Parametrized post-Friedmannian framework for interacting dark energy theories

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We present the most general parametrization of models of dark energy in the form of a scalar field which is explicitly coupled to dark matter. We follow and extend the parametrized post-Friedmannian approach, previously applied to modified gravity theories, in order to include interacting dark energy. We demonstrate its use through a number of worked examples and show how the initially large parameter space of free functions can be significantly reduced and constrained to include only a few nonzero coefficients. This paves the way for a model-independent approach to classify and test interacting dark energy theories.

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I. INTRODUCTION

In recent years, cosmological data from experiments with exquisite precision (cosmic microwave background measurements [1,2], Ia supernovae [3], baryon acoustic oscillation surveys [4]) suggest that ∼96% of the matter/energy content of our Universe is in the form of an exotic dark sector. Approximately a quarter of the dark sector is believed to be weakly interacting cold dark matter, while roughly 70% is in the form of a dark energy component, a substance with negative pressure responsible for the current accelerated expansion of the Universe.

The best candidate for dark energy is the cosmological constant Λ. The concordance model of cosmology, ΛCDM, is currently the best fit to observations, but it comes along with fundamental questions and problems. One of them is the coincidence problem, which poses the question of why the energy densities of the dark sector components are of the same order today, when their cosmological evolution is very different. A possible solution to the coincidence problem is a coupling between the dark energy and the dark matter. The introduction of an appropriate coupling does not violate observational constraints, and it can change the background evolution of the dark sector components in order to offer a solution to the coincidence problem.

A plethora of such dark coupling models can be found in the literature (see, e.g. [5–38]). In most of these models, the choice of the coupling is purely phenomenological. In a recent paper [39], we made further progress at the level of construction of such models by identifying three separate *classes* of models of dark energy in the form of a scalar field (ϕ) coupled to cold dark matter (CDM).

After constructing general models of exotic dark energy or modified gravity and checking their mathematical and physical viability (for example by identifying fundamental problems like ghosts or strong coupling issues), one is interested in testing them against the available data to see if they might offer a viable alternative to ΛCDM.

Currently, there is a pressing need for fast and efficient ways to rule out and constrain the large number of cosmological models available—it would be practically impossible to go through each and every one of them individually. The parametrized post-Friedmannian (PPF) approach offers such a framework and has been applied to modified gravity theories [40–42] (see [43] for a recent overview). In this work we apply the PPF approach to interacting dark energy theories and demonstrate its use through a number of worked examples. In Sec. II we go through the PPF basic principles and general formalism, extending it to the case of coupled dark matter/dark energy. In Sec. III we first demonstrate how a few of the most well-known phenomenological models in the literature fit in to this formalism and then we proceed to apply it to the general classes of models we presented in [39]. We conclude in Sec. IV.

II. FORMALISM

A. Basic concepts

We start by writing the gravitational field equations of a theory as

$$G_{\mu\nu} = 8\pi G(T^{(\mathrm{SM})}_{\mu\nu} + T^{(\mathrm{GDM})}_{\mu\nu} + T^{(\mathrm{DE})}_{\mu\nu}), \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor of the metric $g_{\mu\nu}$, $T^{(\mathrm{SM})}_{\mu\nu}$ is the stress-energy tensor of the *known* forms of matter

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(baryons, photons, neutrinos, etc.) that are part of the Standard Model of particle physics, $T^{(\mathrm{GDM})}_{\mu\nu}$ is the stress-energy tensor of (generalized) dark matter and $T^{(\mathrm{DE})}_{\mu\nu}$ represents the stress-energy tensor of all the unknown modifications to the gravitational field equations that generate the effect of dark energy. Such modifications may be purely due to a dark energy fluid or perhaps due to a modification of gravity. It may be shown that any kind of modification of gravity can be put in the form (1) (see for instance [40]). Let us also note that although we start with a generalized dark matter which may have nonzero pressure and nonzero shear [44], we shall later on specialize to the CDM case where both of these quantities are zero.

The Bianchi identities tell us that the Einstein tensor is divergenceless:

$$\nabla_\mu G^\mu{}_\nu = 0, \tag{2}$$

which in turn implies that $\nabla_\mu(T^{(\mathrm{SM})\mu}{}_\nu + T^{(\mathrm{GDM})\mu}{}_\nu + T^{(\mathrm{DE})\mu}{}_\nu) = 0$. We assume that the Standard Model particles do not explicitly couple to the dark sector so that $\nabla_\mu T^{(\mathrm{SM})\mu}{}_\nu = 0$. This assumption is well justified by observations which strongly constrain such couplings [45]. Furthermore, a coupling of the evolving quintessence field to baryons would lead to time varying constants of nature, which are tightly constrained, see [46] and references therein. This leaves us with $\nabla_\mu(T^{(\mathrm{GDM})\mu}{}_\nu + T^{(\mathrm{DE})\mu}{}_\nu) = 0$ but neither part is assumed to be individually conserved. Thus we have that

$$\nabla_\mu T^{(\mathrm{GDM})\mu}{}_\nu = J_\nu = -\nabla_\mu T^{(\mathrm{DE})\mu}{}_\nu \tag{3}$$

where the coupling current J_ν represents the energy and momentum exchange between the dark sector components.

In what follows we aim to parametrize the coupling current J_ν in terms of metric potentials and their derivatives as well as the scalar modes that are part of the stress-energy tensors of the two dark sector components. We shall do that in such a way so that the resulting field equations contain at most two time derivatives, or equivalently, each dark sector component obeys two first order linearized field equations on a Friedmann-Robertson-Walker (FRW) background resulting from (3). We shall proceed by considering first a FRW background spacetime and finding the relevant equations that describe the dark sector and then considering linear perturbations about this background spacetime and see how this affects the parametrization. Background variables will be signified with an overbar (unless no confusion could arise, e.g. the scale factor a is always a background variable) while typically all perturbed tensors will be preceded by a δ. For instance, we may split J_ν into $J_\nu = \bar{J}_\nu + \delta J_\nu$.

B. FRW background

Consider a FRW background spacetime described by a metric

$$ds^2 = a^2(-d\tau^2 + \gamma_{ij}dx^i dx^j) \tag{4}$$

where a is the scale factor, τ is the conformal time and γ_{ij} is the spatial metric, assumed to be flat. The symmetries of the spacetime impose that the only nonzero components $\bar{T}_{\mu\nu}$ are the energy density $\bar{\rho} = -\bar{T}^0{}_0$ and the pressure $\bar{P}$ such that $\bar{T}^i{}_j = \bar{P}\delta^i{}_j$.

The generalized Einstein equations (1) for this ansatz give

$$3\mathcal{H}^2 = 8\pi G a^2(\bar{\rho}_{\mathrm{SM}} + \bar{\rho}_{\mathrm{GDM}} + \bar{\rho}_{\mathrm{DE}}) \tag{5}$$

and

$$\mathcal{H}^2 - 2\frac{\ddot{a}}{a} = 8\pi G(\bar{P}_{\mathrm{SM}} + \bar{P}_{\mathrm{GDM}} + \bar{P}_{\mathrm{DE}}), \tag{6}$$

where $\mathcal{H} = \frac{\dot{a}}{a}$ is the conformal Hubble parameter and dots denote derivatives with respect to τ.

Turning now to the coupling current J_ν, the symmetries of the spacetime impose that the only nonzero component is

$$Q \equiv \bar{J}_0 \tag{7}$$

while $\bar{J}_i = 0$. The function $Q(\tau)$ is the background coupling function which is for our purposes a phenomenologically free function. Specific models of a coupled dark sector will in general result to specific choices of $Q(\tau)$ (see, for example, [5,25], which are two models we present and parametrize in Sec. III).

The $\nu = 0$ component of (3) gives the field equations for the evolution of a particular component indexed by I as

$$\dot{\bar{\rho}}_I + 3\mathcal{H}\bar{\rho}_I(1 + w_I) = s_I Q, \tag{8}$$

where we have defined the equation of state parameter for each I-component as $w_I \equiv \bar{P}_I/\bar{\rho}_I$ and the constant s_I takes the values

$$s_I = \begin{cases} 1 & \text{DE} \\ 0 & \text{SM fields}\,. \\ -1 & \text{GDM} \end{cases} \tag{9}$$

C. Linear perturbations

1. The perturbed variables

We now turn to linear perturbations about the FRW background. We shall consider only scalar modes. The spacetime metric takes the form

$$ds^2 = -a^2(1+2\Psi)dt^2 - 2a^2\nabla_i\zeta dt dx^i + a^2\left[\left(1+\frac{1}{3}h\right)\gamma_{ij} + D_{ij}\nu\right]dx^i dx^j, \tag{10}$$

where Ψ, ζ, h and ν are four functions of time and space (four scalar modes) and

$$D_{ij} = \nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2 \tag{11}$$

is a derivative operator that projects out the longitudinal, traceless, spatial part of the perturbation.

Let us now consider the perturbed variables of the fluids. These are the density contrast $\delta \equiv \delta\rho/\bar{\rho}$, the scalar mode of the momentum, θ, such that $u_i = a\nabla_i\theta$, the dimensionless pressure perturbation $\Pi \equiv \delta P/\bar{\rho}$ such that $\delta T^i_j = \Pi\bar{\rho}\delta^i{}_j$ and the scalar mode of the shear Σ such that the shear tensor is $\Sigma_{ij} = D_{ij}\Sigma$. Putting it all together, the stress-energy tensor components for a fluid are

$$T^0{}_0 = -\bar{\rho}(1+\delta), \tag{12}$$

$$T^0{}_i = -(\bar{\rho}+\bar{P})\vec{\nabla}_i\theta, \tag{13}$$

$$T^i{}_0 = (\bar{\rho}+\bar{P})\vec{\nabla}^i(\theta-\zeta), \tag{14}$$

$$T^i{}_j = \bar{\rho}(w+\Pi)\delta^i{}_j + (\bar{\rho}+\bar{P})D^i{}_j\Sigma. \tag{15}$$

2. Einstein and fluid equations

The perturbed Einstein equations (1) are

$$\mathcal{H}(\dot{h}+2\vec{\nabla}^2\zeta) - 6\mathcal{H}^2\Psi + 2\vec{\nabla}^2\eta = 8\pi Ga^2\sum_I\bar{\rho}_I\delta_I, \tag{16a}$$

$$2\dot{\eta} + 2\mathcal{H}\Psi = 8\pi Ga^2\sum_I(\bar{\rho}_I+\bar{P}_I)\theta_I, \tag{16b}$$

$$\begin{aligned}&-\ddot{h} - 2\mathcal{H}\dot{h} + 6\mathcal{H}\dot{\Psi} + 6(\mathcal{H}^2+2\dot{\mathcal{H}})\Psi\\ &-\vec{\nabla}^2(2\eta - 2\Psi + 2\dot{\zeta} + 4\mathcal{H}\zeta)\\ &= 24\pi Ga^2\sum_I\bar{\rho}_I\Pi_I\end{aligned} \tag{16c}$$

and

$$\begin{aligned}&\frac{1}{2}\ddot{\nu} + \dot{\zeta} + \mathcal{H}(\dot{\nu}+2\zeta) + \eta - \Psi\\ &= 8\pi Ga^2\sum_I(\bar{\rho}_I+\bar{P}_I)\Sigma_I.\end{aligned} \tag{16d}$$

Turning now to the fluid equations, they are obtained by perturbing (3). To this purpose we define the two scalar mode perturbations q and S by

$$q \equiv \delta J_0 \qquad \vec{\nabla}_i S \equiv \delta J_i. \tag{17}$$

We find

$$\begin{aligned}\dot{\delta}_I &= 3w_I\mathcal{H}\delta_I + (1+w_I)\left[\vec{\nabla}^2\theta_I - \frac{1}{2}\dot{h} - \vec{\nabla}^2\zeta\right]\\ &\quad - 3\mathcal{H}\Pi_I + \frac{s_I}{\bar{\rho}_I}[q - Q\delta_I],\end{aligned} \tag{18a}$$

and

$$\begin{aligned}\dot{\theta}_I &= -\left[\mathcal{H}(1-3w_I) + \frac{\dot{w}_I}{1+w_I}\right]\theta_I + \frac{\Pi_I}{1+w_I}\\ &\quad + \frac{2}{3}\vec{\nabla}^2\Sigma_I + \Psi + \frac{s_I}{\bar{\rho}_I}\left[\frac{S}{1+w_I} - Q\theta_I\right].\end{aligned} \tag{18b}$$

where the index I runs over all species (and once again let us recall that $s_{\rm DE} = 1 = -s_{\rm GDM}$ while $s_I = 0$ for all other species).

D. Dark coupling parametrization

The goal of this article is to parametrize both of the two perturbation variables q and S as linear combinations of all other perturbations, such as the fluid variables δ, θ, Π and Σ for each fluid, as well as the metric variables Ψ, ζ, h and ν. This means 12 variables in total for both q and S. However, this linear combination is not entirely arbitrary, but must obey certain rules regarding gauge transformations. As we shall see, this reduces the number of effective independent variables to ten for both q and S. Before proceeding to the parametrization, let us briefly discuss gauge transformations.

1. Gauge transformations

The metric in (10) is in a form which is not gauge fixed. In other words the four scalar modes are not invariant under gauge transformations $\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \underset{\xi}{\mathcal{L}}\bar{g}_{\mu\nu}$ generated by a vector field ξ^μ. Parametrized as $\xi^\mu = \frac{1}{a}(\xi_T, \vec{\nabla}^i\xi_L)$ for two scalar modes ξ_T and ξ_L the gauge transformations of the metric and fluid perturbations will involve combinations of ξ_T and ξ_L and their first time derivatives.

Consider first the variables q and S. We find that they transform as

$$q \to q + \frac{1}{a}[Q\dot{\xi}_T + (\dot{Q} - \mathcal{H}Q)\xi_T] \tag{19a}$$

and

$$S \to S + \frac{1}{a}Q\xi_T, \tag{19b}$$

respectively. Thus if we write q and S as a linear combination of the metric and fluid variables, variables which involve ξ_L in their transformation must combine together so that ξ_L does not appear overall in the transformation of the entire linear combination.

Now the fluid variables transform only with the gauge variable ξ_T, i.e. as (dropping the obvious I indices)

$$\delta \to \delta - \frac{1}{a}\left[3\mathcal{H}(1+w) - s\frac{Q}{\bar{\rho}}\right]\xi_T, \tag{20a}$$

$$\theta \to \theta + \frac{1}{a}\xi_T, \tag{20b}$$

$$\Pi \to \Pi + \frac{1}{a}\left[\dot{w} - 3\mathcal{H}(1+w)w + sw\frac{Q}{\bar{\rho}}\right]\xi_T, \tag{20c}$$

while Σ is gauge invariant, hence, all four of them are allowed to appear in the q and S parametrization.

However, the metric variables involve ξ_L in their transformation. This means that the metric variables must combine together so that ξ_L is eliminated altogether. Following [40] we can find three linear combinations of the metric perturbations and their first time derivatives which transform only with the gauge variable ξ_T. These are $U \equiv h - \nabla^2\nu$ and $V \equiv \dot{\nu} + 2\zeta$ as well as $\dot{h} + 2\nabla^2\zeta$. The latter one is not independent but is equal to $\dot{U} + \nabla^2 V$. Thus out of the four metric scalar modes, we are left with two combinations, namely U and V, which transform exclusively with ξ_T and Ψ which transforms with $\dot{\xi}_T$. Explicitly, the transformations are

$$U \to U + \frac{6}{a}\mathcal{H}\xi_T, \tag{21a}$$

$$V \to V + \frac{2}{a}\xi_T, \tag{21b}$$

$$\Psi \to \Psi + \frac{\dot{\xi}_T}{a}. \tag{21c}$$

Since q and Ψ contain $\dot{\xi}_T$ in their transformation, we must allow a further metric variable combination which does so, but which does not have higher than second time derivatives. The only possibility is the variable $\dot{U}$.

To summarize, we expect that both q and S can be written as linear combinations of the four fluid variables (for each fluid) plus the four variables U, V, Ψ and $\dot{U}$.

2. Completing the parametrization

Following the discussion above, we start from the parametrization

$$\begin{aligned} q = {} & C_1\Psi + C_2 V + A_1 U + A_2\dot{U} + A_3\delta_{\mathrm{DE}} + A_4\delta_{\mathrm{GDM}} \\ & + A_5\theta_{\mathrm{DE}} + A_6\theta_{\mathrm{GDM}} + A_7\Pi_{\mathrm{DE}} + A_8\Pi_{\mathrm{GDM}} \\ & + A_9\Sigma_{\mathrm{DE}} + A_{10}\Sigma_{\mathrm{GDM}} \end{aligned} \tag{22}$$

and

$$\begin{aligned} S = {} & C_3\Psi + C_4 V + B_1 U + B_2\dot{U} + B_3\delta_{\mathrm{DE}} + B_4\delta_{\mathrm{GDM}} \\ & + B_5\theta_{\mathrm{DE}} + B_6\theta_{\mathrm{GDM}} + B_7\Pi_{\mathrm{DE}} + B_8\Pi_{\mathrm{GDM}} \\ & + B_9\Sigma_{\mathrm{DE}} + B_{10}\Sigma_{\mathrm{GDM}}. \end{aligned} \tag{23}$$

Performing the gauge transformations in (22) we find two constraint equations, namely

$$C_1 = Q - 6\mathcal{H}A_2 \tag{24a}$$

and

$$\begin{aligned} 2C_2 = {} & \dot{Q} - \mathcal{H}Q - 6\mathcal{H}A_1 + 6(\mathcal{H}^2 - \dot{\mathcal{H}})A_2 - \frac{\dot{\bar{\rho}}_{\mathrm{DE}}}{\bar{\rho}_{\mathrm{DE}}}A_3 \\ & - \frac{\dot{\bar{\rho}}_{\mathrm{GDM}}}{\bar{\rho}_{\mathrm{GDM}}}A_4 - A_5 - A_6 - \frac{\dot{\bar{P}}_{\mathrm{DE}}}{\bar{\rho}_{\mathrm{DE}}}A_7 - \frac{\dot{\bar{P}}_{\mathrm{GDM}}}{\bar{\rho}_{\mathrm{GDM}}}A_8. \end{aligned} \tag{24b}$$

Likewise, performing the gauge transformations in (23) we find two further constraint equations, namely

$$C_3 = -6B_2\mathcal{H} \tag{24c}$$

and

$$\begin{aligned} 2C_4 = {} & Q - 6\mathcal{H}B_1 + 6(\mathcal{H}^2 - \dot{\mathcal{H}})B_2 - \frac{\dot{\bar{\rho}}_{\mathrm{DE}}}{\bar{\rho}_{\mathrm{DE}}}B_3 \\ & - \frac{\dot{\bar{\rho}}_{\mathrm{GDM}}}{\bar{\rho}_{\mathrm{GDM}}}B_4 - B_5 - B_6 - \frac{\dot{\bar{P}}_{\mathrm{DE}}}{\bar{\rho}_{\mathrm{DE}}}B_7 - \frac{\dot{\bar{P}}_{\mathrm{GDM}}}{\bar{\rho}_{\mathrm{GDM}}}B_8. \end{aligned} \tag{24d}$$

The two constraints (24a) and (24b) are then used to eliminate C_1 and C_2 from (22) while the two constraints (24c) and (24d) are used to eliminate C_3 and C_4 from (23). The remaining perturbations are written in terms of the gauge-invariant variables listed in Table I by combining them with V. The result is

$$\begin{aligned} q = {} & \frac{1}{2}(\dot{Q} - \mathcal{H}Q)V + Q\Psi - 6A_1\hat{\Phi} - 6A_2\hat{\Gamma} \\ & + A_3\hat{\delta}_{\mathrm{DE}} + A_4\hat{\delta}_{\mathrm{GDM}} + A_5\hat{\theta}_{\mathrm{DE}} + A_6\hat{\theta}_{\mathrm{GDM}} \\ & + A_7\hat{\Pi}_{\mathrm{DE}} + A_8\hat{\Pi}_{\mathrm{GDM}} + A_9\Sigma_{\mathrm{DE}} + A_{10}\Sigma_{\mathrm{GDM}} \end{aligned} \tag{25}$$

and

TABLE I. Gauge-invariant variables.

$\hat{\Phi} \equiv -\frac{1}{6}U + \frac{1}{2}\mathcal{H}V$
$\hat{\Psi} \equiv \Psi - \frac{1}{2}\dot{V} - \frac{1}{2}\mathcal{H}V$
$\hat{\Gamma} \equiv -\frac{1}{6}\dot{U} + \mathcal{H}\Psi + \frac{1}{2}(\dot{\mathcal{H}} - \mathcal{H}^2)V = \dot{\hat{\Phi}} + \mathcal{H}\hat{\Psi}$
$\hat{\delta} \equiv \delta - \frac{1}{2}\frac{\dot{\bar{\rho}}}{\bar{\rho}}V$
$\hat{\theta} \equiv \theta - \frac{1}{2}V$
$\hat{\Pi} \equiv \Pi - \frac{\dot{\bar{P}}}{\bar{\rho}}V$

$$S = \frac{1}{2}QV - 6B_1\hat{\Phi} - 6B_2\hat{\Gamma} + B_3\hat{\delta}_{\mathrm{DE}} + B_4\hat{\delta}_{\mathrm{GDM}} + B_5\hat{\theta}_{\mathrm{DE}} + B_6\hat{\theta}_{\mathrm{GDM}} + B_7\hat{\Pi}_{\mathrm{DE}} + B_8\hat{\Pi}_{\mathrm{GDM}} + B_9\Sigma_{\mathrm{DE}} + B_{10}\Sigma_{\mathrm{GDM}}. \tag{26}$$

Hence, we are left with 20 free functions in total.

3. Special case: Cold dark matter

From now on we will assume that the dark matter fluid is completely cold. This automatically means that $w_{\mathrm{GDM}} = \Pi_{\mathrm{GDM}} = \Sigma_{\mathrm{GDM}} = 0$. We shall further make the assumption that the dark energy fluid has no shear, i.e. $\Sigma_{\mathrm{DE}} = 0$. Furthermore, since there is no possibility of confusion we shall set $w_{\mathrm{DE}} = w$.

In general, the pressure perturbation Π_{DE} would be an independent dynamical degree of freedom (see [47] for an explicit model). However, there are many instances where Π_{DE} is expressed in terms of δ_{DE} and θ_{DE} via equations of state such as the generalized dark matter model [44]. As in [44] we shall also assume that the pressure perturbation Π_{DE} is expressed in terms of δ_{DE} and θ_{DE} via equations of state. However, the usual expression in [44] no longer holds, as it does not transform correctly under gauge transformations. An expression which does is

$$\Pi_{\mathrm{DE}} = c_s^2\delta_{\mathrm{DE}} + (c_s^2 - c_a^2)\left[3(1+w)\mathcal{H} - \frac{Q}{\bar{\rho}_{\mathrm{DE}}}\right]\theta_{\mathrm{DE}} + \mu(\theta_c - \theta_{\mathrm{DE}}) \tag{27}$$

where c_s^2 and c_a^2 are the (gauge-invariant) effective and adiabatic speeds of sound respectively. It may be shown that the divergence of the entropy flux is proportional to $\Pi_{\mathrm{DE}} - c_a^2\delta_{\mathrm{DE}}$ [48], hence, the gauge-invariant "relative entropy" parameter μ measures entropy transfer to dark energy (DE) due to its motion relative to the CDM fluid. The adiabatic speed of sound is fixed by the equation of state w via

$$c_a^2 = w + \frac{\dot{w}}{\frac{Q}{\bar{\rho}_{\mathrm{DE}}} - 3\mathcal{H}(1+w)}. \tag{28}$$

Hence, without loss of generality, we may further set A_7 and B_7 to zero. With these choices, the number of free functions is reduced to 12.

We shall further assume the conformal Newtonian gauge for which $\zeta = \nu = 0$ (so that $V = 0$). With this choice, the gauge-invariant variables we have defined in Table I are equal to the conformal Newtonian gauge variables.

Let us now restate the parametrization as well as the necessary evolution equations. The two parameters q and S are given by

$$q = Q\Psi - 6A_1\Phi - 6A_2(\dot{\Phi} + \mathcal{H}\Psi) + A_3\delta_{\mathrm{DE}} + A_4\delta_c + A_5\theta_{\mathrm{DE}} + A_6\theta_c \tag{29a}$$

and

$$S = -6B_1\Phi - 6B_2(\dot{\Phi} + \mathcal{H}\Psi) + B_3\delta_{\mathrm{DE}} + B_4\delta_c + B_5\theta_{\mathrm{DE}} + B_6\theta_c \tag{29b}$$

for unknown functions A_i and B_i with $i \in 1\ldots6$.

The evolution equations for CDM are

$$\dot{\delta}_c = \vec{\nabla}^2\theta_c + 3\dot{\Phi} + \frac{1}{\bar{\rho}_c}(Q\delta_c - q), \tag{30a}$$

and

$$\dot{\theta}_c = -\mathcal{H}\theta_c + \Psi + \frac{1}{\bar{\rho}_c}(Q\theta_c - S), \tag{30b}$$

while the evolution equations for DE are

$$\dot{\delta}_{\mathrm{DE}} = 3w\mathcal{H}\delta_{\mathrm{DE}} + (1+w)[\vec{\nabla}^2\theta_{\mathrm{DE}} + 3\dot{\Phi}] - 3\mathcal{H}\Pi_{\mathrm{DE}} + \frac{1}{\bar{\rho}_{\mathrm{DE}}}[q - Q\delta_{\mathrm{DE}}], \tag{31a}$$

and

$$\dot{\theta}_{\mathrm{DE}} = -\left[\mathcal{H}(1-3w) + \frac{\dot{w}}{1+w}\right]\theta_{\mathrm{DE}} + \frac{\Pi_{\mathrm{DE}}}{1+w} + \Psi + \frac{1}{\bar{\rho}_{\mathrm{DE}}}\left[\frac{S}{1+w} - Q\theta_{\mathrm{DE}}\right]. \tag{31b}$$

In the following section we are going to investigate the underlying space of models of coupled DM to DE, and show how we can construct a "dictionary" of interacting dark energy theories and their PPF correspondences. The same method was applied to modified gravity theories in [42].

III. WORKED EXAMPLES

As a "warm-up" exercise, we are first going to demonstrate the use of our PPF formalism for interacting dark energy theories by showing that the functions A_i and B_i are severely constrained when one considers specific models which appear often in the literature. These are the "coupled quintessence" model [5], a model where $J_\mu \propto u_\mu$ [25,49] and the elastic scattering of model of dark matter and dark energy [33,50]. In Table II one can see the list of the models we consider with their coefficients displayed.

Following that, we consider the parametrization of the general classes of coupled theories we constructed in [39]. More specifically, in [39] we presented three distinct types of models of dark energy in the form of a scalar field

TABLE II. Specific models and their PPF coefficients. The coupled quintessence model is a subcase of type 1 with $\alpha_\phi = \beta_A$. The elastic scattering model is in fact distinct from type 3 (see text at the end of Sec. III D). For the coefficients A_2, A_3, A_4 and A_5 in the case of type 2 see (70). For the coefficients B_3 and A_5 in the case of type 3 see (86). For the remaining functions the reader is referred to each specific example in the text.

Model/coefficients	Q	A_1	A_2	A_3	A_4	A_5	A_6	B_1	B_2	B_3	B_4	B_5	B_6
Coupled quintessence	$-\beta_A\bar{\rho}_c\dot{\bar{\phi}}$	$\cdots$	$\cdots$	$\frac{Q}{1+w}$	Q	$\beta_A\bar{\rho}_c a^2 V_\phi$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Q	$\cdots$
$J_\mu \propto u_\mu$	$a\Gamma_{int}\bar{\rho}_c$	$\cdots$	$\cdots$	$\cdots$	Q	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Q
Elastic scattering	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$-\bar{\rho}_{\rm DE}(1+w)an_D\sigma_D$	$-B_5$
Type 1	$-\bar{\rho}_c\alpha_\phi\dot{\bar{\phi}}$	$\cdots$	$\cdots$	$\frac{Qc_s^2}{1+w}$	Q	$Q\left[\frac{\alpha_{\phi\phi}}{\alpha_\phi} - \frac{c_s^2\dot{\bar{\phi}}\bar{K}_\phi}{(1+w)\bar{K}}\right]$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Q	$\cdots$
Type 2	$\frac{\bar{Z}\beta_Z\bar{\rho}_c}{1+\bar{Z}\beta}\dot{\bar{Z}}$	$\cdots$	A_2	A_3	A_4	A_5	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	Q	$\cdots$
Type 3	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	B_3	$\cdots$	B_5	$-B_5 + \frac{3\mathcal{H}\bar{Z}F_Z c_s^2}{1-\frac{\bar{Z}\bar{F}_Z}{\bar{\rho}_c}}$

explicitly coupled to dark matter. We used the pull-back formalism for fluids and generalized the standard fluid action in order to include a dark coupling. The general functional form for the combined dark energy and dark matter Lagrangian we considered is

$$L = L(n, Y, Z, \phi), \tag{32}$$

where n is the fluid number density, $Y = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$, and $Z = u^\mu\nabla_\mu\phi$. As an example, within general relativity (GR), a quintessence field and an uncoupled fluid is described by the Lagrangian $L = Y + V(\phi) + f(n)$.

We then considered three distinct ways to reduce the general function (32) giving rise to the three types of coupled models which we now want to parametrize. These are the type 1 models where $L = F(Y, \phi) + f(n, \phi)$ [the coupled quintessence model [5] is a subcase of type 1 with the choice $F = Y + V(\phi)$ and $f = ne^{\beta_A\phi}$], the type 2 models where $L = F(Y, \phi) + f(n, Z)$ and the type 3 models where $L = F(Y, Z, \phi) + f(n)$.

A. Specific models

1. Coupled quintessence

Let us start with the coupled quintessence (CQ) model suggested by Amendola [5], which is a specific subcase of the type 1 class of models we presented in [39]. The scalar field action for this model is

$$S = -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \tag{33}$$

where $V(\phi)$ is the quintessence potential. If a constant coupling parameter β_A is assumed, the coupling current J_μ is found to be (see [39] for details)

$$J_\mu = -\beta_A\rho_c\nabla_\mu\phi. \tag{34}$$

Writing the scalar field as $\phi = \bar{\phi} + \varphi$ for a background field $\bar{\phi}$ and perturbation φ, the components of the stress-energy tensor for this model are (using expressions from [39])

$$\bar{\rho}_{\rm DE} = \frac{1}{2a^2}\dot{\bar{\phi}}^2 + V \qquad \bar{P}_{\rm DE} = \frac{1}{2a^2}\dot{\bar{\phi}}^2 - V, \tag{35}$$

$$c_a^2 = 1 + \frac{2\dot{\bar{\phi}}V_\phi}{3\frac{\dot{\bar{\phi}}^2}{a^2}\mathcal{H} - Q} \qquad Q = -\beta_A\bar{\rho}_c\dot{\bar{\phi}} \tag{36}$$

for the background, where $V_\phi \equiv \frac{dV}{d\phi}$, and

$$\begin{aligned} \delta\rho_{\rm DE} &= \frac{\dot{\bar{\phi}}}{a^2}(\dot{\varphi} - \dot{\bar{\phi}}\Psi) + V_\phi\varphi, \\ \theta_{\rm DE} &= \frac{\varphi}{\dot{\bar{\phi}}}, \end{aligned} \tag{37}$$

$$\delta P_{\rm DE} = \frac{\dot{\bar{\phi}}}{a^2}(\dot{\varphi} - \dot{\bar{\phi}}\Psi) - V_\phi\varphi, \qquad c_s^2 = 1, \qquad \mu = 0 \tag{38}$$

for the perturbations. The required coupling parameters are found to be

$$q = Q\left(\delta_c + \frac{\dot{\varphi}}{\dot{\bar{\phi}}}\right), \tag{39}$$

$$S = Q\frac{\varphi}{\dot{\bar{\phi}}}. \tag{40}$$

Now we read off the coefficients. They are

$$\begin{aligned} &A_1 = A_2 = A_6 = 0 \qquad A_3 = \frac{Q}{1+w} \\ &A_4 = Q \qquad A_5 = \beta_A\bar{\rho}_c a^2 V_\phi \\ &B_5 = Q \qquad B_{i\neq 5} = 0. \end{aligned} \tag{41}$$

2. Model with $J_\mu \propto u_\mu$

In this model, which was introduced in [25] and [49], the energy-momentum transfer vector J^μ is parallel to the dark matter 4-velocity u^μ. In our notation we have $u_\mu = a(1+\Psi, \nabla\theta_c)$ and

$$J_\mu = \Gamma_{\rm int}\bar{\rho}_c(1+\delta_c)u_\mu, \tag{42}$$

with $\Gamma_{\rm int}$ being a local constant interaction rate. The background coupling function is

$$Q = a\Gamma_{\rm int}\bar{\rho}_c \tag{43}$$

while the perturbative coupling parameters q and S are

$$q = Q(\delta_c + \Psi) \quad \text{and} \quad S = Q\theta_c. \tag{44}$$

In comparison with our general parametrization scheme, we find that the only nonzero coefficients are

$$A_4 = B_6 = Q. \tag{45}$$

3. Elastic scattering of dark matter and dark energy

This model was introduced in [33] and it considered an elastic interaction between dark energy and dark matter. It is a pure momentum transfer model and its background cosmology remains unaltered. In our language, this model has

$$Q = 0 \tag{46}$$

in the background while

$$q = 0 \qquad S = (\bar{\rho}_{\rm DE} + \bar{P}_{\rm DE})an_D\sigma_D(\theta_c - \theta_{\rm DE}), \tag{47}$$

at the level of the perturbations, with n_D being the proper number density of dark matter particles and σ_D the scattering cross section between dark matter and dark energy (also note that $w = \text{const}$ and $c_s^2 = 1$ in this model) [33]. We therefore find that the only nonzero coefficients are

$$B_5 = -(\bar{\rho}_{\rm DE} + \bar{P}_{\rm DE})an_D\sigma_D = -B_6. \tag{48}$$

Now we turn our attention to the three general types of models in [39].

B. Type 1 theory of DM coupled to DE

Type 1 models are classified in [39] via

$$L(n, Y, Z, \phi) = F(Y, \phi) + f(n, \phi). \tag{49}$$

For the case where the dark matter is CDM we further have $f(n,\phi) = ne^{\alpha(\phi)}$ where $\alpha(\phi)$ is a free function of the field ϕ.

From [39] we have that the coupling current is

$$J_\mu = -\rho_c\alpha_\phi\nabla_\mu\phi \tag{50}$$

so that

$$\begin{aligned} Q &= -\bar{\rho}_c\alpha_\phi\dot{\bar{\phi}} \\ q &= Q\left(\delta_c + \frac{\dot{\varphi}}{\dot{\bar{\phi}}}\right) - \bar{\rho}_c\alpha_{\phi\phi}\varphi \\ S &= Q\frac{\varphi}{\dot{\bar{\phi}}} \end{aligned} \tag{51}$$

where $\alpha_\phi \equiv \frac{d\alpha}{d\phi}$ and $\alpha_{\phi\phi} \equiv \frac{d^2\alpha}{d\phi^2}$. Note that the expression for S agrees with the one we recovered previously, Eq. (40), and the expression for q agrees with Eq. (39) if $\alpha_{\phi\phi} = 0$. This is expected, as the CQ model we studied is a subcase of type 1.

We now need to express φ and $\dot{\varphi}$ in terms of fluid-type variables. For type 1 theories, we find from [39] that the background energy density and pressure are given by

$$\bar{\rho}_{\rm DE} = -\bar{K} \qquad \bar{P}_{\rm DE} = -\bar{F} \tag{52}$$

where we have introduced the function

$$K(Y, \phi) = 2YF_Y - F \tag{53}$$

which will come in handy below (and also for type 2). Furthermore, the perturbed variables of interest are [39]

$$\delta_{\rm DE} = -\frac{\bar{Z}\bar{K}_Y}{\bar{K}}\delta Z + \frac{\bar{K}_\phi}{\bar{K}}\varphi, \qquad \theta_{\rm DE} = \frac{\varphi}{\dot{\bar{\phi}}}, \tag{54}$$

$$c_a^2 = c_s^2 + \frac{\dot{\bar{\phi}}[\bar{F}_\phi - c_s^2\bar{K}_\phi]}{3(\bar{\rho}_{\rm DE} + \bar{P}_{\rm DE})\mathcal{H} - Q}, \tag{55}$$

$$c_s^2 = \frac{\bar{F}_Y}{\bar{K}_Y}, \qquad \mu = 0, \tag{56}$$

where $\bar{K}_\phi \equiv \frac{\partial\bar{K}}{\partial\phi}$ and similarly for $\bar{F}$. The expression for c_a^2, above (and since also $\mu = 0$), says that if $\bar{F}_\phi = 0$ then the scalar field perturbations are adiabatic.

To express φ and $\dot{\varphi}$ in terms of fluid-type variables we invert the relations (56) to get

$$\delta Z = -\frac{\bar{K}}{\bar{Z}\bar{K}_Y}\delta_{\rm DE} - \frac{a\bar{K}_\phi}{\bar{K}_Y}\theta_{\rm DE}, \qquad \varphi = \dot{\bar{\phi}}\theta_{\rm DE}, \tag{57}$$

which are valid for adiabatic and nonadiabatic perturbations.

Let us now calculate the sought-after coefficients. Using $a\delta Z = \dot{\bar{\phi}}\Psi - \dot{\varphi}$ and (57) we find

$$\frac{\dot{\varphi}}{\dot{\bar{\phi}}} = \Psi + \frac{c_s^2}{1+w}\delta_{\rm DE} - \frac{c_s^2\bar{K}_\phi}{(1+w)K}\dot{\bar{\phi}}\theta_{\rm DE}, \tag{58}$$

so that (51) gives the required coefficients as

$$A_1 = A_2 = A_6 = 0 \qquad A_3 = \frac{Qc_s^2}{1+w}$$
$$A_4 = Q \qquad A_5 = Q\left[\frac{\alpha_{\phi\phi}}{\alpha_\phi} - \frac{c_s^2 \bar{K}_\phi}{(1+w)\bar{K}}\dot{\bar{\phi}}\right]$$
$$B_5 = Q \qquad B_{i\neq 5} = 0. \tag{59}$$

For the case where $\alpha_\phi = \beta_A$ is a constant and furthermore $F(Y,\phi) = Y + V(\phi)$ we recover the coefficients for the coupled quintessence model discussed above, which is in fact a subcase of a type 1 model of coupled dark energy.

C. Type 2 theory of DM coupled to DE

Type 2 models are classified via [39]

$$L(n, Y, Z, \phi) = F(Y, \phi) + f(n, Z), \tag{60}$$

with $f = nh(Z)$ in the case that the scalar field is coupled to CDM. The coupling current in this case is [39]

$$J_\mu = \nabla_\nu(\rho_c \beta u^\nu)\nabla_\mu \phi \tag{61}$$

where $\beta(Z)$ is the function

$$\beta(Z) = \frac{h_Z}{h - Zh_Z}, \tag{62}$$

and $h_Z = \frac{dh}{dZ}$ (and the same when Z is used as a subscript for β). Let us first note that the relations for the fluid variables given by (56) are still valid for the case of type 2 theory and the function K is still defined via (53).

In order to proceed further we need the function Q which at first glance using (61) is given by

$$Q = \bar{Z}[(\dot{\bar{\rho}}_c + 3\mathcal{H}\bar{\rho}_c)\beta + \bar{\rho}_c\beta_Z \dot{\bar{Z}}]. \tag{63}$$

Using (8) to eliminate the terms $\dot{\bar{\rho}}_c$ and $\dot{\bar{Z}}$ we find

$$Q = \frac{\bar{Z}\beta_Z}{1+\bar{Z}\beta}\bar{\rho}_c \dot{\bar{Z}} \tag{64}$$

where $\dot{\bar{Z}}$ is determined from Eq. (70) of [39] as

$$\dot{\bar{Z}} = -\frac{3\bar{Z}\bar{F}_Y \mathcal{H} + a\bar{K}_\phi}{\bar{K}_Y - \frac{\bar{\rho}_c\beta_Z}{1+\bar{Z}\beta}}. \tag{65}$$

The perturbative variables q and S are also found from (61). First, S is easily calculated as

$$S = Q\theta_\phi \tag{66}$$

while q is found to be from (61) as

$$q = Q\Psi + \bar{Z}\Big\{(\delta_c - \vec{\nabla}^2\theta_c - 3\dot{\Phi})\bar{\rho}_c\beta + \bar{\rho}_c\beta_Z \dot{\delta Z}$$
$$+ [(\dot{\bar{\rho}}_c + 3\mathcal{H}\bar{\rho}_c)\beta + \bar{\rho}_c\beta_Z\dot{\bar{Z}}][\delta_c - \Psi]$$
$$+ \left[(\dot{\bar{\rho}}_c + 3\mathcal{H}\bar{\rho}_c)\beta_Z + \bar{\rho}_c\beta_{ZZ}\dot{\bar{Z}} + \frac{Q}{\bar{Z}^2}\right]\delta Z\Big\}. \tag{67}$$

However, using (63) as well as (30) in order to eliminate $\dot{\delta}_c$ the expression for q simplifies to

$$q = Q\delta_c + Q\frac{\dot{\delta Z}}{\dot{\bar{Z}}} + \frac{d}{dZ}\left[\frac{\bar{Z}\beta_Z}{1+\bar{Z}\beta}\right]\bar{\rho}_c\dot{\bar{Z}}\delta Z. \tag{68}$$

What remains is now to eliminate $\dot{\delta Z}$. This can be done using the perturbative version of Eq. (70) of [39] which gives

$$\frac{\dot{\delta Z}}{\dot{\bar{Z}}} = \Psi + \Bigg\{\left[\bar{Z}\bar{K}_{YY} + \bar{\rho}_c\frac{d}{dZ}\left(\frac{\beta_Z}{1+\bar{Z}\beta}\right)\right]\frac{1}{\bar{K}_Y - \frac{\bar{\rho}_c\beta_Z}{1+\bar{Z}\beta}}$$
$$+ \frac{3\bar{K}_Y\mathcal{H} + \bar{K}_{Y\phi}\dot{\bar{\phi}}}{3\bar{Z}\bar{F}_Y\mathcal{H} + a\bar{K}_\phi}\Bigg\}\delta Z - \frac{3\bar{Z}\bar{F}_Y}{3\bar{Z}\bar{F}_Y\mathcal{H} + a\bar{K}_\phi}(\dot{\Phi} + \mathcal{H}\Psi)$$
$$+ \bar{Z}\left[\frac{3\bar{F}_{Y\phi}\mathcal{H}\dot{\bar{\phi}} - a^2\bar{K}_{\phi\phi} - \bar{F}_Y\vec{\nabla}^2}{3\bar{Z}\bar{F}_Y\mathcal{H} + a\bar{K}_\phi} + \frac{a\bar{K}_{Y\phi}}{\bar{K}_Y - \frac{\bar{\rho}_c\beta_Z}{1+\bar{Z}\beta}}\right]\theta_\phi$$
$$+ \frac{\bar{\rho}_c\beta_Z}{(1+\bar{Z}\beta)\bar{K}_Y - \bar{\rho}_c\beta_Z}\delta_c. \tag{69}$$

Using (69) and (57) into (68) we may now determine the coefficients. They are

$$A_1 = A_6 = 0, \qquad A_2 = \frac{\bar{Z}\bar{F}_Y}{6\bar{Z}\bar{F}_Y\mathcal{H} + 2a\bar{K}_\phi}Q$$
$$A_3 = \frac{c_s^2}{(1+w)(\bar{K}_Y - \frac{\bar{\rho}_c\beta_Z}{1+\bar{Z}\beta})}\Bigg\{Q\left[\bar{Z}\bar{K}_Y\frac{d}{dZ}\ln\left(\frac{\bar{Z}\beta_Z}{1+Z\beta}\right)\right.$$
$$\left. + \bar{Z}^2\bar{K}_{YY}\right] - \frac{\bar{\rho}_c\beta_Z[Q + \bar{Z}^2(3\bar{K}_Y\mathcal{H} + \bar{K}_{Y\phi}\dot{\bar{\phi}})]}{1+\bar{Z}\beta}\Bigg\}$$
$$A_4 = \frac{(1+\bar{Z}\beta)\bar{K}_Y}{(1+\bar{Z}\beta)\bar{K}_Y - \bar{\rho}_c\beta_Z}Q$$
$$A_5 = \frac{1}{\bar{K}_Y - \frac{\bar{\rho}_c\beta_Z}{1+\bar{Z}\beta}}\Bigg\{a\bar{K}_\phi Q\left[\frac{\bar{\rho}_c\beta_Z}{\bar{Z}\bar{K}_Y(1+\bar{Z}\beta)} - \frac{\bar{Z}\bar{K}_{YY}}{\bar{K}_Y}\right.$$
$$\left. - \frac{d}{dZ}\ln\left(\frac{\bar{Z}\beta_Z}{1+Z\beta}\right)\right] + \frac{\bar{Z}\bar{\rho}_c\beta_Z(3\bar{K}_Y\mathcal{H} + \bar{K}_{Y\phi}\dot{\bar{\phi}})}{1+\bar{Z}\beta}\frac{a\bar{K}_\phi}{\bar{K}_Y}$$
$$- \frac{\bar{Z}^2\bar{\rho}_c\beta_Z(3\bar{F}_{Y\phi}\mathcal{H}\dot{\bar{\phi}} - a^2\bar{K}_{\phi\phi} - \bar{F}_Y\vec{\nabla}^2)}{1+\bar{Z}\beta} + Q\bar{Z}a\bar{K}_{Y\phi}\Bigg\}$$
$$B_5 = Q \qquad B_{i\neq 5} = 0. \tag{70}$$

D. Type 3 theory of DM coupled to DE

Type 3 models are classified via

$$L(n,Y,Z,\phi) = F(Y,Z,\phi) + f(n). \tag{71}$$

The coupling current in this case is [39]

$$J_\nu = q^\beta{}_\nu\{X\nabla_\beta\phi + F_Z\nabla_\beta Z + ZF_Z u^\mu\nabla_\mu u_\beta\}, \tag{72}$$

where $X \equiv \nabla_\mu(F_Z u^\mu)$. A straightforward calculation gives

$$Q = q = 0 \tag{73}$$

(although there can be second order corrections to J_0). This means that the type 3 case provides for a pure momentum transfer coupling up to linear order in perturbation theory.

To proceed to the coefficients we need S which is found to be

$$S = -(\bar{X}\dot{\bar{\phi}} + \bar{F}_Z\dot{\bar{Z}} + \bar{Z}\bar{F}_Z\mathcal{H})\theta_c - \bar{Z}\bar{F}_Z\dot{\theta}_c - \frac{1}{a}\bar{F}_Z\dot{\varphi} + \bar{X}\varphi \tag{74}$$

where the background value of X is

$$\bar{X} = \frac{1}{a}[(\bar{Z}\bar{F}_{ZY} - \bar{F}_{ZZ})\dot{\bar{Z}} - \bar{F}_{Z\phi}\dot{\bar{\phi}} - 3\mathcal{H}\bar{F}_Z]. \tag{75}$$

We eliminate the $\dot{\theta}_c$ term using (30) to get

$$S = \frac{1}{1-\frac{\bar{Z}\bar{F}_Z}{\bar{\rho}_c}}[\bar{X}\varphi - (\bar{X}\dot{\bar{\phi}} + \bar{F}_Z\dot{\bar{Z}})\theta_c + \bar{F}_Z\delta Z], \tag{76}$$

where to remind the reader $\delta Z = -\frac{1}{a}(\dot{\varphi} - \dot{\bar{\phi}}\Psi)$. Now we need to express φ and $\dot{\varphi}$ in terms of the fluid variables. From [39] we find

$$\bar{\rho}_{\rm DE} = \bar{Z}^2F_Y - \bar{Z}F_Z + F \qquad \bar{P}_{\rm DE} = -F \tag{77}$$

for the background variables while

$$\begin{aligned}\delta\bar{\rho}_{\rm DE} &= \bar{Z}[F_Y - \bar{Z}^2F_{YY} + 2\bar{Z}F_{YZ} - F_{ZZ}]\delta Z \\ &\quad + [\bar{Z}^2F_{Y\phi} - \bar{Z}F_{Z\phi} + F_\phi]\varphi,\end{aligned} \tag{78}$$

$$\delta\bar{P}_{\rm DE} = (\bar{Z}F_Y - F_Z)\delta Z - F_\phi\varphi, \tag{79}$$

$$\theta_{\rm DE} = \frac{\frac{F_Y}{a}\varphi + F_Z\theta_c}{F_Z - \bar{Z}F_Y} \tag{80}$$

for the perturbations. For completeness, the adiabatic sound speed is

$$c_a^2 = \frac{3\mathcal{H}(\bar{Z}F_Y - F_Z) - a[F_\phi + \bar{Z}^2F_{Y\phi} - \bar{Z}F_{Z\phi}]}{3\mathcal{H}\bar{Z}(\bar{F}_Y + 2\bar{Z}\bar{F}_{YZ} - \bar{Z}^2\bar{F}_{YY} - F_{ZZ})} - \frac{aF_\phi}{3\mathcal{H}(\bar{Z}\bar{F}_Y - \bar{F}_Z)} \tag{81}$$

while the effective sound speed c_s^2 is

$$c_s^2 = \frac{\bar{Z}\bar{F}_Y - \bar{F}_Z}{\bar{Z}(\bar{F}_Y + 2\bar{Z}\bar{F}_{YZ} - \bar{F}_{ZZ} - \bar{Z}^2\bar{F}_{YY})}, \tag{82}$$

and the relative entropy parameter is

$$\mu = \frac{3F_Z}{\bar{Z}\bar{F}_Y}(c_s^2 - c_a^2)(\bar{\rho}_{\rm DE} + \bar{P}_{\rm DE})\mathcal{H}. \tag{83}$$

Clearly if $F_\phi = 0$ then the perturbations are adiabatic, i.e. $c_s^2 = c_a^2$ and $\mu = 0$ (so that $\Pi_{\rm DE} = c_a^2\delta_{\rm DE}$).

We can now proceed to find the coefficients. Equations (78) and (80) can be inverted to give

$$\delta Z = \left[\frac{\mu}{\bar{F}_Z} - \frac{aF_\phi}{F_Y}\right]\left[\theta_{\rm DE} + \frac{\bar{Z}F_Z}{\bar{\rho}_{\rm DE} + \bar{P}_{\rm DE}}\theta_c\right] + \frac{c_s^2\bar{Z}}{1+w}\delta_{\rm DE} \tag{84}$$

and

$$\varphi = a\left(\frac{\bar{F}_Z}{\bar{F}_Y} - \bar{Z}\right)\theta_{\rm DE} - \frac{a\bar{F}_Z}{\bar{F}_Y}\theta_c. \tag{85}$$

We also need the equation for $\dot{\bar{Z}}$ which is found to be [Eq. (75) in [39]]

$$\dot{\bar{Z}} = -3\mathcal{H}\bar{Z}\left[c_a^2 + \frac{aF_\phi}{3\mathcal{H}(\bar{Z}\bar{F}_Y - \bar{F}_Z)}\right].$$

The above equations are then inserted into (76) to give the required coefficients as

$$\begin{aligned}B_1 &= B_2 = B_4 = 0 \\ B_3 &= \frac{1}{1-\frac{\bar{Z}\bar{F}_Z}{\bar{\rho}_c}}\frac{\bar{Z}\bar{F}_Z c_s^2}{1+w} \\ B_5 &= \frac{a}{1-\frac{\bar{Z}\bar{F}_Z}{\bar{\rho}_c}}\left[\bar{X}\left(\frac{\bar{F}_Z}{\bar{F}_Y} - \bar{Z}\right) + \bar{F}_Z\left[\frac{\mu}{a\bar{F}_Z} - \frac{F_\phi}{F_Y}\right]\right] \\ B_6 &= -B_5 + \frac{3\mathcal{H}\bar{Z}F_Z c_s^2}{1-\frac{\bar{Z}\bar{F}_Z}{\bar{\rho}_c}}.\end{aligned} \tag{86}$$

It would seem tempting to try model the elastic scattering model [33] discussed above (Sec. III A 3) into the type 3 class. However, this is in fact impossible. As we can easily check, the elastic scattering model requires $B_3 = 0$. Within the type 3 class this is possible only if F is independent of Z

(i.e. $F_Z = 0$). This implies that B_5 and B_6 are also zero, in other words, the model becomes completely uncoupled. Hence, it is impossible to construct a model of elastic scattering between CDM and DE within the type 3 class of coupled dark energy.

IV. CONCLUSIONS

We presented the most general parametrization of models of dark energy which is explicitly coupled to dark matter using the parametrized post-Friedmannian framework, and have shown that it is able to encapsulate a rich variety of theories.

Starting from the linearized Einstein equations and using the Bianchi identities we managed to express the modifications to GR coming from the dark sector coupling as a collection of new terms containing the metric potentials and their derivatives as well as the scalar modes of the two dark sector components, i.e. the fluid variables of (generalized) dark matter and dark energy. Of course, our formalism is based on a few basic assumptions: the background cosmology has a FRW solution, all field equations are at most second order in time derivatives, and the field equations are gauge invariant. Completing the parametrization we were left with 24 free functions, but demanding gauge invariance we derived four constraint equations which eliminated four free functions.

Twenty free functions in our general parametrization is certainly a big number, but by imposing certain well motivated assumptions, for instance that the dark matter is cold, that the dark energy is shearless and that the pressure perturbation is not a dynamical quantity, we reduced the number of free functions to 12. Furthermore, we showed that only a handful of these functions are nonzero when one considers known models. We demonstrated this by investigating a number of specific models in the literature, as well as the classes of theories we constructed in [39]. It is useful to note that, although our theories in [39] are derived from an action, the PPF parametrization does *not* require knowledge of the action, but only knowledge of the field equations. This means that the PPF parametrization is a very useful tool for phenomenological model building (see [42] for further discussion in the context of modified gravity theories). The full list of models we consider in this work is displayed in Table II along with their coefficients.

Our type 1, 2 and 3 theories contain a fairly general coupling function and hence they encapsulate many different models. The parametrization coefficients for these theories can depend, of course, on the chosen coupling function and its derivatives, and other quantities such as the background coupling Q, the background field energy density $\bar{\rho}_\phi$, the quintessence potential $V(\phi)$, the speed of sound c_s^2 etc. For type 1 theories there is only one nonzero B coefficient and three nonzero A coefficients, for type 2 there is one nonzero B coefficient and four nonzero A coefficients, while for type 3 all A's are automatically zero and there are three nonzero B's: different classes of theories correspond to different nonzero functions. In particular, from all the cases we studied, the coefficients A_1, A_6, B_1, B_2 and B_4 were always zero. It would indeed be very interesting to find models for which any of these coefficients is nonzero.

It would also be interesting to consider the inverse problem, i.e. given $Q(t)$ and a set of PPF coefficients A_i and B_i, can we reconstruct the functions that appear in the coupled dark energy Lagrangian for each of the type 1, 2 and 3 theories? For instance, what kinds of functions correspond to constant PPF coefficients? Tackling the inverse problem will help to reduce the free functions into simple functional forms which are parametrized by a set of constants and can make constraining such theories easier and more efficient.

Another important question is how we can further constrain the PPF functions? As discussed in [42], we might expect to find that a subset of the PPF functions can be very well constrained, while another subset cannot. However, this might not pose a serious problem, as the constraining power of a few PPF functions might be sufficient to distinguish between theories. Tackling the inverse problem will certainly help here as it can guide us to which functional form of the PPF coefficients is the most useful. The implementation of the PPF framework presented here in numerical codes for the computation of the cosmological effects of interacting dark energy could provide an answer to these questions. We plan to investigate this in future work.

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[1] E. Komatsu *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **192**, 18 (2011).
[2] P. A. R. Ade *et al.* (Planck Collaboration), Astron. Astrophys. **571**, A16 (2014).
[3] M. Kowalski *et al.* (Supernova Cosmology Project), Astrophys. J. **686**, 749 (2008).
[4] H. Lampeitl, R. Nichol, H. Seo, T. Giannantonio, C. Shapiro *et al.*, Mon. Not. R. Astron. Soc. **401**, 2331 (2010).
[5] L. Amendola, Phys. Rev. D **62**, 043511 (2000).
[6] A. P. Billyard and A. A. Coley, Phys. Rev. D **61**, 083503 (2000).
[7] W. Zimdahl and D. Pavon, Phys. Lett. B **521**, 133 (2001).
[8] G. R. Farrar and P. J. E. Peebles, Astrophys. J. **604**, 1 (2004).
[9] S. Matarrese, M. Pietroni, and C. Schimd, J. Cosmol. Astropart. Phys. 08 (2003) 005.
[10] L. Amendola, Phys. Rev. D **69**, 103524 (2004).
[11] A. V. Maccio, C. Quercellini, R. Mainini, L. Amendola, and S. A. Bonometto, Phys. Rev. D **69**, 123516 (2004).
[12] L. Amendola, M. Gasperini, and F. Piazza, J. Cosmol. Astropart. Phys. 09 (2004) 014.
[13] L. Amendola, G. Camargo Campos, and R. Rosenfeld, Phys. Rev. D **75**, 083506 (2007).
[14] Z.-K. Guo, N. Ohta, and S. Tsujikawa, Phys. Rev. D **76**, 023508 (2007).
[15] T. Koivisto, Phys. Rev. D **72**, 043516 (2005).
[16] S. Lee, G.-C. Liu, and K.-W. Ng, Phys. Rev. D **73**, 083516 (2006).
[17] B. Wang, J. Zang, C.-Y. Lin, E. Abdalla, and S. Micheletti, Nucl. Phys. **B778**, 69 (2007).
[18] R. Mainini and S. Bonometto, J. Cosmol. Astropart. Phys. 06 (2007) 020.
[19] V. Pettorino and C. Baccigalupi, Phys. Rev. D **77**, 103003 (2008).
[20] J.-Q. Xia, Phys. Rev. D **80**, 103514 (2009).
[21] L. P. Chimento, A. S. Jakubi, D. Pavon, and W. Zimdahl, Phys. Rev. D **67**, 083513 (2003).
[22] G. Olivares, F. Atrio-Barandela, and D. Pavon, Phys. Rev. D **71**, 063523 (2005).
[23] H. M. Sadjadi and M. Alimohammadi, Phys. Rev. D **74**, 103007 (2006).
[24] A. Brookfield, C. van de Bruck, and L. M. Hall, Phys. Rev. D **77**, 043006 (2008).
[25] C. G. Boehmer, G. Caldera-Cabral, R. Lazkoz, and R. Maartens, Phys. Rev. D **78**, 023505 (2008).
[26] G. Caldera-Cabral, R. Maartens, and B. M. Schaefer, J. Cosmol. Astropart. Phys. 07 (2009) 027.
[27] J.-H. He and B. Wang, J. Cosmol. Astropart. Phys. 06 (2008) 010.
[28] M. Quartin, M. O. Calvao, S. E. Joras, R. R. Reis, and I. Waga, J. Cosmol. Astropart. Phys. 05 (2008) 007.
[29] J. Valiviita, R. Maartens, and E. Majerotto, Mon. Not. R. Astron. Soc. **402**, 2355 (2010).
[30] S. Pereira and J. Jesus, Phys. Rev. D **79**, 043517 (2009).
[31] R. Bean, E. E. Flanagan, I. Laszlo, and M. Trodden, Phys. Rev. D **78**, 123514 (2008).
[32] M. Gavela, D. Hernandez, L. Lopez Honorez, O. Mena, and S. Rigolin, J. Cosmol. Astropart. Phys. 07 (2009) 034.
[33] F. Simpson, Phys. Rev. D **82**, 083505 (2010).
[34] E. R. Tarrant, C. van de Bruck, E. J. Copeland, and A. M. Green, Phys. Rev. D **85**, 023503 (2012).
[35] T. Koivisto, D. Wills, and I. Zavala, J. Cosmol. Astropart. Phys. 06 (2014) 036.
[36] V. Salvatelli, N. Said, M. Bruni, A. Melchiorri, and D. Wands, Phys. Rev. Lett. **113**, 181301 (2014).
[37] C. G. Boehmer, N. Tamanini, and M. Wright, arXiv:1501.06540.
[38] C. G. Boehmer, N. Tamanini, and M. Wright, arXiv:1502.04030.
[39] A. Pourtsidou, C. Skordis, and E. J. Copeland, Phys. Rev. D **88**, 083505 (2013).
[40] C. Skordis, Phys. Rev. D **79**, 123527 (2009).
[41] T. Baker, P. G. Ferreira, C. Skordis, and J. Zuntz, Phys. Rev. D **84**, 124018 (2011).
[42] T. Baker, P. G. Ferreira, and C. Skordis, Phys. Rev. D **87**, 024015 (2013).
[43] P. G. Ferreira, T. Baker, and C. Skordis, Gen. Rel. Grav. **46**, 1788 (2014).
[44] W. Hu, Astrophys. J. **506**, 485 (1998).
[45] S. M. Carroll, Phys. Rev. Lett. **81**, 3067 (1998).
[46] C. Martins, Nucl. Phys. B, Proc. Suppl. **194**, 96 (2009).
[47] M. Banados, P. Ferreira, and C. Skordis, Phys. Rev. D **79**, 063511 (2009).
[48] R. Durrer, *The Cosmic Microwave Background* (Cambridge University Press, Cambridge, England, 2008).
[49] J. Valiviita, E. Majerotto, and R. Maartens, J. Cosmol. Astropart. Phys. 07 (2008) 020.
[50] M. Baldi and F. Simpson, arXiv:1412.1080.