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Tensor gauge fields of $\mathcal{N}=8$ supergravity

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We study the tensor gauge fields ("notophs") of ungauged $\mathcal{N}=8, D=4$ supergravity in superspace. These are described by 2-form potentials B_2^G in the adjoint representation of $G=E_{7(+7)}$. The consistency of the natural candidates for the superspace constraints for their field strengths H_3^G fixes the form of the generalized Bianchi identities $DH_3^G=\ldots$ and also requires the potentials $B_2^{G/H}$ with indices of $G/H=E_{7(+7)}/SU(8)$ generators to be dual to the scalars of the $\mathcal{N}=8, D=4$ supergravity multiplet. In contrast, the field strengths of the 2-form potentials corresponding to the $SU(8)$ generators are dual to fermionic bilinears, so that these potentials are auxiliary rather than physical fields. Their presence, however, is essential to formulate a tensor hierarchy of $\mathcal{N}=8, D=4$ supergravity consistent with its U-duality group $E_{7(+7)}$.

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I. INTRODUCTION

The action of the maximal $\mathcal{N}=8, D=4$ supergravity was obtained in [1] by dimensional reduction of the $D=11$ supergravity [2] followed by dualization of 7 antisymmetric tensor gauge potentials $B_{\mu\nu}^I$ originating in the 11-dimensional 3-form, called "notophs" in [3],[1] to scalars. Then the complete set of $70(=28+35+7)$ scalars of the $\mathcal{N}=8, D=4$ supergravity multiplet was found to parametrize the coset space $E_{7(+7)}/SU(8)$ [1].

The natural question is whether this duality can be performed in an opposite direction, introducing a dual notoph for each scalar of the theory. In this paper we study this problem in the $\mathcal{N}=8, D=4$ superspace formulation of supergravity. To be more precise, we search for a "duality symmetric" formulation of the theory, containing both the scalar fields and the notophs rather than trying to replace everywhere the former by the latter (which is not possible beyond the linear approximation in fields).

The motivation for such a study is twofold. On one hand, we hope that our results will contribute to a deeper understanding of the U-duality group of the $\mathcal{N}=8$, $D=4$ supergravity, the exceptional Lie group $E_{7(+7)}$. The interest in this symmetry has remained high during the nearly 36 years that have passed since its discovery in [1], and, recently, a relation with the exceptional convergence properties of its loop amplitudes has been proposed (see Refs. [6] and references therein).

On the other hand, the knowledge on existence of $(p+1)$-form gauge potentials in a supergravity superspace might indicate the existence of supersymmetric extended objects, p-branes, coupled electrically to these potentials. In this sense our results imply the possible existence of a family of supersymmetric strings in an $\mathcal{N}=8, D=4$ supergravity superspace.[2] The search for possible worldvolume actions of such hypothetical superstrings is one of the natural applications of our results.

A first result showed by our study is that, to be consistent, one has to introduce a 2-form potential for each of the generators of the $G=E_{7(+7)}$ group, $B_2^G=(B_2^{G/H},B_2^H)$, and not just for the generators of the coset G/H. This result can be generalized to other theories with scalars parametrizing a symmetric space [8]. An early example of how the dualization of scalars requires the introduction of a $(d-2)$-form potential for each generator of the isometry group, even though their numbers do not match, is the dualization of the dilaton and Ramond-Ramond 0-form of $\mathcal{N}=2B, D=10$ supergravity in [9] (see also [10]): the two real scalars parametrize an $SL(2,\mathbb{R})/SO(2)$ coset space and they are dualized into a triplet of 8-forms transforming in the adjoint. The existence of this triplet of 8-forms is required by the symmetry algebra E_{11} [11] and has clear implications in the classification of the possible 7-branes of the theory [9,12–14]. In the context of the embedding tensor formalism for 4-dimensional gauged supergravities [15–18] (bosonic, spacetime) 2-form potentials in the adjoint representation of the duality group have to be introduced for different

[1]"Notoph" is "photon" read from the right to the left. Other, more popular names are Kalb-Ramond field [4], 2-form potential, and even B-field [5].

[2]The BPS branes of the maximal supergravity theories were studied originally in Refs. [7], but their worldvolume actions are, in general, unknown.

technical reasons, unrelated to the dualization of scalar fields, and for the specific case of $\mathcal{N}=8, D=4$ supergravity this was done some years ago in Refs. [18,19]. The general duality rule between scalars and $(d-2)$-forms was established in Refs. [16,20,21] using the embedding tensor formalism, but the results remain valid in the ungauged limit.

The study of supersymmetrization of these and other higher-rank gauge potentials has received much less attention[3] and in this paper we will start filling this gap for the case of the notophs of $\mathcal{N}=8, D=4$ supergravity using the superspace formalism. The knowledge of the gauge and supersymmetry transformations of these fields is a key ingredient in the construction of κ-symmetric worldvolume actions for possible associated supersymmetric string (p-brane) models.

In superspace formalism the problem of duality symmetric formulation, including the scalars and 2-form potentials dual to them on the mass shell, can be posed as searching for a set of constraints for 3-forms $H_3^G = dB_2^G + \cdots$ which are generalized field strengths of the corresponding 2-form potentials B_2^G defined on superspace. Below we present such superspace constraints for the $E_{7(+7)}$-algebra-valued 3-form field strengths on the curved $\mathcal{N}=8$ superspace of maximal $D=4$ supergravity, and study their self-consistency conditions: the generalized Bianchi identities (gBIs) $dH_3^G = \ldots$.

The explicit form of these gBIs are part of the definition of the tensor hierarchy of the Cremmer-Julia (CJ) $\mathcal{N}=8$ supergravity.[4] They reflect the group theoretical structure associated to the $E_{7(+7)}$ symmetry of $\mathcal{N}=8$ supergravity in the dual language. We will recover this piece of the tensor hierarchy starting from the natural candidate for superspace constraints for H_3^G and requiring that the algebraic part of the suitable gBIs, concentrated in their lower-dimensional components, should be satisfied identically when the candidate constraints are taken into account. At this stage we find, in particular, that the standard Bianchi identities $dH_3^G = 0$, if imposed, would lead to inconsistency and also that one cannot formulate a consistent set of constraints for the 3-forms corresponding to the coset generators, $H_3^{G/H}$, without introducing simultaneously the 3-forms H_3^H corresponding to the generators of the stability subgroup $H = SU(8)$ of the coset. In this sense one of the messages of this paper is that the superspace approach can be used in the search for a consistent tensorial hierarchies of supergravity (as well as of the theories invariant under rigid supersymmetry).

After this is done, we further study their higher-dimensional components and show that the duality relations between the field strengths of the notophs, $H_{\mu\nu\rho}^{G/H}$, and of the scalar fields of $\mathcal{N}=8$ supergravity (generalized Cartan forms $P_\mu^{G/H}$) are the consequences of our superspace constraints. The field strengths of the stability subgroup generators, $H_{\mu\nu\rho}^H$, are found to be dual to fermionic bilinears; this reflects the auxiliary character of the corresponding notophs $B_{\mu\nu}^H$.

[3]Some partial results on the supersymmetrization of the 2-forms dual to scalars in 4-dimensional $\mathcal{N}=2,1$ theories can be found in [22,23]. Supersymmetry has, nevertheless, been one of the main tools to find higher-rank potentials that can be added to the 10-dimensional maximal supergravities [11,13,24], in particular for $(d-1)$- and d-form potentials.

[4]The tensor hierarchy arises naturally in the democratic gauging of theories using the embedding-tensor formalism [15–18], but the fields still make sense when the embedding tensor and any other deformation parameters are switched off, in the ungauged, undeformed theory.

II. $\mathcal{N}=8$ SUPERGRAVITY SUPERSPACE

A. Geometry of $\mathcal{N}=8$ superspace and Cartan forms of $E_{7(+7)}$

Let us denote the bosonic and fermionic supervielbein forms of $\mathcal{N}=8, D=4$ superspace $\Sigma^{(4|32)}$ by

$$E^A \equiv (E^a, E^{\underline{\alpha}}) = (E^a, E_i^\alpha, \bar{E}^{\dot{\alpha}i}) = dZ^M E_M^a(Z). \tag{2.1}$$

Here $Z^M = (x^\mu, \theta^{\underline{\check{\alpha}}})$ are local bosonic and fermionic coordinates of $\Sigma^{(4|32)}$, $a = 0,1,2,3$ is Lorentz group vector index, $\alpha = 1,2$ and $\dot{\alpha} = 1,2$ are Weyl spinor indices of different chirality (see Appendix A), $i = 1,\ldots,8$ is the index of the fundamental representation of the $SU(8)$ R-symmetry group, and $\underline{\alpha}$ is the 32-valued cumulative index of $SL(2,\mathbb{C}) \otimes SU(8)$. In the case of world indices, only the counterpart of this cumulative index seems to make sense (until the Wess-Zumino gauge is fixed); it is carried by the fermionic (Grassmann-odd) coordinate $\theta^{\underline{\check{\alpha}}}$. Finally, $\mu = 0,1,2,3$ is the world vector index carried by bosonic (Grassmann-even) coordinate x^μ.

The curved superspace of $\mathcal{N}=8, D=4$ supergravity is endowed with a spin connection $\omega^{ab} = -\omega^{ba} = dZ^M \omega_M^{ab}(Z)$ and with the composite connection of the $SU(8)$ R-symmetry group, $\Omega_i{}^j = -(\Omega_j{}^i)^* = dZ^M \Omega_{M\,i}{}^j(Z)$, $\Omega_i{}^i = 0$; these are used to define the $SL(2,\mathbb{C}) \otimes SU(8)$ covariant derivative D. The exterior covariant derivatives of the supervielbein forms are called bosonic and fermionic torsion 2-forms,

$$T^a \coloneqq DE^a = dE^a - E^b \wedge w_b{}^a = \frac{1}{2} E^B \wedge E^C T_{CB}{}^a, \tag{2.2}$$

$$\begin{aligned} T_i^\alpha &\coloneqq DE_i^\alpha = dE_i^\alpha - E_i^\beta \wedge w_\beta{}^\alpha - \Omega_i{}^j \wedge E_j^\alpha \\ &= \frac{1}{2} E^B \wedge E^C T_{CB\,i}{}^\alpha, \end{aligned} \tag{2.3}$$

$$\begin{aligned} T^{\dot{\alpha}i} &\coloneqq D\bar{E}^{\dot{\alpha}i} = d\bar{E}^{\dot{\alpha}i} - \bar{E}^{\dot{\beta}i} \wedge w_{\dot{\beta}}{}^{\dot{\alpha}} - \bar{E}^{\dot{\alpha}j} \wedge \Omega_j{}^i \\ &= \frac{1}{2} E^B \wedge E^C T_{CB}{}^{\dot{\alpha}i}. \end{aligned} \tag{2.4}$$

Here $\wedge$ denotes the exterior product of differential forms with the basic properties

$$E^a\wedge E^b=-E^b\wedge E^a,\qquad E^{\underline{\alpha}}\wedge E^{\underline{\beta}}=E^{\underline{\beta}}\wedge E^{\underline{\alpha}},$$
$$E^a\wedge E^{\underline{\alpha}}=-E^{\underline{\alpha}}\wedge E^a,$$

and d is exterior derivative which acts from the right (see Appendix B).

By construction, the torsion 2-forms obey the Bianchi identities

$$I_3^a := DT^a + E^b\wedge R_b{}^a = 0, \tag{2.5}$$

$$I_{3_i}{}^\alpha := DT_i^\alpha + E_i^\beta\wedge R_\beta{}^\alpha - R_i{}^j\wedge E_j^\alpha = 0, \tag{2.6}$$

$$I_3^{\dot\alpha i} := DT^{\dot\alpha i} + \bar E^{\dot\beta i}\wedge R_{\dot\beta}{}^{\dot\alpha} + \bar E^{\dot\alpha j}\wedge R_j{}^i = 0, \tag{2.7}$$

which involve the curvature of the spin connection $[\omega_\alpha{}^\beta=\frac14\omega^{ab}\sigma_{ab\,\alpha}{}^\beta=(\omega_{\dot\alpha}{}^{\dot\beta})^*]$,

$$R^{ab}=(d\omega-\omega\wedge\omega)^{ab}=-R^{ba}=\frac12 E^C\wedge E^D R_{DC}{}^{ab}, \tag{2.8}$$

$$R_\alpha{}^\beta=\frac14 R^{ab}\sigma_{ab\,\alpha}{}^\beta=(d\omega-\omega\wedge\omega)_\alpha{}^\beta=\frac12 E^B\wedge E^A R_{AB\,\alpha}{}^\beta, \tag{2.9}$$

$$\begin{aligned}R_{\dot\alpha}{}^{\dot\beta}&=(R_\alpha{}^\beta)^*=-\frac14 R^{ab}\tilde\sigma_{ab}{}^{\dot\beta}{}_{\dot\alpha}=(d\omega-\omega\wedge\omega)_{\dot\alpha}{}^{\dot\beta}\\&=\frac12 E^B\wedge E^A R_{AB\,\dot\alpha}{}^{\dot\beta},\end{aligned} \tag{2.10}$$

and also the curvature of the induced $SU(8)$ connection, $R_i{}^j:=d\Omega_i{}^j-\Omega_i{}^k\wedge\Omega_k{}^j$. The compositeness of $\Omega_i{}^j$ is reflected by the fact that its curvature is expressed as [25]

$$R_i{}^j=-(R_j{}^i)^*=\frac13\mathbb{P}_{iklp}\wedge\bar{\mathbb{P}}^{jklp}, \tag{2.11}$$

where $\mathbb{P}_{ijkl}$ is the covariant Cartan form of the $E_{7(+7)}/SU(8)$ coset and $\bar{\mathbb{P}}^{ijkl}$ is its complex conjugate, which is also its $SU(8)$ dual up to an arbitrary constant phase β,

$$\bar{\mathbb{P}}^{ijkl}=(\mathbb{P}_{ijkl})^*=\frac{1}{4!}e^{-i\beta}\varepsilon^{ijklpqrs}\mathbb{P}_{pqrs}. \tag{2.12}$$

The Cartan forms are covariantly closed,

$$D\mathbb{P}_{ijkl}:=d\mathbb{P}_{ijkl}-4\Omega_{[i|}{}^p\wedge\mathbb{P}_{p|jkl]}=0,\qquad D\bar{\mathbb{P}}^{ijkl}=0. \tag{2.13}$$

Some further properties obeyed by these forms can be found in Appendix B.

Equations (2.13), and (2.11) with $\mathbb{P}_{ijkl}$ obeying (2.12), are structure equations of the $E_{7(+7)}$ Lie group. These can be solved providing the expressions for the covariant Cartan forms $\mathbb{P}_{ijkl}$ and $SU(8)$ connection $\Omega_i{}^j$ in terms of scalar superfields of the $\mathcal{N}=8$ supergravity, the explicit form of which is not needed for our discussion below.

B. $\mathcal{N}=8$, $D=4$ superspace constraints and their consequences

The constraints of $\mathcal{N}=8, D=4$ supergravity [25,26] can be collected in the following expressions for the bosonic and fermionic torsion 2-forms:

$$T^a=-iE_i^\alpha\wedge\bar E^{\dot\beta i}\sigma^a_{\alpha\dot\beta}, \tag{2.14}$$

$$\begin{aligned}T_i^\alpha&=\frac12\bar E^{\dot\beta j}\wedge\bar E^{\dot\gamma k}\epsilon_{\dot\beta\dot\gamma}\chi^\alpha_{ijk}+E^c\wedge E_j^\beta T^j_{\beta\,c}{}^\alpha_i+E^c\wedge\bar E^{\dot\beta j}T_{\dot\beta j\,c}{}^\alpha_i\\&\quad+\frac12 E^c\wedge E^b T_{bc}{}^\alpha_i,\end{aligned} \tag{2.15}$$

$$\begin{aligned}T^{\dot\alpha i}&=-\frac12 E_j^\beta\wedge E_k^\gamma\epsilon_{\beta\gamma}\bar\chi^{\dot\alpha ijk}+E^c\wedge E_j^\beta T^j_{\beta c}{}^{\dot\alpha i}+E^c\wedge\bar E^{\dot\beta j}T_{\dot\beta jc}{}^{\dot\alpha i}\\&\quad+\frac12 E^c\wedge E^b T_{bc}{}^{\dot\alpha i}.\end{aligned} \tag{2.16}$$

Here $\chi^\alpha_{ijk}=(\bar\chi^{\dot\alpha ijk})^*$ is the main fermionic superfield of $\mathcal{N}=8, D=4$ supergravity and the dimension 1 fermionic torsion components have the expressions

$$\begin{aligned}&T^j_{\beta\,b}{}^\alpha_i=\frac14\chi_{ikl\beta}(\bar\chi^{jkl}\tilde\sigma_b)^\alpha,\qquad T_{\dot\beta j\,b}{}^{\dot\alpha i}=\frac14\bar\chi^{ikl}_{\dot\beta}(\tilde\sigma_b\chi_{jkl})^{\dot\alpha},\\&T_{\dot\beta j\,b}{}^\alpha_i=-\frac{i}{2}\sigma_{b\beta\dot\beta}M^{\alpha\beta}_{ij}-\frac{i}{2}\tilde\sigma_b^{\dot\alpha\alpha}\bar N_{\dot\alpha\dot\beta\,ij},\\&T^j_{\beta b}{}^{\dot\alpha i}=-\frac{i}{2}\sigma_{b\beta\dot\beta}\overline{M}^{\dot\alpha\dot\beta\,ij}-\frac{i}{2}\tilde\sigma_b^{\dot\alpha\alpha}N^{i\,j}_{\alpha\beta},\end{aligned} \tag{2.17}$$

in terms of the fermionic bilinears[5]

$$\begin{aligned}N^{i\,j}_{\alpha\beta}&=\frac{e^{-i\beta}}{6\cdot 4!}\varepsilon^{ij[3][3']}\chi_{\alpha[3]}\chi_{\beta[3']},\\\bar N_{\dot\alpha\dot\beta\,ij}&=-\frac{e^{i\beta}}{6\cdot 4!}\varepsilon_{ij[3][3']}\bar\chi^{[3]}_{\dot\alpha}\bar\chi^{[3']}_{\dot\beta},\end{aligned} \tag{2.18}$$

and the bosonic superfields $M_{ij\alpha\beta}=M_{[ij](\alpha\beta)}=(\bar M^{ij}_{\dot\alpha\dot\beta})^*$. These appear as irreducible parts of the fermionic covariant derivatives of the main fermionic superfield,

$$D^i{}_{(\alpha}\chi_{\beta)jkl}=-3\delta^i_{[j}M_{kl]\alpha\beta},\qquad\bar D_{i(\dot\alpha}\bar\chi_{\dot\beta)}{}^{jkl}=-3\delta_i^{[j}\bar M^{kl]}_{\dot\alpha\dot\beta}, \tag{2.19}$$

The other irreducible components of these covariant derivatives of the main superfield are expressed through their bilinears,

[5] $\varepsilon_{ij[3][3']}\bar\chi^{[3]}_{\dot\alpha}\bar\chi^{[3']}_{\dot\beta}\equiv\varepsilon_{ijklmnpq}\bar\chi^{klm}_{\dot\alpha}\bar\chi^{npq}_{\dot\beta}$.

$$D^{\alpha i}\chi_{\alpha jkl} = -\frac{e^{i\beta}}{12}\varepsilon_{jkl[2][3]}\bar{\chi}_{\dot{\alpha}}^{i[2]}\bar{\chi}^{\dot{\alpha}[3]},$$
$$\bar{D}_i^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}^{jkl} = -\frac{e^{-i\beta}}{12}\varepsilon^{jkl[2][3]}\chi^{\alpha}_{i[2]}\chi_{\alpha[3]}. \qquad (2.20)$$

C. $E_{7(+7)}/SU(8)$ Cartan forms in $\mathcal{N}=8$ supergravity superspace

The covariantly constant Cartan 1-forms obey the constraints

$$\mathbb{P}_{ijkl} = 2E^{\alpha}_{[i}\chi_{jkl]\alpha} - 2\frac{e^{i\beta}}{4!}\bar{E}^{\dot{\alpha}p}\varepsilon_{ijklp[3]}\bar{\chi}^{[3]}_{\dot{\alpha}} + E^a\mathbb{P}_{aijkl}, \qquad (2.21)$$

$$\bar{\mathbb{P}}^{ijkl} = 2\frac{e^{-i\beta}}{4!}E^{\alpha}_p\varepsilon^{ijklp[3]}\chi_{\alpha[3]} - 2\bar{E}^{\dot{\alpha}[i}\bar{\chi}^{jkl]}_{\dot{\alpha}} + E^a\bar{\mathbb{P}}^{ijkl}_a. \qquad (2.22)$$

These coincide with those in Refs. [25,26] up to the constant phase parameter β. With the constraints (2.21), (2.22) the Bianchi identities (2.13) imply

$$\bar{D}_{\dot{\alpha}i}\chi_{\alpha jkl} = 2i\sigma^a_{\alpha\dot{\alpha}}\mathbb{P}_{aijkl}, \qquad D^i_{\alpha}\bar{\chi}^{jkl}_{\dot{\alpha}} = -2i\sigma^a_{\alpha\dot{\alpha}}\bar{\mathbb{P}}^{ijkl}_a, \qquad (2.23)$$

The results of Eq. (2.13) are also of help to find the expression Eq. (2.20) for $D^{\alpha i}\chi_{\alpha jkl}$ and the duality relation between the vector $\mathbb{P}_{aijkl}$ and its conjugate $\bar{\mathbb{P}}^{ijkl}_a$,

$$\mathbb{P}_{aijkl} = \frac{e^{i\beta}}{4!}\varepsilon_{ijklpqrs}\bar{\mathbb{P}}^{pqrs}_a. \qquad (2.24)$$

Just after this stage the superspace 1-forms in Eqs. (2.21) and (2.22) become related by Eq. (2.12).

III. 1-FORM GAUGE POTENTIALS IN $\mathcal{N}=8$ SUPERGRAVITY SUPERSPACE

Although the supervielbein forms restricted by the torsion constraints already contain all the fields of supergravity multiplets, including the vector fields and their field strength, it is possible and also convenient to introduce the corresponding 1-form gauge potentials in superspace. As was found already in Ref. [25], to preserve manifest $SU(8)$ R-symmetry, one should introduce the super-1-forms corresponding to both the "electric" gauge fields of the supergravity multiplet and to their magnetic duals, packed in the complex 1-form $A_{ij} = A_{[ij]} = dZ^M A_{M\,ij}(Z)$ in the **28** representation of $SU(8)$, and its complex conjugate $\bar{A}^{ij} = \bar{A}^{[ij]} = dZ^M\bar{A}^{ij}_M(Z) = (A_{ij})^*$ in its $\overline{\mathbf{28}}$ representation.

Their 2-form field strengths, which obey the gBIs

$$DF_{ij} = \mathbb{P}_{ijkl}\wedge\bar{F}^{kl}, \qquad D\bar{F}^{ij} = \bar{\mathbb{P}}^{ijkl}\wedge F_{kl}, \qquad (3.1)$$

are restricted by the constraints

$$F_{ij} = -iE^{\alpha}_i\wedge E^{\beta}_j\epsilon_{\alpha\beta} - \frac{1}{2}E^a\wedge\bar{E}^{\dot{\gamma}k}\sigma_{a\gamma\dot{\gamma}}\chi^{\gamma}_{ijk} + \frac{1}{2}E^c\wedge E^b F_{bc\,ij}, \qquad (3.2)$$

$$\bar{F}^{ij} = -iE^{\dot{\alpha}i}\wedge E^{\dot{\beta}j}\epsilon_{\dot{\alpha}\dot{\beta}} + \frac{1}{2}E^a\wedge E^{\gamma}_k\sigma_{a\gamma\dot{\gamma}}\bar{\chi}^{\dot{\gamma}ijk} + \frac{1}{2}E^c\wedge E^b\bar{F}^{ij}_{bc}. \qquad (3.3)$$

The antisymmetric tensor superfield can be decomposed in the two irreducible parts[6]

$$\sigma^a_{\alpha\dot{\alpha}}\sigma^b_{\beta\dot{\beta}}F_{ab\,ij} = 2\epsilon_{\alpha\beta}F_{\dot{\alpha}\dot{\beta}\,ij} - 2\epsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta\,ij}. \qquad (3.4)$$

The Bianchi identities, including (3.1), imply, in particular,

$$F_{\alpha\beta\,ij} = \frac{i}{2}M_{\alpha\beta\,ij},$$
$$F_{\dot{\alpha}\dot{\beta}\,ij} = \frac{i}{2}\bar{N}_{\dot{\alpha}\dot{\beta}\,ij} = -i\frac{e^{i\beta}}{12\cdot 4!}\varepsilon_{ij[3][3']}\bar{\chi}^{[3]}_{\dot{\alpha}}\bar{\chi}^{[3']}_{\dot{\beta}}. \qquad (3.5)$$

IV. 2-FORM GAUGE POTENTIALS IN $\mathcal{N}=8$ SUPERGRAVITY SUPERSPACE

Now we are ready to turn to the main subject of this paper: 2-form gauge potentials $B_2^{\tilde{\Sigma}}$ (notophs) in the complete supersymmetric description of $\mathcal{N}=8, D=4$ supergravity.

As we discussed in the Introduction, although the appearance of seven 2-form potentials after dimensional reduction from $D=11$ down to $D=4$ is manifest and was already noticed in [1], these were immediately dualized to scalars. Only then does the global $E_{7(+7)}$ duality become manifest. The inverse transformations relating all the scalars of $\mathcal{N}=8$ supergravity, parametrizing $E_{7(+7)}/SU(8)$, to 2-form potential have not been studied, at least in a complete form and especially in superspace; this is our goal here. As we will see, in addition to the 2-forms associated to the coset generators, $B_2^{G/H}$, which were expected as dual to the physical scalars parametrizing $G/H = E_{7(+7)}/SU(8)$ (basically because there are 70 of them), it is necessary to introduce $\mathfrak{su}(8)$-valued 2-form B_2^H. These are auxiliary and do not correspond to any dynamical degrees of freedom of $\mathcal{N}=8, D=4$ supergravity. The general situation will be discussed in the companion paper [8]. Here we adopt a more technical superspace-based approach to establishing the content and the structure of the tensorial hierarchy of $\mathcal{N}=8, D=4$ supergravity.

A. Strategy

Our strategy to search for higher form potentials in maximal supergravity is essentially superspace based: we

[6]Notice that $F_{\dot{\alpha}\dot{\beta}\,ij} = +\frac{1}{4}F_{ab\,ij}\tilde{\sigma}^{ab}_{\dot{\alpha}\dot{\beta}} = -(\bar{F}_{\alpha\beta}{}^{ij})^*$.

begin by searching for an ansatz for possible superspace constraints for 3-form field strengths $H_3^{\tilde{\Sigma}} = dB_2^{\tilde{\Sigma}} + \cdots$ suggested by the indices carried by the potentials. Checking their consistency, we can find whether more forms have to be introduced and what kind of "free differential algebra" (FDA) they have to generate. This is described by a set of gBIs $DF_4^{\mathcal{A}} = \ldots$ The further study of the gBIs (FDA relations) for the constrained field strength should result (provided the constraints are consistent and the potentials are dynamical fields) in equations of motion which, in the case of the 2-form potentials, should have the form of duality of their field strength to the covariant derivatives of the scalar fields. Since, in our case, these are (the bosonic leading components of) the covariant $E_{7(7)}/SU(8)$ Cartan forms, i.e. the complex self-dual 1-forms $P_{ijkl} = \frac{1}{4!}\varepsilon_{ijkli'j'k'l'}\bar{P}^{i'j'k'l'}$ in the **70** of $SU(8)$ ($e^{-i\beta/2}\mathbb{P}_{a\,ijkl} = \frac{1}{4!}\varepsilon_{ijklpqrs}e^{i\beta/2}\bar{\mathbb{P}}_a^{pqrs}$ in terms of bosonic component of superforms), the "physical" 2-form potentials are expected to be $B_{2\,ijkl}$ and its complex conjugate and dual $\bar{B}_2^{ijkl}$.

However, as discussed in the Introduction, experience suggests that when the scalars parametrize a coset space G/H, it is not sufficient to introduce only the dual $(D-2)$-form potentials with indices of the generators of the coset: the $(D-2)$-forms associated to the generators of the subgroup H must be included as well (see [8] for a general discussion). In our case, these correspond to the Hermitian traceless matrix of 2-forms $B_{2i}{}^j = (B_{2j}{}^i)^*$ with the generalized field strength $H_{3i}{}^j = dB_{2i}{}^j + \cdots$.

B. Constraints and generalized Bianchi identities for 3-form field strengths

The natural candidate for the superspace constraints are

$$H_{3\,ijkl} = E^\alpha_{[i} \wedge \sigma^{(2)}{}_\alpha{}^\beta \chi_{jkl]\beta} - \frac{e^{i\beta}}{4!}\varepsilon_{ijkli'j'k'l'}\bar{E}^{\dot{\alpha}i'} \wedge \tilde{\sigma}^{(2)\dot{\beta}}{}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}}^{j'k'l'} + \frac{1}{3!}E^c \wedge E^b \wedge E^a H_{abc\,ijkl}, \tag{4.1}$$

where $\sigma^{(2)}{}_\alpha{}^\beta = \frac{1}{2}E^b \wedge E^a \sigma_{ab\alpha}{}^\beta = -(\tilde{\sigma}^{(2)\dot{\beta}}{}_{\dot{\alpha}})^*$, and

$$H_{3\,i}{}^j = iE^a \wedge E^\alpha_i \wedge E^{\dot{\alpha}j}\sigma_{a\alpha\dot{\alpha}} - \frac{i}{8}\delta_i{}^j E^a \wedge E^\alpha_k \wedge E^{\dot{\alpha}k}\sigma_{a\alpha\dot{\alpha}} + \frac{1}{3!}E^c \wedge E^b \wedge E^a H_{abc\,i}{}^j. \tag{4.2}$$

Clearly, the leading term in the expression for $H_{3\,ijkl}$ should be $dB_{2\,ijkl}$. But the question to be answered is whether other terms are also present, and the answer is affirmative. Indeed, if we assume $H_{3\,ijkl} = dB_{2\,ijkl}$ [or, keeping the $SU(8)$ invariance, $H_{3\,ijkl} = DB_{2\,ijkl}$], the generalized field strength should obey the simplest Bianchi identities $dH_{3\,ijkl} = 0$ (or $DH_{3\,ijkl} = 4R_{[i|}{}^p \wedge H_{3|jkl]p}$), and the constraints (4.1) are not consistent if consistency is expressed by such a simple Bianchi identity.

Similarly one can check that no consistent FDA can be formulated without introducing also the $\mathfrak{su}(8)$ valued field strength $H_{3i}{}^j$. It might also look tempting to omit the tracelessness condition $H_{3i}{}^i = 0$ and thus to consider the $\mathfrak{u}(8)$ rather than $\mathfrak{su}(8)$ valued 3-form field strength, obeying simpler constraints given by (4.2) without the second term in the r.h.s. However, as we have checked, this is also inconsistent with the superspace constraints of $\mathcal{N}=8$ supergravity. Thus the structure of the tensor hierarchy of $\mathcal{N}=8$ supergravity is quite rigid.

To make a long story short, we have found that the constraints (4.1) and (4.2) are consistent with the FDA relations (generalized Bianchi identities)

$$I_{4i}{}^j \coloneqq DH_{3i}{}^j + 2F_{ik} \wedge \bar{F}^{jk} - \frac{1}{4}\delta_i{}^j F_{kl} \wedge \bar{F}^{kl} + \frac{1}{3}H_{3ikpq} \wedge \bar{\mathbb{P}}^{jkpq} + \frac{1}{3}\bar{H}_3^{jkpq} \wedge \mathbb{P}_{ikpq} = 0 \tag{4.3}$$

and

$$I_{4\,ijkl} \coloneqq DH_{3\,ijkl} - 4H_{3[i}{}^{j'} \wedge \mathbb{P}_{jkl]j'} - 3F_{[ij} \wedge F_{kl]} + \frac{3e^{i\beta}}{4!}\varepsilon_{ijkli'j'k'l'}\bar{F}^{i'j'} \wedge \bar{F}^{k'l'} = 0. \tag{4.4}$$

Let us stress that:

(1) As long as $\tilde{H}_{3p}{}^{[i} \wedge \bar{\mathbb{P}}^{jkl]p} = -\frac{e^{-i\beta}}{4!}\varepsilon^{ijkli'j'k'l'}\tilde{H}_{3i'}{}^p \wedge \mathbb{P}_{j'k'l'p}$, the identity (4.4) and the complex conjugate identity for $\bar{H}_3{}^{ijkl} = (H_{3\,ijkl})^*$ are consistent with the duality relation [cf. (2.12); notice the sign]

$$\bar{H}_3^{ijkl} = -\frac{e^{-i\beta}}{4!}\varepsilon^{ijkli'j'k'l'}H_{3\,i'j'k'l'}. \tag{4.5}$$

(2) When this property is taken into account, the traces of last two terms in the r.h.s. of (4.3) cancel one another.

(3) The terms quadratic in 2-form field strengths are those that occur in the $E_{7(+7)}$ Noether-Gaillard-Zumino current [27]. This current, whose components are all conserved, even for the $E_{7(+7)}$ transformations which are not symmetries of the action, may play an important role in the UV finiteness of the theory [6].

To check the consistency of our ansatz for the gBIs (4.3) and (4.4) one has to study the "identities for identities" $I_5^G = DI_4^G = 0$,

$$I_{5\,i}{}^j \coloneqq DI_{4\,i}{}^j = 0, \qquad I_{5ijkl} \coloneqq DI_{4ijkl} = 0, \tag{4.6}$$

taking into account the Ricci identities. In application to our 3-form the latter reads

$$DDH_{3\,i}{}^{j} = R_i{}^p \wedge H_{3\,p}{}^{j} - H_{3\,i}{}^{p} \wedge R_p{}^j,$$
$$DDH_{3ijkl} = 4R_{[i|}{}^p \wedge H_{3p|jkl]}, \tag{4.7}$$

and can be further specified substituting the explicit expression (2.11) for the curvature of induced $SU(8)$ connection. In such a way, after some algebra, one can prove that the proposed gBIs (4.3) and (4.4) are consistent provided the following identity holds:

$$\mathbb{P}_{[3][i|} \wedge \bar{\mathbb{P}}^{[3]q} \wedge H_{3|jkl]q} - \mathbb{P}_{p[ijk|} \wedge \bar{\mathbb{P}}^{p[3]} \wedge H_{3|l][3]} - \mathbb{P}_{p[ijk} \wedge \mathbb{P}_{l][3]} \wedge \bar{H}_3{}^{p[3]} = 0. \tag{4.8}$$

This equation is proven in Appendix B using only the complex self-duality and anti-self-duality of $\mathbb{P}_{ijkl}$ and $H_{3\,ijkl}$, respectively.

C. Superfield duality equations

Substituting Eqs. (4.2) and (4.1) and using the superspace supergravity constraints, we have checked that the dim 2 and 5/2 components of the gBIs (4.4) and (4.3) are satisfied. As far as dim 3 components are concerned, the $\propto E^b \wedge E^a \wedge E^\alpha_p \wedge E^\beta_q$ component of Eq. (4.4) is satisfied identically [due to the basic constraints and properties of main superfields, like (2.19) with (3.5)], while its $\propto E^b \wedge E^a \wedge E^\alpha_p \wedge \bar{E}^{\dot\beta q}$ component shows that $H_{abcijkl}$ is dual to the generalized Cartan form $\mathbb{P}^d_{ijkl}$,

$$H_{abc\,ijkl} = \frac{i}{2}\epsilon_{abcd}\mathbb{P}^d_{ijkl}. \tag{4.9}$$

The $\propto E^b \wedge E^a \wedge E^\alpha_p \wedge \bar{E}^{\dot\beta q}$ component of (4.4) shows that $H_{abci}{}^j$ is dual to a bilinear of fermionic superfields,

$$H_{abci}{}^j \propto \epsilon_{abcd}\left(\chi_{i[2]}\sigma^d\bar\chi^{j[2]} - \frac{1}{8}\delta_i^j\chi_{[3]}\sigma^d\bar\chi^{[3]}\right). \tag{4.10}$$

This reflects the auxiliary character of the $\mathfrak{su}(8)$ (pseudo-) notophs.

D. Identities for identities and the proof of the consistency of the constraints

Instead of studying the higher-dimensional components of the gBIs, we simplify our study by proving that they are dependent and cannot produce independent consequences; this implies that our constraints are consistent and all the dynamical equations are contained as higher components in the superfield duality equations (4.9) and (4.10).

To this end we solve the identities for identities (4.6), $0 = I_5^G = (I_{5\,ijkl}, I_{5i}{}^j) = DI_4^G$, with respect to the (l.h.s. of the) gBIs, I^G_{ABCD}, in the same manner as we solve Bianchi identities for the torsion and curvature tensors (and also gBIs for the 3-forms above), expressing them in terms of the main superfields (see Ref. [28]).

As we have already said, the lower dimensional, dim 2 and 5/2, components of the 4-form gBIs are satisfied algebraically, without any involvement of superfields. Setting these to zero, $I^G_{\underline{\alpha}\underline{\beta}\underline{\gamma}A} = 0$, we obtain a counterpart of the torsion constraints of supergravity. Substituting

$$I_4^G = \frac{1}{4}E^b \wedge E^a \wedge E^{\underline{\alpha}} \wedge E^{\underline{\beta}} I^G_{\underline{\beta}\underline{\alpha}ab} + \frac{1}{3!}E^c \wedge E^b \wedge E^a \wedge E^{\underline{\alpha}} I^G_{\underline{\alpha}abc} + \frac{1}{4!}E^d \wedge E^c \wedge E^b \wedge E^a I^G_{abcd} \tag{4.11}$$

into Eq. (4.6) and using the torsion constraints of $\mathcal{N} = 8$, $D = 4$ supergravity, Eqs. (2.14), (2.15) and (2.16), we find

$$0 = I_5^G = -\frac{i}{2}E^b \wedge E^\alpha_p \wedge E^{\dot\alpha q} \wedge E^{\underline{\beta}} \wedge E^{\underline{\gamma}} \delta_q^p \sigma^a_{\alpha\dot\alpha} I^G_{\underline{\beta}\underline{\gamma}\,ab} + \propto E^b \wedge E^a. \tag{4.12}$$

Thus, the lowest-dimensional (dim 3) nontrivial components of the identities for identities imply the following algebraic equations for the l.h.s. of the dim 3 gBIs:

$$0 = \delta_q^p \sigma^a_{\alpha\dot\alpha} I^{Gk}{}_{\beta\dot\gamma lab} + \delta_q^k \sigma^a_{\beta\dot\alpha} I^{Gp}{}_{\alpha\dot\gamma lab} + (\dot\alpha q \mapsto \dot\gamma l), \tag{4.13}$$

$$0 = \delta_q^p \sigma^a_{\alpha\dot\alpha} I^{Gk\,l}{}_{\beta\gamma\,ab} + \delta_q^k \sigma^a_{\beta\dot\alpha} I^{Gl\,p}{}_{\gamma\alpha\,ab} + \delta_q^l \sigma^a_{\gamma\dot\alpha} I^{Gp\,k}{}_{\alpha\beta\,ab}, \tag{4.14}$$

plus the complex conjugate of Eq. (4.14). It is not difficult to find that the latter as well as Eq. (4.14) have only trivial solutions $I^{Gk\,l}{}_{\beta\gamma\,ab} = 0$. In contrast, the general solution of Eq. (4.13) reads $I^{Gi}{}_{\alpha\dot\alpha jbc} = \delta^i_j \sigma^a_{\alpha\dot\alpha}\tilde{I}^G_{abc}$ with an arbitrary antisymmetric $\tilde{I}^G_{abc} = \tilde{I}^G_{[abc]}$. This implies that the only independent consequences for the superfields can be obtained from $I^{Gj}{}_{\alpha\dot\alpha j[bc}\tilde\sigma_{a]}^{\dot\alpha\alpha} = 0$.

This is exactly what we have observed in the explicit calculations of the dimension 3 Bianchi identities for $H_{3\,ijkl}$ (see Sec. IV C). Namely, we have found that

$$0 = (I_{4\,ijkl})^p_{\alpha\dot\alpha q\,ab} \equiv -i\delta_q^p \sigma^c_{\alpha\dot\alpha}\left(H_{abc\,ijkl} - \frac{i}{2}\epsilon_{abcd}\mathbb{P}^d_{ijkl}\right), \tag{4.15}$$

which implies the superfield duality equation (4.9).

The above general statement allows one to escape the exhausting algebraic calculations necessary to check explicitly the cancellation of different terms in the equation $I^G{}_{\underline{\alpha}\underline{\beta}\,ab} = 0$.

Furthermore, the higher-dimensional components of identities for identities Eq. (4.12) show the dependence of higher-dimensional Bianchi identities $I^G_{\underline{\alpha}abc} = 0$ and $I^G_{abcd} = 0$. This implies that their results can be obtained by applying covariant derivatives to the results of the dimension 3 gBIs, this is to say to the superfield duality equations (4.9) and (4.10), with the use of the superspace constraints for torsion, Cartan forms and 2-form field strength of the 1-form gauge fields and of their consequences. The latter include the equations of motion of $\mathcal{N}=8$, $D=4$ CJ supergravity.

E. Scalar (super)field equation of motion and duality equation

To illustrate this statement let us consider the dimension 4 Bianchi identity corresponding to $E_{7(+7)}/SU(8)$ generators,

$$\begin{aligned}0 = I_{ijkl\,abcd} &= 4D_{[a}H_{bcd]ijkl} + 16H_{[abc|[i}{}^{p}\mathbb{P}_{jkl]p|d]}\\ &- 18F_{[ij|[ab}F_{cd]|kl]} + \frac{3e^{i\beta}}{4}\varepsilon_{ijkli'j'k'l'}\bar{F}^{i'j'}_{[ab}\bar{F}'^{k'l'}_{cd]}\\ &+ 6T_{[ab|}{}^{\alpha}_{[i|}(\sigma_{|cd]}\chi_{|jkl]})_{\alpha}\\ &- \frac{e^{i\beta}}{4}\varepsilon_{ijkli'j'k'l'}T_{[ab|}{}^{\dot{\alpha}i'}(\bar{\chi}^{j'k'l'}\tilde{\sigma}_{|cd]})_{\dot{\alpha}}. \end{aligned} \tag{4.16}$$

Using (4.9) we can equivalently write this as

$$\begin{aligned}D^{a}\mathbb{P}_{a\,ijkl} &= -\frac{4i}{3}\varepsilon^{abcd}H_{abc[i}{}^{p}\mathbb{P}_{jkl]pd} - \frac{3i}{2}\varepsilon_{abcd}F^{ab}_{[ij}F^{cd}_{kl]}\\ &+ \frac{ie^{i\beta}}{16}\varepsilon_{ijkli'j'k'l'}\varepsilon^{abcd}\bar{F}^{i'j}_{ab}\bar{F}'^{k'l'}_{cd} + T_{ab}{}^{\alpha}_{[i}(\sigma^{ab}\chi_{jkl]})_{\alpha}\\ &+ \frac{e^{i\beta}}{4!}\varepsilon_{ijkli'j'k'l'}T_{ab}{}^{\dot{\alpha}i'}(\bar{\chi}^{j'k'l'}\tilde{\sigma}^{ab})_{\dot{\alpha}}. \end{aligned} \tag{4.17}$$

After using Eq. (4.10), this expression acquires the usual form of the scalar (super)field equation of $\mathcal{N}=8$, $D=4$ supergravity,

$$\begin{aligned}D^{a}\mathbb{P}_{a\,ijkl} &= -\frac{3i}{2}\varepsilon_{abcd}F^{ab}_{[ij}F^{cd}_{kl]}\\ &+ \frac{ie^{i\beta}}{16}\varepsilon_{ijkli'j'k'l'}\varepsilon^{abcd}\bar{F}^{i'j'}_{ab}\bar{F}'^{k'l'}_{cd} + \cdots, \end{aligned} \tag{4.18}$$

where the dots stand for the terms bilinear in fermions.

To reflect the dependence of the higher-dimensional Bianchi identities proved in the previous section (Sec. IV D), the above line should be read in the opposite direction: the results of the dimension 4 Bianchi identity Eq. (4.16) can be obtained by taking the bosonic covariant derivative of the duality equation (4.9) and using the scalar (super)field equation (as obtained from the torsion constraints of [25,26]) and Eq. (4.10).

Thus, the results of Sec. IV C and the arguments of Sec. IV D allow us to conclude that our constraints for the 3-form field strength are consistent and describe a set of notophs dual to the scalar fields of $\mathcal{N}=8$, $D=4$ supergravity.

V. CONCLUSION AND OUTLOOK

In this paper we have provided the complete supersymmetric description of the notophs (2-form gauge potentials) of the Cremmer-Julia $\mathcal{N}=8$, $D=4$ supergravity [1]. More specifically, we have presented the set of superspace constraints for the 3-form field strengths of the 2-form gauge potentials defined on $\mathcal{N}=8$, $D=4$ supergravity superspace [25] and we have shown that these are consistent and produce the duality relation between the field strengths of the physical notophs and the scalar fields of the $\mathcal{N}=8$, $D=4$ CJ supergravity parametrizing the $G/H=E_{7(+7)}/SU(8)$ coset. We have found that the consistency, expressed by the generalized Bianchi identities, requires us to introduce also the auxiliary 2-form potentials corresponding to the generators of the stability subgroup $H=SU(8)$ of the coset. In the companion paper [8] we will discuss the reasons for this in detail. Here we have adopted a purely superspace approach and arrived at this conclusion starting from the natural candidate for the superspace constraints and searching for their consistency. The generalized Bianchi identities for the 3-form field strengths of the notophs, which define the tensorial hierarchy (or free differential algebra) of the $\mathcal{N}=8$, $D=4$ CJ supergravity, have been also obtained in this manner.

The list of natural directions of development of our approach includes the studies of the superfield description of the notophs of gauged $\mathcal{N}=8$, $D=4$ supergravity [15,29,30] using the torsion constraints of [31] and of the supersymmetric aspects of the generalized notophs of the exceptional field theories [32–34] in $\mathcal{N}=8$, $D=4$ superspace enlarged by 56 bosonic "central charge" coordinates (see [35]). Another obvious extension of this work is the search for worldvolume actions of possible superstring models carrying the "electric" charges with respect to the antisymmetric tensor gauge fields. Probably the correct posing of this problem may require us to work in the Howe-Linmdstöm enlarged $\mathcal{N}=8$, $D=4$ superspace.

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Note added.—Recently, the superspace description of higher-form gauge fields in D-dimensional maximal and half-maximal supergravities has been discussed in [36], where the cases of D = 11 and D = 10 are elaborated explicitly. For cases where $3 \leq D < 10$ the representations carried by higher forms in maximal and half-maximal superspaces and their generalized Bianchi identities have been tabulated in Appendix A of [36]. Higher forms in maximal and half-maximal D = 3 dimensional superspaces were studied in [37].

APPENDIX A: 4D WEYL SPINORS AND SIGMA MATRICES

We use the relativistic Pauli matrices $\sigma^a_{\beta\dot{\alpha}} = \epsilon_{\beta\alpha}\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\sigma}^{a\dot{\beta}\alpha}$ which obey

$$\sigma^a\tilde{\sigma}^b = \eta^{ab} + \frac{i}{2}\epsilon^{abcd}\sigma_c\tilde{\sigma}_d, \qquad \tilde{\sigma}^a\sigma^b = \eta^{ab} - \frac{i}{2}\epsilon^{abcd}\tilde{\sigma}_c\sigma_d, \tag{A1}$$

where $\eta^{ab} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric and $\epsilon^{abcd} = \epsilon^{[abcd]}$ is the antisymmetric tensor with $\epsilon^{0123} = 1 = -\epsilon_{0123}$.

The spinorial $[SL(2,\mathbb{C})]$ indices are raised and lowered by $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = i\tau_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right) = -\epsilon_{\alpha\beta}$ obeying $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$: $\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta$ and $\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta$. The antisymmetrized products $\sigma^{ab}{}_\beta{}^\alpha = \sigma^{[a}\tilde{\sigma}^{b]} := \frac{1}{2}(\sigma^a\tilde{\sigma}^b - \sigma^b\tilde{\sigma}^a)$ and $\tilde{\sigma}^{ab\dot{\alpha}}{}_{\dot{\beta}} = \tilde{\sigma}^{[a}\sigma^{b]}$ are self-dual and anti-self-dual, $\sigma^{ab} = \frac{i}{2}\epsilon^{abcd}\sigma_{cd}$, $\tilde{\sigma}^{ab} = -\frac{i}{2}\epsilon^{abcd}\tilde{\sigma}_{cd}$.

APPENDIX B: MORE ON DIFFERENTIAL FORMS IN CURVED $\mathcal{N} = 8$, $D = 4$ SUPERSPACE

1. Exterior derivative

The exterior derivative d acts on a q-form

$$\Omega_q = \frac{1}{q!}dZ^{M_q}\wedge \ldots \wedge dZ^{M_1}\Omega_{M_1\ldots M_q}(Z) = \frac{1}{q!}E^{A_q}\wedge \ldots \wedge E^{A_1}\Omega_{A_1\ldots A_q}(Z)$$

as

$$d\Omega_q = \frac{1}{q!}dZ^{M_q}\wedge\ldots\wedge dZ^{M_1}\wedge d\Omega_{M_1\ldots M_q}(Z) = \frac{1}{(q+1)!}dZ^{M_{q+1}}\wedge\ldots\wedge dZ^{M_2} \wedge dZ^{M_1}(q+1)\partial_{[M_1}\Omega_{M_2\ldots M_{q+1}\}}(Z). \tag{B1}$$

In action on the product of differential forms, e.g. the q-form Ω_q and the p-form Ω_p, it obeys the Leibniz rule

$$d(\Omega_q\wedge\Omega_p) = \Omega_q\wedge d\Omega_p + (-)^p d\Omega_q\wedge\Omega_p. \tag{B2}$$

The mixed brackets $[\cdots\}$ denote the graded antisymmetrization of the enclosed indices with the weight unity, so that $(q+1)!\partial_{[M_1}\Omega_{M_2\ldots M_{q+1}\}}(Z) = \partial_{M_1}\Omega_{M_2\ldots M_{q+1}}(Z) - (-)^{\varepsilon(M_1)\varepsilon(M_2)}\partial_{M_2}\Omega_{M_1M_3\ldots M_{q+1}}(Z) + \cdots$, where $\varepsilon(M) := \varepsilon(Z^M)$ is the Grassmann parity (fermionic number), $\varepsilon(\mu) := \varepsilon(x^\mu) = 0$, $\varepsilon(\underline{\alpha}) = \varepsilon(\theta^{\underline{\alpha}}) = 1$.

2. On $E_{7(+7)}$ Cartan forms

Using the complex self-duality of $\mathbb{P}_{ijkl}$ Eq. (2.12) and the antisymmetry of the exterior product of $\mathbb{P}$ one finds

$$\mathbb{P}_{[4]}\wedge\bar{\mathbb{P}}^{[4]} = 0. \tag{B3}$$

Then, using Eq. (2.12) and this last property Eq. (B3) in $\mathbb{P}_{ij[2]}\wedge\bar{\mathbb{P}}^{kl[2]}$ one finds

$$\mathbb{P}_{ijpq}\wedge\bar{\mathbb{P}}^{klpq} = \frac{2}{3}\delta_{[i}{}^{[k}\mathbb{P}_{j][3]}\wedge\bar{\mathbb{P}}^{l][3]}. \tag{B4}$$

The Ricci identity

$$DD\mathbb{P}_{ijkl} = -4R_{[i}{}^p\wedge\mathbb{P}_{jkl]p} = -\frac{4}{3}\mathbb{P}_{[3][i}\wedge\bar{\mathbb{P}}^{[3]p}\wedge\mathbb{P}_{jkl]p} = 0 \tag{B5}$$

is satisfied because, by virtue of Eq. (B4), the r.h.s. is equivalent to

$$\mathbb{P}_{pq[ij}\wedge\bar{\mathbb{P}}^{pqrs}\wedge\mathbb{P}_{kl]rs}, \tag{B6}$$

which vanishes automatically on account of the antisymmetry of the wedge product and the symmetry under the interchange of pairs of the $SU(8)$ indices.

From Eq. (B4) it follows that the first term in Eq. (4.8) can be reexpressed as

$$\mathbb{P}_{[3][i|}\wedge\bar{\mathbb{P}}^{[3]q}\wedge H_{3|jkl]q} = -\frac{3}{2}\mathbb{P}_{[2][ij|}\wedge\bar{\mathbb{P}}^{[2][2']}\wedge H_{3|kl][2']}. \tag{B7}$$

Using again the complex self-duality of $\mathbb{P}_{ijkl}$ and the complex anti-self-duality of $H_{3\,ijkl}$, the third term in Eq. (4.8) can be reexpressed as

$$\mathbb{P}_{p[ijk}\wedge\mathbb{P}_{l][3]}\wedge\bar{H}_3{}^{p[3]} = -\frac{1}{8}\mathbb{P}_{ijkl}\wedge\mathbb{P}_{[4]}\wedge H_3{}^{[4]} - \frac{3}{4}\mathbb{P}_{[2][ij|}\wedge\bar{\mathbb{P}}^{[2][2']}\wedge H_{3|kl][2']}. \tag{B8}$$

The same properties and this last identity allow us to rewrite the second term in Eq. (4.8) as

$$\mathbb{P}_{p[ijk|}\wedge\bar{\mathbb{P}}^{p[3]}\wedge H_{3|l][3]} = \frac{1}{8}\mathbb{P}_{ijkl}\wedge\mathbb{P}_{[4]}\wedge H_3{}^{[4]} - \frac{3}{4}\mathbb{P}_{[2][ij|}\wedge\bar{\mathbb{P}}^{[2][2']}\wedge H_{3|kl][2']}. \tag{B9}$$

After rewriting the three terms of Eq. (4.8) using the above identities, we find that Eq. (4.8) is identically satisfied.

3. On $E_{7(+7)}$ Cartan forms in $\mathcal{N}=8$ supergravity superspace

Equations (2.20) can be derived also from (2.13) with (2.21). To this end it is useful to notice the trivial identity

$$\bar{\chi}^{\dot{\alpha}pq[i}\bar{\chi}_{\dot{\alpha}}^{jkl]} = \frac{5}{2}\bar{\chi}^{\dot{\alpha}p[qi}\bar{\chi}_{\dot{\alpha}}^{jkl]} - \frac{3}{2}\bar{\chi}^{\dot{\alpha}p[ij}\bar{\chi}_{\dot{\alpha}}^{kl]q}. \quad \text{(B10)}$$

Its l.h.s. is antisymmetric, while the second term in its r.h.s is symmetric. Hence

$$\bar{\chi}^{\dot{\alpha}p[ij}\bar{\chi}_{\dot{\alpha}}^{kl]q} = \frac{5}{6}(\bar{\chi}^{\dot{\alpha}p[qi}\bar{\chi}_{\dot{\alpha}}^{jkl]} + \bar{\chi}^{\dot{\alpha}q[pi}\bar{\chi}_{\dot{\alpha}}^{jkl]}), \quad \text{(B11)}$$

and

$$\bar{\chi}^{\dot{\alpha}pq[i}\bar{\chi}_{\dot{\alpha}}^{jkl]} = \frac{5}{4}(\bar{\chi}^{\dot{\alpha}p[qi}\bar{\chi}_{\dot{\alpha}}^{jkl]} - \bar{\chi}^{\dot{\alpha}q[pi}\bar{\chi}_{\dot{\alpha}}^{jkl]}). \quad \text{(B12)}$$

As a consequence

$$\varepsilon^{ijkli'j'k'l'}\bar{\chi}^{\dot{\alpha}}_{pq[i'}\chi_{\dot{\alpha}[j'k'l']} = -2\delta^{[p}_{[i}\varepsilon^{jkl[2][3]}\bar{\chi}^{\dot{\alpha}}_{q][2]}\chi_{\dot{\alpha}[3]}. \quad \text{(B13)}$$

4. Curvature 2-forms of $\mathcal{N}=8$ superspace

The study of the Bianchi identities results in the following expressions for the curvature of the spin connection (the "Riemann" curvature 2-form) (see [25,26]):

$$\sigma^a_{\alpha\dot{\alpha}}\tilde{\sigma}^{\dot{\beta}\beta}_b R_a{}^b = 2\delta_\alpha{}^\beta R_{\dot{\alpha}}{}^{\dot{\beta}} + 2\delta_{\dot{\alpha}}{}^{\dot{\beta}} R_\alpha{}^\beta, \quad \text{(B14)}$$

$$\begin{aligned} R^{\alpha\beta} &= \frac{1}{4}R^{ab}\sigma^{\alpha\beta}_{ab} = \frac{1}{2}E^\gamma_i \wedge E^\delta_j(\epsilon_{\gamma\delta}N^{\alpha\beta ij} + 2\delta_{(\gamma}{}^\alpha\delta_{\delta)}{}^\beta S^{ij}) \\ &\quad + \frac{1}{2}\bar{E}^{\dot{\beta}i}\wedge\bar{E}^{\dot{\gamma}j}\epsilon_{\dot{\gamma}\dot{\delta}}M^{\alpha\beta}_{ij} + E^\gamma_i \wedge \bar{E}^{\dot{\gamma}j}R^i_{\gamma\dot{\gamma}j}{}^{\alpha\beta} \\ &\quad + E^c \wedge E^{\underline{\beta}}R_{\underline{\beta}c}{}^{\alpha\beta} + \frac{1}{2}E^c\wedge E^b R_{bc}{}^{\alpha\beta}, \end{aligned} \quad \text{(B15)}$$

$$\begin{aligned} R^{\dot{\alpha}\dot{\beta}} &= -\frac{1}{4}R^{ab}\tilde{\sigma}^{\dot{\alpha}\dot{\beta}}_{ab} \\ &= -\frac{1}{2}E^\gamma_i \wedge E^\delta_j \epsilon_{\gamma\delta}\bar{M}^{\dot{\alpha}\dot{\beta}\,ij} \\ &\quad -\frac{1}{2}\bar{E}^{\dot{\beta}i}\wedge\bar{E}^{\dot{\gamma}j}(\epsilon_{\gamma\delta}\bar{N}^{\dot{\gamma}\dot{\delta}}_{ij} + 2\delta_{(\dot{\gamma}}{}^{\dot{\alpha}}\delta_{\dot{\delta})}{}^{\dot{\beta}}\bar{S}_{ij}) \\ &\quad + E^\gamma_i\wedge\bar{E}^{\dot{\gamma}j}R^i_{\gamma\dot{\gamma}j}{}^{\dot{\alpha}\dot{\beta}} + E^c\wedge E^{\underline{\beta}}R_{\underline{\beta}c}{}^{\dot{\alpha}\dot{\beta}} + \frac{1}{2}E^c\wedge E^b R_{bc}{}^{\dot{\alpha}\dot{\beta}}. \end{aligned} \quad \text{(B16)}$$

Equations (3.4) and (3.5) can be combined as

$$\begin{aligned} F_{ab\,ij} &= \frac{1}{2}\sigma^{\alpha\beta}_{ab}F_{\alpha\beta\,ij} + \frac{1}{2}\tilde{\sigma}^{\dot{\alpha}\dot{\beta}}_{ab}F_{\dot{\alpha}\dot{\beta}\,ij} \\ &= \frac{i}{4}\sigma^{\alpha\beta}_{ab}M_{\alpha\beta\,ij} + \frac{ie^{i\beta}}{12\cdot 4!}\varepsilon_{ij[3][3']}\bar{\chi}^{[3]}\tilde{\sigma}_{ab}\bar{\chi}^{[3']}. \end{aligned} \quad \text{(B17)}$$

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