

Third order equations of motion and the Ostrogradsky instability

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It is known that any nondegenerate Lagrangian containing time derivative terms higher than first order suffers from the Ostrogradsky instability, pathological excitation of positive and negative energy degrees of freedom. We show that, within the framework of analytical mechanics of point particles, any Lagrangian for third order equations of motion, which evades the nondegeneracy condition, still leads to the Ostrogradsky instability. Extension to the case of higher odd order equations of motion is also considered.

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I. INTRODUCTION

Nature prefers to describe the laws of physics by the second-order differential equations. Newton's equation of motion (EOM), Maxwell equations and Einstein equations, etc. are all the second-order differential equations for positions of particles or fields. Why the laws of physics are like this is a fundamental question of physics.

It is known that any nondegenerate Lagrangian containing time derivative terms higher than the first order yields a Hamiltonian which is not bounded from below [1]. This means that the energy of the system can take an arbitrarily negative value. While such a system for free theory is not pathological, when it is coupled to a normal system with positive energy, the total system will quickly develop into excitation of positive and negative energy degrees of freedom.¹ This instability, recently called the Ostrogradsky instability,² is quite general and discussed in detail in [1]. Here we briefly explain the main point of the Ostrogradsky instability by following [1]. In order to have the essence, we focus on a Lagrangian of N variables given by $L(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$, where $\mathbf{x} = (x_1, x_2, \dots, x_N)$. The Euler-Lagrange equations are given by

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{\mathbf{x}}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}} + \frac{\partial L}{\partial \mathbf{x}} = 0. \quad (1)$$

¹For a possible loophole, see [2].

²Although it has been commonly referred to as ‘‘Ostrogradsky instability’’ recently and we will follow the convention in the present paper, it is not clear if Ostrogradsky himself made this statement in his original paper [3]. Furthermore, we are also not sure if the unboundedness of the Hamiltonian for higher derivative Lagrangians, although it is referred to as ‘‘theorem of Ostrogradsky’’ in [1], was shown by Ostrogradsky in Ref. [3]. As far as we know, Ref. [4] by Pais and Uhlenbeck is one of the oldest work in which unboundedness of the Hamiltonian for a special type of higher derivative Lagrangian was shown explicitly.

We require nondegeneracy, i.e., $\det(\partial^2 L / \partial \ddot{x}_i \partial \ddot{x}_j) \neq 0$. Then the equations of motion are the fourth-order differential equations and the number of the degrees of freedom as the number of independent initial conditions is $4N$.³ In order to move to the canonical formalism, we need to define $4N$ canonical coordinates. They are given by [3,5],

$$\mathbf{Q}_1 = \mathbf{x}, \quad \mathbf{Q}_2 = \dot{\mathbf{x}}, \quad \mathbf{P}_1 = \frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{\mathbf{x}}}, \quad \mathbf{P}_2 = \frac{\partial L}{\partial \ddot{\mathbf{x}}}. \quad (2)$$

With the assumption of nondegeneracy, we can solve the above equations for $\{\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \mathbf{x}^{(3)}\}$. Then the Hamiltonian is defined in the standard manner,

$$H = \mathbf{P}_1 \cdot \dot{\mathbf{Q}}_1 + \mathbf{P}_2 \cdot \dot{\mathbf{Q}}_2 - L. \quad (3)$$

We can confirm that the Hamilton's equations are equivalent to the Euler-Lagrange equation. Thus the Hamiltonian represents the generator of the time evolution and can be interpreted as the energy of the system. It is immediate to understand that $\ddot{\mathbf{x}}$ is written in terms of $\{\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{P}_2\}$. This means that \mathbf{P}_1 appears only in the first term and the Hamiltonian depends on \mathbf{P}_1 linearly. Since there is no constraint among each element of \mathbf{P}_1 , they can independently take any value. Thus, the Hamiltonian can take an arbitrarily positive value or negative value, leading to the Ostrogradsky instability.

As the above argument shows, the Ostrogradsky instability is a quite generic feature of higher derivative theories and answers why nature is described by the second-order EOMs. Due to its simplicity and generality, the

³There are literatures where the number of the degrees of freedom denotes the half of the number of the initial conditions, but this notation assumes the second-order EOM. Here, we denote that the number of the degrees of freedom as the number of the initial conditions.

Ostrogradsky instability has also played a powerful role in constructing a sensible scalar-tensor theory (for instance, [6–9]), mainly for the purpose of modifying gravity at infrared to explain the accelerated expansion of the Universe.

Now, it is intriguing to ask if the Ostrogradsky instability still persists when the EOMs (1) are third-order differential equations. Since the nondegenerate Lagrangian always yields the fourth-order EOMs, the third-order EOMs must be obtained from the degenerate Lagrangian. More generally, any nondegenerate Lagrangian yields the even order equations while the odd order EOMs are obtained from a degenerate Lagrangian. Clearly, the existence of the Ostrogradsky instability in the latter case is not covered by the argument provided above. As far as we know, it has not been discussed in the literature whether the third-order EOMs (and higher odd order EOMs) are inevitably associated with the Ostrogradsky instability or not. The aim of this paper is to answer this question.

Finally, it is important to make a distinction between the degeneracy of Lagrangian and the third-order EOMs. Third-order EOMs are always derived from a degenerate Lagrangian but the opposite statement is not necessarily true (see also [10] for relevant discussion). For instance, although a Lagrangian of a single variable ($N = 1$) given by $L = \ddot{x}f(\dot{x}, x)$ is degenerate, its EOM is the second-order differential equation and the Ostrogradsky instability does not appear. Thus, a statement that any degenerate Lagrangian suffers from the Ostrogradsky instability is incorrect. But a statement that any degenerate Lagrangian giving third-order EOMs suffers from the Ostrogradsky instability is correct, as we shall show below.

This paper is organized as follows. In Sec. II, we construct the most general Lagrangian for the third-order EOMs in analytical mechanics of point particles. In Sec. III, we show that the Ostrogradsky instability always appears for the theory by performing the Hamiltonian analysis. Extension to the case of higher odd order EOMs is also considered in the end of Sec. III, and Sec. IV is devoted to conclusion. Extension to the field theory is beyond the scope of this paper, although it has interesting applications, for instance, like the Chern-Simons gravity [11] for which the gravitational field equations are third-order differential equations.

II. LAGRANGIAN FOR THIRD-ORDER EOMS

Let us try to construct the most general Lagrangian that yields the third-order EOMs. We will address general odd order EOMs after completing an analysis for third-order EOMs. We again consider a Lagrangian which depends on N variables $\mathbf{x} = (x_1, \dots, x_N)$, and contains up to n th time derivative:

$$L = L(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}). \quad (4)$$

At this moment, $n \geq 2$ is an arbitrary positive integer.

We determine n as well as the form of the Lagrangian by requiring that L yields the third-order EOMs for \mathbf{x} . Since our primary concern is the Ostrogradsky instability of variables having the third-order EOMs, we require that all the variables obey the third-order EOMs independently. We do not consider their couplings to variables obeying at most the second-order EOMs. Such coupling does not affect the existence of the Ostrogradsky instability. The Euler-Lagrange equation for (4) is given by

$$\frac{d^n}{dt^n} \frac{\partial L}{\partial \mathbf{x}^{(n)}} - \frac{d^{n-1}}{dt^{n-1}} \frac{\partial L}{\partial \mathbf{x}^{(n-1)}} + \frac{d^{n-2}}{dt^{n-2}} \frac{\partial L}{\partial \mathbf{x}^{(n-2)}} - \dots = 0. \quad (5)$$

For a generic Lagrangian, the highest order derivative is $\mathbf{x}^{(2n)}$, that comes only from the first term of Eq. (5). For EOMs (5) not to have $\mathbf{x}^{(2n)}$, we require that

$$\frac{\partial^2 L}{\partial x_i^{(n)} \partial x_j^{(n)}} = 0, \quad (6)$$

for any $i, j = 1, \dots, N$. Thus the Lagrangian should be written as

$$L = \sum_{j=1}^N x_j^{(n)} f_j(\mathbf{x}, \dots, \mathbf{x}^{(n-1)}) + g(\mathbf{x}, \dots, \mathbf{x}^{(n-1)}), \quad (7)$$

with arbitrary functions f_j and g . The Euler-Lagrange equation then reads

$$\sum_{j=1}^N x_j^{(2n-1)} \left(\frac{\partial f_i}{\partial x_j^{(n-1)}} - \frac{\partial f_j}{\partial x_i^{(n-1)}} \right) + (\text{lower deriv.}) = 0. \quad (8)$$

To obtain the third-order EOMs, we impose that the coefficient for $x_j^{(2n-1)}$ vanishes:

$$\frac{\partial f_i}{\partial x_j^{(n-1)}} = \frac{\partial f_j}{\partial x_i^{(n-1)}}. \quad (9)$$

By using the Green's theorem, there exists F that satisfies

$$f_i = \frac{\partial}{\partial x_i^{(n-1)}} F(\mathbf{x}, \dots, \mathbf{x}^{(n-1)}). \quad (10)$$

Therefore we can rewrite the Lagrangian as

$$L = - \sum_{j=1}^N \sum_{k=1}^{n-1} x_j^{(k)} \frac{\partial F}{\partial x_j^{(k-1)}} + g, \quad (11)$$

up to a difference of a total derivative dF/dt . This means that, without a loss of generality, we could have started from a Lagrangian containing at most $\mathbf{x}^{(n-1)}$ in Eq. (4). By repeating the same procedure, we can reduce the original Lagrangian (4) to the one that contains at most $\ddot{\mathbf{x}}$. From the

absence of fourth-order derivative in the EOMs, the Lagrangian should be written as

$$L = \sum_{j=1}^N \ddot{x}_j f_j(\mathbf{x}, \dot{\mathbf{x}}) + g(\mathbf{x}, \dot{\mathbf{x}}). \quad (12)$$

The EOM is then given by

$$\sum_{j=1}^N M_{ij}^{(\dot{x})} \ddot{x}_j + (\text{lower derivatives.}) = 0, \quad (13)$$

where we have defined

$$M_{ij}^{(X)} \equiv \frac{\partial f_i}{\partial X_j} - \frac{\partial f_j}{\partial X_i}. \quad (14)$$

Since $M^{(\dot{x})}$ is an $N \times N$ antisymmetric matrix, we can block-diagonalize it to arrive at

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 & \cdots \\ 0 & \Lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \text{with} \quad \Lambda_a = \begin{pmatrix} 0 & \lambda_a \\ -\lambda_a & 0 \end{pmatrix}. \quad (15)$$

If N is a odd number, $N = 2I - 1$ with $I \geq 1$, we have $\det \Lambda = 0$ due to the Jacobi's theorem. Thus, there exists a J such that, $1 \leq J \leq I - 1$ and $(2J - 1)$ rows and columns of Λ are occupied by zero. Hence, not all the variables obey the third-order differential equations independently. In other words, the degrees of freedom as the number of independent initial conditions is smaller than $3N$ and we can introduce a new set of $2(I - J)$ variables obeying the third-order EOMs independently and a new set of $2J$ variables obeying at most the second-order EOMs. As we stated before, couplings to variables obeying the second-order EOMs are not essential for the existence of the Ostrogradsky instability. Therefore, we only need to consider the even N case. For even N , if $M^{(\dot{x})}$ has vanishing λ_a , we can further reduce the number of variables and make all the λ_a are nonvanishing. Therefore, without loss of generality, we can set N to be even in (12) and all the λ_a are nonvanishing, i.e., $\det M^{(\dot{x})} \neq 0$.

In conclusion, the most general Lagrangian that yields independent third-order EOMs for all the variables is the Lagrangian (12) for even number variables:

$$L = \sum_{j=1}^{2N} \ddot{x}_j f_j(\mathbf{x}, \dot{\mathbf{x}}) + g(\mathbf{x}, \dot{\mathbf{x}}), \quad (16)$$

with $\det M_{ij}^{(\dot{x})} \neq 0$. Here, we have changed the notation of the number of variables from N to $2N$. Thus, N is still an arbitrary positive integer. With this setup, we need $6N$ initial conditions for $x_i, \dot{x}_i, \ddot{x}_i$ for $i = 1, \dots, 2N$ at given initial time $t = t_{\text{ini}}$ to determine the time evolution of the system.

III. HAMILTONIAN ANALYSIS

We proceed to the Hamiltonian analysis of the most general Lagrangian for the third-order EOMs (16). Among several different approaches for constructing the canonical formalism for higher derivative theories [12–17], we adopt the one used in [12,16], which reduces the higher derivative Lagrangian to the standard one with constraints by introducing the Lagrange multipliers. The advantage of this method is that we can use the standard Dirac's formalism for the canonical formalism with constraints. With the Lagrange multiplier λ_i , the Lagrangian (16) can be rewritten as

$$L = \sum_{j=1}^{2N} [\dot{y}_j f_j(\mathbf{x}, \mathbf{y}) + \lambda_j (\dot{x}_j - y_j)] + g(\mathbf{x}, \mathbf{y}). \quad (17)$$

Variation with respect to λ_i yields $\dot{x}_i = y_i$, with which (17) reproduces the original Lagrangian (16).

Now the Lagrangian (17) is written up to the first-order derivatives of x_i, y_i, λ_i . This implies that its EOMs are the second-order differential equations and we need $12N$ initial conditions in total. We will see that $6N$ initial conditions out of $12N$ initial conditions are fixed by second-class constraints and we are finally left with $6N$ initial conditions, which matches the number of initial conditions needed for the original Lagrangian (16).

The canonical momentum for x_i, y_i, λ_i is given by

$$p_{xi} \equiv \frac{\partial L}{\partial \dot{x}_i} = \lambda_i, \quad (18)$$

$$p_{yi} \equiv \frac{\partial L}{\partial \dot{y}_i} = f_i(\mathbf{x}, \mathbf{y}), \quad (19)$$

$$p_{\lambda i} \equiv \frac{\partial L}{\partial \dot{\lambda}_i} = 0. \quad (20)$$

Since all the momenta do not contain $\dot{x}, \dot{y}, \dot{\lambda}$, we cannot solve them in terms of momenta. This implies that we have $6N$ primary constraints:

$$\phi_{xi} \equiv p_{xi} - \lambda_i = 0, \quad (21)$$

$$\phi_{yi} \equiv p_{yi} - f_i(\mathbf{x}, \mathbf{y}) = 0, \quad (22)$$

$$\phi_{\lambda i} \equiv p_{\lambda i} = 0. \quad (23)$$

Following the Hamiltonian formalism for constrained systems [18–20], we incorporate these constraints by adding them into the Lagrangian with Lagrange multipliers. Thus we consider the variation of the action

$$S = \int_{t_1}^{t_2} dt \left[\sum_{j=1}^{2N} \sum_{q=x,y,\lambda} (p_{qj} \dot{q}_j - \mu_{qj} \phi_{qj}) - H \right], \quad (24)$$

where the Hamiltonian H is given by

$$\begin{aligned} H(\mathbf{p}, \mathbf{q}) &= \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} p_{qj} \dot{q}_j - L, \\ &= \sum_{j=1}^{2N} \lambda_j y_j - g(\mathbf{x}, \mathbf{y}), \end{aligned} \quad (25)$$

and $\mu_{xi}, \mu_{yi}, \mu_{\lambda i}$ are the Lagrange multipliers for the primary constraints. The variation of (24) yields the canonical equation

$$\dot{q}_i = \frac{\partial H}{\partial p_{qi}} + \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} \mu_{qj} \frac{\partial \phi_{qj}}{\partial p_{qi}}, \quad (26)$$

$$\dot{p}_{qi} = -\frac{\partial H}{\partial q_i} - \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} \mu_{qj} \frac{\partial \phi_{qj}}{\partial q_i}. \quad (27)$$

Therefore, the time evolution of any function $\xi(p, q)$ is governed by

$$\begin{aligned} \frac{d\xi}{dt} &= \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} \left(\frac{\partial \xi}{\partial q_j} \dot{q}_j + \frac{\partial \xi}{\partial p_{qj}} \dot{p}_{qj} \right), \\ &= \{\xi, H\} + \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} \mu_{qj} \{\xi, \phi_{qj}\}, \\ &\approx \{\xi, H_T\}, \end{aligned} \quad (28)$$

where the total Hamiltonian H_T is given by

$$H_T = H + \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} \mu_{qj} \phi_{qj}, \quad (29)$$

and $\{\}$ denotes the Poisson bracket

$$\{\xi, \eta\} = \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} \left(\frac{\partial \xi}{\partial q_j} \frac{\partial \eta}{\partial p_{qj}} - \frac{\partial \xi}{\partial p_{qj}} \frac{\partial \eta}{\partial q_j} \right). \quad (30)$$

The weak equality \approx expresses an equality that holds after performing the Poisson bracket and then imposing that all the constraints vanishes.

To make the constraints satisfied through the time evolution, we impose a consistency condition

$$\frac{d\phi_{qi}}{dt} = \{\phi_{qi}, H\} + \sum_{j=1}^{2N} \sum_{q=x,y,\lambda} \mu_{qj} \{\phi_{qi}, \phi_{qj}\} \approx 0. \quad (31)$$

For $q = x, y, \lambda$, this reads

$$\frac{d\phi_{xi}}{dt} = \sum_{j=1}^{2N} \mu_{yj} \frac{\partial f_j}{\partial x_i} - \mu_{\lambda i} + \frac{\partial g}{\partial x_i}, \quad (32)$$

$$\frac{d\phi_{yi}}{dt} = -\lambda_i - \sum_{j=1}^{2N} \left(\mu_{xj} \frac{\partial f_j}{\partial x_i} + \mu_{yj} M_{ij}^{(y)} \right) - \frac{\partial g}{\partial y_i}, \quad (33)$$

$$\frac{d\phi_{\lambda i}}{dt} = -y_i + \mu_{xi}. \quad (34)$$

Since $\det M^{(y)} \neq 0$ by definition, there exists an inverse matrix $M_{(y)}^{-1}$. Thus we can solve the consistency condition to determine all the Lagrangian multipliers

$$\mu_{xi} = y_i, \quad (35)$$

$$\mu_{yi} = -\sum_{j=1}^{2N} (M_{(y)}^{-1})_{ij} \left(\lambda_j + \frac{\partial g}{\partial y_j} + \sum_{k=1}^{2N} y_k \frac{\partial f_j}{\partial x_k} \right), \quad (36)$$

$$\begin{aligned} \mu_{\lambda i} &= -\sum_{j=1}^{2N} \sum_{k=1}^{2N} (M_{(y)}^{-1})_{jk} \left(\lambda_k + \frac{\partial g}{\partial y_k} + \sum_{\ell=1}^{2N} y_\ell \frac{\partial f_k}{\partial x_\ell} \right) \frac{\partial f_j}{\partial x_i} \\ &\quad + \frac{\partial g}{\partial y_i}, \end{aligned} \quad (37)$$

which means that we have exhausted all the constraints and there is no additional (secondary) constraint.

Since all the Lagrange multipliers are determined, (31) should be invertible, i.e., $\det \Delta \neq 0$ where $\Delta_{qiq'j} \equiv \{\phi_{qi}, \phi_{q'j}\}$. Indeed, the Poisson brackets between constraints are

$$\{\phi_{yi}, \phi_{yj}\} = M_{ji}^{(y)}, \quad (38)$$

$$\{\phi_{xi}, \phi_{yj}\} = \frac{\partial f_j}{\partial x_i}, \quad (39)$$

$$\{\phi_{\lambda i}, \phi_{xj}\} = \delta_{ij}, \quad (40)$$

and all the other combinations vanish. Thus $\det \Delta = \det M^{(y)} \neq 0$, which is exactly the condition that allows us to solve for the Lagrange multipliers. Therefore, all $6N$ constraints are second class, and removes $6N$ initial conditions. Consequently, we are left with $12N - 6N = 6N$ initial conditions, as expected.

With these Lagrange multipliers, we obtain the self-consistent total Hamiltonian H_T by (29), with which all the constraints are satisfied all the time if they are satisfied initially. Considering (29) on the constraint surface, we note that $H_T \approx H$ and (25) suggests that its first term is linear in p_{xi} , which is not restricted by any primary constraints and can take an arbitrary value dynamically. Therefore, the Hamiltonian can vary from $-\infty$ to $+\infty$ and the system suffers from the Ostrogradsky instability.

We finally mention that the Hamiltonian coincides with the Noether's conserved quantity corresponding to the invariance of the action under the time translation when (\mathbf{p}, \mathbf{q}) satisfies the canonical equations and constraints.

Hitherto we have focused on the Lagrangian for the third-order EOM. Our analysis also applies to systems with general odd order EOM. We can prove that the Lagrangian for $(2d - 1)$ th order EOM is given by

$$L = \sum_{j=1}^{2N} x_j^{(d)} f_j(\mathbf{x}, \dots, \mathbf{x}^{(d-1)}) + g(\mathbf{x}, \dots, \mathbf{x}^{(d-1)}), \quad (41)$$

with $\det M^{(x^{(d-1)})} \neq 0$. We can remove $\ddot{\mathbf{x}}, \dots, \mathbf{x}^{(d-1)}$ from the Lagrangian (41) by invoking Lagrange multipliers as we have done in the third-order case, and proceed to the Hamiltonian formalism with constraints. We can then confirm that all the constraints are second class and the Hamiltonian is unbounded.

IV. CONCLUSION

We considered the Lagrangian that yields odd order EOMs in analytical mechanics. We explicitly demonstrated how to construct the most general Lagrangian for odd order EOMs. Using the Hamiltonian formalism for constrained systems, we proved that for this class of theories the Hamiltonian is unbounded. Thus, the Ostrogradsky instability persists even in this case.

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