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Magnetized relativistic stellar models in Eddington-inspired Born-Infeld gravity

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We consider the structure of the magnetic fields inside the neutron stars in Eddington-inspired Born-Infeld (EiBI) gravity. In order to construct the magnetic fields, we derive the relativistic Grad-Shafranov equation in EiBI and numerically determine the magnetic distribution in such a way that the interior magnetic fields should be connected to the exterior distribution. Then, we find that the magnetic distribution inside the neutron stars in EiBI is qualitatively similar to that in general relativity, where the deviation of magnetic distribution in EiBI from that in general relativity is almost comparable to uncertainty due to the equation of state for the neutron star matter. However, we also find that the magnetic fields in the crust region are almost independent of the coupling constant in EiBI, which suggests a possibility of obtaining the information about the crust equation of state independent from the gravitational theory via the observations of the phenomena associated with the crust region. In any case, since the imprint of EiBI gravity on the magnetic fields is weak, the magnetic fields could be a poor probe of gravitational theories, considering the many magnetic uncertainties.

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I. INTRODUCTION

The magnetic field is one of the principal properties in the phenomena of astrophysical objects. In fact, it is believed that magnetic fields can play an important role during supernova explosions, gamma-ray bursts, jets from active galactic nuclei, and so on. The existence of strongly magnetized neutron stars, the so-called magnetars, is also suggested via the measurements of spin period and its down rate of the central objects in soft gamma repeaters and anomalous x-ray pulsars. According to the magnetic dipole model, the strength of the surface magnetic fields of the magnetars is considered to be as large as 10^{14}–10^{15} G [1,2]. From the soft gamma repeaters, sporadic radiations of γ and x rays are observed, while fierce flare activities called giant flares are also detected on rare occasions. In particular, the quasiperiodic oscillations discovered in the afterglow of giant flares provide evidence for the oscillations of magnetized neutron stars [3]. To theoretically explain the quasiperiodic oscillation frequencies, there are many attempts in terms of crustal oscillations [4–8] and/or magnetic oscillations [9–13]. In any case, in addition to the equation of state (EOS) for neutron star matter, the structure of magnetic fields inside the neutron star is crucial to understanding such phenomena.

On the other hand, the gravitational theory must also be imperative in discussing the relativistic objects. General relativity is a mathematically beautiful theory of gravity, and its validity has been probed through a lot of experiments and astronomical observations. However, most of the verifications of general relativity have been done in a weak field regime, such as the Solar System [14], while the tests in a strong field regime are very poor. Perhaps, the gravitational theory describing the astronomical phenomena in a strong field regime might differ from general relativity. This is a reason why modified theories of gravity are proposed. Since the observable properties could depend on gravitational theory, one would see the imprint of gravitational theory as an inverse problem [15]. In fact, the science technology is developing increasingly, which will enable us to observe the relativistic objects and the phenomena around such objects with high precision. The gravitational waves radiating from such a system are probably also one of them. Through these observations, it is possible to probe gravitational theory [16–18].

As a modified theory of gravity, Eddington-inspired Born-Infeld (EiBI) gravity has recently drawn attention in the context of the avoidance of big bang singularity [19,20]. This theory was originally proposed by Bañados and Ferreira [21], based on the gravitational action proposed by Eddington [22] and on the nonlinear electrodynamics of Born and Infeld [23]. EiBI was developed according to a Paratini approach, where the connection is considered as an independent field because the field equations contain ghosts in the metric approach [24]. The deviation of EiBI from general relativity can be seen only when matter exists, i.e., EiBI in vacuum is completely equivalent to general relativity, and the deviation becomes significant in a high density region. Thus, the compact objects are good candidates to display such a deviation. To date, compact objects in EiBI have been discussed on several occasions and have shown a deviation in stellar properties from the

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expectations of general relativity [25–31]. Perhaps, via the direct observations of such stellar properties, one would distinguish EiBI from general relativity. We remark that in EiBI the curvature singularity can appear at the stellar surface for polytropic EOSs [32], which presents a problem to solve even though this theory is attractive.

However, in spite of the importance of magnetic effects in astronomical phenomena, the magnetic fields on the neutron stars in EiBI have not been considered. There are solely the considerations of electrically charged black holes in EiBI [21,33,34]. Thus, in this paper, we consider the magnetized relativistic stellar models in EiBI. This discussion could serve as a first step in examining the phenomena associated with neutron stars in EiBI. Actually, there are many uncertainties regarding the magnetic properties, such as their geometry and the currents supporting them, even for a given fixed EOS. So, it must be quite difficult to see the imprints of gravitational theory on the magnetic properties if neutron stars would have different magnetic geometry and/or crust properties irrespective of the theory of gravity. In this paper, to see how magnetic fields depend on gravitational theory, we focus especially on the axisymmetric dipole configuration of magnetic fields because such a configuration could be dominant in old neutron stars. Additional factors in determining magnetic properties should be taken into account, but we neglect such effects here to simplify the problem. In this paper, we adopt the geometric units $c = G = 1$, where c and G denote the speed of light and the gravitational constant, respectively, and the metric signature is $(-,+,+,+)$.

II. MAGNETIZED STELLAR MODELS IN EiBI

Before considering the stellar models in EiBI, we briefly mention EiBI. This gravitational theory is obtained from the action S given by

$$S = \frac{1}{16\pi}\frac{2}{\kappa}\int d^4x\left(\sqrt{|g_{\mu\nu}+\kappa R_{\mu\nu}|} - \lambda\sqrt{-g}\right) + S_{\rm M}[g,\Psi_{\rm M}], \tag{1}$$

where g and $|g_{\mu\nu}+\kappa R_{\mu\nu}|$ denote the determinants of the physical metric $g_{\mu\nu}$ and $(g_{\mu\nu}+\kappa R_{\mu\nu})$, $R_{\mu\nu}$ is the Ricci tensor constructed from the connection $\Gamma^{\mu}_{\alpha\beta}$, and $S_{\rm M}$ denotes the matter action depending on the metric $g_{\mu\nu}$ and the matter field $\Psi_{\rm M}$. That is, the matter field is assumed to minimally couple to the metric tensor, $g_{\mu\nu}$; i.e., the matter action depends on $g_{\mu\nu}$, independent from the connection Γ. This theory also has a dimensionless constant λ and the Eddington parameter κ, which are related to the cosmological constant as $\Lambda = (\lambda-1)/\kappa$. Since we especially focus on the asymptotically flat solutions ($\Lambda = 0$) in this paper, hereafter we take $\lambda = 1$. On the other hand, κ is constrained from observations of the Solar System, big bang nucleosynthesis, and the existence of neutron stars [21,25,35,36]. The existence of neutron stars can give us a strong constraint on κ, i.e., $|\kappa| \lesssim 1\ \mathrm{m^5\,kg^{-1}\,s^{-2}}$ [25]. Recently, the possibility of constraining κ with the terrestrial measurements of the neutron skin thickness of ^{208}Pb and the astronomical observations of the radius of the $0.5M_\odot$ neutron star has also been suggested [30]. In this paper, we adopt a normalized coupling constant such as $8\pi\kappa\varepsilon_s$, where ε_s denotes the saturation density, i.e., $\varepsilon_s = 2.68\times 10^{14}\ \mathrm{g/cm^3}$. We remark that $8\pi\kappa\varepsilon_s$ becomes a dimensionless parameter.

EiBI is characterized by two independent fields, i.e., the physical metric $g_{\mu\nu}$ and the connection $\Gamma^{\mu}_{\alpha\beta}$. So, varying the action with respect to $\Gamma^{\mu}_{\alpha\beta}$ and $g_{\mu\nu}$, one can obtain the field equations for $\lambda = 1$:

$$q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}, \tag{2}$$

$$\sqrt{-q}q^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - 8\pi\kappa\sqrt{-g}T^{\mu\nu}, \tag{3}$$

where q is a determinant of $q_{\mu\nu}$ and $q_{\mu\nu}$ is an auxiliary metric associated with the connection as $\Gamma^{\mu}_{\alpha\beta} = q^{\mu\sigma}(q_{\sigma\alpha,\beta} + q_{\sigma\beta,\alpha} - q_{\alpha\beta,\sigma})/2$. $T^{\mu\nu}$ denotes the energy-momentum tensor, which is given by $T^{\mu\nu} = (\delta S_{\rm M}/\delta g_{\mu\nu})/\sqrt{-g}$. Equation (3) shows that the auxiliary metric $q_{\mu\nu}$ becomes equivalent to the physical metric $g_{\mu\nu}$ if $T^{\mu\nu} = 0$. That is, EiBI without matter reduces to general relativity in vacuum [21]. In addition to the above field equations, the energy-momentum tensor should satisfy the conservation law, i.e., $\nabla_\mu T^{\mu\nu} = 0$, where the covariant derivative ∇_μ is defined by $g_{\mu\nu}$. As far as we know, unfortunately, there is no explicit proof that the conservation law of $\nabla_\mu T^{\mu\nu} = 0$ is directly derived from field equations (2) and (3). However, since the matter field is minimally coupled to the metric $g_{\mu\nu}$, the conservation law might be obtained as in Ref. [37] if the argument in [37] is applicable even for a bimetric theory like EiBI.

Now, we consider the neutron star models in EiBI. In general, magnetized neutron stars could deform due to nonspherically symmetric magnetic pressure. However, the magnetic energy in the neutron star is much smaller than the gravitational binding energy even for a magnetar, which is a strongly magnetized neutron star. That is, the deformation due to the magnetic pressure is quite small and the shape of the star is almost spherically symmetric. Thus, in this paper, we neglect the stellar deformation induced by the existence of a magnetic field. Under such an assumption, the equilibrium stellar model can be determined as a solution of Tolman-Oppenheimer-Volkov equations in EiBI [25–30]. The metric describing the stellar models is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + f(r)d\Omega^2, \tag{4}$$

$$q_{\mu\nu}dx^\mu dx^\nu = -e^{\beta(r)}dt^2 + e^{\alpha(r)}dr^2 + r^2 d\Omega^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. In this paper, we consider stellar models composed of perfect fluid, i.e., $T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + pg^{\mu\nu}$, where ε, p, and u^μ are the energy density, the pressure, and the four velocity of matter given by $u^\mu = (e^{-\nu/2}, 0, 0, 0)$. Then, one can show that $abf = r^2$ from Eq. (3), where a and b are given by $a = \sqrt{1 + 8\pi\kappa\varepsilon}$ and $b = \sqrt{1 - 8\pi\kappa p}$, respectively [30]. In addition to the Tolman-Oppenheimer-Volkov equations, one needs to prepare the EOS for neutron star matter to construct the stellar models. In particular, we adopt the Friedman-Pandharipande-Skyrme (FPS) [38] and SLy4 EOSs [39], which are based on the Skyrme-type effective interaction (see also [40] for the adopted EOSs). Figure 1 shows the neutron star models constructed with the FPS EOS, where the left panel corresponds to the stellar mass as a function of the central density normalized by ε_s, while the right panel corresponds to the stellar mass as a function of the stellar radius. In this figure, the solid line denotes the results in general relativity and the other lines denote the results in EiBI with various values of $8\pi\kappa\varepsilon_s$. From this figure, one can easily observe that the mass and the radius of neutron stars depend strongly on the coupling constant in EiBI, even if the EOS of neutron star matter is fixed. In fact, the stellar radii of $1.4M_\odot$ neutron stars in EiBI become 9.3% smaller for $8\pi\kappa\varepsilon_s = -0.02$, 7.6% larger for $8\pi\kappa\varepsilon_s = 0.02$, and 16.5% larger for $8\pi\kappa\varepsilon_s = 0.05$ compared to that in general relativity.

On such a neutron star model, we consider an axisymmetric magnetic field generated by a four currency J^μ, adopting an ideal magnetohydrodynamics approximation. The electromagnetic field is governed by the Maxwell equations with the physical metric $g_{\mu\nu}$,

$$F_{[\mu\nu;\alpha]} = 0, \tag{6}$$

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu, \tag{7}$$

where $F_{\mu\nu}$ is the Faraday tensor and the covariant derivative would be calculated with the physical metric $g_{\mu\nu}$. Equation (6) automatically holds by introducing a vector potential, A_μ, associated with $F_{\mu\nu}$ as $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. In order to determine the geometry of the magnetic field, one also needs the equations of motion in addition to Eq. (7), which is obtained by projecting $T^{\mu\nu}{}_{;\nu} = 0$ onto the hypersurface normal to u^μ. With the ideal magnetohydrodynamics approximation, the equations of motion becomes

$$(\varepsilon + p)u_{\mu;\nu}u^\nu + p_{,\mu} + u_\mu u^\nu p_{,\nu} = F_{\mu\nu}J^\nu. \tag{8}$$

Now, assuming the appropriate gauge condition, A_μ can be described as $A_\mu = (0, A_r, 0, A_\phi)$. In general, A_ϕ can be expanded, such that

$$A_\phi(r,\theta) = a_\ell(r)\sin\theta\partial_\theta P_\ell(\cos\theta), \tag{9}$$

where $P_\ell(\cos\theta)$ is the Legendre polynomial of order ℓ. Furthermore, we focus particularly on the dipole magnetic field, i.e., $\ell = 1$, because the dipole fields could be dominant in the neutron stars. Then, in the same way as in Refs. [41,42], one can derive the equation to determine the vector potential a_1:

$$a_1'' + \left(\frac{\nu'}{2} - \frac{\lambda'}{2}\right)a_1' + \left(\zeta^2 e^{-\nu} - \frac{2}{f}\right)e^\lambda a_1 = -4\pi e^\lambda j_1, \tag{10}$$

where the prime denotes partial derivative with respect to r, $j_1 = c_0 f(\varepsilon + p)$, and c_0 is constant. We remark that the constant ζ in Eq. (10) is associated with the radial component of the vector potential, i.e., $A_r = \zeta e^{-\nu/2+\lambda/2}a_\ell P_\ell$. The procedure on how to derive Eq. (10) is detailed in the Appendix. Consequently, since the magnetic field can be given by $B_\mu = \varepsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}/2$, the components of the magnetic field B_μ are expressed as

$$B_r = \frac{2a_1}{f}e^{\lambda/2}\cos\theta, \tag{11}$$

$$B_\theta = -a_1' e^{-\lambda/2}\sin\theta, \tag{12}$$

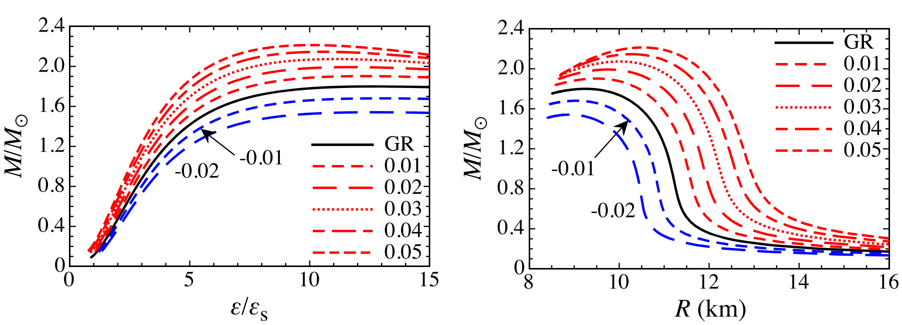

FIG. 1 (color online). Neutron star models in EiBI with FPS EOS. The left panel corresponds to the stellar mass as a function of the central density normalized by the saturation density, while the right panel corresponds to the stellar mass as a function of the stellar radius. The solid line denotes the result in general relativity ($\kappa = 0$) and the other lines denote the results in EiBI with various normalized coupling constant $8\pi\kappa\varepsilon_s$.

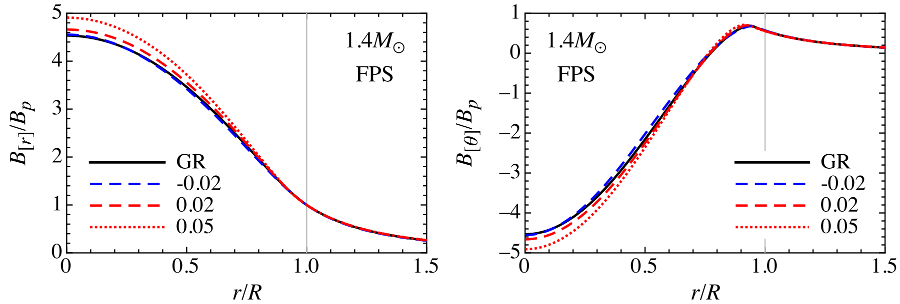

FIG. 2 (color online). For the stellar models with $M = 1.4M_\odot$ for FPS EOS, the tetrad components of the pure poloidal magnetic fields are plotted as a function of r/R, where the left and right panels correspond to the radial component on the symmetry axis ($\theta = 0$) and the θ component on the equatorial plane ($\theta = \pi/2$), respectively. The both components are normalized by B_p, which is the magnetic field strength at the stellar surface of the poles. The solid line corresponds to the result in general relativity, while the broken and dotted lines correspond to the results in EiBI with various values of $8\pi\kappa\varepsilon_s$. The vertical lines denote the position of the stellar surface.

$$B_\phi = -\zeta a_1 e^{-\nu/2}\sin^2\theta, \tag{13}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ denotes the totally antisymmetric tensor and $\varepsilon_{tr\theta\phi} = \sqrt{-g}$. From these expressions, one can see that the constant ζ corresponds to the strength of the toroidal magnetic field. Additionally, the tetrad components of the magnetic fields are given by

$$B_{[r]} = 2a_1 f^{-1}\cos\theta, \tag{14}$$

$$B_{[\theta]} = -a_1' f^{-1/2} e^{-\lambda/2}\sin\theta, \tag{15}$$

$$B_{[\phi]} = -\zeta a_1 f^{-1/2} e^{-\nu/2}\sin\theta. \tag{16}$$

Since we consider that the exterior region of the star is in vacuum, as mentioned before, the spacetime outside the star becomes the same as that in general relativity, which can be described as the Schwarzschild metric. In such a spacetime, the poloidal magnetic field ($\zeta = 0$) is analytically given by

$$a_1^{(\mathrm{ex})} = -\frac{3\mu_b r^2}{8M^3}\left[\ln\left(1-\frac{2M}{r}\right) + \frac{2M}{r} + \frac{2M^2}{r^2}\right], \tag{17}$$

where μ_b is the magnetic dipole moment observed at infinity [43]. Thus, at the stellar surface, the interior solution determined from Eq. (10) should be connected to the exterior solution [Eq. (17)] in such a way that a_1 and a_1' become continuous. In practice, from Eq. (10), one can show that the behavior of a_1 in the vicinity of the stellar center is expressed as $a_1 = \alpha_0 r^2 + \mathcal{O}(r^4)$, where α_0 is an arbitrary constant. So, the arbitrary constants α_0 and c_0, which is a constant in the four currency j_1, are determined so that a_1 and a_1' should be continuous at the stellar surface.

III. NUMERICAL RESULTS

The magnetic field strength B is calculated by $B = (B_\mu B_\nu g^{\mu\nu})^{1/2}$, which can be expressed as

$$B = f^{-1}[4a_1^2\cos^2\theta + a_1'^2 f e^{-\lambda}\sin^2\theta + \zeta^2 a_1^2 f e^{-\nu}\sin^2\theta]^{1/2}. \tag{18}$$

Thus, one can show that the magnetic field strength at the stellar center is $B_0 = 2\alpha_0 a_0 b_0$, where $a_0 = \sqrt{1+8\pi\kappa\varepsilon_0}$ and $b_0 = \sqrt{1-8\pi\kappa p_0}$, while ε_0 and p_0 denote the central values of ε and p. In the limit of $\kappa = 0$, this expression reduces to that in general relativity [44]. The concrete structure of magnetic fields is discussed below, where we separately examine pure poloidal magnetic fields ($\zeta = 0$) in Sec. III A and mixed magnetic fields ($\zeta \neq 0$) in Sec. III B.

A. Pure poloidal magnetic fields ($\zeta = 0$)

First, one can show that the magnetic distribution is scaled by the magnetic field strength at the stellar surface of the poles ($\theta = 0$), B_p, if the stellar model is fixed. That is, the distributions of $B_{[i]}/B_p$ for $i = r$, θ, and ϕ are independent from B_p for each stellar model. In Fig. 2, we show the distributions of $B_{[r]}/B_p$ on the symmetry axis ($\theta = 0$) in the left panel and $B_{[\theta]}/B_p$ on the equatorial plane ($\theta = \pi/2$) in the right panel for the stellar models with $M = 1.4M_\odot$ contracted with FPS EOS, where the solid line corresponds to the result in general relativity and the other lines correspond to the results in EiBI with various values of $8\pi\kappa\varepsilon_s$. From this figure, one can observe that the magnetic distributions in EiBI are qualitatively the same as that in general relativity. In fact, the deviation between the results in general relativity and in EiBI is not much. In Fig. 3, we show the relative deviation of $B_{[r]}/B_p$ in EiBI from that in general relativity for the stellar models with $M = 1.4M_\odot$ constructed with FPS EOS. From this figure, we find that the deviation from general relativity is, at most, 10% from the coupling constant in EiBI adopted in this paper. In particular, the magnetic distribution in the crust region depends weakly on the coupling constant in EiBI, which is less than 0.5%. That is, apart from the gravitational theory, one might be able to discuss the magnetic properties in the crust region of neutron stars. In addition, we show the magnetic configurations on

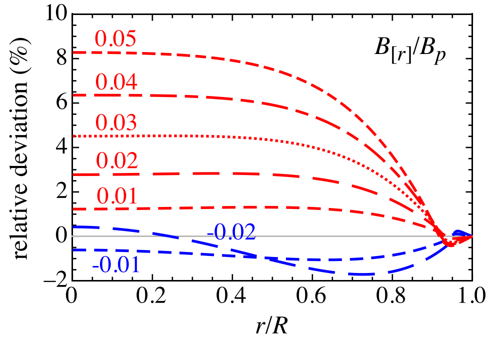

FIG. 3 (color online). Relative deviation of $B_{[r]}/B_p$ in EiBI from that in general relativity for the stellar models with $M = 1.4M_\odot$ constructed with FPS EOS, which corresponds to the left panel in Fig. 2. The labels in the figure denote the values of the coupling constant in EiBI.

the meridional plane for the stellar models with $M = 1.4M_\odot$ for FPS EOS in Fig. 4, where the middle panel corresponds to that in general relativity ($\kappa = 0$), while the left and right panels correspond to those in EiBI with $8\pi\kappa\varepsilon_s = -0.02$ and 0.05. The magnetic field strength is normalized by the magnetic dimple moment. As shown in Fig. 2, the magnetic configurations in EiBI are quite similar to that in general relativity. As with Fig. 2, we also show the magnetic distributions for the stellar models with $M = 1.4M_\odot$ constructed with SLy4 EOS in Fig. 5. Comparing Figs. 2 and 5, we find that the dependence of magnetic distribution on the coupling constant in EiBI is comparable to that on EOS for neutron star matter.

Moreover, the magnetic field strength at the stellar center can be also scaled by B_p for each stellar model, such that

$$B_0 = \beta B_p, \tag{19}$$

where β is a proportionality constant [44]. In Fig. 6, we show the proportionality factor β as a function of the stellar mass with various values of the coupling constant in EiBI, where the left and right panels correspond to the results for the stellar models constructed with FPS and SLy4 EOSs, respectively. From this figure, one can see that the value of β is almost independent of the adopted EOS for neutron star matter, which is ~5. Additionally, for the stellar models whose masses are smaller than a critical value depending

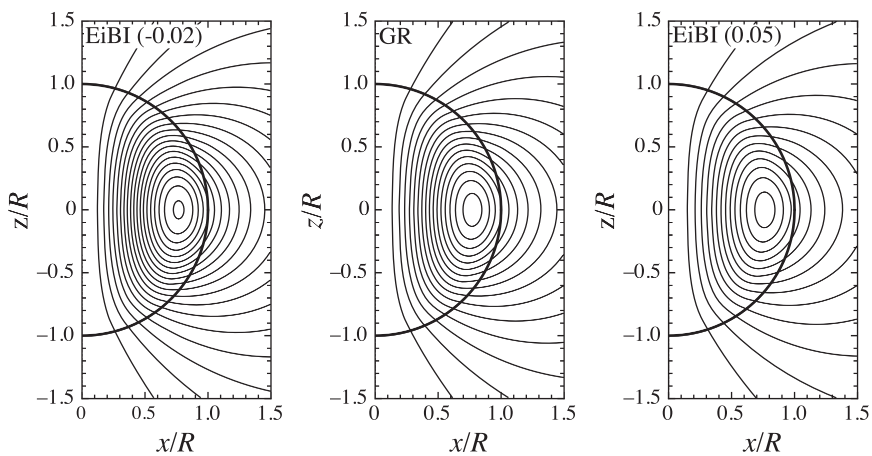

FIG. 4. Magnetic configurations on the meridional plane for the stellar models with $M = 1.4M_\odot$ for FPS EOS in general relativity (middle panel) and in EiBI with $8\pi\kappa\varepsilon_s = -0.02$ (left panel) and 0.05 (right panel). The magnetic field strength is normalized by the magnetic dipole moment μ_b.

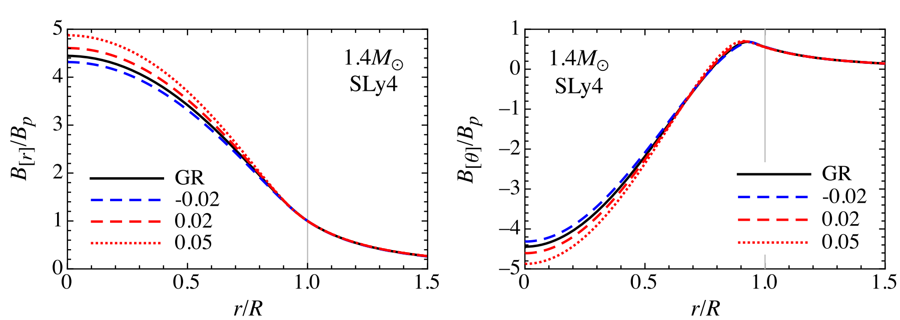

FIG. 5 (color online). Similar to Fig. 2, but for the stellar models constructed with SLy4 EOS.

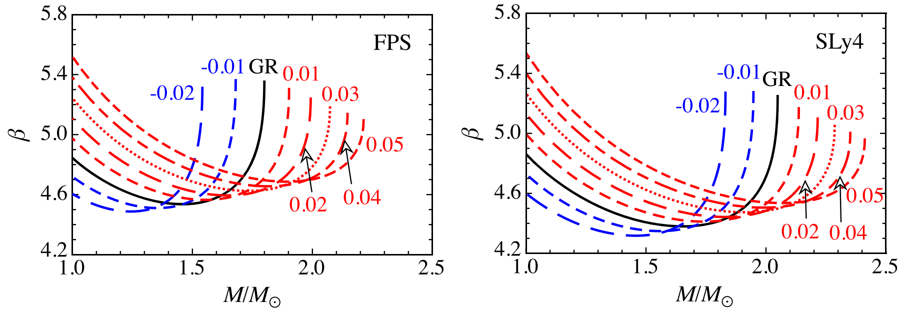

FIG. 6 (color online). The proportionality factor β for the stellar models with various values of coupling constant in EiBI are shown as a function of the stellar mass, where the left and right panels correspond to the results for FPS and SLy4 EOSs, respectively. The labels in the figure denote the values of the coupling constant in EiBI.

on the adopted EOS, β for the fixed stellar mass is almost proportional to the coupling constant in EiBI, at least in the range adopted in this paper. This statement is clear from Fig. 7, where the proportionality factor in Eq. (19) for the fixed stellar mass is shown as a function of the coupling constant in EiBI. From this figure, such a critical stellar mass would be around $1.2M_\odot$ for FPS EOS and $1.4M_\odot$ for SLy4 EOS. Furthermore, from this figure, we find that β for each coupling constant $8\pi\kappa\varepsilon_s$ depends weakly on the EOS for neutron star matter if the mass of the neutron star is very low, for instance $M \simeq M_\odot$.

B. Mixed magnetic fields ($\zeta \neq 0$)

As in the case of the pure poloidal magnetic fields shown in the previous subsection, the distribution of the mixed magnetic fields is also scaled by B_p, and the profiles of $B_{[i]}/B_p$ for $i = r$, θ, and ϕ are independent of the strength of B_p for each stellar model. For reference, we first show the magnetic distributions in general relativity for the stellar models with $M = 1.4M_\odot$ constructed with FPS EOS in Fig. 8. In this figure, the left, middle, and right panels correspond to $B_{[r]}/B_p$ on the symmetry axis ($\theta = 0$),

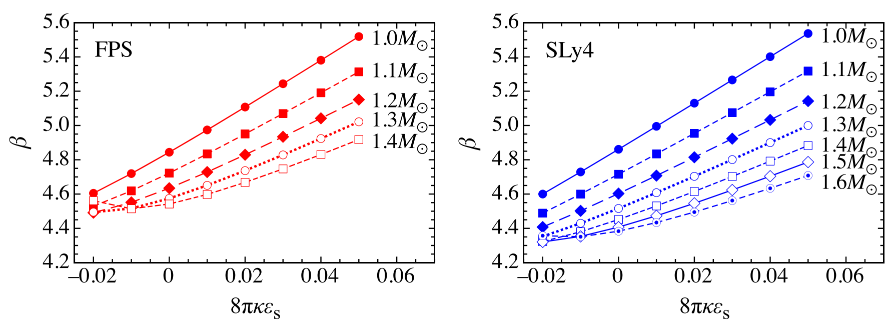

FIG. 7 (color online). The proportionality factor β as a function of the coupling constant in EiBI for the fixed stellar mass, where the left and right panels correspond to the results for FPS and SLy4 EOSs, respectively.

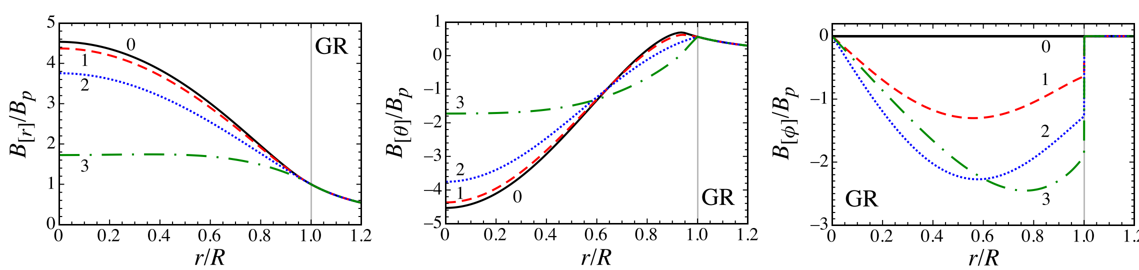

FIG. 8 (color online). For the stellar models in general relativity with $M = 1.4M_\odot$ constructed with FPS EOS, the tetrad components of the mixed magnetic fields normalized by B_p are plotted as a function of r/R, where the left, middle, and right panels are $B_{[r]}/B_p$ on the symmetry axis ($\theta = 0$), $B_{[\theta]}/B_p$ on the equatorial plane ($\theta = \pi/2$), and $B_{[\phi]}/B_p$ on the equatorial plane, respectively. The different lines in the figure correspond to the magnetic field distributions with different values of ζR, and the labels in the figure denote the value of ζR.

$B_{[\theta]}/B_p$ on the equatorial plane ($\theta = \pi/2$), and $B_{[\phi]}/B_p$ on the equatorial plane, respectively. The solid line denotes the magnetic distribution for the pure poloidal field, while the other lines denote those for the mixed fields. As mentioned before, the toroidal magnetic component is characterized by the parameter ζ, as in Eq. (16). In Fig. 8, we show the magnetic distributions with the variable values of ζ normalized by $1/R$ because ζ is a parameter with the dimension of inverse of length and ζR then becomes a dimensionless parameter. From this figure, one can observe that the distributions of $B_{[r]}/B_p$ and $B_{[\theta]}/B_p$ are also changed due to the existence of the toroidal magnetic field, where the central field strengths of $B_{[r]}/B_p$ and $B_{[\theta]}/B_p$ decrease with the value of ζ. This result suggests the existence of the maximum of ζ where the central values of $B_{[r]}/B_p$ and $B_{[\theta]}/B_p$ become zero. Hereafter, such a maximum of ζ denotes $\zeta_{\max}$, and $\zeta_{\max} R = 3.30$ for the case of the neutron star model in Fig. 8. In practice, with ζ more than $\zeta_{\max}$, the direction of the magnetic field can be opposite inside the star [41,42]. Additionally, from Fig. 8, one can see that the position where $|B_{[\phi]}/B_p|$ becomes maximum is shifting outward with the value of ζ.

On the other hand, the magnetic distributions of $B_{[r]}/B_p$, $B_{[\theta]}/B_p$, and $B_{[\phi]}/B_p$ for the stellar models in EiBI with various coupling constants are shown in Fig. 9, where the upper, middle, and lower panels correspond to the results with $8\pi\kappa\varepsilon_s = -0.02$, 0.02, and 0.05, respectively. Comparing this figure with Fig. 8, one can see that the profiles of the magnetic distributions in EiBI are basically similar to that of general relativity, where the distributions of $B_{[r]}/B_p$ and $B_{[\theta]}/B_p$ depend strongly on that of $B_{[\phi]}/B_p$. We also find that, as the coupling constant becomes smaller, the magnetic distributions are more sensitive to the value of ζR. For example, for $\zeta R = 3$, one can see that $B_{[r]}/B_p$ and $|B_{[\theta]}/B_p|$ with $8\pi\kappa\varepsilon_s = -0.02$ in the vicinity of the stellar center become smaller than in general relativity. As a result, it is expected that the value of $\zeta_{\max} R$ for the stellar model with a smaller coupling constant in EiBI could be smaller. In fact, we find that $\zeta_{\max} R = 3.07$, 3.30, 3.41, and 3.51 for the stellar models with $M = 1.4M_\odot$ constructed with FPS EOS with $8\pi\kappa\varepsilon_s = -0.02$, 0, 0.02, and 0.05, respectively.

Furthermore, in Fig. 10, we show the values of $\zeta_{\max} R$ for the stellar models with various stellar masses from $M = M_\odot$ up to the maximum mass, where the left and right

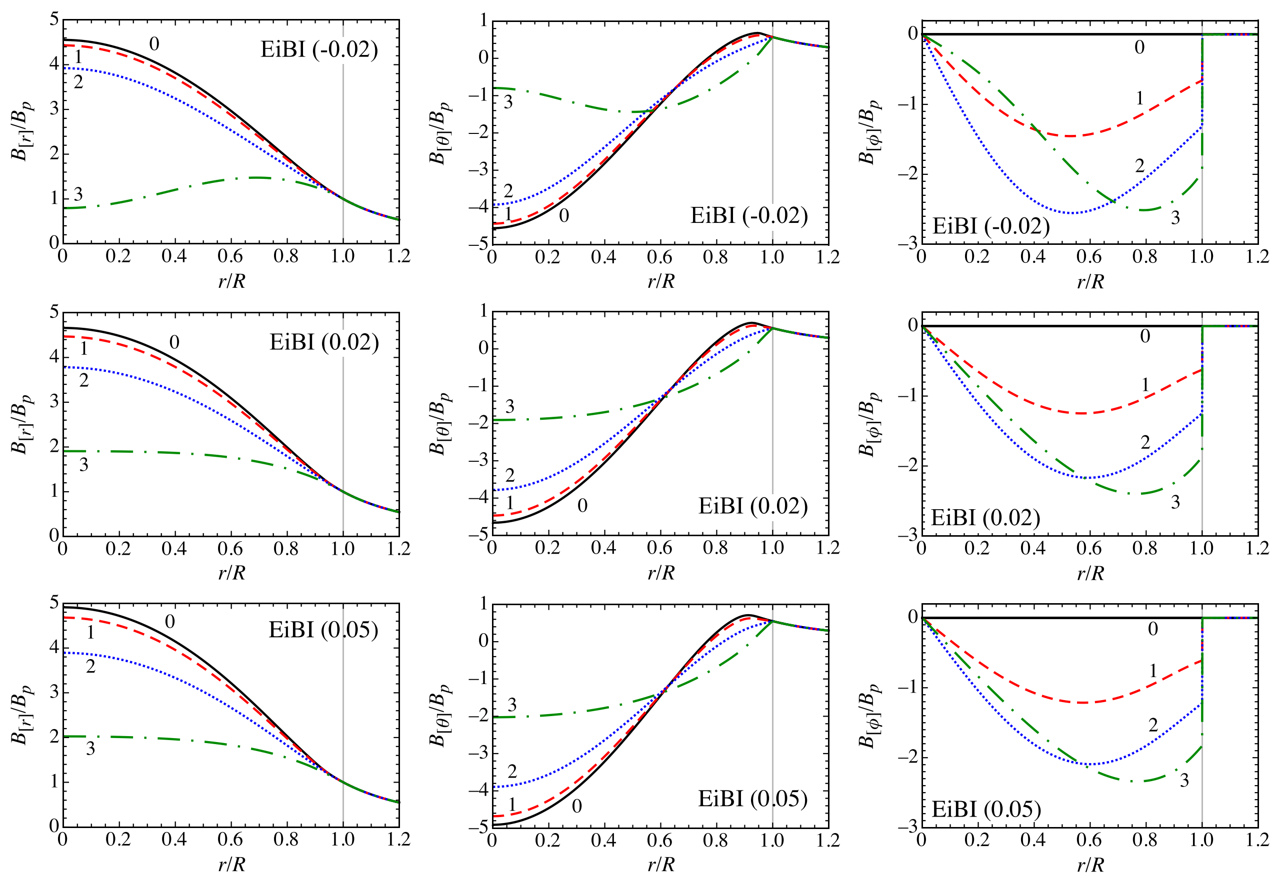

FIG. 9 (color online). Similar to Fig. 8, but in EiBI with various coupling constants. Upper, middle, and lower panels correspond to the results in EiBI with $8\pi\kappa\varepsilon_s = -0.02$, 0.02, and 0.05, respectively.

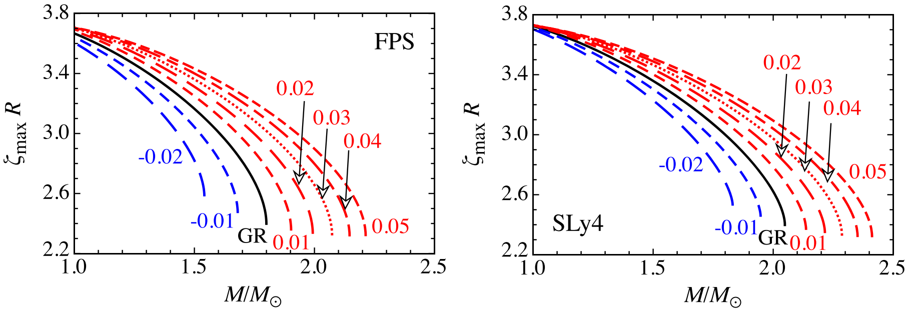

FIG. 10 (color online). Maximum value of ζ allowed for the stellar models with various coupling constants in EiBI are shown as a function of the stellar mass, where the left and right panels correspond to the results for FPS and SLy4 EOSs, respectively. The labels in the figure denote the values of the coupling constant in EiBI.

panels correspond to the results for the stellar models constructed with FPS and SLy4 EOSs, respectively, and the labels in the figure denote the values of the coupling constant in EiBI. From this figure, one can see that the value of $\zeta_{\max}R$ for the low-mass neutron star model is almost independent of not only the EOS but also the coupling constant in EiBI. On the other hand, the magnetic fields for the stellar models with canonical mass are more or less dependent on both the EOS and the coupling constant in EiBI. That is, an uncertainty due to the EOS for neutron star matter is degenerate into that due to the coupling constant in EiBI. Thus, only the measurement of the magnetic properties for the neutron star with canonical mass might be insufficient to observationally distinguish EiBI from general relativity. In any case, the additional observations of the relativistic objects must become important in probing the gravitational theory in the strong field regime.

IV. CONCLUSION

We consider the magnetic fields in the neutron stars in EiBI, where we especially focus on the dipole magnetic fields because such fields must be dominant in old neutron stars. To construct magnetic fields inside the neutron star, we derive the relativistic Grad-Shafranov equation in EiBI. Since the spacetime in vacuum in EiBI is equivalent to that in general relativity, i.e., the Schwarzschild spacetime, the magnetic field in EiBI outside the star is also equivalent to that in general relativity. In such a way that the interior magnetic fields should be connected to the exterior solution, the structure of magnetic fields is determined. Then, we find that the magnetic geometry inside the neutron stars in EiBI is qualitatively similar to that of general relativity. The deviation of magnetic fields in EiBI from that in general relativity is not much, which is almost comparable to the uncertainty due to the EOS for neutron star matter. Therefore, it might be difficult to distinguish EiBI from general relativity by only using the observations of the magnetic properties in neutron stars. However, the magnetic fields in the crust region for the neutron star with canonical mass depend weakly on the coupling constant in EiBI, while the crust properties such as the crust thickness depend strongly on the EOS for neutron star matter [9]. That is, independent from gravitational theory, one might be able to see the information about the EOS in the crust region through the observations associated with the phenomena in the crust region, such as the stellar oscillations. In any case, there are many uncertainties regarding magnetic properties even in general relativity, such as the magnetic geometry and the current distribution supporting the fields, although we consider only dipole magnetic fields in this paper. Compared to such uncertainties, the imprint of EiBI gravity on magnetic fields is weak, which suggests that the magnetic field could be a poor probe for gravitational theories.

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APPENDIX: DERIVATION OF EQ. (10)

According to Refs. [41,42], we briefly show in this appendix how to derive Eq. (10), which is the equation for determining the distribution of the magnetic field inside the star. One can obtain the following equations from Eq. (7):

$$4\pi J^r = -\frac{1}{f}e^{-\lambda}\left(A_{r,\theta\theta} + A_{r,\theta}\frac{\cos\theta}{\sin\theta}\right), \tag{A1}$$

$$4\pi J^\theta = \frac{1}{f}e^{-\lambda}\left[A_{r,\theta r} + A_{r,\theta}\left(\frac{\nu'}{2} - \frac{\lambda'}{2}\right)\right], \tag{A2}$$

$$4\pi J^\phi \sin^2\theta = -\frac{1}{f}e^{-\lambda}\left[A_{\phi,rr} + A_{\phi,r}\left(\frac{\nu'}{2} - \frac{\lambda'}{2}\right)\right] - \frac{1}{f^2}A_{\phi,\theta\theta} + \frac{1}{f^2}\frac{\cos\theta}{\sin\theta}A_{\phi,\theta}, \tag{A3}$$

while from Eq. (8),

$$-A_{r,\theta}J^{\theta} + A_{\phi,r}J^{\phi} = (\varepsilon + p)\frac{\nu'}{2} + p', \tag{A4}$$

$$A_{r,\theta}J^{r} + A_{\phi,\theta}J^{\phi} = 0, \tag{A5}$$

$$A_{\phi,r}J^{r} + A_{\phi,\theta}J^{\theta} = 0. \tag{A6}$$

Equation (A6) with Eqs. (A1) and (A2) can be written as

$$-\eta_{,\theta}A_{\phi,r} + \eta_{,r}A_{\phi,\theta} = 0, \tag{A7}$$

where $\eta \equiv e^{\nu/2-\lambda/2}A_{r,\theta}\sin\theta$. Thus, η should depend only on A_ϕ, as $\eta = \zeta A_\phi$ with a constant ζ. As a result, A_r is expressed as $A_r = \zeta e^{-\nu/2+\lambda/2}a_\ell P_\ell$ if A_ϕ is expanded as in Eq. (9).

On the other hand, Eqs. (A4) and (A5) become

$$\chi_{,r} = A_{\phi,r}\mathcal{J}, \tag{A8}$$

$$\chi_{,\theta} = A_{\phi,\theta}\mathcal{J}, \tag{A9}$$

where χ and $\mathcal{J}$ are defined as

$$\chi_{,r} = (\varepsilon + p)\frac{\nu'}{2} + p', \tag{A10}$$

$$\mathcal{J} = \frac{1}{f(\varepsilon + p)\sin^2\theta}\left(J_\phi - \frac{\zeta^2}{4\pi}e^{-\nu}A_\phi\right). \tag{A11}$$

Owing to the relation $\chi_{,r\theta} = \chi_{,\theta r}$ with Eqs. (A8) and (A9), one can obtain

$$A_{\phi,r}\mathcal{J}_{,\theta} - A_{\phi,\theta}\mathcal{J}_{,r} = 0. \tag{A12}$$

Therefore, $\mathcal{J}$ also depends only on A_ϕ, i.e., $\mathcal{J} = -c_0 - c_1 A_\phi$, where c_0 and c_1 are constants. That is,

$$J_\phi = \frac{\zeta^2}{4\pi}e^{-\nu}A_\phi - (c_0 + c_1 A_\phi)f(\varepsilon + p)\sin^2\theta. \tag{A13}$$

Finally, substituting Eqs. (9) and (A13) into Eq. (A3), one can obtain Eq. (10), which describes the function of a_1. We remark that the term c_1 in Eq. (A13) is neglected because we focus on the dipole ($\ell = 1$) magnetic distribution in this paper and the term c_1 can contribute as the multipole higher than $\ell = 3$ [41,42].

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