Entangled entropy between real and virtual particles in ϕ^4 quantum field theory

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The aim of this work is to compute the entanglement entropy of real and virtual particles by rewriting the generating functional of ϕ^4 theory as a mean value between states and observables defined through the correlation functions. Then the von Neumann definition of entropy can be applied to these quantum states and in particular, for the partial traces taken over the internal or external degrees of freedom. This procedure can be done for each order in the perturbation expansion showing that the entanglement entropy for real and virtual particles behaves as $\ln(m_0)$. In particular, entanglement entropy is computed at first order for the correlation function of two external points showing that mutual information is identical to the external entropy and that conditional entropies are negative for all the domain of m_0. In turn, from the definition of the quantum states, it is possible to obtain general relations between total traces between different quantum states of a ϕ^r theory. Finally, discussion about the possibility of taking partial traces over external degrees of freedom is considered, which implies the introduction of some observables that measure space-time points where an interaction occurs.

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I. INTRODUCTION

Entanglement entropy associated to a region of the space V has been extensively studied, where the degrees of freedom localized in that region are only taken into account and the rest is traced out. By using the von Neumann definition of entanglement entropy $S = -\mathrm{Tr}[\rho \ln(\rho)]$, it is possible to quantify the inaccessibility to the full system, ρ being the quantum state that results from the partial trace. This quantity has been widely used in several branches of physics, for example, quantum field theory (QFT) [1–5], condensed matter physics and black hole thermodynamics (see [6–15]) and in particular for free quantum field theory with temperature (see [16–18]), in curved space-time (see [19–22]), with excited states (see [23–25]) and non-Lorentz covariant QFT [26]. In the context of quantum field theory, geometric entropy of the free Klein-Gordon field has been related to the Bekenstein-Hawking black hole [27,28] and in general, free models can reveal features that are common to all quantum field theories, in particular, the interacting ones. In d dimensions it is shown that entropy behaves as a Laurent series starting in $\epsilon^{-(d-1)}$, where ϵ is a short-distance cutoff and the coefficients are functions on the boundary ∂V. The leading coefficient that multiplies to $\epsilon^{-(d-1)}$ is proportional to the $d-1$ power of the size of V, which is the area law for the entanglement entropy. In this work, the entanglement between real particles and virtual particles is investigated, that is, the subsystems considered are not partial traces over a region of space, but over the intermediate states that are necessarily introduced in the perturbation expansion. In this work, the interacting ϕ^4 field theory will be considered and the entanglement between external and internal propagators will be studied. Although the virtual states are a mathematical artifact of the perturbative expansion of the correlation functions, these states contribute to the physical mass, the vacuum energy and the coupling constant. On the other hand, if virtual states are not real or they do not exist (see [29]), then it is feasible to trace out these states from the correlation function. In QFT, the particles that are created in these vertices are virtual particles because they are off shell; that is, they do not obey the conservation laws. In this sense, the conceptual meaning of the partial trace of the internal degrees of freedom is to neglect the virtual nonphysical modes. This is consistent with the experiments of scattering because basically what is measured are the in and out states. In turn, the interpretation of the integration of the internal vertices is to sum over all points where this process can occur (see [30], p. 94). From the point of view of this work, the integration over the internal vertices reflects the fact that the virtual degrees of freedom are eliminated. Before computing the entanglement entropy, the quantum field theory formalism for a self-interacting system must be rewritten in a suitable way, which has been done in [31,32] with an application to nonrenormalizable theories in [33]. In [31,32] the model has been applied to the renormalization of ϕ^4 theory, showing that the renormalization procedure is equivalent to a projector that neglects the diagonal part of the quantum state defined through the correlation function, which is in turn a specific representation of the operator $\mathcal{K}$ defined in Eq. (9.76) of [34]. In [33], the model has been applied to nonrenormalizable theories, showing

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that a renormalization group equation can be obtained. For the sake of simplicity, a short introduction to the main idea of papers [31,32] will be given in this section. In QFT, some (symmetric) n-point functions $\tau^{(n)}(x_1, \ldots, x_n)$ (like Feynman or Euclidean functions) can be considered; then the corresponding generating functional [[35], Eq. (II.2.21), [36], Eq. (3.2.11)] can be defined as

$$W[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \tau^{(n)}(x_1, \ldots, x_n) J(x_1) \ldots J(x_n) \prod_{i=1}^{n} d^4 x_i \quad (1)$$

where

$$\tau^{(n)}(x_1, \ldots, x_n) = \langle \Omega | \phi(x_1) \ldots \phi(x_n) | \Omega \rangle \quad (2)$$

and $J(x_i)$ are external sources. A convenient way to eliminate trivial contributions of single-particle propagators is by introducing a modified generating functional $Z[J]$ for irreducible Green's functions that is defined as

$$W[J] = e^{iZ[J]}. \quad (3)$$

The new generating functional $Z[J]$ satisfies the normalization condition $Z[0] = 0$ and it reads

$$iZ[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \tau_c^{(n)}(x_1, \ldots, x_n) J(x_1) \ldots J(x_n) \prod_{i=1}^{n} d^4 x_i \quad (4)$$

where in this case $\tau_c^{(n)}(x_1, \ldots, x_n)$ are connected n-point functions that can be obtained by differentiation:

$$\tau_c^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^{n-1}} \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \Big|_{J=0}. \quad (5)$$

In turn, the connected n-point functions can be written in terms of the Lagrangian interaction density as [see Eq. (II.2.33) of [35]][1]

$$\tau_c^{(n)}(x_1, \ldots, x_n)^{(p)} = \frac{i^p}{p!} \int \langle \Omega_0 | T \phi_0(x_1) \ldots \phi_0(x_n) \mathcal{L}_I^0(y_1) \ldots \mathcal{L}_I^0(y_p) | \Omega_0 \rangle \prod_{i=1}^{p} d^4 y_i. \quad (6)$$

Introducing (6) in (4) we have

$$iZ[J] = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{i^n}{n!} \frac{i^p}{p!} \int \langle \Omega_0 | T \phi_0(x_1) \ldots \phi_0(x_n) \mathcal{L}_I^0(y_1) \ldots \mathcal{L}_I^0(y_p) | \Omega_0 \rangle J(x_1) \ldots J(x_n) \prod_{i=1}^{n} d^4 x_i \prod_{i=1}^{p} d^4 y_i. \quad (7)$$

This equation can be written as a mean value of an observable defined through the $J(x_n)$ sources in a quantum state defined by the correlation function $\langle \Omega_0 | T \phi(x_1) \ldots \phi(x_n) \mathcal{L}_I^0(y_1) \ldots \mathcal{L}_I^0(y_p) | \Omega_0 \rangle$.[2] This procedure can be done for each correlation function of n external points. To define the quantum state we can consider some operator function $\mathbf{F}$ that depends on a set of vertices $y_1, \ldots, y_p$ and some new coodinates $w_1, \ldots, w_p$ in such a way that

$$\int \mathbf{F}(y_1, \ldots, y_p, w_1, \ldots, w_p) \prod_{i=1}^{p} \delta(y_i - w_i) \prod_{i=1}^{p} d^4 w_i = \mathcal{L}_I^0(y_1) \ldots \mathcal{L}_I^0(y_p) \quad (8)$$

where $\mathcal{L}_I^0(y_p)$ is the Lagrangian that appears in Eq. (7). In [31] we have studied the ϕ^4 theory for two external points and the corresponding operator can be represented by two different functional forms,

$$\mathbf{F}_1(y_1, \ldots, y_p, w_1, \ldots, w_p) = \prod_{i=1}^{p} \frac{\lambda_0}{4!} \phi^3(y_i) \phi(w_i) \qquad \mathbf{F}_2(y_1, \ldots, y_p, w_1, \ldots, w_p) = \prod_{i=1}^{p} \frac{\lambda_0}{4!} \phi^2(y_i) \phi^2(w_i). \quad (9)$$

In both cases, Eq. (8) holds. Then, inserting Eq. (8) into Eq. (7) we obtain

$$iZ[J] = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{i^n}{n!} \frac{i^p}{p!} \int \langle \Omega_0 | T \phi(x_1) \ldots \phi(x_n) \mathbf{F}(y_1, \ldots, y_p, w_1, \ldots, w_p) | \Omega_0 \rangle$$
$$J(x_1) \ldots J(x_n) \prod_{i=1}^{p} \delta(y_i - w_i) \prod_{i=1}^{n} d^4 x_i \prod_{i=1}^{p} d^4 y_i d^4 w_i. \quad (10)$$

[1]In Eq. (6) we have introduced the perturbative expansion of the correlation function, where the y_i are the internal vertices.
[2]In some sense, these observables will be recorded at the particle detector [see [37], p. 6, below Eq. (2.6)].

Now we can define two quantum operators in the following way:

$$\varrho^{(n,p)} = \int \langle\Omega_0|T\phi(x_1)\,\ldots\,\phi(x_n)\mathbf{F}(y_1,\ldots,y_p,w_1,\ldots,w_p)|\Omega_0\rangle$$
$$|x_1,\ldots,x_{\frac{n}{2}},y_1,\ldots,y_p\rangle\langle x_{\frac{n}{2}+1},\ldots,x_n,w_1,\ldots,w_p|\prod_{i=1}^{n}d^4x_i\prod_{i=1}^{p}d^4y_id^4w_i \tag{11}$$

$$O^{(n,p)} = \int J(x_1)\,\ldots\,J(x_n)\prod_{i=1}^{p}\delta(y_i-w_i)|x_1,\ldots,x_{\frac{n}{2}},y_1,\ldots,y_p\rangle\langle x_{\frac{n}{2}+1},\ldots,x_n,w_1,\ldots,w_p|$$
$$\times\prod_{i=1}^{n}d^4x_i\prod_{i=1}^{p}d^4y_id^4w_i \tag{12}$$

then, Eq. (10) can be written as

$$iZ[J] = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{i^n}{n!}\frac{i^p}{p!}\mathrm{Tr}(\varrho^{(n,p)}O^{(n,p)}) \tag{13}$$

The quantum operator of Eq. (12) has the following form:

$$O^{(n,p)} = O^{(n)}_{\mathrm{ext}}\otimes I^{(p)}_{\mathrm{int}} \tag{14}$$

where

$$O^{(n)}_{\mathrm{ext}} = \int J(x_1)\,\ldots\,J(x_n)|x_1,\ldots,x_{\frac{n}{2}}\rangle\langle x_{\frac{n}{2}+1},\ldots,x_n|\prod_{i=1}^{n}d^4x_i \tag{15}$$

and

$$I^{(p)}_{\mathrm{int}} = \int\prod_{i=1}^{p}\delta(y_i-w_i)|y_1,\ldots,y_p\rangle\langle w_1,\ldots,w_p|\prod_{i=1}^{p}d^4y_id^4w_i$$
$$= \int|y_1,\ldots,y_p\rangle\langle y_1,\ldots,y_p|\prod_{i=1}^{p}d^4y_i \tag{16}$$

is an identity operator acting on the y_i vertices that appear in the perturbation expansion. The Dirac delta that appears as the coefficient of the identity operator can be considered as a particular choice of observable that physically implies no measurement.[3] The subscript ext in Eq. (14) refers to the external points x_i and the subscript int to the internal vertices y_i. Then, the generating functional of Eq. (7) can be written as the mean value of the quantum operator O_{ext} on the reduced operator $\bar{\varrho}_{\mathrm{ext}}$ as

$$\mathrm{Tr}(\varrho^{(n,p)}O^{(n,p)}) = \mathrm{Tr}(\bar{\varrho}^{(n,p)}_{\mathrm{ext}}O^{(n)}_{\mathrm{ext}}) \tag{17}$$

where

$$\bar{\varrho}^{(n,p)}_{\mathrm{ext}} = \mathrm{Tr}_{\mathrm{int}}(\varrho^{(n,p)})$$
$$= \int\langle y_1,\ldots,y_p|\varrho^{(n,p)}|y_1,\ldots,y_p\rangle\prod_{i=1}^{p}d^4y_i$$
$$= \int\left(\int\langle\Omega_0|T\phi(x_1)\ldots\phi(x_n)\mathcal{L}^0_I(y_1)\ldots\mathcal{L}^0_I(y_p)|\Omega_0\rangle\right.$$
$$\left.\times\prod_{i=1}^{p}d^4y_i\right)|x_1,\ldots,x_{\frac{n}{2}}\rangle\langle x_{\frac{n}{2}+1},\ldots,x_n|\prod_{i=1}^{n}d^4x_i$$
$$= \int\tau^{(n)}(x_1,\ldots,x_n)|x_1,\ldots,x_{\frac{n}{2}}\rangle\langle x_{\frac{n}{2}+1},\ldots,x_n|\prod_{i=1}^{n}d^4x_i \tag{18}$$

where the subscript int refers to the partial trace over the degrees of freedom that represents the internal vertices of the perturbation expansion. This reduced state contains the divergences of QFT due to the correlation function $\tau^{(n)}(x_1,\ldots,x_n)$ that appears as the coefficient of the external reduced operator. What basically this model does is to duplicate the number of internal vertices. The new vertices are linked to the old vertices in such a way that identification one to one gives the Feynman diagram again. In turn, the way in which the observable in Eq. (12) is written is suitable for a generalization, where the Dirac delta distributions are replaced by some nicer well-behaved distributions. This could be interpreted as if the interaction has been smeared out [see Eq. (8)], although this implies that a nonlocal interaction has been introduced.[4] This way of introducing the perturbation expansion of the generating functional of QFT allows to discriminate the internal

[3]This point deserves more attention because a generalization of the Dirac delta can be introduced in such a way as to parametrize the partial trace, for example by using the Dirac delta representation $\delta(x) = \lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2}$ or any other representation.

[4]An interesting point is that if the second operator function $\mathbf{F}_2$ is considered, then the interaction term can be written as a mass term in the Lagrangian, where the coefficient is $\lambda_0\phi^2(\omega)$, which acts in another space-time point. This means that the mass is not constant and depends on what happens with the quantum field in another point where the interaction occurs. This point will be considered in future works.

vertices from the external ones in a simple way through the observable of Eq. (14). If the quantum operator $\varrho^{(n,p)}$ is normalized, the generating functional can be interpreted as a mean value between an observable defined by Eq. (14) and a quantum density operator defined in Eq. (11). This quantum density operator represent the probability amplitude for a particle to propagate from x_1 to x_2, that is a distribution, that belongs to a C^* algebra and where all the machinery of the algebraic approach of QFT can be applied to it. Nevertheless it is not the purpose of this work to discuss the algebraic and analytical properties of these states, although some issues will be considered.

As was shown in Eq. (17), the mean value can be computed through a reduced quantum operator, where a partial trace over the internal degrees of freedom has been taken. This suggests that the partial trace over the external degrees of freedom can be considered as well. Both partial traces can be used to compute the von Neumann entropy defined as $S_{\mathrm{ext/int}} = -\mathrm{Tr}[\varrho_{\mathrm{ext/int}} \ln(\varrho_{\mathrm{ext/int}})]$, where $\varrho_{\mathrm{ext/int}}$ are partial traces with respect the internal/external vertices respectively. This result could be important to understand the physical effects of virtual particles in real particles in the successive orders in the perturbation expansion and to obtain theoretical values of entanglement between these states. Although the procedure introduced in this section is simple, an important point has to be considered and is the ambiguity in the choice of the operator function $\mathbf{F}$ for a ϕ^4 theory [see Eq. (9)]. As an example, for the case of the second order in the perturbation expansion with two external vertices and one internal vertex, the two operator functions imply the following open-loop connected Feynman diagrams:

$$\varrho_1(x_1,x_2,y_1,w_1) \sim \langle\Omega_0|T\phi(x_1)\phi(x_2)\mathbf{F}_1(y_1,w_1)|\Omega_0\rangle \sim \Delta(x_1-y_1)\Delta(x_2-y_1)\Delta(y_1-w_1) \quad (19)$$

and

$$\varrho_2(x_1,x_2,y_1,w_1) \sim \langle\Omega_0|T\phi(x_1)\phi(x_2)\mathbf{F}_2(y_1,w_1)|\Omega_0\rangle \sim \Delta(x_1-y_1)\Delta(y_1-w_1)\Delta(w_1-x_2) \quad (20)$$

which represent two different Feynman diagrams, the first one with two external points and the second one with three external points. If we take the partial trace over the internal degrees of freedom we obtain the same reduced quantum state:

$$\bar{\varrho}_{\mathrm{ext}}(x_1,x_2) \sim \mathrm{Tr}_{\mathrm{int}}\varrho_1 \sim \int d^4y_1 \langle\Omega_0|T\phi(x_1)\phi(x_2)\mathbf{F}_1(y_1,y_1)|\Omega_0\rangle \sim \Delta(0)\int d^4y_1 \Delta(x_1-y_1)\Delta(x_2-y_1) \quad (21)$$

$$\bar{\varrho}_{\mathrm{ext}}(x_1,x_2) \sim \mathrm{Tr}_{\mathrm{int}}\varrho_2 \sim \int d^4y_1 \langle\Omega_0|T\phi(x_1)\phi(x_2)\mathbf{F}_2(y_1,y_1)|\Omega_0\rangle \sim \Delta(0)\int d^4y_1 \Delta(x_1-y_1)\Delta(x_2-y_1) \quad (22)$$

which is the first order contribution to the correlation function of two external points. This does not occur with the partial trace over the external degrees of freedom where two different results are obtained,

$$\bar{\varrho}^{(1)}_{\mathrm{int}}(y_1,w_1) \sim \mathrm{Tr}_{\mathrm{ext}}\varrho_1 \sim \int d^4x_1 \langle\Omega_0|T\phi(x_1)\phi(x_1)\mathbf{F}_1(y_1,w_1)|\Omega_0\rangle \sim \int d^4x_1 \Delta^2(x_1-y_1)\Delta(y_1-w_1) \quad (23)$$

and

$$\bar{\varrho}^{(2)}_{\mathrm{int}}(y_1,w_1) \sim \mathrm{Tr}_{\mathrm{ext}}\varrho_2 \sim \int d^4x_1 \langle\Omega_0|T\phi(x_1)\phi(x_2)\mathbf{F}_2(y_1,w_1)|\Omega_0\rangle \sim \int d^4x_1 \Delta(x_1-y_1)\Delta(y_1-w_1)\Delta(w_1-x_1). \quad (24)$$

In turn, the trace of both quantum states gives the same result (see Fig. 1):

$$\mathrm{Tr}\varrho = \mathrm{Tr}_{\mathrm{ext}}(\mathrm{Tr}_{\mathrm{int}}\varrho) = \int d^4x_1 d^4y_1 \langle\Omega_0|T\phi(x_1)\phi(x_1)\mathbf{F}_{1/2}(y_1,y_1)|\Omega_0\rangle \sim \Delta(0)\int d^4x_1 d^4y_1 \Delta^2(x_1-y_1). \quad (25)$$

With these results, the only ambiguity is located in the partial trace over the external degrees of freedom which define the quantum state that represent only the internal vertices, which physically represent virtual propagation states. This could be related to the fact that in principle, the virtual states are an artifact of the perturbation expansion of the correlation function or perhaps that there are different ways of rearranging the internal vertices and links in such a way that the loop expansion of ϕ^4 theory is obtained when the identification of internal and external vertices is done. This is particularly important when observables that depend on the quantum states are taken into account, for example the entanglement entropy. In this case, different values will be obtained for the external/internal entanglement entropies. In this work $\mathbf{F}_1$ will be considered only and

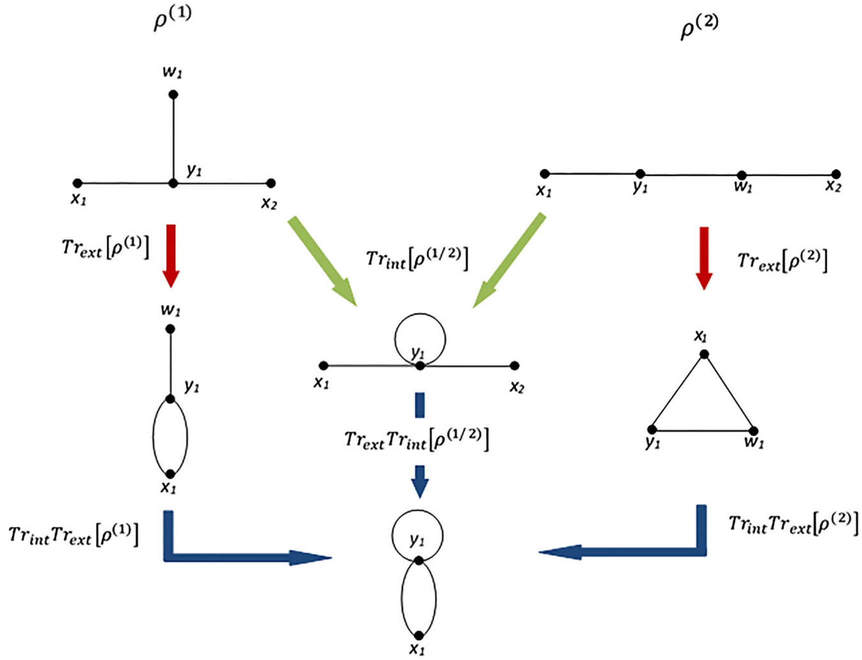

FIG. 1 (color online). Scheme of partial traces over the possible quantum states.

entanglement and relative entropies will be computed at linear order in λ_0.

A different point of view for the idea behind the manuscript is that the correlation function of ϕ^4 theory is in fact the coefficient of a quantum density operator which is in turn a partial trace over a larger quantum operator. This quantum density operator is the correlation function for a nonlocal interaction $\phi^3(y_1)\phi(w_1)$ or $\phi^2(y_1)\phi^2(w_1)$. In the first case, the interaction is ϕ^3 which contains its own Feynman propagators. In other words, the following process can be considered for the operator function $\mathbf{F}_1$: a particle in a definite momentum is prepared in the infinite past. When the interaction is turned on, this particle annihilates and two more particles are created. If the propagation of one of the resulting particles is not taken into account, then the propagator becomes a loop and the first order in the perturbation expansion for the correlation function of two external points is obtained. Ignoring one of the particles is identical to putting two observables, one in the infinite past and one in the infinite future that measure plane waves, that has some definite values of the momentum operator, or in terms of group theory, two possible values of the eigenvalues of the mass operator in the reference frame at rest of the Poincaré group, which is valid in the in and out states because the interaction goes to zero in those stages. One of the observables is the one which prepares the initial state and the second is the one that measures one of the particles that is a product of the interaction. In this sense, to take partial traces over the internal degrees of freedom is identical to make no measurement over the remaining particle, although it is there propagating in space-time. This point cannot clarify the discussion about the reality of virtual states, but it gives a physical reason of divergences in QFT: when intermediate states are not measured they must be traced out, then loops appear and infinities proliferate. In turn, the model introduced above is suitable for a generalization of observables, in particular, those that prepare and measure in and out states and simultaneously have an effect over intermediate states in such a way as to avoid divergences.

On the other hand, the partial trace over the external degrees of freedom can be interpreted as the following process: a particle is prepared in a definite momentum in the infinite past and in a particular space-time point, it annihilates and two other particles are created. If this product of particles is not measured but an observable that measures the exact space-time point where the interaction occurs is introduced, that is, where the initial particle annihilates in two other particles, then the resulting quantum state is that of Eq. (23). In the other case, where the trace is taken over the internal degrees of freedom, the observables only prepare and measure the in and out states, but in this case, one observable prepares a quantum state and the other observable measures the annihilation of the particle, which implies a measurement at one space-time point which would necessitate infinite energy.

Another important point is to note that by applying a Wick rotation in the time coordinate, the generating functional of correlation functions is identical to the partition function of statistical mechanics. Then it would be possible to study the entanglement of intermediate states, for m_0^{-1} the Compton wavelength of the quanta which is the correlation length of statistical fluctuations and relate the results with those found in [38], where area law for entanglement is obtained by considering partial traces over regions of space dictated by the decay of the correlations.

II. ϕ^4 THEORY

Before computing the quantum entropy of the reduced quantum states, we must take into account the algebraic structure of the Hilbert space. The quantum states can be written as

$$\varrho^{(n)} = \frac{1}{\mathrm{Tr}(\varrho^{(n)})}[\varrho^{(n,0)} \oplus \varrho^{(n,1)} \oplus \ldots \oplus \varrho^{(n,i)} \ldots] = \frac{1}{\mathrm{Tr}(\varrho^{(n)})} \bigoplus_{j=0}^{+\infty} \varrho^{(n,i)} \tag{26}$$

where the supperscript n indicates the number of external points and i indicates the order in the perturbation expansion. The coefficient of each quantum state will be of the form

$$\varrho^{(n,i)}(x_1, \ldots, x_n, y_1, \ldots, y_p, w_1, \ldots, w_p) = \langle \Omega_0 | T\phi_0(x_1) \ldots \phi_0(x_n) \mathbf{F}(y_1, \ldots, y_p, w_1, \ldots, w_p) | \Omega_0 \rangle. \tag{27}$$

The trace reads[5]

$$\mathrm{Tr}(\varrho^{(n)}) = \sum_{j=0}^{+\infty} (-i\lambda_0)^j W_{(n,j)} \mathrm{Tr}(\rho^{(n,j)}) \tag{28}$$

where $W_{(n,i)}$ is the weight factor (see [34], Chap. 3) corresponding to the connected Feynman diagram and $\rho^{(n,j)}$ is an operator that depends on the propagator of the respective Feynman diagram.[6] The total quantum entropy can be computed as

$$S^{(n)} = -\mathrm{Tr}[\varrho^{(n)} \ln(\varrho^{(n)})] \tag{29}$$

where S will be a function of λ_0 and some factor which will depend on the regularization scheme is chosen. Up to first order in λ_0, the quantum entropy in terms of ρ reads

$$S^{(n)} = \ln(\beta^{(n,0)}) - \frac{1}{\beta^{(n,0)}} \mathrm{Tr}[\rho^{(n,0)} \ln(\rho^{(n,0)})] - \frac{\lambda_0 W_{(n,1)}}{(\beta^{(n,0)})^2 W_{(n,0)}} [\beta^{(n,1)} \mathrm{Tr}[\rho^{(n,0)} \ln(\rho^{(n,0)})] - \beta^{(n,0)} \mathrm{Tr}[\rho^{(n,1)} \ln(\rho^{(n,0)})]] + O(\lambda_0^2) \tag{30}$$

where $\beta^{(n,i)} = \mathrm{Tr}(\rho^{(n,i)})$.

A. $n = 2$, zeroth order in the perturbation expansion

In the case of two external points, at zero order in λ_0, $\beta^{(2,0)} = \mathrm{Tr}[\rho_{\mathrm{ext}}^{(2,0)}]$ and $\mathrm{Tr}[\rho_{\mathrm{ext}}^{(2,0)} \ln(\rho_{\mathrm{ext}}^{(2,0)})]$ must be computed and where $W_{(2,0)} = 1$ and $W_{(2,1)} = 1/2$. The quantum state at zero order is the free propagator

$$\rho_{\mathrm{ext}}^{(2,0)} = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x_1 - x_2)}}{p^2 - m_0^2} |x_1\rangle\langle x_2| d^4x_1 d^4x_2. \tag{31}$$

Taking the Fourier transform by writing $|x_1\rangle = \int \frac{d^4q_1}{(2\pi)^4} e^{-iq_1x_1} |q_1\rangle$ and $\langle x_2| = \int \frac{d^4q_2}{(2\pi)^4} e^{iq_2x_2} \langle q_2|$ the quantum state $\rho_0^{(2)}$ in momentum space is diagonal and reads

$$\rho_{\mathrm{ext}}^{(2,0)} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_0^2} |p\rangle\langle p| \tag{32}$$

and the trace reads $\beta^{(2,0)} = \mathrm{Tr}[\rho_{\mathrm{ext}}^{(2,0)}] = i2TV\Delta_0$, where

$$\Delta_j = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m_0^2)^{j+1}} \tag{33}$$

and $2TV = \int d^4x = \delta^4(p = 0)$ (see [30], p. 96). Because $\rho^{(2,0)}$ is diagonal in the momentum basis, $\ln[\rho^{(2,0)}]$ reads

$$\ln[\rho_{\mathrm{ext}}^{(2,0)}] = \int \frac{d^4p}{(2\pi)^4} \ln\left(\frac{i}{p^2 - m_0^2}\right) |p\rangle\langle p| \tag{34}$$

and $\mathrm{Tr}[\rho_{\mathrm{ext}}^{(2,0)} \ln(\rho_{\mathrm{ext}}^{(2,0)})]$ reads

$$\mathrm{Tr}[\rho_{\mathrm{ext}}^{(2,0)} \ln(\rho_{\mathrm{ext}}^{(2,0)})] = -2TV\left(\frac{\pi}{2}\Delta_0 + i\chi_0\right) \tag{35}$$

where χ_0 reads

$$\chi_j = \int \frac{d^4p}{(2\pi)^4} \frac{\ln(p^2 - m_0^2)}{(p^2 - m_0^2)^{j+1}}. \tag{36}$$

Taking into account all the terms and using Eq. (30) at zero order,

$$S_{\mathrm{ext}}^{(2)} = \ln(2TV\Delta_0) + \frac{\chi_0}{\Delta_0}. \tag{37}$$

[5]It should be clear that the quantum states $\varrho^{(n)}$ that depend only on the two external points are the partial traces over the internal degrees of freedom.

[6]What this means is that the weight factor of the Feynman diagram which corresponds to the true perturbation expansion must be taken into account. For example, for $n = 2$ and $i = 1$, the quantum state is proportional to $i^3\Delta(x_1 - y_1)\Delta(y_1 - x_2)\Delta(y_1 - w_1)$ which has a weight factor of $1/4$ and the trace over the internal degrees of freedom gives a quantum state with a weight factor of $1/2$, which is the corresponding tadpole diagram. The factor $1/2$ is the one that must be take into account in Eq. (28).

At this point it is crucial to compute Δ_0 and χ_0 with some regularization. Using dimensional regularization [see Appendix A, Eqs. (A3) and (A4)] the entropy $S^{(2)}$ at order $O(\lambda_0^0)$ reads

$$S_{\rm ext}^{(2)} = -\frac{2}{\epsilon} - 1 + \ln\left(\frac{m_0^4 TV}{4\pi^2\epsilon}\right) + O(\epsilon) \tag{38}$$

where $\epsilon = d - 4$ can be considered as a microscopic cutoff. In turn, the appearance of the logarithm of the microscopic cutoff ϵ has been obtained in several works [24,39–42]. The entropy is proportional to the dimensionless coefficient $\frac{m_0^4 TV}{4\pi^2\epsilon}$ and reflects the fact that higher values of space-time volume or higher values of the mass of the propagating state increase the entropy.

B. $n = 2$, first order in the perturbation expansion

In this case the total quantum state at first order in λ_0 and for $n = 2$ reads

$$\rho^{(2,1)} = \int \Delta(x_1 - y_1)\Delta(x_2 - y_1)\Delta(y_1 - w_1)|x_1, y_1\rangle \times \langle x_2, w_1| d^4y_1 d^4w_1 d^4x_1 d^4x_2. \tag{39}$$

To compute the external entropy $S_{\rm ext}^{(2,1)}$ we can take the Fourier transform of Eq. (39):

$$\rho_{\rm ext}^{(2,1)} = -i\Delta_0 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m_0^2)^2} |p\rangle\langle p|. \tag{40}$$

This quantum state is diagonal in the momentum basis, so $\ln(\rho_{\rm ext}^{(2,1)})$ can be computed as in the last section. Using Eq. (30) at order λ_0, the entropy contribution reads

$$\begin{aligned} S_{\rm ext}^{(2,1)} &= -i\frac{\lambda_0}{2}\left(\chi_1 - \frac{\chi_0\Delta_1}{\Delta_0}\right) \\ &= \frac{\lambda_0}{2}\left[\frac{1}{4\pi^2\epsilon} + \frac{1}{16\pi^2}\left(2\gamma - 1 + \ln\left(\frac{m_0^4}{16\pi^2\mu^4}\right)\right)\right] \end{aligned} \tag{41}$$

where we have used $\beta^{(2,1)} = -i2TV\Delta_0\Delta_1$ and we have introduced a mass factor $\mu^{-\epsilon}$ to maintain the coupling constant dimensionless. From the last equation, the contribution at first order in λ_0 contains a microscopic divergence. Considering the last result and Eq. (38) the total contribution up to order λ_0 reads

$$\begin{aligned} S_{\rm ext}^{(2)} &= \frac{1}{\epsilon}\left(\frac{\lambda_0}{2}\frac{1}{4\pi^2} - 1\right) - \ln(\epsilon) - \frac{1}{2} + \ln\left(\frac{m_0^4 TV}{4\pi^2}\right) \\ &\quad + \frac{\lambda_0}{32\pi^2}\left[2\gamma - 1 + \ln\left(\frac{m_0^4}{16\pi^2\mu^4}\right)\right]. \end{aligned} \tag{42}$$

This result implies that the reduced state at first order in λ_0 increases the entropy when virtual states are traced out. This point can be detailed as follows: if the quantum state of Eq. (39) normalized by dividing it by $\mathrm{Tr}(\rho^{(2,1)}) = -i2TV\Delta_0\Delta_1$ is considered only, then both quantum internal and external entropies read

$$\begin{aligned} S_{\rm ext}^{(2,1)} &= -\mathrm{Tr}[\rho_{\rm ext}^{(2)}\ln(\rho_{\rm ext}^{(2)})] \\ &= \frac{2\chi_1}{\Delta_1} + \ln(2TV\Delta_1) \\ &= -\frac{4}{\epsilon} + 2 + \ln\left(\frac{m_0^4 TV}{4\pi^2\epsilon}\right) + O(\epsilon) \end{aligned} \tag{43}$$

and

$$\begin{aligned} S_{\rm int}^{(2,1)} &= -\mathrm{Tr}[\rho_{\rm int}^{(2,1)}\ln(\rho_{\rm int}^{(2,1)})] \\ &= \frac{\chi_0}{\Delta_0} + \ln(2TV\Delta_0) \\ &= -\frac{2}{\epsilon} - 1 + \ln\left(\frac{m_0^4 TV}{4\pi^2\epsilon}\right) + O(\epsilon) \end{aligned} \tag{44}$$

where we have used

$$\begin{aligned} \rho_{\rm int}^{(2,1)} &= \mathrm{Tr}_{\rm ext}\left[\frac{\rho^{(2,1)}}{\mathrm{Tr}(\rho^{(2,1)})}\right] \\ &= \frac{1}{2TV\Delta_0}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 - m_0^2}|p\rangle\langle p| \end{aligned} \tag{45}$$

and we have used the results of Appendix A. The result of Eq. (44) is identical to the result of Eq. (37) for the entropy of $\rho_{\rm ext}^{(2,1)}$, which is a particular case of a general structure that will be described in Sec. III. In turn, by applying the Fourier transform to the quantum state $\rho^{(2,1)}$ it can be shown that it is not diagonal in the momentum basis:

$$\begin{aligned} \rho^{(2,1)} &= -\frac{1}{2TV\Delta_0\Delta_1}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{1}{p_1^2 - m_0^2} \\ &\quad \times \frac{1}{p_2^2 - m_0^2}\frac{1}{p_3^2 - m_0^2}|p_1, p_2 + p_3 - p_1\rangle\langle p_2, p_3|. \end{aligned} \tag{46}$$

That is, the last equation implies a momentum entanglement as studied in [43]. Then, $\ln(\rho^{(2,1)})$ cannot be applied unless the quantum state is diagonalized, then the von Neumann entropy must be computed in a different way. In particular, a family of functions called the Renyi entropies S_n where the limit $n = 1$ reproduce the von Neumann entropy is defined as

$$S = -\frac{\partial}{\partial n}\ln(\mathrm{Tr}[\rho^n])|_{n=1} \tag{47}$$

and can be used to compute $S^{(2,1)}$. To compute the nth power or $\rho^{(2,1)}$ it can be noted that

$$(\rho^{(2,1)})^2 = \frac{1}{(2TV\Delta_0\Delta_1)^2}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{1}{p_1^2-m_0^2} \times \frac{1}{p_2^2-m_0^2}\frac{\eta(p_2+p_3)}{p_3^2-m_0^2}|p_1,p_2+p_3-p_1\rangle\langle p_2,p_3| \tag{48}$$

where

$$\eta(p_2+p_3) = \int \frac{d^4p_1}{(2\pi)^4}\frac{1}{(p_1^2-m_0^2)^2}\frac{1}{(p_2+p_3-p_1)^2-m_0^2}. \tag{49}$$

In turn

$$(\rho^{(2,1)})^3 = \frac{1}{(2TV\Delta_0\Delta_1)^3}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{1}{p_1^2-m_0^2} \times \frac{1}{p_2^2-m_0^2}\frac{\eta^2(p_2+p_3)}{p_3^2-m_0^2}|p_1,p_2+p_3-p_1\rangle\langle p_2,p_3|. \tag{50}$$

In this way, the nth power or $\rho^{(2,1)}$ can be written as

$$(\rho^{(2,1)})^n = \frac{1}{(2TV\Delta_0\Delta_1)^n}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{1}{p_1^2-m_0^2} \times \frac{1}{p_2^2-m_0^2}\frac{\eta^{n-1}(p_2+p_3)}{p_3^2-m_0^2}|p_1,p_2+p_3-p_1\rangle\langle p_2,p_3|. \tag{51}$$

Taking the trace,

$$\mathrm{Tr}[(\rho^{(2,1)})^n] = \frac{2TV}{(2TV\Delta_0\Delta_1)^n}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{1}{(p_1^2-m_0^2)^2} \times \frac{\eta^{n-1}(p_1+p_2)}{p_2^2-m_0^2}. \tag{52}$$

Introducing the following change of variable $p_1+p_2=r$ and $d^4p_2=d^4r$, the last equation can be written as

$$\mathrm{Tr}[(\rho^{(2,1)})^n] = \frac{2TV}{(2TV\Delta_0\Delta_1)^n}\int \frac{d^4r}{(2\pi)^4}\eta^n(r). \tag{53}$$

Computing $\frac{\partial}{\partial n}\ln(\mathrm{Tr}[(\rho^{(2,1)})^n])$,

$$\frac{\partial}{\partial n}\ln(\mathrm{Tr}[(\rho^{(2,1)})^n]) = -\ln(2TV\Delta_0\Delta_1) + \frac{\int\frac{d^4r}{(2\pi)^4}\ln[\eta(r)]\eta(r)}{\int\frac{d^4r}{(2\pi)^4}\eta^n(r)} \tag{54}$$

where a derivative under the integral sign has been taken. Finally, taking the limit $n\to 1$,

$$S^{(2,1)} = \frac{A}{B} + \ln(2TV\Delta_0\Delta_1) = \tau + \ln\left(\frac{m_0^4TV}{32\pi^4\epsilon^2}\right) + O(\epsilon) \tag{55}$$

where

$$A = \int \frac{d^4r}{(2\pi)^4}\ln[\eta(r)]\eta(r) \tag{56}$$

and

$$B = \int \frac{d^4r}{(2\pi)^4}\eta(r). \tag{57}$$

Using Eq. (8.19) of [34], $\eta(r)$ reads

$$\eta(r) = -i\frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx(1-x)[r^2x(1-x)+m_0^2]^{\frac{d}{2}-3} = -i\frac{\arctan\left(\frac{r}{\sqrt{-m_0^2-r^2}}\right)}{8\pi^2r\sqrt{-m_0^2-r^2}} + O(d-4) \tag{58}$$

and the A and B coefficients have been computed in Appendix B. Taking into account the results of Eqs. (55), (43) and (44), mutual information can be computed and reads

$$I(\rho^{(2,1)}) = S(\rho^{(2,1)}_{\mathrm{ext}}) + S(\rho^{(2,1)}_{\mathrm{int}}) - S(\rho^{(2,1)}) = \frac{2\chi_1}{\Delta_1} + \frac{\chi_0}{\Delta_0} - \frac{A}{B} + \ln(2TV) = -\frac{6}{\epsilon} + 1 - \tau + \ln(2m_0^4TV). \tag{59}$$

In Fig. 2, the finite part of total entropy $S^{(2,1)}$, entanglement entropy $S^{(2,1)}_{\mathrm{ext}}$ and $S^{(2,1)}_{\mathrm{int}}$, mutual information $I(\rho^{(2,1)})$ and the sum of external and internal entropy $S^{(2,1)}_{\mathrm{ext}}+S^{(2,1)}_{\mathrm{int}}$ is plotted as a function of bare mass m_0 using $TV=1$. The subadditivity of a bipartite system (see [44]) can be seen to be obeyed by noting that the dashed curve corresponding to $S^{(2,1)}_{\mathrm{ext}}+S^{(2,1)}_{\mathrm{int}}$ is larger than $S^{(2,1)}$. In turn, entanglement entropy and total entropy go as $\sim 4\ln(m_0)$ as occurs for entanglement entropy for different geometries (see [45]). Mutual information and external entropy are similar only for low values of m_0, where a difference can be seen in the inset of Fig. 2. Internal and total entropy behaves similarly for larger values of m_0, although differences can be obtained for values of m_0 near $m_0=1$. These results imply that the partial trace over the internal degrees of freedom at first order in the perturbation expansion has a negligible effect on mutual information and the total entropy of the system. Of course, as is expected, entanglement entropy for external and internal states are larger than total entropy,

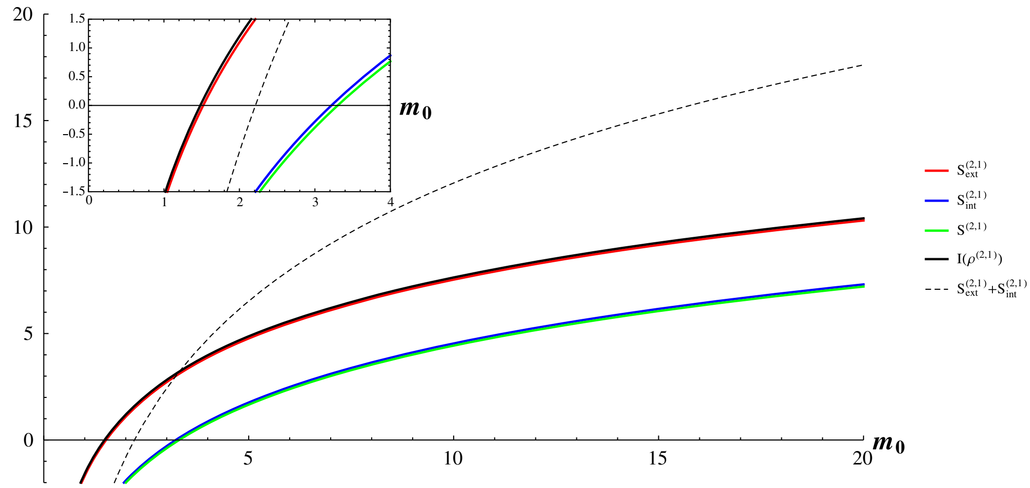

FIG. 2 (color online). Total and entanglement entropies, mutual information for the first order perturbation expansion of the two correlation function.

although it is bigger for $S_{\rm ext}^{(2,1)}$ than $S_{\rm int}^{(2,1)}$. In turn, the finite part of the conditional entropy for $\rho_{\rm ext}^{(2,1)}$ and $\rho_{\rm int}^{(2,1)}$ can be computed and both results do not depend on mass m_0:

$$S(\rho_{\rm ext}^{(2,1)}/\rho_{\rm int}^{(2,1)}) = S(\rho^{(2,1)}) - S(\rho_{\rm int}^{(2,1)}) = \tau + 1 - \ln(8\pi^2) \sim -0.102 \quad (60)$$

and

$$S(\rho_{\rm int}^{(2,1)}/\rho_{\rm ext}^{(2,1)}) = S(\rho^{(2,1)}) - S(\rho_{\rm ext}^{(2,1)}) = \tau - 2 - \ln(8\pi^2) \sim -3.102. \quad (61)$$

Both results have negative values for the whole domain of m_0 at first order in the perturbation expansion which reflect the fact of the quantum nonseparability of the total system and the coherent information.

1. *Nonperturbative approach*

General properties can be obtained by not taking into account the perturbative expansion in λ_0, but considering the spectral representation of the two-point correlation function $\langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle$ where Ω and $\phi(x)$ are the vacuum state and the field operator of the interacting theory. Following the procedure of the Introduction, a quantum state, which is the partial trace over the internal degrees of freedom, can be defined as

$$\varrho^{(2)} = \frac{1}{\eta}\int \langle\Omega|T\phi(x_1)\phi(x_2)|\Omega\rangle|x_1\rangle\langle x_2|d^4x_1 d^4x_2 \quad (62)$$

where $\eta = \mathrm{Tr}(\varrho^{(2)})$ reads

$$\eta = \int \langle\Omega|T\phi(x_1)\phi(x_1)|\Omega\rangle d^4x_1 \quad (63)$$

and is introduced in the definition of the quantum state of Eq. (62) to have a normalized quantum state $\mathrm{Tr}(\varrho^{(2)}) = 1$. Using Eq. (7.6) of [30] we can write the last equation as

$$\varrho^{(2)} = \frac{1}{\eta}\int\int_0^{+\infty}\frac{dM^2}{2\pi}\frac{i\sigma(M^2)e^{-ip(x_1-x_2)}}{p^2-M^2+i\epsilon}|x_1\rangle\langle x_2|d^4x_1d^4x_2 \quad (64)$$

where $\sigma(M^2)$ is a positive spectral density function [see Eq. (7.7) of [30]] which contains one-particle and multiparticle states. Taking the Fourier transform, the quantum state $\varrho^{(2)}$ in momentum space is diagonal and reads

$$\varrho^{(2)} = \frac{1}{\eta}\int\frac{d^4p}{(2\pi)^4}\int_0^{+\infty}\frac{dM^2}{2\pi}\frac{i\sigma(M^2)}{p^2-M^2+i\epsilon}|p\rangle\langle p| \quad (65)$$

then

$$\eta = i2TV\gamma \quad (66)$$

where

$$\gamma = \int_0^{+\infty}\frac{dM^2}{2\pi}\sigma(M^2)\Delta_0(M^2) \quad (67)$$

and where

$$\Delta_0(M^2) = \int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-M^2+i\epsilon}. \quad (68)$$

The entropy of the quantum state defined above reads

$$S^{(2)} = \ln(i2TV\gamma) - \frac{1}{\gamma}\int\frac{d^4p}{(2\pi)^4}\int_0^{+\infty}\frac{dM^2}{2\pi}\frac{i\sigma(M^2)}{p^2-M^2+i\epsilon} \times\ln\left[\int_0^{+\infty}\frac{dM^2}{2\pi}\frac{i\sigma(M^2)}{p^2-M^2+i\epsilon}\right]. \quad (69)$$

Considering only the contribution to the spectral density of the one-particle states,

$$\sigma(M^2) = 2\pi Z\delta(M^2 - m^2) + \text{multiparticle states} \quad (70)$$

where $Z = |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ is the field-strength renormalization and m is the physical mass, the entropy reads

$$S = \ln[2TV\Delta_0(m^2)] + \frac{\chi_0(m^2)}{\Delta_0(m^2)} \quad (71)$$

where $\Delta_0(m^2)$ and $\chi_0(m^2)$ are the functions defined in Eqs. (33) and (36) with m_0^2 replaced by the physical renormalized mass m^2. This last result is in fact identical to the result obtained in Eq. (37). The first integral of the last equation is the known free propagator and the second integral has been computed in Appendix A. Although Eq. (71) does not contains any approximations, its depends on m which is a renormalized constant that depends on λ_0 due to the renormalization group equation.

Finally, as was said in the first section, by considering $O \in \mathcal{B}(\mathcal{H})$, where $\mathcal{B}$ is the set of all bounded operators that form an algebra $\mathcal{A}$ acting in a Hilbert space $\mathcal{H}$, a linear form φ over this set can be defined. In particular, the Gelfand-Naimark-Segal theorem [46] implies that each positive linear form φ has a representation in the set of bounded operators $\mathcal{A}$. This allows one to define a scalar product on $\mathcal{A}$ as $\varphi(O) = \mathrm{Tr}(\pi(\varphi)O)$ where $\pi(\varphi) = \rho$ is the representation of φ over $\mathcal{A}$. For two external points, the quantum state $\rho^{(2)}$ can be used and the trace with an observable defined as

$$O = \int J(x_1)J^*(x_2)|x_1\rangle\langle x_2|d^4x_1d^4x_2 \quad (72)$$

reads

$$\mathrm{Tr}(\rho^{(2)}O) = \frac{1}{2TV\gamma}\int_0^{+\infty}\frac{dM^2}{2\pi}\rho(M^2)\int\frac{d^4p}{(2\pi)^4}\frac{|\tilde{J}(p)|^2}{p^2 - M^2 + i\epsilon} \quad (73)$$

where $\tilde{J}(p)$ is the Fourier transform of $J(x)$ and γ is defined in Eq. (67), which is the normalization factor introduced as having $\mathrm{Tr}(\rho^{(2)}) = 1$. In the case of plane waves, $|\tilde{J}(p)|^2 = 1$ and $\mathrm{Tr}(\rho^{(2)}O) = (2TV)^{-1} > 0$. It is not difficult to show that $\varphi(O^*O) = \mathrm{Tr}(\rho O^*O) > 0$ and that $\varphi(O^*) = \mathrm{Tr}(\rho O^*) = \overline{\mathrm{Tr}(\rho O)} = \overline{\varphi(O)}$ where the top bar indicates conjugation (see p. 122 of [35]). In this sense, the positive linear form can be considered a state because $\varphi(I) = \mathrm{Tr}(\rho) = 1$ and all the machinery for entanglement entropy can be applied.

C. $n = 0$, first order in the perturbation expansion

In a similar way, we can compute the total entropy of the quantum state related to the vacuum-vacuum amplitude. We can consider the inner product between $\langle\Omega|$ and $|\Omega\rangle$ [see p. 87 of [30] or Eq. (29) of [32]],

$$\langle\Omega|\Omega\rangle = 1 = \frac{e^{iE_0 2T}\mathrm{Tr}(\rho^{(0)})}{|\langle\Omega_0|\Omega\rangle|^2}, \quad (74)$$

where

$$\begin{aligned}\mathrm{Tr}(\rho^{(0)}) &= \langle\Omega_0|\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]|\Omega_0\rangle \\ &= 1 + \frac{(-i\lambda_0)}{4!}\int d^4y_1\langle\Omega_0|\phi_0^4(y_1)|\Omega_0\rangle \\ &\quad + \left(\frac{-i\lambda_0}{4!}\right)^2\int d^4y_1d^4y_2\langle\Omega_0|\phi_0^4(y_1)\phi_0^4(y_2)|\Omega_0\rangle \\ &\quad + \cdots \end{aligned} \quad (75)$$

where we are considering that the quantum states up to first order in the perturbation expansion read[7]

$$\begin{aligned}\rho^{(0,0)} &= \frac{1}{2TV}\int |y_1\rangle\langle w_1|d^4y_1d^4w_1 \\ \rho^{(0,1)} &= \frac{1}{4}\int d^4y_1d^4w_1\langle\Omega_0|\phi_0^3(y_1)\phi_0(w_1)|\Omega_0\rangle|y_1\rangle\langle w_1|.\end{aligned} \quad (76)$$

In this case, the quantum state, which represents the generating functional of the $n = 0$ external points, contains no real particles; then, the entropy will be related to the process of creation and annihilation of virtual particles. Using Eqs. (30) and (76), the contribution at first order in λ_0 to the entropy reads

$$S^{(0)} = \ln(2TV)\left[1 - \frac{i\lambda_0}{4}TV\Delta_0\left(\Delta_0 + \frac{1}{m_0^2}\right)\right] \quad (77)$$

which, by using dimensional regularization, becomes

$$\begin{aligned}\frac{S^{(0)}}{\ln(2TV)} = 1 - \frac{\lambda_0}{4}TV\Bigg(&\frac{m_0^4}{64\pi^4}\epsilon^{-2} \\ &+ \left[\frac{m_0^4}{64\pi^4}\left(\gamma - 1 - \ln\left(\frac{4\pi\mu}{m_0^2}\right)\right) - 8\pi^2\right]\epsilon^{-1} \\ &+ \frac{1}{1536\pi^4}[A + Bm_0^4 + f(m_0^2)] + O(\epsilon)\Bigg)\end{aligned} \quad (78)$$

where

$$A = 96\pi^2[1 - \gamma + \ln(4\pi\mu^2)] \quad (79)$$

and

[7]Equation (74) implies that $\mathrm{Tr}(\rho^{(0)}) = |\langle\Omega_0|\Omega\rangle|^2 e^{-iE_0 2T}$.

$$B = 18 + 12(\gamma - 2)\gamma + \pi^2 + 12\ln(4\pi\mu)[\ln(4\pi\mu) + 2 - 2\gamma] \tag{80}$$

and

$$f(m_0^2) = 48m_0^4 \ln(m_0)\left[-1 + \gamma + \ln\left(\frac{m_0}{4\pi\mu}\right)\right]. \tag{81}$$

In Fig. 3, the finite contribution at first order in the perturbation expansion for the entropy of virtual processes is plotted against m_0 for different values of mass factor μ, which is a renormalization scale, that is, the renormalized couplings depend on the μ value in such a way as to obey the renormalization group equations. Because m_0^2 depends on m^2, then the entropy computed in Eqs. (42) and (78) depends on $2TV$ and the mass scale μ considered. In all the cases, entropy has a minimum for a particular value of m_0 which decreases when μ increases. It should be clear that the lack of knowledge that leads to the entropy contribution at first order for the zero point correlation function comes from the absence of measurement of the propagation of a real particle from one space-time to another as can be seen from Eq. (76). This measurement is not done at all because what is being considered is the process of vacuum to vacuum amplitude that occurs in short periods in time. In this case, the quantum entropy computed is not important. But it is useful in the case in which the virtual process converts to a real process, for example see [47], where the effect on the entanglement entropy of real particles coming from a common virtual pair is considered or the opposite process where a massive pseudoscalar particle decays into a particle-antiparticle pair (see [48]). For example, if the first order contribution to the vacuum correlation function is considered and one of the loop propagators is cut, then we obtain the partial trace over the internal degrees of freedom of the first order contribution of the two-point correlation function. If it is possible that some physical process that involves these two quantum states, for example a loop propagator converting in a real particle propagating in space-time then the entropy of both states can be computed separately as was done in Sec. II and then compared.

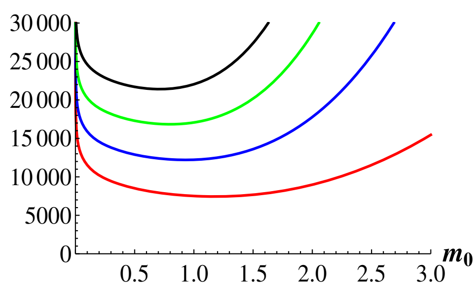

FIG. 3 (color online). Entropy of the zero point correlation function at first order in the perturbation expansion for different values of mass factor μ.

III. GENERAL CONSIDERATIONS

A relation between traces of $\rho^{(0)}$, $\rho^{(2)}$ and $\rho^{(4)}$ can be found by noting that the trace over external points in $\rho^{(2)}$ or $\rho^{(4)}$ gives contributions to the perturbative expansion of $\rho^{(0)}$. As an example, the first order contribution of $\rho^{(2)}_{\text{ext}}$ can be considered

$$\rho^{(2,1)}_{\text{ext}} = -i\lambda_0\Delta_0\int \Delta(x_1 - y_1)\Delta(x_2 - y_1)d^4y_1|x_1\rangle\langle x_2|d^4x_1d^4x_2 \tag{82}$$

and we take the trace on the reduced state we obtain the trace

$$\begin{aligned}\text{Tr}(\rho^{(2,1)}) &= \text{Tr}_{\text{ext}}(\rho^{(2,1)}_{\text{ext}}) \\ &= -i\lambda_0\Delta_0\int \Delta(x_1 - y_1)\Delta(x_1 - y_1)d^4y_1d^4x_1\end{aligned} \tag{83}$$

which is in fact one of the second order contributions to $\text{Tr}(\rho^{(0)})$ divided by Δ_0 (see Fig. 4, blue boxes):

$$\text{Tr}(\rho^{(0,2)}_1) \sim -\lambda_0^2\Delta_0^2\int \Delta^2(y_2 - y_1)d^4y_1d^4y_2 \tag{84}$$

that is

$$\frac{\text{Tr}(\rho^{(0,2)}_1)}{\text{Tr}(\rho^{(2,1)})} \sim -i\lambda_0\Delta_0. \tag{85}$$

In turn, the other contribution to $\text{Tr}(\rho^{(0,2)})$,

$$\text{Tr}(\rho^{(0,2)}_2) \sim -\Delta_0^2\int \Delta^4(y_2 - y_1)d^4y_1d^4y_2, \tag{86}$$

is in fact proportional to the total trace of $\rho^{(4,1)}$ (see Fig. 4, black boxes),

$$\text{Tr}(\rho^{(4,1)}) = -i\lambda_0\int \Delta^4(x_1 - y_1)d^4y_1d^4x_1, \tag{87}$$

that is[8]

$$\frac{\text{Tr}(\rho^{(0,2)}_2)}{\text{Tr}(\rho^{(4,1)})} \sim -i\lambda_0 \tag{88}$$

In a similar way, a relation between the connected Feynman diagrams of the $n = 2$, $n = 4$ and $n = 0$ correlation functions can be done, showing that a general expression can be obtained for a ϕ^4 interaction in terms of the quantum state traces

$$\text{Tr}(\rho^{(0)}) = 1 - i\lambda_0[\text{Tr}(\rho^{(4)}) + \Delta_0\text{Tr}(\rho^{(2)}) - \Delta_0^2] \tag{89}$$

where the product of loop propagators Δ_0^2 corresponds to the trace of a free propagator multiplied by Δ_0, which is the

[8]It must be said that the trace over $\rho^{(4)}$ is taken by considering identical arguments, that is $\text{Tr}(\rho^4) = \int\langle z, z|\rho^{(4)}|z, z\rangle d^4z$.

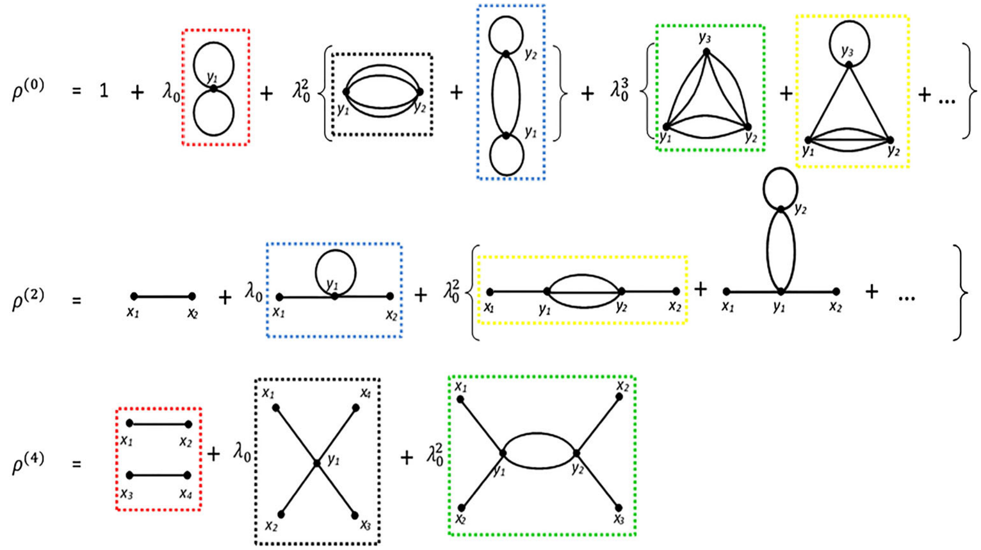

FIG. 4 (color online). Scheme of partial traces over the possible quantum states.

zero order in λ_0 of the $n = 2$ correlation function [see Fig. 4, the first Feynman diagram of $\mathrm{Tr}(\rho^{(2)})$]. The last equation can be related to the generation of correlation functions from vacuum diagrams (see Sec. 5.5 of [34], p. 68), where for example, by cutting one line to the first order vacuum diagram we obtain the first order contribution to the two-point function. In terms of quantum states, the "vacuum to two-point or four-point functions" way cannot be done because cutting a line implies introducing a new propagator, which implies the introduction of at least two quantum fields that act as external points in the correlation function and this, in algebraic terms, implies to introduce a new Hilbert space in the algebraic structure. Then, because the enlarged quantum state is not a tensor product of quantum states of each Hilbert space, then there is no operation that allows finding the correlation functions from vacuum diagrams. The opposite way is the one introduced in Eq. (82) to Eq. (88), where by using correlation functions it is possible to obtain the vacuum diagrams. In fact, because the procedure is the same and because the weight factors in each diagram match as is pointed out in [34], then the weight factors match as in Eqs. (82)–(88), which enables us to obtain Eq. (89). This equation can be generalized for a general ϕ^r interaction:

$$\mathrm{Tr}(\rho_0^{(r)}) = 1 - i\lambda_0\left[-\left(\frac{r}{2}-1\right)\Delta_0^{\frac{r}{2}} + \sum_{j=0}^{\frac{r}{2}-1}\Delta_0^j \mathrm{Tr}(\rho_{r-2j}^{(r)})\right] \quad (90)$$

which puts all the ϕ^l theories on an equal footing. From Eqs. (89), (74) and (66), $\mathrm{Tr}(\rho^{(4)})$ can be obtained as

$$\mathrm{Tr}(\rho^{(4)}) = \frac{i}{\lambda_0}[|\langle\Omega_0|\Omega\rangle|^2 e^{-iE_0 2T} - 1] + \Delta_0(m_0^2)[\Delta_0(m_0^2) - 2TVZ\Delta_0(m^2)] \quad (91)$$

where m_0^2 in turn can be written in terms of renormalized mass m^2. In Eq. (4.538) of [49], the general relation between m^2 and m_0^2 is shown where the coefficients of the expansion depend on the renormalization prescription, for example the mass-independent prescription (p. 137 of [49]). In this prescription, m_0, λ_0 and Z depend on the renormalized coupling constant λ, so then the rhs of the last equation can be written as a function of λ, the vacuum energy E_0 and the space-time volume $2TV$.

IV. CONCLUSIONS

In this work, the entanglement entropy between real and virtual propagating states has been computed by rewriting the generating functional of the ϕ^4 theory in such a way to separate the internal propagators from the external ones by considering the initial and end points as labels of position states belonging to particular Hilbert spaces. This formalism does not introduce new mathematical elements, but allows us to introduce the idea of observables that measure the in and out states and do not measure the intermediate states. In this way, the divergences of quantum field theory appear from the partial traces taken over the internal vertices of the respective Feynman diagrams. Entanglement entropy has been computed for two external points at first order in the perturbation expansion. It is shown that mutual information between real and virtual particles increase with bare mass and that the conditional entropy is negative, which implies that virtual and real particles are highly entangled. In turn, the entropy of a virtual process such as a vacuum to vacuum amplitude has been computed showing that minimum values are found for particular choices of mass factor μ. Finally, general results can be found for total traces of quantum states that represent zero, two and four external point correlation functions. These results imply that the formalism introduced in this work naturally relates different correlation functions for a general ϕ^r theory.

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APPENDIX A

In this appendix χ_j is computed using dimensional regularization, where χ_j is that of Eq. (36). In the first case,

$$\chi_j = \int \frac{d^d p}{(2\pi)^d} \frac{\ln(p^2 - m_0^2)}{(p^2 - m_0^2)^{j+1}}. \tag{A1}$$

By applying the Wick rotation to perform the four-dimensional integral in four-dimensional spherical coordinates, where p_0 is switched with ip_0, then $p^2 = -p_E^2$ and by using $\frac{d^d p_E}{(2\pi)^d} = i \frac{d\Omega_d p_E^{d-1} dp_E}{(2\pi)^d}$ and introducing the following change of coordinates, $x = \frac{m_0^2}{p_E^2 + m_0^2}$, the last equation can be written as

$$\chi_j = i \frac{(-1)^{j+1} m_0^{d-2(j+1)}}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^1 \ln\left(-\frac{m_0^2}{x}\right) x^{j-\frac{d}{2}} (1-x)^{\frac{d}{2}-1} dx. \tag{A2}$$

The result reads

$$\chi_j = i \frac{(-1)^{j+1} m_0^{d-2(j+1)}}{(4\pi)^{d/2}} \frac{(-1)^j \Gamma(j+1-d/2)}{\Gamma(j+1)} \times [H_j - H_{j-\frac{d}{2}} + \ln(-m_0^2)] \tag{A3}$$

where H_j is the Harmonic number of order j. In a similar way, the functions Δ_j can be computed for an arbitrary dimension d. In several textbooks these functions are shown (see Appendix A.4 of [30]):

$$\Delta_j = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m_0^2)^{j+1}} = \frac{i(-1)^{j+1} m_0^{d-2(j+1)}}{(4\pi)^{d/2}} \frac{\Gamma(j+1-d/2)}{\Gamma(j+1)}. \tag{A4}$$

These results will be used in the main sections of the manuscript.

APPENDIX B

In Eq. (55), the A and B coefficients as a function of m_0 must be obtained. To do so, the function $\eta(r)$ has been obtained in Eq. (58). Then, by introducing the following change of variable, $s = \frac{r}{i\sqrt{m_0^2 + r^2}}$, Eq. (57) can be written as

$$B = b m_0^2 \tag{B1}$$

where

$$b = \frac{1}{8\pi^2} \int_0^{-i} \frac{s^2 \arctan(s)}{(1+s^2)^2} ds \tag{B2}$$

where the last term is not bounded for $x \to -i$. In a similar way, by using the same change of variables, the A coefficient reads

$$A = m_0^2 [a + b \ln(m_0^2)] \tag{B3}$$

where

$$a = \frac{1}{8\pi^2} \int_0^{-i} \frac{s^2 \arctan(s)}{(1+s^2)^2} \ln\left[\frac{i(1+s^2)\arctan(s)}{8\pi^2 s}\right] ds. \tag{B4}$$

In this way

$$\frac{A}{B} = \frac{m_0^2 [a_1 + b \ln(m_0^2)]}{b m_0^2} = \frac{a}{b} + \ln(m_0^2) \tag{B5}$$

where a/b is not bounded in the limit $s \to -i$. Nevertheless, this divergence can be grouped with the microscopic cutoff divergences in the result of Eq. (55). For the $s \to 0$ limit,

$$\lim_{s\to 0} \frac{a}{b} = \tau = \frac{1}{4}[-2\gamma - \ln(4) + 12 + 3\zeta(3)] \sim 3.2663 \tag{B6}$$

where $\zeta(z)$ is the zeta function.

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