Radiated power and radiation reaction forces of coherently oscillating charged particles in classical electrodynamics

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For the foreseeable future, the analysis and design of the complex systems needed to generate intense beams of radiation via the process of coherent emission into free-space will depend on the principles and methods of classical electrodynamics (CED). But the fields and forces predicted by the currently accepted CED theory are manifestly incompatible with Maxwell's equations' energy integral as applied to the process of coherent emission into free-space. It is the purpose of this paper to review the evidence for these limitations of conventional CED, to identify an alternative formulation of CED that does not suffer from these defects, and to describe how the predictions of this more physically realistic formulation of electrodynamics, including the role of the advanced interactions allowed by Maxwell's equations and thermodynamics, might be tested by experiment and applied to enhance the capabilities of devices and systems employing the mechanism of "radiation into free-space."

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I. INTRODUCTION

As the technology of electron beam-based sources of electromagnetic radiation has advanced, the significance of the physics of the process of coherent radiation, in which a number of charges within a fraction of a wavelength are induced by their synchronized motion to emit electromagnetic radiation, has increased. In such coherently radiating systems, the amplitude of the emitted electric and magnetic fields increases linearly with the number of radiating charges, leading to radiated powers that increase as the square of that number, in contrast to powers that increase linearly with total charge applicable to the emission of radiation by charges moving in uncorrelated random motion.

Historically, the first applications of this coherent radiation process involved the radiation emitted by such coherently moving charges within the reflecting boundary conditions of radio-frequency, microwave, or optical cavities [1]. The presence of these reflecting boundaries made it necessary to implement forms of analysis that explicitly included the effects of these boundaries on the fields within the cavities and hence on the dynamics of the moving charges, typically via the reformulation of the systems' equations of motion in terms of the amplitudes of the cavities' normal modes as developed by Slater [2]. By these means, it is now understood that the dynamics of these systems reflect at every time and spatial scale the interactions between the moving charges within each of these cavities and the currents induced by these charges in the walls of the cavities as determined by the relevant boundary conditions.

More recently, technology has advanced to the point at which sources based on the radiation emitted into free-space by intense, coherently oscillating relativistic electron beams in free electron laser systems without the resonator mirrors employed in the earlier cavity-based FEL oscillators have been used to generate unprecedented levels of power at ultraviolet and x-ray wavelengths [3–8]. But the presently available field-based classical electrodynamics (CED) theory available to analyze the dynamics of these intense coherently radiating charge distributions can be shown to be deficient, yielding fields and forces that are incompatible with Maxwell's equations [9] and relying on the imposition of boundary conditions that are manifestly incompatible with the actual physical boundary conditions for these systems. The existence of these problems was first described by Kimel and Elias, who also described a phenomenological but fully covariant modification to the electrons equations of motion to insure the conservation of energy in such radiating systems [10,11].

As related in further detail below, the failings of conventional, field-based CED are most easily demonstrated through consideration of the fields and forces predicted by conventional CED for an elementary system consisting of a pair of coherently oscillating charges separated by a distance on the order of a wavelength at the frequency of oscillation, including wavelengths and spacings extending out to the physically macroscopic wavelengths relevant to radiation at microwave and radio frequencies. The persistence of the failings of conventional field-based CED [12] at these meter-scale distances makes it clear that the problem is not the consequence of the limitations of the models used to analyze the self-forces needed to conserve energy in the case of single oscillating particles or their equivalent continuous "fuzz ball" [13–16] charge distributions, but

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rather must be found in the nonlocal forces attributable to the coherently oscillating charges subject to the relevant boundary conditions for "radiation into free, unbounded space."

It is the first purpose of this paper to establish the limitations of conventional field-based CED by demonstrating that the approach is manifestly incompatible with the requirements implicit in the energy integral of Maxwell's equations in the simplest possible case of the coherent radiation emitted by pairs of synchronously oscillating charges spaced by distances of the order of a wavelength.

We will also observe in this section that one of the potentially relevant limitations of conventional field-based CED theory is its reliance on the imposition at all spatial and temporal scales of Sommerfeld's "radiation condition" restricting the Green's function for radiation by isolated accelerated elementary charges to the "retarded" solutions allowed, but not mandated, by Maxwell's equations [17]. The restriction to the retarded solutions of the wave equation in the case of radiation into free-space stands in direct contrast to the successful solutions of the radiation problem in closed, conducting or reflecting cavities in which there can be no distinction on physical grounds between the advanced and retarded solutions to the wave equation in the absence of the irreversible thermodynamic processes (like dissipation) postulated by Einstein as a prerequisite for the determination of the direction of the "arrow of time" [18,19].

Whether it is simply the reliance on Sommerfeld's radiation condition that is responsible for the manifest failings of conventional field-based CED has yet to be established. But what has recently been established is that a competing model of CED, the action-at-a-distance model of Wheeler and Feynman [20], (1) allows for the time-symmetric inclusion of both the advanced and retarded interactions allowed by Maxwell's equations, (2) includes a plausible if not unique statement of the physical boundary conditions for the case of radiation into free-space, and (3) provides a description of the forces generated through the process of coherent emission that is fully compatible with the energy integral of Maxwell's equations in contrast to the fundamentally flawed solutions of conventional field-based CED as restricted by Sommerfeld's radiation condition.

It is the second key purpose of this paper to demonstrate that the solutions developed by Wheeler and Feynman in their model of "radiation into free-space" successfully predict the forces needed to insure compliance with Maxwell's equations. The success of this model does not necessarily imply the need to abandon the field-based electrodynamics, but demonstrates the need to reformulate it. In a mathematical sense it demonstrates the need to include both Lienard and Wiechert's "special" solutions to the inhomogeneous wave equation [21,22] and the homogeneous solutions needed to satisfy boundary conditions.

The intriguing implications of the revisions to conventional field-based CED theory (needed to insure compliance with Maxwell's equations) suggest an experimental test be undertaken of the effects predicted by inclusion of both the advanced and retarded fields allowed by Maxwell's equations and Wheeler and Feynman's assumed "absorbing" boundary conditions. It is the third purpose of this paper to describe one such possible experiment.

Finally, we conclude this paper with a description of how such a revised version of CED might be employed to optimize the operation of the newly developed free-space self-amplified spontaneous emission (SASE) FELs and other devices or systems whose operation may be subtly altered by the advanced forces of the boundary conditions that may emerge as the basis of a more physically realistic version of classical electrodynamics.

II. HISTORY

By the early decades of twentieth century, attempts to describe radiation had favored classical field theory over the idea of action-at-a-distance. In the early 1900s the derivations by Lienard and Wiechert offered the first detailed covariant picture of radiation by accelerating point particles. The boundary conditions applicable to this solution have traditionally been taken to follow from Sommerfeld's analysis of the uniqueness of solutions to the wave equation. In 1912, Sommerfeld established the radiation condition [23], a boundary condition applied outside the volume occupied by radiating charges, which he applied to analyze the implications of the advanced solutions to the wave equation. Sommerfeld's analysis concluded by requiring that advanced solutions outside this far off boundary must be excluded to achieve uniqueness [23], and that result has continued to be relied upon through present date.

Although the Sommerfeld radiation condition holds a very strong position in the classical theory of radiation, there have been cases where it has been enforced unnecessarily. For example, the advanced solutions for cases of radiation in conducting cavities were ignored until Einstein argued that the advanced and retarded solutions to Maxwell equations were critical to conservation of energy and momentum for a radiating particle in a "box" (conducting cavity) [24] which served as the intellectual basis for the subsequent treatment of radiation into microwave and laser cavities by Slater, Lamb and Siegman [25–27]. It is of great interest to us that Einstein's original remarks and the subsequent cavity analyses were not intended to deal with the case of radiation into free-space [28], therefore most classical analysis of radiating sources completely ignored the effect of an absorbing media in the system until the analysis by Wheeler and Feynman.

Similarly, it has also been brought to our attention that the Wheeler and Feynman analysis has been criticized for not agreeing with Sommerfeld's radiation condition [29]. However it must be noted that a main requirement of Wheeler and Feynman's analysis is that the radiation fields of all the accelerated charges in the Universe are to be specified as one half of the advanced plus one half of the retarded solutions to the inhomogeneous wave equation. Since the relative contributions of the advanced and retarded solutions are specified in this model, there can be no ambiguity regarding the fields that are actually present in the system, and as further shown by Wheeler and Feynman, superposition of these fields yields the retarded field of experience at large distances from the radiating charge.

Sommerfeld's condition was intended to restrict solutions to the wave equation to yield only the retarded fields of experience but Wheeler and Feynman's assumption of half-advanced plus half-retarded fields, while leading to the same retarded field of experience at large distances from the source, also provides a nonlocal field in the vicinity of accelerated charges attributable to the advanced fields of the absorbing boundaries needed to achieve compliance with Maxwell's equations. The key failing of Sommerfeld's radiation condition is that it rules out this nonlocal field and thus cannot generate the fields needed to satisfy Maxwell's equations for the case of coherent radiation.

The continuing efforts to identify an approach to the forces needed to insure compliance with Maxwell's equations in the case of accelerated single charges, which began with Abraham and Lorentz [13,14] and continues through current date [12], also appears to have been limited by the imposition of Sommerfeld's radiation condition as exemplified by the failure of Larmor's theorem to account for the restriction of radiation by such elementary charges to the eigenfrequencies of the closed conducting or reflecting cavities in which they find themselves [30], an effect that cannot be understood without consideration of the effects of the advanced forces and boundary conditions in these systems.

Thus, it seems to have been Wheeler and Feynman's exploration of the action-at-a-distance model of electrodynamics that has helped to clarify the physical effects needed to resolve these problems. Though starting from a model of the interactions of charged particles that makes no references to electric or magnetic fields, only to the forces acting between these particles, it was established by Gauss [31], Schwarzschild [32], Tetrode [33] and Fokker [34–36] that descriptions of electrodynamics based on such action-at-a-distance models could be developed that were self-consistent, causal, and fully consistent with Maxwell's equations and thus consistent with the experimental basis of these equations.

To explain the instantaneous loss of energy that must occur in the course of radiation by an accelerated elementary charged particle, Wheeler and Feynman noted in their paper that this objective could only be achieved through the introduction of an advanced force acting between the emitting and absorbing particles [37]. In further consideration of Einstein's arguments concerning the role of thermodynamics in determining the direction of the arrow of time in electrodynamics, Wheeler and Feynman also postulated that the fields generated by isolated charges should be time symmetric, including equal components of the advanced and retarded solutions to interactions governed by Maxwell's equations [38]. Wheeler and Feynman's final step was to evaluate the forces that would appear in the vicinity of an accelerated charged particle as a consequence of its interactions with a surrounding ensemble of passively accelerated—and thus absorbing—charged particles whose motion responded to the forces attributable to a single, periodically accelerated test charge [39].

It has long been known that Wheeler and Feynman's approach leads to an expression for a force at the position of a periodically accelerated charge that is in phase with its velocity and has an amplitude precisely equal to the amplitude needed to insure consistency with Maxwell's equations and conservation of energy in such systems without the fundamental divergences of the earlier model of Abraham and Lorentz [40]. Wheeler and Feynman also pointed out that the interference of the advanced and retarded force-fields in such a system would lead to a final result in which an observer at large distances from the accelerated charge (compared to a wavelength at the frequency of oscillation) would see that the propagating force fields in the system would coalesce to form what appeared to be a single, retarded force field propagating towards the absorbing charges with an amplitude and phase corresponding to the retarded solution of Lienard and Wiechert. Thus consistency is achieved both with "every day" experience [41] and Einstein's hypothesis that the introduction of an irreversible thermodynamics process—dissipation—into such an otherwise time-symmetric system would unambiguously establish a direction to the arrow of time in the evolution of the dynamics of the system [18].

What was noted [41] but not identified as significant by Wheeler and Feynman at the time was the existence in the vicinity of the periodically oscillating charge of a nonlocal force field owing its origins to the incomplete interference of the multiple advanced and retarded forces attributable to the multiple interacting particles in the system whose spatially dependent amplitude and phase is precisely that needed to insure compliance with Maxwell's equations in the case of the coherent radiation emitted by pairs of spatially separated but synchronously oscillating pairs of the elementary charges.

It is only now with the emergence of powerful new ebeam radiation sources based on the mechanism of coherent emission into free-space that we have come to appreciate this last consequence of Wheeler and Feynman's model. In contrast, absent the advanced forces and boundary conditions incorporated within Wheeler and Feynman's model of electrodynamics, the fields computed following conventional field-based CED as further restricted by Sommerfeld's radiation condition have neither the spatially dependent amplitudes or phases to achieve compliance with Maxwell's equations in the case of coherent radiation.

While Wheeler and Feynman subsequently discarded their model based on their inability to find a means by which it could serve as the basis of a self-consistent quantized version of electrodynamics [42] along the lines of the approach subsequently followed by Schwinger, Feynman and Tomonaga in developing the modern theory of quantum electrodynamics, that decision may have been premature given Hoyle and Narlikar's recent demonstration that Wheeler and Feynman's action-at-a-distance model could in fact serve as the basis of an alternative, divergence-free version of QED [28,43].

There is obviously motivation in these developments for further critical analyses and reviews that go far beyond the objective of this paper, defined pursuant to the abstract and introduction above to explore by means of analysis and experiment whether the alternative model of electrodynamics of Wheeler and Feynman can provide a solution to the description of coherent emission into free-space that is consistent with Maxwell's equations, but has so far eluded the efforts based on the principles and methods of conventional, field-based CED.

III. RADIATED POWER AND ELECTRIC FIELDS OF PERIODICALLY OSCILLATING CHARGES

When in equilibrium, the electromagnetic time-averaged energy stored within a spherical shell surrounding a periodically oscillating charge is constant, requiring that the time-averaged surface integral of the Poynting vector must be equal to the time-averaged volume integral of $\mathbf{E}\cdot\mathbf{j}$ within the sphere.

$$-\int_A^B dt\frac{d}{dt}\int\left[\frac{E^2}{2}+\frac{H^2}{2}\right]dV = \int_A^B dt\left[\int\mathbf{E}\times\mathbf{H}d\mathbf{a}+\int\mathbf{E}\cdot\mathbf{j}dV\right], \tag{1}$$

if A and B are separated by an integral number of periods of the oscillation of the charges, then

$$\int_A^B dt\frac{d}{dt}\int\left[\frac{E^2}{2}+\frac{H^2}{2}\right]dV = 0 \Rightarrow \int_A^B dt\int\mathbf{E}\times\mathbf{H}d\mathbf{a} = -\int_A^B dt\int\mathbf{E}\cdot\mathbf{j}dV. \tag{2}$$

The surface integral of the Poynting vector can thus be used to infer the amplitude of the electric field acting on the electrons in phase with their velocities. The equilibrium between the time-averaged radiated power as calculated using Poynting's vector and the work done on the radiating charges by the electric field they generate subject to the relevant boundary conditions is thus a requirement that follows directly from Maxwell's equations [44,45], and requires the rigorous detailed balance of these two quantities.

Equations (1) and (2) are exact and complete without the addition of any other fields or forces. More general statements of energy conservation can be and have been developed through the use of these equations in combination with the Lorentz force law. Although in this paper we do not rely on these more general statements, we have cited several authors who have used such theorems in their analysis.

Consider the geometry of the case presented in Fig. 1, where two charged particles are oscillating in phase and are separated by distance r in the direction parallel to the direction of their oscillation, λ (wavelength of the oscillating motion) is approximately a micron, [$\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are the amplitude of of oscillation, and r is the magnitude of $\mathbf{r}$]. The power radiated by this system, $P=\int\mathbf{E}\times\mathbf{H}\cdot d\mathbf{a}$, integrated over the surface of a large diameter sphere centered on midpoint between the two charges is

$$P = \frac{e^2\ddot{x}_0^2}{c^3}\int_0^\pi \sin^3(\theta)\cos^2\left[\frac{\omega r}{c}\frac{\cos(\theta)}{2}\right]d\theta, = \frac{e^2\ddot{x}_0^2}{c^3}\left[\frac{2}{3}-\frac{2\cos[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^2}+\frac{2\sin[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^3}\right]. \tag{3}$$

When the separation $r=0$, the total time-averaged power radiated by the two electrons is $\frac{4e^2\ddot{x}_0^2}{3c^3}$ which is used to normalize Eq. (3) in the subsequent plots. The normalized power radiated by these two coherently oscillating charged particles is shown in Fig. 2 as a function of their separation.

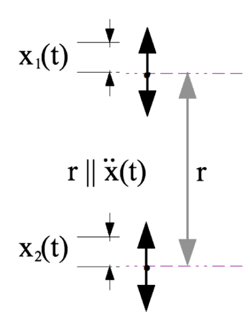

FIG. 1 (color online). Two coherently oscillating charged particles with distance r between the centers of their oscillation, and amplitude of oscillation of $x(t)$, where $\mathbf{r}$ is parallel to the direction of oscillation [$\mathbf{x}_1(t)=\mathbf{x}_2(t)=\mathbf{x}(t)=x_0\cos(\omega t)\hat{\mathbf{y}}$].

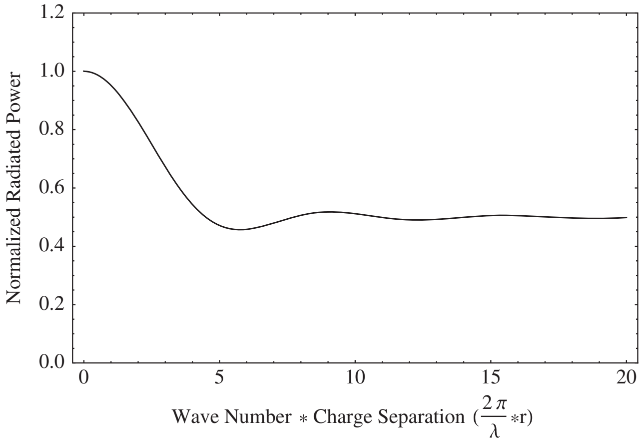

FIG. 2. Plot of normalized total radiated power from two coherently oscillating charged particles showing dependence on the separation between the center of oscillation of two charges. When $r \ll \lambda$ the radiated power has twice the magnitude compared to when $r \gg \lambda$.

Since, as shown in Fig. 2, the time-averaged power radiated by the particles varies with their separation, there must be an electric field acting on the two electrons that oscillates in phase with their velocities and has an amplitude that varies with spacing to match the variation with spacing of their radiated power [46]. We can then ask the question of whether any of the existing theories for the radiation reaction field provide an estimate of that force, which agrees with the electrons' radiated power and has the same variation of power with the vector separation of the two charges.

A. Role of the induction field

According to Sommerfeld's *retarded only* formulation of electrodynamics, the electric fields generated by the periodically oscillating charges in the model of coherent radiation shown in Fig. 1 include the charges' individual Abraham-Lorentz-Dirac self-fields, their retarded Lienard-Wiechert radiation and the induction fields. For the assumed case of radiation into free-space, the only charges whose fields can contribute to the integrals of Eqs. (1) and (2) are the two charges shown in Fig. 1, with no contribution from any of the possible boundaries which might be present in a more physically realistic model.

Since, for the alignment of the charges shown in Fig. 1, there is no component of the particles' radiation fields along the axis on which they are positioned, the particles' radiation fields cannot contribute to the radiation reaction force needed to conserve energy for the case of radiation into free-space. The retarded electric induction field, however, does include an oscillating component when evaluated for oscillations of small amplitude compare to the wavelength and the distance separating the oscillating charges ($x_0 \ll \lambda$ and $x_0 \gg r$). The retarded induction field acting on one of the particles due to the other is

$$\mathbf{E}(\mathbf{x}, t) = e\left[\frac{\hat{\mathbf{n}} - \beta}{\gamma^2(1 - \beta.n)^3 R^2}\right]_{\text{ret}} \tag{4}$$

[47] where

$$R = r + \delta r, \tag{5a}$$

$$\delta\mathbf{r} = \mathbf{x_1}(t) - \mathbf{x_2}(t_{\text{ret}}), \tag{5b}$$

$$t_{\text{ret}} = t - \frac{R}{c} = t - \frac{r + \delta r}{c}. \tag{5c}$$

In Eq. (5a), R is the retarded distance of the two particles from the perspective of the first particle at present time and the second particle at $t_{\text{ret}} = t - \frac{R}{c}$. Expanding terms in the denominator of Eq. (4) and keeping only the lowest order terms in x_0 and δr we evaluate the volume integral of $\mathbf{E} \cdot \mathbf{j}$ (see Appendix B),

$$P_{\text{ind}} = \langle e\mathbf{E}_{2_{\text{ret}}} \cdot \dot{\mathbf{x}}_1(t)\rangle, \tag{6}$$

which has a dependence on particle spacing r and is plotted in Fig. 3. Here and in the rest of the paper, we use the $\langle\rangle$ notation instead of the $\int_A^B dt$ for integer intervals in Eqs. (1) and (2).

As is evident from Fig. 3, the force due to the induction field fails to comprise a component of the radiation reaction force, not only because it is much weaker than the required radiation reaction force as inferred by Fig. 2, but also because it diverges as the particles' separation approaches zero. The only way to cancel this divergence is to add a component to the field equal to the advanced component of

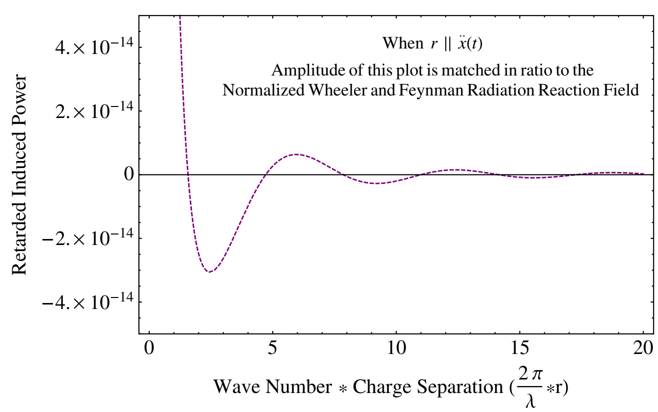

FIG. 3 (color online). The figure shows the volume integral of $\mathbf{E} \cdot \mathbf{j}$ attributable to the component of the retarded induction field generated by one charge (in Fig. 1) and oscillating in phase with the velocity of the other charge. This result diverges as $1/r^2$ at small separations. The amplitude of this plot is matched to the amplitudes in Figs. 2 and 4–7, showing that the amplitude of the volume integral of $\mathbf{E} \cdot \mathbf{j}$ generated by the induction field is far too small compared to the amplitude of the time-averaged surface integral of the Poynting vector.

the Lienard-Wiechert induction fields which has a divergence of the opposite sign. However, if that advanced component is also added to the two particles' radiation fields, the two particle system no longer emits any induced radiation (see Appendix C).

In principle, Sommerfeld's *retarded only* formulation of electrodynamics could be extended to include the fields generated by reflection of the outgoing radiation fields of the oscillating charges in Fig. 1 from the physical boundaries that might exist in more physically realistic situations. But those fields could not appear at the position of the oscillating charges in Fig. 1 until the retarded radiation fields of the oscillating charges in Fig. 1 had enough time to propagate outwards (towards those boundaries), and the retarded radiation fields of the charges and currents in the physical boundaries had the time needed to propagate back towards the oscillating charges in Fig. 1. For distant boundaries, the "out and back" propagation delay times would foreclose the possibility of satisfying the constraints that follow from Eqs. (1) and (2) at intermediate times. For such shorter times, the outcome would still manifestly violate the strict equivalence of the time-averaged volume integral of $\mathbf{E} \cdot \mathbf{j}$ and the time-averaged surface integral of $\mathbf{E} \times \mathbf{H}$. Sommerfeld's *retarded only* formulation of electrodynamics is thus fundamentally incapable of achieving consistency with Maxwell's equations for the case of coherent emission.

B. Dirac's Solution

In his 1938 "Classical Theory of Radiating Electrons" [48], Dirac proposed an expression for the "radiation reaction field (rrf)" of a single particle, needed to insure energy conservation:

$$\mathbf{F}_{\mathrm{Dirac-rrf}} \propto \frac{[\exp(iu) - \exp(-iu)]}{u}, \tag{7}$$

where $u = kr$ for our physical interpretation. While this seems to have been a key step in the right direction for explaining the phenomena of radiation in free-space, Dirac did not disclose the physical basis for his radiation reaction field. Dirac's motivation for expressing Eq. (7) as half the difference of the advanced and retarded components of the field seems purely mathematical.

Dirac noted that the real part of the radiated Lienard-Wiechert field at the position of the particle is 90 deg out of phase with the particle's velocity. The imaginary part, on the other hand, is in phase with the particle's velocity and is therefore capable of reducing the particle's kinetic energy during the process of radiation. By evaluating half the difference of the advanced and retarded solutions for the field equations he was able to demonstrate that the electric field at $r = 0$ had precisely the value required to insure conservation of energy. Although Dirac's radiation reaction field at finite values of r in the near field is also defined by

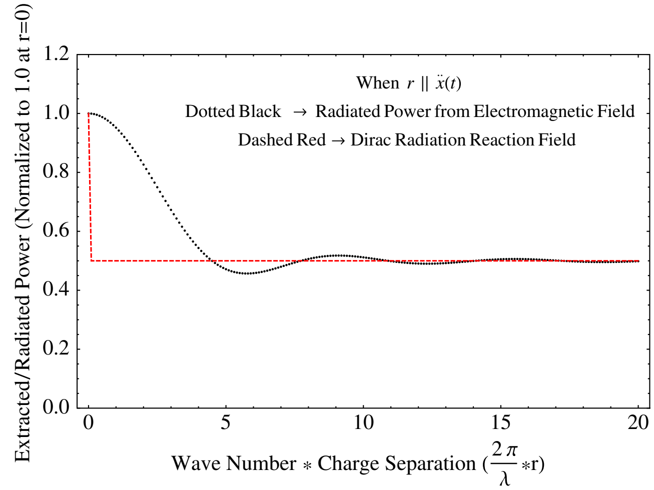

FIG. 4 (color online). Comparison of the volume integral of $\mathbf{E} \cdot \mathbf{j}$ attributable to the Dirac coherent radiation force with the power radiated by the two oscillating charged particles' for displacements parallel to their vector accelerations. The nonlocal component of the Dirac coherent radiation reaction force falls to zero for any finite displacement of the two charges along this direction, leaving only each particle's single point radiation reaction force to oppose their oscillating velocities.

Eq. (7), these values are only close to what is required for energy balance for coherent emission in vicinity of the particle.

Dirac's field fails in two ways to account for the power actually radiated by a pair of coherently oscillating charged particles. Since such charges emit no electric fields along the direction of their acceleration, the Dirac model is not capable of satisfying Eq. (2) for vector displacements parallel to the direction of the charges' oscillation as in Figs. 1 and 2. So in the case of two particle oscillating coherently with r being parallel to direction of oscillation, Dirac's formula does not provide a suitable solution as demonstrated in Fig. 4 (see Appendix D). Also as stated, the Dirac field for two coherently oscillating charges,

$$P_{\mathrm{Dirac-rrf},2e} = \langle 2\mathbf{F}_{\mathrm{Dirac-rrf}}(0,t) \cdot \dot{\mathbf{x}}(t) \rangle + \left\langle 2\mathbf{F}_{\mathrm{Dirac-rrf}}\left(\frac{\omega r}{c}, t\right) \cdot \dot{\mathbf{x}}(t) \right\rangle \propto \frac{1 + \frac{\sin(\omega r/c)}{\frac{\omega r}{c}}}{2}, \tag{8}$$

does not match the variation of coherently radiated power by particles with vector separations r normal to the direction of acceleration as shown in Fig. 5.

C. Wheeler and Feynman approach

Fortunately, Dirac's radiation reaction field inspired Wheeler and Feynman to more directly explore the physical basis of the radiation reaction field for point particles. In

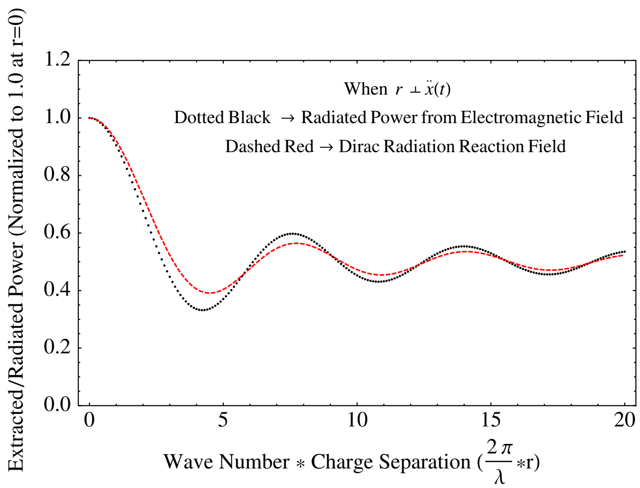

FIG. 5 (color online). Comparison of the volume integral of $\mathbf{E}\cdot\mathbf{j}$ attributable to the Dirac coherent radiation force with the power radiated by the two oscillating charged particles' for displacements perpendicular to their vector accelerations.

their 1945 paper, Wheeler and Feynman were able to demonstrate that, when formulated in the language of covariant action-at-a-distance, the solution of the boundary value problem corresponding to an oscillating particle within a spherical absorbing shell (representing free-space) of arbitrary composition and at an arbitrary distance from the radiating charge was dominated by the interference of the retarded and advanced forces originating in the accelerated and absorbing particles. By including both advanced and retarded solution for the source and absorber they were able to define a radiation reaction force exactly equal to that needed to match the power carried by radiation to the particles in the absorbing shell (see Appendix A for details) that was independent of the positions of the absorbing boundaries, in contrast to the out and back delay that characterizes the interactions of radiating charges and boundary conditions in Sommerfeld's *retarded only* formulation of electrodynamics.

Wheeler and Feynman identify Eq. (A1a) as the force field for the rrf for a single particle which is also finite when $r = 0$. They explicitly note that the adjunct field they had derived converged to Dirac's radiation reaction field at $r = 0$ and also at large distances from the accelerated particle. However they do not mention that their adjunct radiation reaction field diverged from Dirac's radiation reaction field at distances between a few tenths of a wavelength and 4–5 wavelengths. Nevertheless, based on Eq. (A1a), the total power of $\mathbf{F}_{WF-\mathrm{rrf}}$ for the system of two coherently oscillating charged particles is

$$P_{WF-\mathrm{rrf},2e} = \langle 2\mathbf{F}_{WF-\mathrm{rrf}}(0,t)\cdot\dot{\mathbf{x}}(t)\rangle + \left\langle 2\mathbf{F}_{WF-\mathrm{rrf}}\left(\frac{\omega r}{c},t\right)\cdot\dot{\mathbf{x}}(t)\right\rangle. \qquad (9)$$

To evaluate $P_{WF-\mathrm{rrf},2e}(\frac{\omega r}{c})$ explicitly for the case shown in Figs. 1 and 2, we start from Eq. (A1a). In this case, $P_2(\cos(\mathbf{a},r)) = 1$. Because $\cos(\mathbf{a},d)$ is the cos of the angle between $\mathbf{a}$ (acceleration of the oscillating charges with amplitude a_0) and r and since the particles are oscillating parallel to the direction of motion $\cos(\mathbf{a},r) = 1$ and $P_2(x) = \frac{1}{2}(-1+3x^2) \Rightarrow P_2(0) = 1$, therefore Eq. (A1a) becomes

$$F_{WF-\mathrm{rrf}}\left(\frac{\omega r}{c},t\right) = \frac{2e^2}{3c^3}(-i\omega a_0)\exp(-i\omega t) \times \left[F_0\left(\frac{\omega r}{c}\right) - F_2\left(\frac{\omega r}{c}\right)\right]. \qquad (10)$$

Evaluating Eqs. (A1b) and (A1c) and substituting in Eq. (10), we will have

$$F_{WF-\mathrm{rrf}}\left(\frac{\omega r}{c},t\right) = \frac{2e^2}{3c^3}(-i\omega a_0)\exp(-i\omega t) \times \left[\frac{\sin[\frac{\omega r}{c}]}{(\frac{\omega r}{c})} - \frac{3\cos[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^2} + \frac{(3-(\frac{\omega r}{c})^2)\sin[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^3}\right], \qquad (11)$$

which simplifies to

$$F_{WF-\mathrm{rrf}}\left(\frac{\omega r}{c},t\right) = \frac{2*3*e^2}{3c^3}(-i\omega a_0)\exp(-i\omega t) \times \left[-\frac{\cos[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^2} + \frac{\sin[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^3}\right]. \qquad (12)$$

The term in the bracket at $r = 0$ converges to $\lim_{x\to 0}[\frac{\sin[x]}{x^3} - \frac{\cos[x]}{x^2}] = \frac{1}{3}$. Therefore, for a periodic oscillation with $\dot{x} = -\omega x_0 \sin(\omega t)$

$$\langle 2\mathbf{F}_{WF-\mathrm{rrf}}(0,t)\cdot\dot{\mathbf{x}}(t)\rangle = \frac{2e^2\omega a_0}{3c^3}\omega x_0 = \frac{2e^2\ddot{x}_0^2}{3c^3}, \qquad (13)$$

and for any value of r we have

$$\left\langle 2\mathbf{F}_{WF-\mathrm{rrf}}\left(\frac{\omega r}{c},t\right)\cdot\dot{\mathbf{x}}(t)\right\rangle = \frac{2e^2\ddot{x}_0^2}{c^3}\left[-\frac{\cos[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^2} + \frac{\sin[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^3}\right]. \qquad (14)$$

For comparison, the total power for the system of two coherently oscillating charged particles separated by the distance r is

$$P_{WF-\mathrm{rrf},2e} = \frac{e^2\ddot{x}_0^2}{c^3}\left[\frac{2}{3} - \frac{2\cos[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^2} + \frac{2\sin[\frac{\omega r}{c}]}{(\frac{\omega r}{c})^3}\right], \qquad (15)$$

which is identical to the result from Eq. (3). Both Eq. (3) and Eq. (15) are compared in Fig. 6. The results for the total

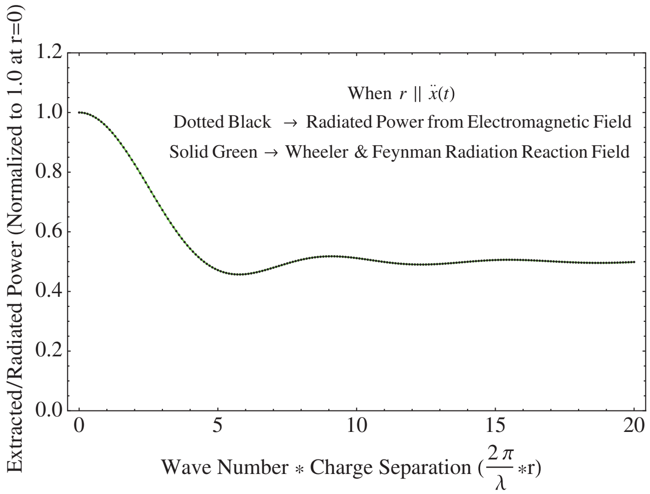

FIG. 6 (color). Comparison of the volume integral of $\mathbf{E} \cdot \mathbf{j}$ attributable to the Wheeler-Feynman coherent radiation force with the dependence of the two oscillating charged particles' radiated power for displacements parallel to their vector accelerations.

radiated power of two oscillating charged particles with displacements at right angles to their vector accelerations is evaluated and plotted in Fig. 7.

Figures 6 and 7 suggest that the Wheeler-Feynman model is significant with respect to the process of coherent radiation in free-space, for in contrast to Dirac's field, the Wheeler-Feynman adjunct radiation reaction field—owing its existence to the radiating particle's interactions with the advanced fields of the absorbing boundaries—accounts exactly for the force needed to satisfy Eq. (2) when compared with the strongly enhanced radiated power emitted by two coherently oscillating charged particles at varying spacings and angular displacements in their mutual near fields.

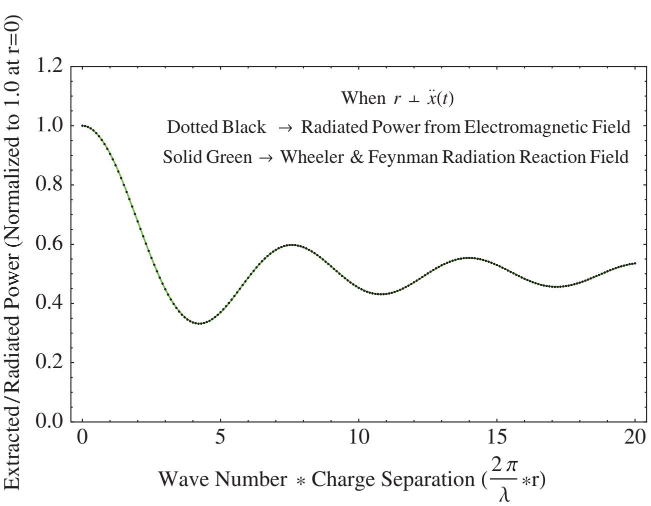

FIG. 7 (color). Comparison of the volume integral of $\mathbf{E} \cdot \mathbf{j}$ attributable to the Wheeler-Feynman coherent radiation force with the dependence of the two oscillating charged particles' radiated power for displacements at right angles to their vector accelerations.

It is also important to note that their approach is not offensive to our naive understanding of causality. In the analysis of Wheeler and Feynman, the advanced field of the absorber must be added to half-retarded and half-advanced field of the source. But in their analysis [41], the advanced field of absorber is equal to the difference between the half-retarded and half-advanced fields of the source. Therefore, the superposition of the half-advanced, half-retarded fields of the source and advanced field of the absorber produces the fully retarded field familiar from experience at large distances from the radiating charge.

IV. EXPERIMENTAL VALIDATION

Based on past critical reviews, the Wheeler-Feynman model of radiation into free-space has been found to be fully compatible with Maxwell's equations, quantum electrodynamics and causality. Therefore, there can be no objection on theoretical grounds to its implication for the reformulation of the more widely accepted field-based CED theory to include the model's half-advanced, half-retarded time-symmetric interactions as required to assure consistency with Maxwell's equations. Objections to this reformulation can only be based on experiment. As we describe below, the elementary process of coherent emission provides a good opportunity to perform such a test. We describe one such a test in Sec. IV A (below).

From the theoretical standpoint, the results demonstrated in Sec. III C (Figs. 6 and 7) strongly suggest that the limitations of established CED radiation theory with respect to the case of coherent radiation in free-space are due to the omission of the radiating particles' interactions via the advanced terms included in the general solution of the inhomogeneous wave equation with the distant charges and currents that absorb that radiation.

This failure is fully consistent with our understanding of the nature of the solutions to inhomogeneous linear differential equations like Maxwell's equations. The general solutions to such inhomogeneous problems are composed of a particular solution to the inhomogeneous equation plus the homogeneous solution needed to satisfy boundary conditions [49]. Accordingly, in field theory, the general solution to radiation problems requires the inclusion of the boundary conditions that fix the values of the solution at the boundaries of the region surrounding the source. Those conditions will differ depending on the nature of the boundary. In the case of conducting cavities, the discrete spectrum of solutions to the homogeneous wave equation which need to be added to the solution of the inhomogeneous wave equation to satisfy the boundary conditions at the conducting walls of the cavity define the nature of the radiation that can be emitted by oscillating charge distributions in the cavity.

The fields which have traditionally been ascribed to the radiation emitted by a single oscillating charge into free-space constitute a "special solution" to the inhomogeneous wave equation for that problem, but do not include any of the solutions to the homogeneous wave equation needed to fulfill the boundary conditions applicable to the problem, presumably those appropriate to the distant absorbing shell assumed in the Wheeler-Feynman model for free-space. It is thus not a surprise that the fields traditionally attributed to a single oscillating charge without consideration of the relevant boundary conditions, either those of the conducting microwave or reflecting optical cavities long familiar from cavity electrodynamics, or the absorbing walls the anechoic chambers used to simulate radiation into free-space should manifestly fail to comply with the test represented by Maxwell's energy integral in the case of coherent radiation into free-space. Absent the inclusion of the specific homogeneous terms needed to satisfy the boundary conditions applicable to radiation into free-space, there obviously can be no confidence that any special solution to the wave equation for this case can accurately define the fields acting on other oscillating charges in the vicinity of the assumed source charge [50].

Although fully consistent with the generally accepted means of solutions for such inhomogeneous linear differential equations, the results presented in this paper, if verified by experiment, will require a significant extension of our understanding of the nature of the radiative interactions that occur in the limit of classical electrodynamics. In particular, since the influence of the absorber at the time the source has just started to radiate is what is responsible for compliance with Maxwell's energy integral, we need to imagine that at the very moment the source begins to oscillate, it is also subject to a connection between itself and its surrounding medium through which energy can be transmitted or shared. Although suggested by theory as elaborated by Wheeler and Feynman, such a radical revision of our understanding of the process of radiation can only be based on experiment.

A. Measurement of the advanced radiation reaction field

The radiation emitted by macroscopic antennas bears many similarities to the radiation emitted by elementary oscillating charges [51]. Further, it has long been known that the driving point impedance of an antenna is affected by presence of other antenna systems (resonant or nonresonant) surrounding them. In particular, it has recently been shown in general that if both advanced and retarded Green's functions are included for the case of a single antenna, a single antenna in free-space will not radiate [52] analogous to the case of the isolated oscillating charges considered by Wheeler and Feynman. Radiation only becomes possible when dissipative elements are introduced into the corresponding boundary value problem [53].

In Sec. III we discussed radiation from two oscillating charged particles and showed the variation of the radiation reaction fields with distance between the oscillating charged particles in the presence of absorbing boundary conditions. Now if we consider a small dipole antenna, it will generate a field attributable to the superposition of the many oscillating elementary charges set in motion along the surfaces of its conducting elements by its signal source [54]. We assume that the current along the elements of the dipole decreases linearly from a maximum at the feed point to zero at the ends of the dipole as is conventional in analysis of short antennas [55].

For the purpose of the proposed experiment, it is important to note that if the elementary oscillating charged particles comprising the superposition are not closely spaced or not limited to a small volume in space, the interference of their individual coherent radiation reaction forces will generally result in a net force field that bears little resemblance to the force fields of the antenna's individual oscillating charged particles. To minimize this distortion, we have determined that the functional form of the Wheeler-Feynman radiation reaction field for an antenna (modeled by the superposition of single oscillating particles) converges to the form of the single-particle radiation reaction field as the dimensions of the antenna are reduced to $\lambda/10$ or less. The current distribution of the antenna can be constructed by adjusting the amplitudes and positions of the single oscillating particles to match the known current distribution for such a short dipole antenna as shown in Fig. 8. The advanced

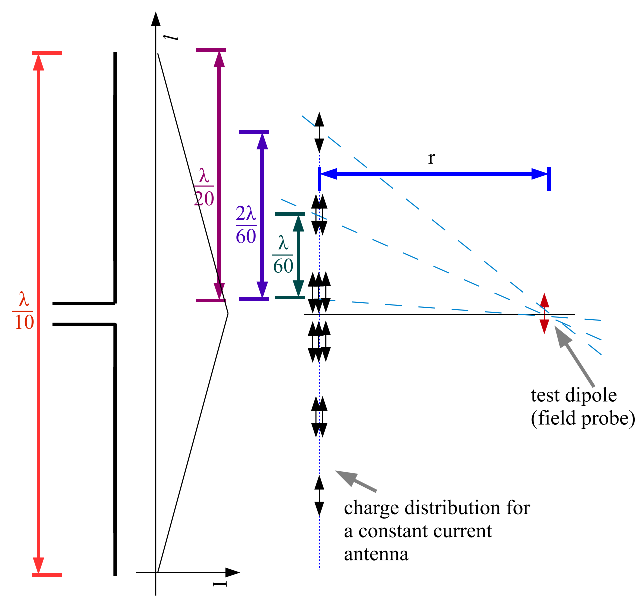

FIG. 8 (color online). Drawing showing the distribution of elementary oscillating charged particles used to derive the form of the Wheeler-Feynman coherent radiation reaction field for the short dipole antenna with constant current I distributed along the length l of the antenna which is located at distance r from the field probe or test charge. Here r is the distance between the probe and the antenna and the axes of the probe and antenna are parallel.

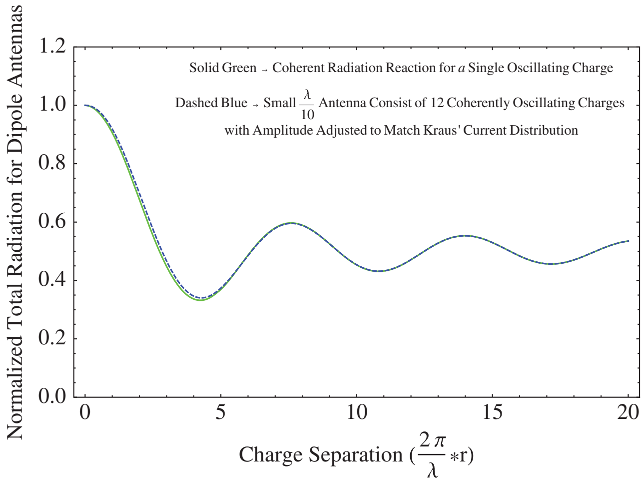

FIG. 9 (color). Comparison of the amplitude of the Wheeler-Feynman coherent radiation reaction force for a single oscillating particle with the superposed and scaled Wheeler-Feynman fields of an array of individual oscillating particles distributed along the length of the elements of a short dipole antenna shown in Fig. 8 according to the known distribution of current in these elements.

radiation reaction field predicted for the antenna from the radiated power and that of single oscillating particles as a function of distance r perpendicular to the direction of the current for absorbing boundaries is shown in Fig. 9; they are nearly identical. By measurement of the power attributed to these fields present in this antenna's vicinity it should therefore be possible to test the existence and functional form of the nonlocal coherent radiation reaction field of the Wheeler-Feynman model.

Although we live an era of technology where minimally perturbing multiaxis electric field probes are now commercially available for use in the characterization of new antenna systems, the additional means needed to measure only that component of the field that oscillates in phase with the velocity of the oscillating charged particles in the nearby radiation source appears to represent a new requirement for these field probes. We nonetheless believe that the design of such a phase sensitive field probe is within the current state of the art, though requiring a significant commitment with respect to engineering and commissioning.

Also, it will be required in any such measurement to implement the boundary conditions surrounding the transmitting antenna and field probe which should match as closely as possible the absorbing boundary conditions assumed in the Wheeler-Feynman analysis. This challenge can be overcome through the use of a high quality anechoic chamber with absorbing walls of the kind used in the field of antenna research and development.

B. Advanced interactions in SASE FELs

Although Kimel and Elias have previously described the derivation of a covariant force field capable of preserving energy conservation when added to the locally sourced electric and magnetic fields in free electron lasers emitting coherent radiation into free-space [10,11], the Wheeler-Feynman model is the only known analysis to date to explain that force field on the basis of clearly defined and physically plausible first principles. The interaction of radiating charges and the "targets" that absorb that radiation in the Wheeler-Feynman model suggest, in particular, the possibility of enhancing the capabilities of the new SASE FELs by altering the structure of their "targets" to alter the nature of the advanced forces acting on the radiating electrons and optimize the spectrum of the emitted coherent radiation, for example, by "backing up" a nonresonant target with a strongly dispersive multilayer x-ray mirror or Mössbauer reflector to subject the radiating electrons to the highly coherent advanced field of these mirrors during the critical process of bunch formation [56].

Such a resonant target reflects only those Fourier components that fall within its high reflectance passband. When illuminated by a beam containing a broader range of Fourier components, only the components within the target's high reflectance passband are reflected. In a time-symmetric world, the advanced field of the target converges on the target with the same amplitude, phase, and wavefront curvature as the retarded reflected wave. And that narrow-band advanced wave will pass through the electrons moving through the undulator as the electrons generate their broad band spontaneous radiation. The fields that impart the velocity modulation to the electrons that result in their subsequent bunching and strong coherent radiation therefore would include both the electrons' broad band spontaneous radiation and the more coherent advanced field of the target. Although the amplitude of the field attributable to the electrons spontaneous radiation would clearly be stronger, the advanced field of the target would be more coherent and hence potentially more effective in inducing the velocity modulation needed for bunching. Such a resonant target might therefore constitute the means needed to improve the coherence of the now intense but only partially coherent radiation generated by SASE FELs.

V. CONCLUSIONS

We have shown that the Wheeler-Feynman analysis of the radiation emitted by a moving charge interacting with absorbing boundary conditions provides a perfect match to the radiated power and radiation reaction fields for the case of two coherently oscillating particles. Therefore, even though some aspects of the model remain controversial and it has been considered a conceptually demanding theory [57], it is unique amongst the possible physical models in providing for compliance with Maxwell's energy integral for coherently radiating pairs of particles. This suggests that the action-at-a-distance concept presented by Tetrode, Wheeler, Feynman and others may be capable of providing a valid and more intrinsically realistic picture for the problem of radiation in classical electrodynamics. We have also shown that it is possible to experimentally test the ideas presented in this paper with an extension of existing technology.

In any event, the manifest failure of conventional retarded electrodynamics make it clear that existing retarded CED theory must be reformulated to achieve consistency with Maxwell's equations in the general case including the further specific requirements imposed by Maxwell's equations on the fields in the vicinity of the radiating particles in the case of the coherent emission of radiation by multiple moving charges.

Classical electrodynamics still serves as basis of our understanding of the dynamics of complex, macroscopic systems of charges and currents in the strong signal regime subject to complex, realistic boundary conditions. Classical electrodynamic's treatment of radiation will therefore remain the foundation for theoretical advancements in the future development of light sources where systems including as many as 10^{10} radiating electrons may need to be considered. We hope that by experimentally testing the relevant physics for these sources presented here we can take a step toward the more physically valid analysis of these sources as well as better understanding of the most fundamental aspects of radiation.

ACKNOWLEDGMENTS

The authors would like to thank Professors Isidoro Kimel and Luis Elias for their pioneering theoretical explorations of the coherent radiation reaction force, Ming Xie for his contribution to theory of high gain FELs, Ian Howe for critical discussions regarding the power radiated by a pair of oscillating charges, Julius Madey for his advice regarding the engineering issues for the proposed experiment, Eric Reckwerdt for elucidating remarks concerning the Sommerfeld radiation condition and Nicholas Wisniewski for valuable discussion regarding the Wheeler and Feynman model.

APPENDIX A: SUMMARY OF THE WHEELER AND FEYNMAN APPROACH AND THE RESULTANT FORMULAS

Wheeler and Feynman's 1945 paper [20] illustrates how advanced forces from a far off absorber can be employed to build an action-at-a-distance model of radiation that intrinsically includes the radiation reaction force in a covariant and causal formulation consistent with observation. Wheeler and Feynman present four increasingly complex derivations using an absorber of arbitrary density. First, they derive an expression for the radiation reaction force on a nonrelativistic accelerated source charge. Second, they derive the fields responsible for the radiation reaction force on that source charge and show how the advanced forces cancel everywhere except at the source charge leading to *retarded only* radiation at long distances from the source, consistent with experience. Third, they consider the source charge to be moving with arbitrary velocity, and forth they take a completely general approach. The result established in the second derivation, Eq. (A1a), constitutes the analytic basis of our analysis of the coherent radiation reaction problem for this paper. Here we summarize how Wheeler and Feynman arrive at this formula.

To calculate the effect of distant absorbers on the forces in the vicinity of an accelerating source charge, first the assumed retarded field of the charge traveling outbound is used to calculate the motions of the absorber particles, then the sum of the advanced forces from the absorber near the source is calculated. It is shown that addition of this field to the actual half-advanced plus half-retarded field of the source gives the assumed fully retarded field of the source while producing the correct radiation reaction force at the position of the source. Using their formulation, the force on a particle of charge e at distance d from the source charge is

$$\mathbf{F}_{WF-\mathrm{rrf}}\left(\frac{\omega d}{c},t\right)=\frac{2e^2}{3c^3}(-i\omega a_0)\exp(-i\omega t)\times\left[F_0\left(\frac{\omega d}{c}\right)-P_2(\cos(\mathbf{a},d))F_2\left(\frac{\omega d}{c}\right)\right] \tag{A1a}$$

$$F_0(u)=\frac{1}{2}\int_{-1}^{1}\exp(iu\cos\theta)d\cos\theta \tag{A1b}$$

$$F_2(u)=\frac{1}{2}\int_{-1}^{1}\exp(iu\cos\theta)P_2(\cos\theta)d\cos\theta. \tag{A1c}$$

For a large d both the F_0 and F_2 terms reduce to

$$\frac{[\exp(iu)-\exp(-iu)]}{2iu} \tag{A2}$$

and $\mathbf{F}_{WF-\mathrm{rrf}}$ becomes $\mathbf{F}_{\mathrm{Dirac-rrf}}$ indicating that the advanced field of the absorber in the vicinity of a source charge is equal to the half-advanced minus half-retarded field of the source itself.

APPENDIX B: DERIVATION OF THE ELECTRIC INDUCTION FIELD FOR SMALL OSCILLATIONS

To calculate and plot the volume integral of $\mathbf{E}\cdot\mathbf{j}$ at the position of particle 1 due to the induction field of particle 2, we use Eqs. (4) and (5a)–(5c). Since $\ddot{x}\|r$, $(\beta\|n)$ and $\beta\ll 1$, $(\gamma^{-2}\approx 1)$, Eq. (4) can be rewritten as

$$\mathbf{E_2}(\mathbf{x},t)=e\left[\frac{\hat{\mathbf{n}}(1+2\beta)}{R^2}\right]_{\mathrm{ret}}. \tag{B1}$$

The first order approximation of Eq. (5a) also reduces the R^{-2} term to $r^{-2}(1-2\frac{\delta r}{r})$, which reduces Eq. (B1) to

$$\mathbf{E_2}(\mathbf{x},t) = e\left[\frac{\hat{\mathbf{n}}(1+2\beta)(1-2\frac{\delta r}{r})}{r^2}\right]_{\mathrm{ret}}, \tag{B2}$$

where $\beta = \frac{\dot{\mathbf{x}}}{c}$. Since now we have $\delta t = (r+\delta r)/c$, Eq. (5b) can be rewritten as

$$\delta \mathbf{r} = \mathbf{x_1}(t) - \mathbf{x_2}\left(t-\frac{r}{c}\right) + \frac{\delta r}{c}\dot{\mathbf{x}}_\mathbf{2}\left(t-\frac{r}{c}\right). \tag{B3}$$

Since $\delta\mathbf{r}$, $\mathbf{x_1}$, $\mathbf{x_2}$, and $\dot{\mathbf{x}}_\mathbf{2}$ are all in the same direction ($\hat{\mathbf{y}}$) we drop the vectors notation. Solving for δr:

$$\delta r = \frac{x_1(t) - x_2(t-\frac{r}{c})}{1-\frac{1}{c}\dot{x}_2(t-\frac{r}{c})} \approx \left(x_1(t) - x_2\left(t-\frac{r}{c}\right)\right)\left(1+\frac{1}{c}\dot{x}_2\left(t-\frac{r}{c}\right)\right), \tag{B4}$$

and keeping only the first order term, we have

$$\delta r = x_1(t) - x_2\left(t-\frac{r}{c}\right), \tag{B5}$$

and

$$\delta t = \frac{1}{c}\left(r + x_1(t) - x_2\left(t-\frac{r}{c}\right)\right). \tag{B6}$$

So Eq. (B2), the retarded induced field due to the 2nd particle will become

$$\mathbf{E_{2_{ret}}}(\mathbf{x},t) = \frac{e\hat{\mathbf{n}}(1+2\frac{\dot{x}_2}{c})(1-2\frac{x_1(t)-x_2[t-(r/c)]}{r})}{r^2}, \tag{B7}$$

where $\dot{x}_2$ is evaluated at t_{ret}. We consider the general oscillating motion where $u_o = \omega x_o$ and $a_o = \omega^2 x_o$:

$$x = x_o \cos(\omega t) \tag{B8a}$$

$$u = -u_o \sin(\omega t) \tag{B8b}$$

$$a = a_o \cos(\omega t). \tag{B8c}$$

Then explicitly evaluating the retarded E_2 in terms of x_1 and x_2, we have

$$\mathbf{E_{2_{ret}}}(\mathbf{x},t) = \frac{e\hat{\mathbf{n}}}{r^2}\left[1 - \frac{2\omega x_o}{c}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) + \frac{2x_0}{r}\left[\cos(\omega t) - \cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right]\right]. \tag{B9}$$

Using Eq. (6), we evaluate the induced power using

$$\begin{aligned}
\langle e\mathbf{E_{2_{ret}}}.\dot{\mathbf{x}}_1(t)\rangle &= \left\langle \frac{e^2\omega x_o\sin(\omega t)}{r^2}\left[1-\frac{2\omega x_o}{c}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) + \frac{2x_0}{r}\left[\cos(\omega t) - \cos\left(\omega\left(t-\frac{r}{c}\right)\right)\right]\right]\right\rangle \\
&= \left\langle \frac{e^2\omega x_o\sin(\omega t)}{r^2}\left[1-\frac{2\omega x_o}{c}\sin\left(\omega\left(t-\frac{r}{c}\right)\right) + \frac{2x_0}{r}\left[\cos(\omega t) - \cos(\omega t)\cos\left(\frac{\omega r}{c}\right) - \sin(\omega t)\sin\left(\frac{\omega r}{c}\right)\right]\right]\right\rangle \\
&= \left\langle \frac{e^2\omega x_o\sin(\omega t)}{r^2}\left[1-\frac{2\omega x_o}{c}\left[\sin(\omega t)\cos\left(\frac{\omega r}{c}\right) - \cos(\omega t)\sin\left(\frac{\omega r}{c}\right)\right] - \frac{2x_0}{r}\left[\sin(\omega t)\sin\left(\frac{\omega r}{c}\right)\right]\right]\right\rangle \\
&= \left\langle \frac{e^2\omega x_o\sin(\omega t)}{r^2}\left[1-\frac{2\omega x_o}{c}\left[\sin(\omega t)\cos\left(\frac{\omega r}{c}\right)\right] - \frac{2x_0}{r}\left[\sin(\omega t)\sin\left(\frac{\omega r}{c}\right)\right]\right]\right\rangle
\end{aligned} \tag{B10}$$

and after expanding and dropping all the $\cos(\omega t)$ terms in $\mathbf{E_{2_{ret}}}$ that result from $\langle\cos(\omega t)\sin(\omega t)\rangle = 0$ for those terms, shown by (B10) we have

$$P_{\mathrm{ind,ret}} = \frac{2\pi e^2\omega x_o^2}{r^2}\left[-\frac{\omega}{c}\cos\left(\frac{\omega r}{c}\right) - \frac{1}{r}\sin\left(\frac{\omega r}{c}\right)\right], \tag{B11}$$

which is plotted in Fig. 3.

APPENDIX C: ADVANCED COMPONENT OF THE RADIATION REACTION FORCE FROM THE INDUCTION FIELD

Now consider the advanced term:

$$\mathbf{E_2}(\mathbf{x},t) = e\left[\frac{\hat{\mathbf{n}}(1+2\beta)(1-2\frac{\delta r}{r})}{r^2}\right]_{\mathrm{adv}}. \tag{C1}$$

For

$$t_{\mathrm{adv}} = -t - \delta t, \tag{C2}$$

the advanced induced field due to the 2nd particle is

$$\mathbf{E_{2_{adv}}}(\mathbf{x},t) = \frac{e\hat{\mathbf{n}}(1+2\frac{\dot{x}_2}{c})(1-2\frac{x_1(t)-x_2[-t-(r/c)]}{r})}{r^2}, \tag{C3}$$

where $\dot{\mathbf{x}}_2$ is evaluated at t_{adv}. Then $P_{\mathrm{ind,adv}}$ can be calculated just as $P_{\mathrm{ind,ret}}$ was in (B10).

$$P_{\mathrm{ind,adv}} = \frac{2\pi e^2 \omega x_o^2}{r^2}\left[\frac{\omega}{c}\cos\left(\frac{\omega r}{c}\right) + \frac{1}{r}\sin\left(\frac{\omega r}{c}\right)\right]. \tag{C4}$$

Since $P_{\mathrm{ind,adv}}$ yields the exact value as the $P_{\mathrm{ind,ret}}$ except with an opposite sign ($P_{\mathrm{ind,adv}} = -P_{\mathrm{ind,ret}}$) therefore $P_{\mathrm{ind,total}} = 0$. So when including $\mathbf{E_{2_{adv}}}$ in calculating $\langle e\mathbf{E}_2 \cdot \dot{\mathbf{x}}_1(t)\rangle$, the power extracted from the oscillating charge is zero.

APPENDIX D: FORMULA FOR THE RADIATED POWER BY TWO COHERENTLY OSCILLATING CHARGED PARTICLES DISPLACED BY DISTANCE *r* IN AN ARBITRARY DIRECTION

If the two coherently oscillating charged particles are displaced by distance r at angle α to their direction of motion, the integral of Poynting vector becomes Eq. (D1). When the charges oscillate perpendicular to the direction of their separation vector, $\alpha = \pi/2$, and the power is given by Eq. (D2),

$$P_{\mathrm{Radiated}}(\alpha) \propto 2\int_0^{2\pi}\int_0^{\pi} \sin^3(\theta)\cos^2 \times \left(\frac{kr(\cos(\theta)\cos(\alpha)+\sin(\theta)\sin(\alpha)\cos(\phi))}{2}\right) d\theta d\phi \tag{D1}$$

$$P_{\mathrm{Radiated}}(\alpha=\pi/2) \propto 2\int_0^{2\pi}\int_0^{\pi} \sin^3(\theta)\cos^2 \times \left(\frac{kr(\sin(\theta)\cos(\phi))}{2}\right) d\theta d\phi \tag{D2}$$

where $k = \frac{\omega}{c}$.

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