0. \quad \text{(D5b)}$$

As usual one can construct gauge-invariant quantities from these variables [59]. We shall use in the ordinary sector

$$\Phi = A + B' - E'' + \mathcal{H}(B - E'), \quad \text{(D6a)}$$

$$\Psi = -C - \mathcal{H}(B - E'), \quad \text{(D6b)}$$

$$X = A - C - (C/\mathcal{H})', \quad \text{(D6c)}$$

$$\Phi_i = E'_i - B_i. \quad \text{(D6d)}$$

Note that the scalar X so defined is not independent from the two other scalars Φ and Ψ:

$$X = \Phi + \Psi + \left(\frac{\Psi}{\mathcal{H}}\right)'. \quad \text{(D7)}$$

Note also that E_{ij} is already a gauge-invariant quantity. The same definitions apply of course to the dark sector $\underline{g}_{\mu\nu}$, for the gauge-invariant quantities $\underline{\Phi}$, $\underline{\Psi}$, $\underline{X}$, $\underline{\Phi}_i$ and $\underline{E}_{ij}$. From Eqs. (2.14) we see that the second-rank tensor field $h_{\mu\nu} = \frac{1}{2}(\alpha^{-1}g_{\mu\nu} - \alpha\underline{g}_{\mu\nu})$ can be written as

$$h_{00} = -a\underline{a}dA, \quad \text{(D8a)}$$

$$h_{0i} = \frac{a\underline{a}}{2}d\tilde{h}_i = \frac{a\underline{a}}{2}(\mathrm{D}_i dB + dB_i), \quad \text{(D8b)}$$

$$h_{ij} = \frac{a\underline{a}}{2}d\tilde{h}_{ij} = a\underline{a}(dC\gamma_{ij} + \mathrm{D}_i\mathrm{D}_j dE + \mathrm{D}_{(i}dE_{j)} + dE_{ij}), \quad \text{(D8c)}$$

where for any spatial scalar, vector or tensor P (gauge invariant or not) we denote the *difference* between P in the ordinary sector and the corresponding quantity $\underline{P}$ in the dark sector by

$$dP \equiv P - \underline{P}. \quad \text{(D9)}$$

It is evident that the difference of gauge-invariant quantities is gauge invariant, but notice that the difference of any quantities (scalar, vector or tensor) *is* a gauge-invariant

[17] In this section we omit indicating that second-order perturbations $\mathcal{O}(2)$ are systematically neglected. Our notation $\tilde{h}_i$, $\underline{\tilde{h}}_i$ and $\tilde{h}_{ij}$, $\underline{\tilde{h}}_{ij}$ is to avoid confusion with the components of the covariant tensor $h_{\mu\nu}$.

quantity. Thus in the following we extensively use the fact that dA, dB, dC, dE, dB_i, dE_i and dE_{ij} are gauge invariant. In addition we have also at our disposal the differences of gauge-invariant variables $d\Phi$, $d\Psi$, dX and $d\Phi_i$ defined similarly to Eqs. (D6).

2. Matter perturbations

We have in our model three fluids: two fluids of dark matter described by scalar densities ρ and $\underline{\rho}$ and four-velocities u^μ and $\underline{u}^\mu$ and the fluid of baryons described by ρ_b and u^μ_b. On the other hand, we have learned from Sec. II B [see Eqs. (2.15) and also (B3) and (B4)] how to relate the densities and four-velocities of the two dark matter fluids via an auxiliary fluid described by ρ_0, u_0^μ corresponding to some equilibrium configuration and a spacelike vector $\xi^\mu_\perp$ called the dipole moment. In addition, we have the fluid associated with the cosmological constant Λ. We already pointed out that in order to have a true cosmological constant, even at first order in perturbations (in agreement with the Λ-CDM model), we must relate the three initial constants λ, $\underline{\lambda}$ and λ_f in the action (2.1) in the way specified by Eq. (2.26), and that Λ then denotes the observed cosmological constant. At perturbative level the four-velocities of the two dark matter fluids read $u^\mu = \overset{\circ}{u}{}^\mu + \delta u^\mu$ and $\underline{u}^\mu = \overset{\circ}{\underline{u}}{}^\mu + \delta\underline{u}^\mu$, with a similar notation for the baryons. The background quantities are given in Eqs. (3.5). Recalling that the fluids ρ, u^μ and $\underline{\rho}$, $\underline{u}^\mu$ are defined with respect to the metrics $g_{\mu\nu}$ and $\underline{g}_{\mu\nu}$, respectively, their first-order perturbed velocities read

$$u^\mu = \frac{1}{a}(1-A,\beta^i), \qquad \underline{u}^\mu = \frac{1}{\underline{a}}(1-\underline{A},\underline{\beta}^i). \tag{D10}$$

We perform the usual SVT decomposition

$$\beta^i = \mathrm{D}^i v + v^i, \qquad \mathrm{D}_i v^i = 0, \tag{D11}$$

and introduce the gauge-invariant variables

$$V = v + E', \tag{D12a}$$

$$V^i = v^i + B^i. \tag{D12b}$$

Obviously we have similar definitions for the dark sector, e.g. $\underline{V} = \underline{v} + \underline{E}'$, and for the baryons, e.g. $V_\mathrm{b} = v_\mathrm{b} + E'$. One can then express the four-acceleration $a^\mu = u^\nu\nabla_\nu u^\mu$ in terms of these gauge-invariant quantities:

$$a^\mu = \frac{1}{a^2}(0, \mathrm{D}^i(V' + \mathcal{H}V + \Phi) + V'^i + \mathcal{H}V^i), \tag{D13}$$

and similarly for $\underline{a}^\mu = \underline{u}^\nu\underline{\nabla}_\nu\underline{u}^\mu$. The scalar densities of dark matters read $\rho = \overset{\circ}{\rho}(1+\delta)$ and $\underline{\rho} = \overset{\circ}{\underline{\rho}}(1+\underline{\delta})$, where δ and $\underline{\delta}$ are the density contrasts. We choose to express the density contrasts in the "flat slicing" gauge (indicated by the superscript F), defined by

$$\delta^\mathrm{F} = \delta + 3C, \qquad \underline{\delta}^\mathrm{F} = \underline{\delta} + 3\underline{C}, \tag{D14}$$

and which obey the equations (Δ being the Laplacian associated with the metric γ_{ij})

$$\delta^{\mathrm{F}\prime} + \Delta V = 0, \qquad \underline{\delta}^{\mathrm{F}\prime} + \Delta\underline{V} = 0. \tag{D15}$$

Similarly for the baryons, we define $\delta^\mathrm{F}_\mathrm{b} = \delta_\mathrm{b} + 3C$ and get $\delta^{\mathrm{F}\prime} + \Delta V_\mathrm{b} = 0$. We now turn to the equilibrium configuration ρ_0, u_0^μ defined with respect to the metric $f_{\mu\nu}$; see Eqs. (2.15) or (B3) and (B4). The background quantities have been given in (3.9). At linear order we have $u_0^\mu = \overset{\circ}{u}{}^\mu_0 + \delta u^\mu_0$, which reads explicitly

$$u_0^\mu = \frac{1}{(a\underline{a})^{1/2}}\left(1 - \frac{1}{2}(A+\underline{A}), \beta_0^i\right). \tag{D16}$$

The SVT decomposition and gauge-invariant variables proceed in the same way:

$$\beta_0^i = \mathrm{D}^i v_0 + v_0^i, \qquad \mathrm{D}_i v_0^i = 0, \tag{D17a}$$

$$V_0 = v_0 + \frac{1}{2}(E' + \underline{E}'), \tag{D17b}$$

$$V_0^i = v_0^i + \frac{1}{2}(B^i + \underline{B}^i). \tag{D17c}$$

For the scalar density we have $\rho_0 = \overset{\circ}{\rho}_0(1+\delta_0)$ and adopt the gauge-invariant definition

$$\delta_0^\mathrm{F} = \delta_0 + \frac{3}{2}(C + \underline{C}), \qquad \delta_0^{\mathrm{F}\prime} + \Delta V_0 = 0. \tag{D18}$$

Now the relations (B3) and (B4) translate immediately to linear cosmological perturbations. With our choice of equilibrium configuration the two displacement vectors read $y^\mu = \frac{1}{2}\xi^\mu$ and $\underline{y}^\mu = -\frac{1}{2}\xi^\mu$, and the fluid at equilibrium obeys the equation of motion (2.22). Since $\xi^\mu_\perp = \perp^\mu_\nu\xi^\nu$ is spacelike it necessarily belongs to first-order perturbations, because a nonvanishing background dipole moment would break the isotropy of space. Then the constraint $\overset{\circ}{u}_{0\mu}\xi^\mu_\perp = 0$ implies $\xi^0_\perp = 0$, so that we have the SVT form

$$\xi^\mu_\perp = (0, \lambda^i), \tag{D19a}$$

$$\text{with} \quad \lambda^i = \mathrm{D}^i z + z^i, \qquad \mathrm{D}_i z^i = 0, \tag{D19b}$$

where z and z^i are by definition the SVT variables. Since the background value is zero, they are directly gauge invariant. Using Eqs. (B4), in which the acceleration a_0^μ can be neglected since it is of first order [$a_0^\mu = \mathcal{O}(1)$; see (2.22)], the variables V, V^i and $\underline{V}$, $\underline{V}^i$ defined in (D12) are related to their partners V_0, V_0^i by[18]

$$V = V_0 + \frac{1}{2}(dE' + z'), \tag{D20a}$$

$$\underline{V} = V_0 - \frac{1}{2}(dE' + z'), \tag{D20b}$$

$$V^i = V_0^i + \frac{1}{2}(dB^i + z'^i), \tag{D20c}$$

$$\underline{V}^i = V_0^i - \frac{1}{2}(dB^i + z'^i). \tag{D20d}$$

From Eqs. (B3) the corresponding gauge-invariant density contrasts are related by

$$\delta^{\mathrm{F}} = \delta_0^{\mathrm{F}} - \frac{1}{2}\Delta(dE + z), \tag{D21a}$$

$$\underline{\delta}^{\mathrm{F}} = \delta_0^{\mathrm{F}} + \frac{1}{2}\Delta(dE + z). \tag{D21b}$$

Let us now deal with the dynamical equations of motion (2.19), in which the four-accelerations in the SVT formalism are given by e.g. (D13). Thus,

$$V' + \mathcal{H}V + \Phi = -4\pi\overset{\circ}{\rho}a^2 z, \tag{D22a}$$

$$\underline{V}' + \mathcal{H}\underline{V} + \underline{\Phi} = 4\pi\overset{\circ}{\underline{\rho}}\underline{a}^2 z, \tag{D22b}$$

$$V'^i + \mathcal{H}V^i = -4\pi\overset{\circ}{\rho}a^2 z^i, \tag{D22c}$$

$$\underline{V}'^i + \mathcal{H}\underline{V}^i = 4\pi\overset{\circ}{\underline{\rho}}\underline{a}^2 z^i, \tag{D22d}$$

with $\overset{\circ}{\underline{\rho}}\underline{a}^2 = \alpha\overset{\circ}{\rho}a^2$. Similarly the equation of motion of the equilibrium fluid found in Eq. (2.22) reads

$$V_0' + \mathcal{H}V_0 + \frac{1}{2}(\Phi + \underline{\Phi}) = -2\pi(1-\alpha)\overset{\circ}{\rho}a^2 z, \tag{D23a}$$

$$V_0'^i + \mathcal{H}V_0^i = -2\pi(1-\alpha)\overset{\circ}{\rho}a^2 z^i. \tag{D23b}$$

Note that the latter equations are in fact implied by (D22) when making use of the relations (D20). Finally, by computing the differences dV and dV^i from Eqs. (D22) and using (D20) together with the definition of $d\Phi$, we get

$$z'' + \mathcal{H}z' + 4\pi(1+\alpha)\overset{\circ}{\rho}a^2 z = -dA - dB' - \mathcal{H}dB, \tag{D24a}$$

$$z''^i + \mathcal{H}z'^i + 4\pi(1+\alpha)\overset{\circ}{\rho}a^2 z^i = -dB'^i - \mathcal{H}dB^i, \tag{D24b}$$

which constitute the SVT form of the equation of evolution (2.20) of the dipole moment. An alternative form of these equations is provided in Appendix E; see (E4).

[18] In order to prove the following relations we used the useful formulas

$$h = dA + 3dC + \Delta dE, \quad h_{\mu\nu}\overset{\circ}{u}{}_0^{\mu}\overset{\circ}{u}{}_0^{\nu} = -dA.$$

APPENDIX E: COSMOLOGICAL PERTURBATIONS IN THE DARK SECTOR

In Sec. III B we investigated the cosmological perturbations of the ordinary sector with metric $g_{\mu\nu}$. In this Appendix we deal with the perturbation equations for the dark sector with metric $\underline{g}_{\mu\nu}$. Actually it is simpler to consider the equations for the *differences* between the perturbation variables in the two sectors. We shall prove that these equations permit one to determine all the variables in the model, even those which cannot be measured by traditional cosmological observations taking place in the ordinary sector.

We write the perturbation equations for the difference of the two metrics in a way similar to Eqs. (3.18). We limit ourselves to the three equations with sources since the other ones are trivial. We get

$$\Delta d\Psi - 3\mathcal{H}^2 dX = 4\pi a^2\overset{\circ}{\rho}[-(p+q)\delta_{\mathrm{b}}^{\mathrm{F}} + p\delta^{\mathrm{F}} + q\underline{\delta}^{\mathrm{F}} + r(\Delta dE + dA)], \tag{E1a}$$

$$d\Psi' + \mathcal{H}d\Phi = -4\pi a^2\overset{\circ}{\rho}\left[-(p+q)V_{\mathrm{b}} + pV + q\underline{V} + r\left(-dE' + \frac{1}{2}dB\right)\right], \tag{E1b}$$

$$(\Delta + 2K)d\Phi^i = -16\pi a^2\overset{\circ}{\rho}\left[-(p+q)V_{\mathrm{b}}^i + pV^i + q\underline{V}^i - \frac{1}{2}r dB^i\right], \tag{E1c}$$

with coefficients

$$p = \frac{4\alpha(\varepsilon+\alpha)}{1+\alpha^2+2\alpha\varepsilon}, \qquad q = -\frac{4\alpha(1+\alpha\varepsilon)}{1+\alpha^2+2\alpha\varepsilon}, \qquad r = \frac{2\alpha(2\alpha+\varepsilon+\alpha^2\varepsilon)}{(\alpha+\varepsilon)(1+\alpha^2+2\alpha\varepsilon)}. \tag{E2}$$

From the right-hand sides of Eqs. (E1), one may define some effective variables for the matter fields, in a way similar to (3.17).

Then the equations of continuity and of motion associated with these matter variables are consequences of the equations themselves (via the Bianchi identities). The coefficients (E2) manage to simplify to give

$$dA' + \frac{1}{2}\Delta dB = 0, \tag{E3a}$$

$$\frac{1}{2}(dB' + \mathcal{H}dB) + dA = -8\pi(\alpha+\varepsilon)a^2\overset{\circ}{\rho}z, \tag{E3b}$$

$$\frac{1}{2}(dB'^i + \mathcal{H}dB^i) = -8\pi(\alpha+\varepsilon)a^2\overset{\circ}{\rho}z^i. \tag{E3c}$$

Gladly, we see that these equations guarantee consistency between Eqs. (D15)–(D22) and Eqs. (3.19) and (3.20). Combining the two last equations (E3b) and (E3c) with the equations of motion (D24), we obtain two further equations for the dipole moment z and z^i:

$$z'' + \mathcal{H}z' + 4\pi(1-3\alpha-4\varepsilon)\overset{\circ}{\rho}a^2 z = dA, \tag{E4a}$$

$$z''^i + \mathcal{H}z'^i + 4\pi(1-3\alpha-4\varepsilon)\overset{\circ}{\rho}a^2 z^i = 0. \tag{E4b}$$

From all these equations we can determine dA, dB, dB^i and the dipole components z, z^i. Next, using $d\Phi^i = dE'^i - dB^i$ one gets from the vector and tensor mode differences

$$dE''^i + 2\mathcal{H}dE'^i = dB'^i + 2\mathcal{H}dB^i, \tag{E5a}$$

$$dE''^{ij} + 2\mathcal{H}dE'^{ij} + (2K-\Delta)dE^{ij} = 0. \tag{E5b}$$

The second equation is simply the difference of Eqs. (3.18g). This then permits one to determine dE^i and dE^{ij}. Then, from the equality $d\Psi = d\Phi$ together with (E3b) we have

$$dF = dE' - \frac{1}{2}dB, \tag{E6a}$$

$$dC = dF' + 2\mathcal{H}dF - \frac{1}{2}\mathcal{H}dB + 8\pi(\alpha+\varepsilon)\overset{\circ}{\rho}a^2 z, \tag{E6b}$$

where dF is a convenient intermediate notation. Thus, dC is known once dF is known. Finally we combine the differences of (3.18d) and (D7) together with $d\Psi = d\Phi$ to obtain[19]

$$(a(ad\Psi)')' = -4\pi(q+r)a^4\overset{\circ}{\rho}d\Psi, \tag{E7}$$

which can be transformed, via

$$d\Psi = -\frac{1}{a}(adF)' - 8\pi(\alpha+\varepsilon)a^2\overset{\circ}{\rho}z, \tag{E8}$$

into the following evolution equation which permits determining dF and hence dE and dC:

$$\begin{aligned}&(a(adF)'')' + 4\pi k(q+r)(adF)'\\&\quad = -8\pi k(\alpha+\varepsilon)[(az')' + 4\pi k(q+r)z],\end{aligned} \tag{E9}$$

where $k \equiv \overset{\circ}{\rho}a^3$. The differences of gauge-invariant velocity variables are also computed from

$$dV' + \mathcal{H}dV + d\Phi = -4\pi(1+\alpha)\overset{\circ}{\rho}a^2 z, \tag{E10a}$$

$$dV'^i + \mathcal{H}dV^i = -4\pi(1+\alpha)\overset{\circ}{\rho}a^2 z^i. \tag{E10b}$$

Finally we conclude that all variables in our model can be fully and consistently determined by solving linear evolution equations.

[19]The background equation $\mathcal{H}'' - 2\mathcal{H}\mathcal{H}' = -4\pi(q+r)\mathcal{H}\overset{\circ}{\rho}a^2$ is also used in this calculation. We recall from Eq. (3.12) that the matter density observed in cosmology is $\overset{\circ}{\rho}_{\mathrm{M}} = \frac{2\alpha\varepsilon}{\alpha+\varepsilon}\overset{\circ}{\rho}$.

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