Geodesic deviation equation in $f(T)$ gravity

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In this work, we show that it is possible to study the notion of a geodesic deviation equation in $f(T)$ gravity, in spite of the fact that in teleparallel gravity there is no notion of geodesics, and the torsion is responsible for the appearance of gravitational interaction. In this regard, we obtain the general relativity equivalent equations for $f(T)$ gravity, which are in the modified gravity form such as $f(R)$ gravity. Then, we obtain the geodesic deviation equation within the context of this modified gravity. In this way, the obtained geodesic deviation equation will correspond to the $f(T)$ gravity. Eventually, we extend the calculations to obtain the modification of the Mattig relation.

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I. INTRODUCTION

The fundamental equation of Einstein geometrodynamics and other metric theories of gravity is the geodesic deviation equation (GDE) [1]. It connects the spacetime curvature described by the Riemann tensor with a measurable physical quantity, namely, the relative acceleration between two nearby test particles. This equation describes the tendency of free falling particles to approach or recede from one another while moving under the influence of a spatially varying gravitational field. Actually, the presence of this kind of tidal force will cause the trajectories to bend towards or away from each other, which produces relative acceleration [2–3]. Moreover, the important Raychaudhuri equation and Mattig relation may be obtained by considering the GDE for timelike and null congruences.

One extended gravity theory beyond general relativity (GR) is *teleparallel gravity* (TG). The birth of this gravity theory refers back to 1928 [4]. At that time Einstein was trying to redefine the unification of gravity and electromagnetism by introducing the notion of tetrad (vierbin) field together with the suggestion of absolute parallelism. In this theory the metric $g_{\mu\nu}$ is not the dynamical object, instead we have a set of tetrad fields $e_a(x^\mu)$, and instead of the well-known torsionless Levi-Civita connection of GR theory, we work with a Weitzenbök connection to introduce the covariant derivative [5]. Furthermore, the role of curvature scalar in GR is played by torsion scalar T in the teleparallel gravity.

Although at the background and perturbation levels TG is completely equivalent to general relativity, $f(T)$ gravity has new structural and phenomenological features. Especially, at the cosmological background it has various cosmological solutions which are consistent with the observational data [6–12]. In addition, by taking spherical geometry one can consider the spherical solutions for $f(T)$ gravity [13–16]. According to these features, $f(T)$ gravity is assumed as a viable theory both at cosmological and at astrophysical aspects. Thus, the cosmological and spherical solutions in $f(T)$ gravity lead to various viable models that support cosmological observations [11,17] along with Solar System tests [12] in which $f(T)$ must be close to the linear form. In [18–20] the authors have studied the Noether symmtery approach in the Friedmann-Lemaître-Robertson-Walker (FLRW) geometry to construct some viable $f(T)$ functional forms [21]. Regarding this line of progress in the context of $f(T)$ gravity, it seems there is still room to study some motivating and interesting gravitational and cosmological aspects of $f(T)$ gravity which have not yet been studied. Following this idea, in this work we consider GDE in the context of $f(T)$ gravity.

The GDE has been studied in $f(R)$ gravity theory [22,23], so it is appealing to study the GDE in the context of $f(T)$ gravity, too. However, there are conceptual differences between GR and TG. In GR, which is fundamentally based on the weak equivalence principle, curvature is used to geometrize the gravitational interaction and the spinless particles follow the curvature of spacetime. In other words, the concept of force is replaced by geometry and the particle trajectories are determined by geodesics, rather than the force equation. In TG, on the other hand, the torsion is responsible for the appearance of gravitational interaction as a real force. Hence, there is no notion of geodesics in TG. In spite of this conceptual difference, one can show that the teleparallel description of the gravitational interaction is completely equivalent to that of general relativity [24]. Therefore, it is possible to cast the force equation in TG into the form of a geodesic equation in GR and obtain the corresponding GDE in TG.

In the present work, we use another approach to obtain the GDE in $f(T)$ gravity. In this regard, first we use the method introduced in Refs. [25–28] to obtain the GR equivalent of $f(T)$ gravity where the field equations are in the modified gravity form, such as $f(R)$ gravity. Then, we

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can benefit from the approach followed in [22] to obtain the GDE within the context of $f(R)$ gravity. In this way, we actually obtain the GDE within the context of $f(T)$ gravity.

II. FIELD EQUATIONS IN $f(T)$ GRAVITY

Instead of using the torsionless Levi-Civita connection in general relativity, we use the curvatureless Weitzenböck connection in teleparallelism [5], whose non-null torsion $T^{\rho}{}_{\mu\nu}$ and contorsion $K^{\rho}{}_{\mu\nu}$ are defined, respectively, by

$$T^{\rho}{}_{\mu\nu} \equiv \tilde{\Gamma}^{\rho}_{\nu\mu} - \tilde{\Gamma}^{\rho}_{\mu\nu} = e^{\rho}_{A}(\partial_{\mu} e^{A}_{\nu} - \partial_{\nu} e^{A}_{\mu}), \tag{1}$$

$$K^{\rho}{}_{\mu\nu} \equiv \tilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}), \tag{2}$$

where $\Gamma^{\rho}{}_{\mu\nu}$ is the Levi-Civita connection. Moreover, instead of the Ricci scalar R for the Lagrangian density in general relativity, the teleparallel Lagrangian density is described by the torsion scalar T as follows:

$$T \equiv S_{\rho}{}^{\mu\nu} T^{\rho}{}_{\mu\nu}, \tag{3}$$

where

$$S_{\rho}{}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}{}_{\rho} + \delta^{\mu}_{\rho} T^{\alpha\nu}{}_{\alpha} - \delta^{\nu}_{\rho} T^{\alpha\mu}{}_{\alpha}). \tag{4}$$

The modified teleparallel action for $f(T)$ gravity is given by [29]

$$\mathcal{S} = \frac{1}{2\kappa}\int d^4x|e|f(T) + \int d^4x|e|\mathcal{L}_M, \tag{5}$$

where $|e| = \det(e^{A}_{\mu}) = \sqrt{-g}$, $8\pi G = \kappa$ and $c = 1$. Varying the action (1) with respect to the vierbein vector field e^{μ}_{A}, we obtain the equation [30]

$$\frac{1}{e}\partial_{\mu}(eS_{A}{}^{\mu\nu})f_T(T) - e^{\lambda}_{A} T^{\rho}{}_{\mu\lambda} S_{\rho}{}^{\nu\mu} f_T(T) + S_{A}{}^{\mu\nu}\partial_{\mu}(T) f_{TT}(T) + \frac{1}{4}e^{\nu}_{A} f = \kappa\Theta^{\nu}_{A}, \tag{6}$$

where a subscript T denotes differentiation with respect to T and Θ^{ν}_{A} is the matter energy-momentum tensor; meanwhile, all indices on the manifold run over 0,1,2,3, and e^{μ}_{A} form the tangent vector on the tangent space over which the metric η_{AB} is defined.

On the other hand, from the relation between the Weitzenböck connection and the Levi-Civita connection given by Eq. (2), one can write the Riemann tensor for the Levi-Civita connection in the form

$$\begin{aligned} R^{\rho}{}_{\mu\lambda\nu} &= \partial_{\lambda}\Gamma^{\rho}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\lambda} + \Gamma^{\rho}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\rho}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda} \\ &= \nabla_{\nu} K^{\rho}{}_{\mu\lambda} - \nabla_{\lambda} K^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\sigma\nu} K^{\sigma}{}_{\mu\lambda} - K^{\rho}{}_{\sigma\lambda} K^{\rho}{}_{\mu\nu}, \end{aligned} \tag{7}$$

whose associated Ricci tensor can then be written as

$$R_{\mu\nu} = \nabla_{\nu} K^{\rho}{}_{\mu\rho} - \nabla_{\rho} K^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\sigma\nu} K^{\sigma}{}_{\mu\rho} - K^{\rho}{}_{\sigma\rho} K^{\sigma}{}_{\mu\nu}. \tag{8}$$

Now, by using $K^{\rho}{}_{\mu\nu}$ given by Eq. (2) along with the relations $K^{(\mu\nu)\sigma} = T^{\mu(\nu\sigma)} = S^{\mu(\nu\sigma)} = 0$ and considering that $S^{\mu}{}_{\rho\mu} = 2K^{\mu}{}_{\rho\mu} = -2T^{\mu}{}_{\rho\mu}$ one has [25–28]

$$\begin{aligned} R_{\mu\nu} &= -\nabla^{\rho} S_{\nu\rho\mu} - g_{\mu\nu}\nabla^{\rho}\Gamma^{\sigma}{}_{\rho\sigma} - S^{\rho\sigma}{}_{\mu} K_{\sigma\rho\nu}, \\ R &= -T - 2\nabla^{\mu} T^{\nu}{}_{\mu\nu}, \end{aligned} \tag{9}$$

and thus obtains

$$G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = -\nabla^{\rho} S_{\nu\rho\mu} - S^{\sigma\rho}{}_{\mu} K_{\rho\sigma\nu}, \tag{10}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor.

Finally, by using Eq. (10), the field equations for $f(T)$ gravity in terms of GR quantities, namely, Eq. (6), can be rewritten in the following form [25–28]:

$$f_T G_{\mu\nu} + \frac{1}{2}(Tf_T - f(T))g_{\mu\nu} + B_{\mu\nu} f_{TT}(T) = \kappa\Theta_{\mu\nu}, \tag{11}$$

where $f_T = \frac{df(T)}{dT}$, $f_{TT}(T) = \frac{df_T}{dT}$, $B_{\mu\nu} = S_{\nu\mu}{}^{\sigma}\nabla_{\sigma}T$, and $\Theta_{\mu\nu}$ is the matter energy-momentum tensor. Now, this equation is in the form of a field equation in modified gravity, such as $f(R)$ gravity.

III. GEODESIC DEVIATION EQUATION IN GR

Here we start with a little discussion about the geodesic deviation equation in general relativity. The geometrical meaning of the Riemann tensor is best explained by examining the behavior of neighborhood geodesics. Imagine C_1 and C_2 are two adjacent geodesics with an affine parameter ν on 2-surface S (see Fig. 1). The vector field $V^{\alpha} = \frac{dx^{\alpha}}{d\nu}$ is the normalized tangent vector of geodesic C_1, and $\eta^{\alpha} = \frac{dx^{\alpha}}{ds}$ is the deviation vector of these two adjacent geodesics. In total we describe these geodesics with $x^{\alpha}(\nu, s)$.

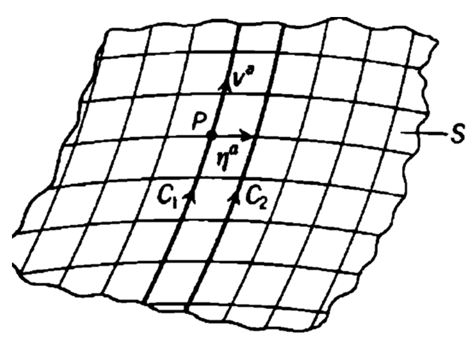

FIG. 1. Geodesic deviation.

Starting with $\pounds_V \eta^\alpha = \pounds_\eta V^\alpha$ ($[V,\eta]^\alpha = 0$) which leads to $\nabla_V \nabla_V \eta^\alpha = \nabla_V \nabla_\eta V^\alpha$ and using $\nabla_X \nabla_Y Z^\alpha - \nabla_Y \nabla_X Z^\alpha - \nabla_{[X,Y]} Z^\alpha = R^\alpha{}_{\beta\gamma\delta} Z^\beta X^\gamma Y^\delta$ in which $Y^\alpha = \eta^\alpha$ and $X^\alpha = Z^\alpha = V^\alpha$, we can obtain the GDE as follows [3]:

$$\frac{D^2\eta^\alpha}{D\nu^2} = -R^\alpha{}_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta. \tag{12}$$

As an introduction, here we review briefly the results of finding GDE in GR. We take the energy-momentum tensor in the form of a perfect fluid

$$\Theta_{\mu\nu} = (\rho + p)u_\alpha u_\beta + p g_{\alpha\beta}, \tag{13}$$

where ρ is the energy density and p is the pressure. The trace of the energy-momentum tensor is

$$\Theta = 3p - \rho. \tag{14}$$

Now, we are supposed to calculate R and $R_{\mu\nu}$ by looking at the standard form of the Einstein field equations in GR (with cosmological constant)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \Theta_{\mu\nu}. \tag{15}$$

The Ricci scalar and Ricci tensor are obtained as follows:

$$R = \kappa(\rho - 3p) + 4\Lambda, \tag{16}$$

$$R_{\mu\nu} = \kappa(\rho + p)u_\alpha u_\beta + \frac{1}{2}[\kappa(\rho - p) + 2\Lambda] g_{\mu\nu}. \tag{17}$$

Considering these expressions and using the following equation [3]:

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(g_{\alpha\gamma}R_{\delta\beta} - g_{\alpha\delta}R_{\gamma\beta} + g_{\beta\delta}R_{\gamma\alpha} - g_{\beta\gamma}R_{\delta\alpha}) - \frac{R}{6}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}) + C_{\alpha\beta\gamma\delta}, \tag{18}$$

the right-hand side of GDE is

$$R^\alpha{}_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta = \left[\frac{1}{3}(\kappa\rho + \Lambda)\epsilon + \frac{1}{2}\kappa(\rho + p)E^2\right]\eta^\alpha, \tag{19}$$

where $\epsilon = V^\alpha V_\alpha$ and $E = -V_\alpha u_\alpha$. This equation is known as the *Pirani equation* [2]. It is worth mentioning that one may derive the Pirani equation for every metric for which the Weyl tensor vanishes. A great deal of important results from this equation have been obtained including some solutions for spacelike, timelike, and null congruences [31].

IV. GEODESICS DEVIATION EQUATION IN $f(T)$ GRAVITY

In this section, we shall obtain the GDE in the GR equivalent theory of $f(T)$ gravity. Before going through the details of the calculations, we extract R by taking the trace of (11) which results in

$$R = \frac{1}{f_T}[2Tf(T) - 2f(T) + f_{TT} S^\mu{}_{\mu\rho} \nabla^\rho T - \kappa\Theta]. \tag{20}$$

Inserting this Ricci scalar into (11) we obtain

$$R_{\mu\nu} = \frac{1}{f_T}\left[\frac{1}{2} g_{\mu\nu}(Tf_T - f + f_{TT} S^\mu{}_{\mu\rho} \nabla^\rho T - \kappa\Theta) - f_{TT} S_{\nu\mu\rho} \nabla^\rho T + \kappa\Theta_{\mu\nu}\right]. \tag{21}$$

By applying Eq. (18) and considering the zero value of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ we will find

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2f_T}[\kappa(g_{\alpha\gamma}\Theta_{\delta\beta} - g_{\alpha\delta}\Theta_{\gamma\beta} + g_{\beta\delta}\Theta_{\gamma\alpha} - g_{\beta\gamma}\Theta_{\delta\alpha}) + (-f - \kappa\Theta + Tf_T + S^\mu{}_{\mu\rho}\nabla^\rho T f_{TT})(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}) + (g_{\alpha\gamma}D_{\delta\beta} - g_{\alpha\delta}D_{\gamma\beta} + g_{\beta\delta}D_{\gamma\alpha} - g_{\beta\gamma}D_{\delta\alpha})f_T] - \frac{1}{6f_T}[-2f + 2Tf_T + f_{TT}S^\mu_{\mu\rho}\nabla^\rho T - \kappa\Theta](g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}), \tag{22}$$

where

$$D_{\mu\nu} = -S_{\nu\mu\rho}\nabla^\rho T \partial_T. \tag{23}$$

After raising the α index in the Riemann tensor and contracting with $V^\beta \eta^\gamma V^\delta$, we will have

$$R^\alpha{}_{\beta\gamma\delta} V^\beta \eta^\gamma V^\delta = \frac{1}{2f_T}[\kappa(\delta^\alpha_\gamma \Theta_{\delta\beta} - \delta^\alpha_\delta \Theta_{\gamma\beta} + g_{\beta\delta}\Theta^\alpha_\gamma - g_{\beta\gamma}\Theta^\alpha_\delta) + (-f - \kappa\Theta + Tf_T + S^\mu{}_{\mu\rho}\nabla^\rho T f_{TT})(\delta^\alpha_\gamma g_{\delta\beta} - \delta^\alpha_\delta g_{\gamma\beta}) + (\delta^\alpha_\gamma D_{\delta\beta} - \delta^\alpha_\delta D_{\gamma\beta} + g_{\beta\delta}D^\alpha_\gamma - g_{\beta\gamma}D^\alpha_\delta)f_T] V^\beta \eta^\gamma V^\delta - \frac{1}{6f_T}[-2f + 2Tf_T + f_{TT}S^\mu{}_{\mu\rho}\nabla^\rho T - \kappa\Theta] \times (g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}) V^\beta \eta^\gamma V^\delta. \tag{24}$$

Note that Eqs. (22) and (24) hold only if the Weyl tensor vanishes. By considering Eq. (13) in Eq. (22) we find

$$R^{\alpha}{}_{\beta\gamma\delta} = \frac{1}{2f_T}\bigg[\kappa(\rho+p)(g_{\alpha\gamma}u_\delta u_\beta - g_{\alpha\delta}u_\gamma u_\beta + g_{\beta\delta}u_\gamma u_\alpha - g_{\beta\gamma}u_\delta u_\alpha) + (g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta})\left(\frac{2\kappa\rho}{3} + Tf_T/3 + \frac{2}{3}S^{\mu}{}_{\mu\rho}\nabla^{\rho}Tf_{TT} - \frac{f}{3}\right) + (g_{\alpha\gamma}D_{\delta\beta} - g_{\alpha\delta}D_{\gamma\beta} + g_{\beta\delta}D_{\gamma\alpha} - g_{\beta\gamma}D_{\delta\alpha})f_T\bigg]. \tag{25}$$

Under the condition of vector field normalization, we have $V^\alpha V_\alpha = \epsilon$ and

$$R_{\alpha\beta\gamma\delta}V^\beta V^\delta = \frac{1}{2f_T}\bigg[\kappa(\rho+p)(g_{\alpha\gamma}(u_\beta V^\beta)^2 - 2(u_\beta V^\beta)V_{(\alpha}u_{\gamma)} + \epsilon u_\alpha u_\gamma) + \left(\frac{2\kappa\rho}{3} + \frac{Tf_T}{3} + \frac{2}{3}S^{\mu}{}_{\mu\rho}\nabla^{\rho}Tf_{TT} - \frac{f}{3}\right)(\epsilon g_{\alpha\gamma} - V_\alpha V_\gamma) + [(g_{\alpha\gamma}D_{\delta\beta} - g_{\alpha\delta}D_{\gamma\beta} + g_{\beta\delta}D_{\gamma\alpha} - g_{\beta\gamma}D_{\delta\alpha})f_T]V^\beta V^\delta\bigg]. \tag{26}$$

Again, we raise the first index and then contract with η^γ,

$$R^{\alpha}{}_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \frac{1}{2f_T}\bigg[\kappa(\rho+p)((u_\beta V^\beta)^2\eta^\alpha - (u_\beta V^\beta)V^\alpha(u_\gamma\eta^\gamma) - (u_\beta V^\beta)u^\alpha(V_\gamma\eta^\gamma) + \epsilon u_\alpha u_\gamma\eta^\gamma) + \left(\frac{2\kappa\rho}{3} + \frac{Tf_T}{3} + \frac{2}{3}S^{\mu}{}_{\mu\rho}\nabla^{\rho}Tf_{TT} - \frac{f}{3}\right)(\epsilon\eta^\alpha - V_\alpha(V_\gamma\eta^\gamma)) + [(\delta^\alpha_\gamma D_{\delta\beta} - \delta^\alpha_\delta D_{\gamma\beta} + g_{\beta\delta}D^\alpha_\gamma - g_{\beta\gamma}D^\alpha_\delta)f_T]V^\beta V^\delta\eta^\gamma\bigg]. \tag{27}$$

By considering $E = -V_\alpha u^\alpha$ and $\eta_\alpha u^\alpha = \eta_\alpha V^\alpha = 0$, Eq. (27) converts into

$$R^{\alpha}{}_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \frac{1}{2f_T}\left[\kappa(\rho+p)E^2 + \epsilon\left(\frac{2\kappa\rho}{3} + \frac{Tf_T}{3} + \frac{2}{3}S^{\mu}{}_{\mu\rho}\nabla^{\rho}Tf_{TT} - \frac{f}{3}\right)\right]\eta^\alpha + \frac{1}{2f_T}[(\delta^\alpha_\gamma D_{\delta\beta} - \delta^\alpha_\delta D_{\gamma\beta} + g_{\beta\delta}D^\alpha_\gamma - g_{\beta\gamma}D^\alpha_\delta)f_T]V^\beta V^\delta\bigg]\eta^\gamma. \tag{28}$$

It should be mentioned that all the results obtained in this section do not require the Friedmann-Robertson-Walker ansatz, but rather are valid as long as the Weyl tensor vanishes. In the next section, we will use these results to obtain the GDE in the GR equivalent of the $f(T)$ gravity model for the FLRW metric whose Weyl tensor is zero. Obviously, our final result will be acceptable provided that the GR limit of this model is checked.

A. GR equivalent method with FLRW background

We take the standard model line element (FLRW universe) including $a(t)$ and k, respectively, as the scale factor and spatial curvature of the Universe, as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2sin^2\theta d\phi^2\right], \tag{29}$$

whose Weyl tensor is zero because of conformal flatness. According to the second section, the expression for the torsion scalar for the flat FLRW metric reduces to

$$T = -6H^2, \tag{30}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Note that Eq. (30) holds only if a diagonal Friedmann-Robertson-Walker tetrad is chosen, since different tetrads giving the same metric lead to different results in $f(T)$ gravity [32]. Apparently, the torsion scalar is thoroughly time dependant; thus, we are supposed to be concerned just about time derivatives of T in $D_{\mu\nu}$.

The vector field normalization implies that $V_\alpha V^\alpha = \epsilon$ and also we have $E = -V_\alpha u^\alpha$, $\eta_\alpha u^\alpha = \eta_\alpha V^\alpha = 0$, $\eta_0 u^0 = 0$. Not only these mentioned conditions but also the nonvanishing components of the S tensor will be used to extract the final result for the action of operator $D_{\mu\nu}$ on f_T. Therefore, after cumbersome calculations we obtain

$$\frac{1}{2f_T}((\delta^\alpha_\gamma D_{\delta\beta} - \delta^\alpha_\delta D_{\gamma\beta} + g_{\beta\delta}D^\alpha_\gamma - g_{\beta\gamma}D^\alpha_\delta)f_T)V^\beta V^\delta \eta^\gamma = -24H^2\dot{H}f_{TT}(E^2 + 2\epsilon)\eta^\alpha, \tag{31}$$

where have used $S^1_{10} = S^2_{20} = S^3_{30} = -2H(t)$. Thus, we can write the reduced expression for $R^\alpha{}_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta$ as follows:

$$R^\alpha{}_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \frac{1}{2f_T}\left[(\kappa(\rho+p) - 24H^2\dot{H}f_{TT})E^2 + \epsilon\left(\frac{2\kappa\rho}{3} - \frac{f}{3} + \frac{Tf_T}{3}\right)\right]\eta^\alpha, \tag{32}$$

which is the generalized Pirani equation. Now, we can write the GDE in the $f(T)$ gravity model:

$$\frac{D^2\eta^\alpha}{D\nu^2} = -\frac{1}{2f_T}\left[(\kappa(\rho+p) - 24H^2\dot{H}f_{TT})E^2 + \epsilon\left(\frac{2\kappa\rho}{3} - \frac{f}{3} + \frac{Tf_T}{3}\right)\right]\eta^\alpha. \tag{33}$$

Apparently, as a result of homogeneity and isotropy of the FLRW metric, in this equation only the magnitude of the deviation vector η^α is changed along the geodesics. Whereas in the anisotropic universe, like Bianchi I, we can also infer a change in the direction of the deviation vector, as described in [33].

B. Direct method with a FLRW background

Once we assume a FLRW metric (tetrad), then the Riemann tensor can be calculated. This means that the geodesic deviation equation (12) can be straightforwardly written in terms of H and its derivative. We can then use the $f(T)$ gravity cosmological equations (43) and (44) (see below) to connect H and $\dot{H}$ to the cosmological sources. Following this way, we can recover Eq. (32) as follows. By taking a radial spatial part, for a suitable choice of local coordinates, we can write the lhs of Eq. (32) for $\alpha = r$ as follows:

$$R^r{}_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = R^r{}_{t\gamma t}V^t\eta^\gamma V^t + R^r{}_{r\gamma r}V^r\eta^\gamma V^r + R^r{}_{\theta\gamma\theta}V^\theta\eta^\gamma V^\theta + R^r{}_{\phi\gamma\phi}V^\phi\eta^\gamma V^\phi. \tag{34}$$

By considering nonvanishing components of the Riemann tensor for the FLRW metric, we must take $\gamma = r$. Thus, Eq. (34) can be written as

$$R^r{}_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = R^r{}_{trt}V^tV^t\eta^r + R^r{}_{rrr}g^{rr}V_rV^r\eta^r + R^r{}_{\theta r\theta}g^{\theta\theta}V_\theta V^\theta\eta^r + R^r{}_{\phi r\phi}g^{\phi\phi}V_\phi V^\phi\eta^r = (-\dot{H}E^2 + \epsilon H^2)\eta^r, \tag{35}$$

where we have used $V^tV^t = E^2$, $V_i\eta^i = 0$, $R^r{}_{rrr} = 0$, $R^r{}_{\theta r\theta} = r^2\dot{a}^2$, $R^r{}_{\phi r\phi} = \dot{a}^2r^2sin^2\theta$, and $R^r{}_{ttr} = \frac{\ddot{a}}{a}$. Note that similar equations are also obtained for $\alpha = \theta$ and $\alpha = \phi$.

According to Eqs. (43) and (44) (see below) we have

$$H^2 = \frac{1}{2f_T}\left(\frac{\kappa\rho}{3} - \frac{f}{6}\right), \tag{36}$$

and

$$\dot{H} = -\frac{1}{2f_T}(\kappa(\rho+p) + 4\dot{H}Tf_{TT}). \tag{37}$$

By substituting Eqs. (36) and (37) into Eq. (35) we can obtain the generalized Pirani equation (32) by means of the direct approach as

$$R^\alpha{}_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \frac{1}{2f_T}\left[(\kappa(\rho+p) - 24H^2\dot{H}f_{TT})E^2 + \epsilon\left(\frac{2\kappa\rho}{3} - \frac{f}{3} + \frac{Tf_T}{3}\right)\right]\eta^\alpha,$$

which results in the same geodesic deviation equation (33). This means that we have found the same results by two different approaches which confirms the validity of the obtained geodesic deviation equation.

C. Fundamental observers with a FLRW background

Here, we are going to limit ourselves to the fundamental observers. In this particular case, we interpret V^α and ν (the affine parameter) as the four-velocity of the fluid u^α and t (the proper time), respectively. Since we are treating with temporal geodesics we have $\epsilon = -1$ and also we fix the vector field normalization by $E = 1$ which leads to

$$R^\alpha{}_{\beta\gamma\delta}u^\beta\eta^\gamma u^\delta = \frac{1}{2f_T}\left[\frac{2\kappa\rho}{3} + \kappa p - 24H^2\dot{H}f_{TT} + \frac{f}{6}\right]\eta^\alpha. \tag{38}$$

We know that if $\eta_\alpha = \ell e_\alpha$, where e_α is parallel propagated along t, then the isotropy results in

$$\frac{De^\alpha}{Dt} = 0, \tag{39}$$

from which we have

$$\frac{D^2\eta^\alpha}{Dt^2} = \frac{d^2\ell}{dt^2}e^\alpha. \tag{40}$$

By using (12) and (38) we can write

$$\frac{d^2\ell}{dt^2} = -\frac{1}{2f_T}\left[\frac{2\kappa\rho}{3} + \kappa p - 24H^2\dot{H}f_{TT} + \frac{f}{6}\right]\ell, \tag{41}$$

which for the particular case $\ell = a(t)$ leads to

$$\frac{\ddot{a}}{a} = \frac{1}{f_T}\left[-\frac{\kappa\rho}{3} - \frac{\kappa p}{2} + 12H^2\dot{H}f_{TT} - \frac{f}{12}\right]. \tag{42}$$

This equation is nothing but a special case of the generalized Raychaudhuri equation. It is worth mentioning here that the above generalized Raychaudhuri equation can be obtained by the standard forms of the modified Friedmann equations in the $f(T)$ gravity model for a flat universe [30]

$$H^2 = \frac{\kappa}{3}\left(\rho + \frac{1}{2\kappa}(2Tf_T - f - T)\right), \tag{43}$$

$$2\dot{H} + 3H^2 = -\kappa p - 4\dot{H}Tf_{TT} - 2\dot{H}f_T + 2\dot{H} + Tf_T - \frac{f}{2} - \frac{T}{2}. \tag{44}$$

The consistency between the modified Friedmann equations in $f(T)$ gravity for a flat universe [30] and the generalized Raychaudhuri equation for a flat universe (42) confirms the fact that the approach followed here is a correct one.

D. Null vector fields with a FLRW background

In this section, we consider the null past directed vector fields, namely, $V^\alpha = k^\alpha$, $k_\alpha k^\alpha = 0$, for which Eq. (32) reduces to

$$R^\alpha{}_{\beta\gamma\delta}k^\beta\eta^\gamma k^\delta = \frac{1}{2f_T}(\kappa(\rho + p) - 24H^2\dot{H}f_{TT})E^2\eta^\alpha. \tag{45}$$

Actually, this is *Ricci focusing* in $f(T)$ gravity as is explained in the following. By considering $\eta^\alpha = \eta e^\alpha$, $e_\alpha e^\alpha = 1$, $\epsilon_\alpha u^\alpha = e_\alpha k^\alpha = 0$, and also writing an aligned base that is parallel propagated $\frac{De^\alpha}{D\nu} = k^\beta\nabla_\beta e^\alpha = 0$, we obtain the null GDE (33) new form as follows:

$$\frac{d^2\eta}{d\nu^2} = -\frac{1}{2f_T}(\kappa(\rho + p) - 24H^2\dot{H}f_{TT})E^2\eta. \tag{46}$$

According to the GR studied in [2], all classes of past-directed null geodesics experience focusing if we have $\kappa(\rho + p) > 0$. Therefore, in a particular case with the equation of state $p = -\rho$ (cosmological constant) we cannot recognize any focusing effect. Obviously (46) shows the focusing condition for the $f(T)$ gravity model provided that

$$\frac{\kappa(\rho + p)}{f_T} > \frac{24H^2\dot{H}f_{TT}}{f_T}. \tag{47}$$

Now, we have an expression (46) which can be written in terms of the redshift parameter z. To do this, we may write

$$\frac{d}{d\nu} = \frac{dz}{d\nu}\frac{d}{dz}, \tag{48}$$

which results in

$$\frac{d^2}{d\nu^2} = \left(\frac{d\nu}{dz}\right)^{-2}\left[-\left(\frac{d\nu}{dz}\right)^{-1}\frac{d^2\nu}{dz^2}\frac{d}{dz} + \frac{d^2}{dz^2}\right]. \tag{49}$$

Let us consider the null geodesics for which we have

$$(1 + z) = \frac{a_0}{a} = \frac{E}{E_0} \to \frac{dz}{1 + z} = -\frac{da}{a}. \tag{50}$$

Choosing $a_0 = 1$ (the current day value of the scale factor), leads to the following result for the past-directed case:

$$dz = (1 + z)\frac{1}{a}\frac{da}{d\nu}d\nu = (1 + z)\frac{\dot{a}}{a}Ed\nu = E_0H(1 + z)^2 d\nu. \tag{51}$$

Thus, we obtain

$$\frac{d\nu}{dz} = \frac{1}{E_0H(1 + z)^2}, \tag{52}$$

and so

$$\frac{d^2\nu}{dz^2} = -\frac{1}{E_0H(1 + z)^3}\left[\frac{1}{H}(1 + z)\frac{dH}{dz} + 2\right], \tag{53}$$

where

$$\frac{dH}{dz} = \frac{d\nu}{dz}\frac{dt}{d\nu}\frac{dH}{dt} = -\frac{1}{H(1 + z)}\frac{dH}{dt}. \tag{54}$$

where use has been made of $\frac{dt}{d\nu} = E = E_0(1 + z)$. From the definition of the Hubble parameter we can write

$$\dot{H} = \frac{\ddot{a}}{a} - H^2. \tag{55}$$

Using (42), $\dot{H}$ becomes

$$\dot{H} = \frac{1}{f_T}\left[\frac{-\kappa\rho}{3} - \frac{\kappa p}{2} + 12\dot{H}H^2f_{TT} - \frac{f}{12}\right] - H^2, \tag{56}$$

thus

$$\frac{d^2\nu}{dz^2} = -\frac{3}{E_0 H(1+z)^3}\left[1+\frac{1}{3H^2 f_T}\left(\frac{\kappa\rho}{3}+\frac{\kappa p}{2}-12\dot{H}H^2 f_{TT}+\frac{f}{12}\right)\right]. \tag{57}$$

Putting this result in (49), we obtain

$$\frac{d^2\eta}{d\nu^2} = E_0 H(1+z)^2\left[\frac{d^2\eta}{dz^2}+\frac{3}{(1+z)}\left[1+\frac{1}{3H^2 f_T}\left(\frac{\kappa\rho}{3}+\frac{\kappa p}{2}-12\dot{H}H^2 f_{TT}+\frac{f}{12}\right)\right]\frac{d\eta}{dz}\right]. \tag{58}$$

Finally, using (46) the null GDE takes the following form:

$$\frac{d^2\eta}{dz^2}+\frac{3}{(1+z)}\left[1+\frac{1}{3H^2 f_T}\left(\frac{\kappa\rho}{3}+\frac{\kappa p}{2}-12\dot{H}H^2 f_{TT}+\frac{f}{12}\right)\right]\frac{d\eta}{dz}+\frac{\kappa(\rho+p)-24H^2\dot{H}f_{TT}}{2H^2(1+z)^2 f_T}\eta = 0. \tag{59}$$

Matter and radiation contributions to ρ and p can be written respectively as

$$\kappa\rho = 3H_0^2\Omega_{m0}(1+z)^3+3H_0^2\Omega_{r0}(1+z)^4, \qquad \kappa p = H_0^2\Omega_{r0}(1+z)^4, \tag{60}$$

where we have used $p_m = 0$ and $p_r = \frac{1}{3}\rho_r$. By using Eq. (60) the null GDE (59) can be written as

$$\frac{d^2\eta}{dz^2}+P(H,\dot{H},z)\frac{d\eta}{dz}+Q(H,\dot{H},z)\eta = 0, \tag{61}$$

where

$$P(H,\dot{H},z) = \frac{\Omega_{m0}(1+z)^3+\frac{3}{2}\Omega_{r0}(1+z)^4+\frac{f}{12H_0^2}+(3f_T+\frac{\dot{T}f_{TT}}{H})[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE}]}{f_T(1+z)[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE}]}, \tag{62}$$

$$Q(H,\dot{H},z) = \frac{3\Omega_{m0}(1+z)^3+4\Omega_{r0}(1+z)^4+2\frac{\dot{T}f_{TT}}{H}(\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE})}{2f_T(1+z)^2[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE}]}, \tag{63}$$

in which we have applied the following new form of (43):

$$H^2 = H_0^2[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE}], \tag{64}$$

where Ω_{DE} has been defined as

$$\Omega_{DE} = \frac{1}{H_0^2}\left(\frac{Tf_T}{3}-\left(\frac{f+T}{6}\right)\right). \tag{65}$$

Note that in solving Eq. (61), we must use (30). In order to check for the agreement of the above results with those of GR, we take the particular case $f(T) = T-2\Lambda$. As a result of this choice we have $f_T = 1$ and $f_{TT} = 0$. Furthermore, Ω_{DE} reduces to

$$\Omega_{DE} = \frac{1}{H_0^2}\left(\frac{T-2\Lambda}{3}-\left(\frac{T-2\Lambda+T}{6}\right)\right) = \frac{\Lambda}{3H_0^2} \equiv \Omega_\Lambda, \tag{66}$$

which can be written as the Friedmann equation in GR

$$H^2 = H_0^2[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_\Lambda]. \tag{67}$$

Hence, we find the reduced expressions P and Q as follows:

$$P(z) = \frac{\frac{7}{2}\Omega_{m0}(1+z)^3+4\Omega_{r0}(1+z)^4+2\Omega_\Lambda}{(1+z)[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_\Lambda]}, \tag{68}$$

$$Q(z) = \frac{3\Omega_{m0}(1+z)+4\Omega_{r0}(1+z)^2}{2[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_\Lambda]}. \tag{69}$$

Eventually, the GDE for null vector fields becomes

$$\frac{d^2\eta}{dz^2}+\frac{\frac{7}{2}\Omega_{m0}(1+z)^3+4\Omega_{r0}(1+z)^4+2\Omega_\Lambda}{(1+z)[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_\Lambda]}\frac{d\eta}{dz} + \frac{3\Omega_{m0}(1+z)+4\Omega_{r0}(1+z)^2}{2(\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_\Lambda)}\eta = 0. \tag{70}$$

In order to obtain the Mattig relation in GR [34], we have to fix $\Omega_\Lambda = 0$, $\Omega_{r0}+\Omega_{m0} = 1$ which leads to

$$\frac{d^2\eta}{dz^2}+\frac{\frac{7}{2}\Omega_{m0}(1+z)^3+4\Omega_{r0}(1+z)^4}{(1+z)[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4]}\frac{d\eta}{dz}+\frac{3\Omega_{m0}(1+z)+4\Omega_{r0}(1+z)^2}{2(\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4)}\eta=0. \tag{71}$$

So, Eq. (61) gives us an opportunity to generalize the Mattig relation in $f(T)$ gravity. By considering the last result, we can infer the following expression for the observer area distance $r_0(z)$ [34]:

$$r_0(z)=\sqrt{\left|\frac{dA_0(z)}{d\Omega}\right|}=\left|\frac{\eta(z')|_z}{d\eta(z')/d\ell|_{z'=0}}\right|, \tag{72}$$

where A_0 is the area of the object and also Ω is the solid angle. Equipped with the $d/d\ell=E_0^{-1}(1+z)^{-1}d/d\nu=H(1+z)d/dz$ and setting the deviation at $z=0$ to zero, clearly we have

$$r_0(z)=\left|\frac{\eta(z)}{H(0)d\eta(z')/dz'|_{z'=0}}\right|, \tag{73}$$

where $H(0)$ is the evaluated modified Friedmann equation (66) at $z=0$.

V. NUMERICAL SOLUTIONS OF GDE FOR $f(T)$ GRAVITY

To solve the null GDE (61) in $f(T)$ gravity we should consider $f(T)$ functional forms. One simple assumption in simplifying Eq. (61) is $\dot{T}=0$. However, this reduces our model to Lambda cold dark matter (ΛCDM) since $\Omega_{DE}=$ Const, as can be seen from Eq. (65). To find new interesting features of $f(T)$ modified gravity, we should consider models beyond the $\dot{T}=0$ assumption. Actually, there are a variety of $f(T)$ functional forms in the literature, each of which has some interesting specific features. Even, some studies have been done to obtain viable $f(T)$ functional forms, using the Noether symmetry approach [18,19].

Following the above discussion, and for simplicity, first we assume $\dot{T}=0$ which is equivalent to a constant $T=-6H^2=T_0$. Thus, Eq. (62) can be rewritten as follows:

$$\frac{d^2\eta}{dz^2}+P(H,T_0,z)\frac{d\eta}{dz}+Q(H,T_0,z)\eta=0, \tag{74}$$

where

$$P(H,T_0,z)=\frac{\Omega_{m0}(1+z)^3+\frac{3}{2}\Omega_{r0}(1+z)^4+\frac{f(T_0)}{12H_0^2}+3f_T(T_0)[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE}(T_0)]}{f_T(T_0)(1+z)[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE}(T_0)]}, \tag{75}$$

$$Q(H,T_0,z)=\frac{3\Omega_{m0}(1+z)+4\Omega_{r0}(1+z)^2}{2f_T(T_0)[\Omega_{m0}(1+z)^3+\Omega_{r0}(1+z)^4+\Omega_{DE}(T_0)]}, \tag{76}$$

and

$$\Omega_{DE}(T_0)=\frac{1}{H_0^2}\left(\frac{T_0f_T(T_0)}{3}-\left(\frac{f(T_0)+T_0}{6}\right)\right). \tag{77}$$

To solve Eq. (74), one can choose the functional form of the $f(T_0)$ as follows:

$$f(T_0)=\alpha T_0+\beta T_0^2, \tag{78}$$

where α and β are constants. We can solve Eq. (74) numerically which results in $\eta(z)$ and $r_0(z)$ plotted in Fig. 2, as a function of z.

Now, we assume $\dot{T}\neq 0$. Since solving the GDE (61) is not an easy task for any given $f(T)$ functional form, and in order to benefit from numerical calculations in solving this complicated equation, we just consider the following power law form as

$$f(T)=aT+bT^n+c, \tag{79}$$

which is well suited in most of the cosmological implications of $f(T)$ gravity [18,19], or in a slightly modified form as

$$f(T)=aT+\frac{bT^n}{1+T^{(n+1)}}+c, \tag{80}$$

where a, b, c, and n are constants. One can rewrite Eq. (44) as

$$\frac{\dot{T}}{12H}=\frac{\kappa p+3H^2-Tf_T+\frac{f}{2}+\frac{T}{2}}{4Tf_{TT}+2f_T}. \tag{81}$$

Thus, by plugging Eq. (81) in Eqs. (62) and (63), we can solve Eq. (61) numerically for suggested $f(T)$ forms (79) and (80) which results in $\eta(z)$ and $r_0(z)$ plotted, respectively, in Fig. 3 for $f(T)=aT+bT^n+c$ and in Fig. 4 for $f(T)=aT+\frac{bT^n}{1+T^{(n+1)}}+c$, as a function of z.

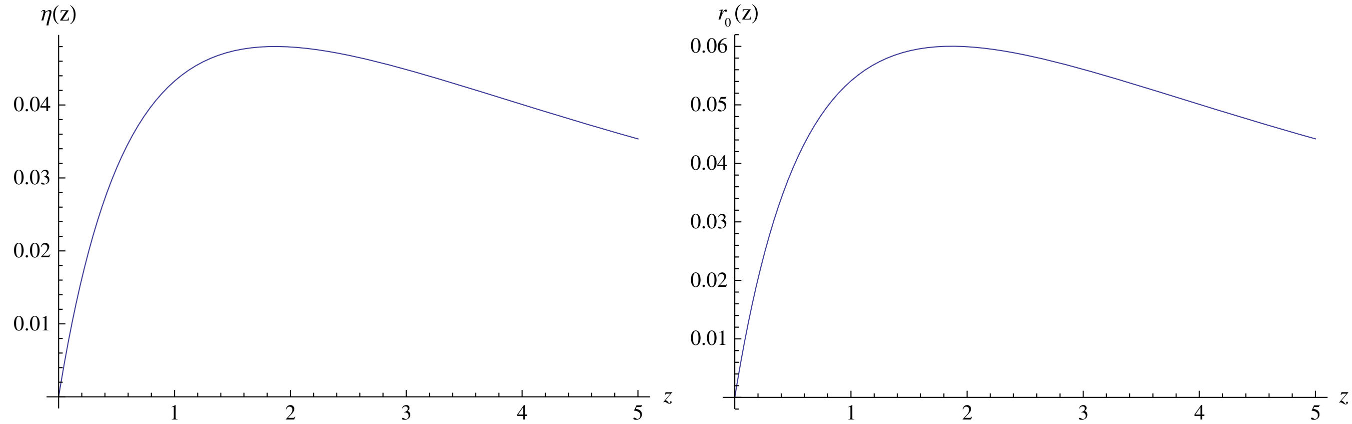

FIG. 2 (color online). Plot of the deviation vector magnitude $\eta(z)$ (left) and plot of the observer area distance $r_0(z)$ (right). The parameter values chosen are $H_0 = 80$ km/s/Mpc, $\Omega_{m0} = 0.3$, $\Omega_{r0} = 0$, $\Omega_\Lambda = 0.7$, and $d\eta(z)/dz|_{z=0} = 0.1$.

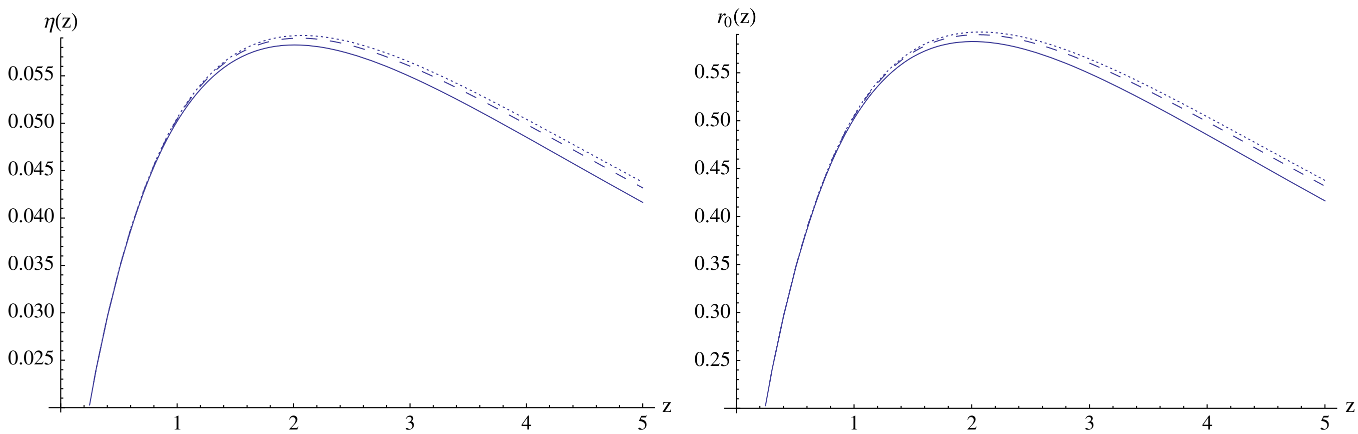

FIG. 3 (color online). Plot of the deviation vector magnitude $\eta(z)$ (left) and plot of the observer area distance $r_0(z)$ (right). The parameter values chosen are $H_0 = 80$ km/s/Mpc, $\Omega_{m0} = 0.3$, $a = 0.9$, $b = 0.2$, $c = 0.7$, $n = -1$ (solid), $n = -2$ (dotted), and $n = 2$ (dashed), $\Omega_{r0} = 0$, and $d\eta(z)/dz|_{z=0} = 0.1$.

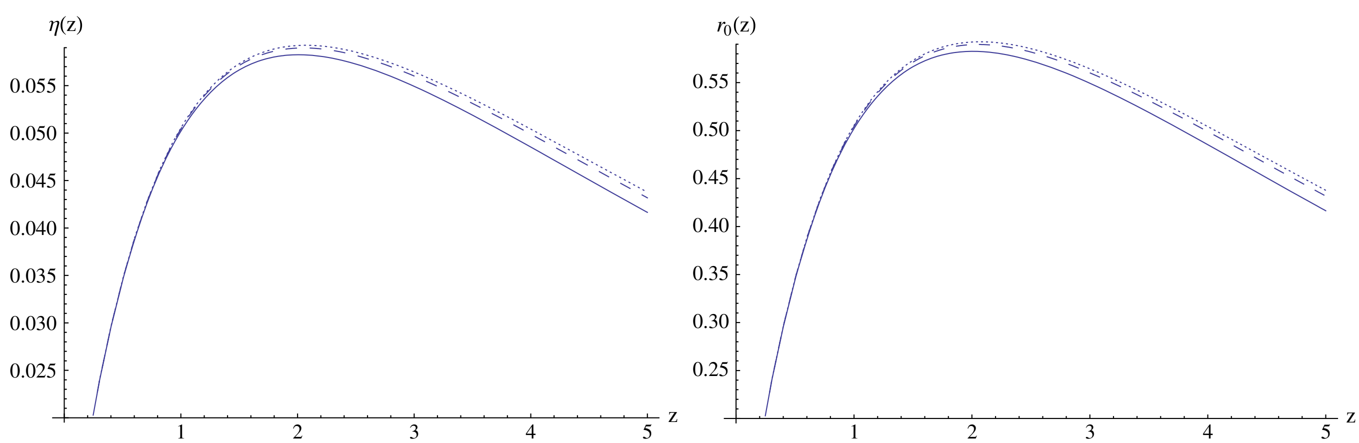

FIG. 4 (color online). Plot of the deviation vector magnitude $\eta(z)$ (left) and plot of the observer area distance $r_0(z)$ (right). The parameter values chosen are $H_0 = 80$ km/s/Mpc, $\Omega_{m0} = 0.3$, $a = 0.9$, $b = 0.2$, $c = 0.7$, $n = 1$ (solid), $n = 1.2$ (dotted), and $n = 1.8$ (dashed), $\Omega_{r0} = 0$, and $d\eta(z)/dz|_{z=0} = 0.1$.

For the model $T = -6H^2 = T_0$, the behavior of null geodesic deviation and observer area distance are the same as those of the ΛCDM model. This is expected because for $\dot{T} = 0$ our $f(T)$ model is reduced to the ΛCDM model. For the other suggested $f(T)$ models, namely, $f(T) = aT + bT^n + c$ and $f(T) = aT + \frac{bT^n}{1+T^{(n+1)}} + c$, the general behavior of null geodesic deviation and observer area distance are similar to those of ΛCDM. At small values of $z < 0.2$, the relative deviations and relative observer area distance with respect to the ΛCDM model are almost ignorable. However, for redshifts $z \gtrsim 0.2$, the geodesic deviation $\eta(z)$ and the area distance $r_0(z)$ corresponding to the suggested $f(T)$ models are rather larger than those of ΛCDM model. This indicates that the suggested $f(T)$ models predict more acceleration than the ΛCDM model, for large values of redshift.

VI. CONCLUSIONS

In this paper, we have considered the GDE in the GR equivalent of the $f(T)$ gravity model. First, we have calculated the Ricci tensor and the Ricci scalar with the modified field equations in $f(T)$ gravity theory. Then, in the FLRW universe, the geodesic deviation equation corresponding to these GR equivalent quantities of $f(T)$ gravity is obtained. To show the consistency of our approach in constructing the GR equivalent of $f(T)$ gravity, the generalized GDE and Pirani equations are recovered for $f(T) = T - 2\Lambda$. We restricted our attention to extract the GDE for two special cases, namely the fundamental observers and past directed null vector fields. In these two cases we have found the Raychaudhuri equation, the generalized Mattig relation and the diametral angular distance differential for $f(T)$ gravity theory. We have also obtained the geodesic deviation $\eta(z)$ and the area distance $r_0(z)$ corresponding to two suggested $f(T)$ models and compared them with those of the ΛCDM model.

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