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Inflationary universe from higher-derivative quantum gravity

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We consider higher-derivative quantum gravity where the renormalization-group-improved effective action beyond the one-loop approximation is derived. Using this effective action, the quantum-corrected Friedmann-Robertson-Walker (FRW) equations are analyzed. The de Sitter universe solution is found. It is demonstrated that such a de Sitter inflationary universe is unstable. The slow-roll inflationary parameters are calculated. The contribution of the renormalization-group-improved Gauss-Bonnet term to the quantum-corrected FRW equations as well as to the instability of the de Sitter universe is estimated. It is demonstrated that in this case, the spectral index and tensor-to-scalar ratio are consistent with Planck data.

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I. INTRODUCTION

Recent more precise observational WMAP data [1] as well as corrected Planck constraints [2] increased the interest in the theoretical models for the inflationary universe. There are a large variety of inflationary models (for review, see, for instance, Refs. [3]) which may comply with observational data, at least to some extent (see also Ref. [4] about the BICEP experiment).

In fact, during recent years there was much activity concerning the quantum effects of general relativity in the construction of the inflationary universe (for an introduction and review, see Ref. [5]). Furthermore, recent study [6] indicates that the quantum effects of specific models of (nonrenormalizable) higher-derivative $F(R)$ gravity may give consistent inflation which complies with Planck data. The next natural step is the extension of the quantum-corrected inflationary scenario for the multiplicatively renormalizable higher-derivative gravity (for a general review, see Ref. [7]). A very interesting attempt in this direction has been recently made in Ref. [8]. Note that since it is multiplicatively renormalizable, higher-derivative quantum gravity is based on the use of a higher-derivative propagator. As a result, such a theory eventually leads to a problem with unitarity which is related to the well-known Ostrogradski instability of higher-derivative theories. In fact, some attempts to resolve this problem were made by proposing that unitarity may be restored at the nonperturbative level. However, there is no complete proof of the nonperturbative restoration of unitarity. Hence, so far this theory may be considered as an effective theory teaching us different general aspects of quantum gravity.

The purpose of the current work is the study of the inflationary universe in general higher-derivative quantum gravity [7]. Making use of the fact that one-loop beta functions of such a theory are well known and their asymptotically free regime is well investigated, we apply the renormalization group (RG) considerations to get the RG-improved effective action in general higher-derivative gravity. This technique is well developed in quantum field theory in curved spacetime [9]. It permits us to get the effective action beyond one-loop approximation, making a sum of all leading logs of the theory.

The paper is organized as follows. In Sec. II, we present the renormalization-group-improved effective action of the multiplicatively renormalizable higher-derivative gravity. In order to do so, the one-loop effective coupling constants are used. Subsequently, the quantum-corrected equations of motion are derived on the flat Friedmann-Robertson-Walker spacetime. In Sec. III, using the asymptotic behavior of the gravitational running constants, the de Sitter inflationary universe is constructed. The asymptotically free regime is discussed in detail. Section IV is devoted to the study of the dynamics of such quantum-corrected inflation. It is shown that de Sitter space is unstable and can lead to a large amount of inflation. Slow-roll conditions are discussed and the expressions for slow-roll parameters are found. In Sec. V, we consider the contribution from total derivative and surface terms (the topological Gauss-Bonnet term and the d'Alembertian of the curvature) to the RG-improved effective action. It is demonstrated that with these terms the spectral index can be compatible with Planck data. Conclusions and final remarks are given in Sec. VI.

II. RENORMALIZATION-GROUP-IMPROVED EFFECTIVE ACTION AND QUANTUM-CORRECTED FRW EQUATIONS

In this section we start from the general action of the higher-derivative gravity which is known to be a multiplicatively renormalizable theory (see Ref. [7] for a general introduction and review). The starting action has the following form,[1]

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g} \times \left(\frac{R}{\kappa^2} - \Lambda + aR_{\mu\nu}R^{\mu\nu} + bR^2 + cR_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} + d\Box R\right), \tag{2.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{M}$ is the spacetime manifold, $R, R_{\mu\nu}, R_{\mu\nu\xi\sigma}$ are the Ricci scalar, the Ricci tensor, and the Riemann tensor, respectively, and $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian, with ∇_μ being the covariant derivative operator associated with the metric $g_{\mu\nu}$. Here, $\kappa^2 > 0$, Λ, a, b, c, and d are constants which characterize the gravitational interaction. The above Lagrangian contains some terms not important in four dimensions. First of all, we note that $\Box R$ is a surface term which does not give any contribution to the dynamical equations. Second, we have

$$R_{\mu\nu}R^{\mu\nu} = \frac{C^2}{2} - \frac{G}{2} + \frac{R^2}{3},$$
$$R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} = 2C^2 - G + \frac{R^2}{3}, \tag{2.2}$$

where G and C^2 are the Gauss-Bonnet term and the "square" of the Weyl tensor,

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\sigma\xi},$$
$$C^2 = \frac{1}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\xi\sigma\mu\nu}R^{\xi\sigma\mu\nu}. \tag{2.3}$$

The Gauss-Bonnet term is a topological invariant in four dimensions, and we can drop it from the action. Thus, we can rewrite the higher-derivative terms with the help of the Weyl squared tensor.

Let us express the constants which appear in the starting action in terms of more convenient coupling constants which stress that the theory under consideration is an asymptotically free one. In order to do it, we follow the notations of Ref. [7]. To take into account quantum gravity effects, we use the RG-improved effective action. The calculation of the RG-improved effective action has been developed in multiplicatively renormalizable quantum field theory in curved spacetime. In general terms, this technique is described in detail in Refs. [7,9]. Recently, the RG-improved scalar potential in curved spacetime has been applied in the study of inflation [11]. In the simplest version [9], the RG-improved effective action follows from the solution of the RG equation applied to the complete effective action of the multiplicatively renormalizable theory. The final result is very simple: one has to replace constants in the classical action by one-loop effective coupling constants where the corresponding RG parameter is defined as a log term of the characteristic mass scale in the theory.

Applying the above considerations to higher-derivative quantum gravity, one can get the RG-improved effective action as the following:

$$I = \int_{\mathcal{M}} d^4\sqrt{-g}\left[\frac{R}{\kappa^2(t')} - \frac{\omega(t')}{3\lambda(t')}R^2 + \frac{1}{\lambda(t')}C^2 - \Lambda(t')\right]. \tag{2.4}$$

The effective coupling constants $\lambda \equiv \lambda(t')$, $\omega \equiv \omega(t')$, $\kappa^2 \equiv \kappa^2(t')$, and $\Lambda \equiv \Lambda(t')$ obey the one-loop RG equations [12]:

$$\frac{d\lambda}{dt'} = -\beta_2\lambda^2 \equiv -\left(\frac{133}{10}\right)\lambda^2, \tag{2.5}$$

$$\frac{d\omega}{dt'} = -\lambda(\omega\beta_2 + \beta_3) \equiv -\lambda\left(\frac{10}{3}\omega^2 + \frac{183}{10}\omega + \frac{5}{12}\right), \tag{2.6}$$

$$\frac{d\kappa^2}{dt'} = \kappa^2\gamma \equiv \kappa^2\lambda\left(\frac{10}{3}\omega - \frac{13}{6} - \frac{1}{4\omega}\right), \tag{2.7}$$

$$\frac{d\Lambda}{dt'} = \frac{\beta_4}{(\kappa^2)^2} - 2\gamma\Lambda(t') \equiv \frac{\lambda^2}{(\kappa^2)^2}\left(\frac{5}{2} + \frac{1}{8\omega^2}\right) + \lambda\Lambda\left(\frac{28}{3} + \frac{1}{3\omega}\right). \tag{2.8}$$

Note that $\kappa^2(t')$ is positive defined, and in general $\lambda(t')$ and $\Lambda(t')$ are also positive defined to have a positive contribution to the Weyl tensor and a positive effective cosmological constant in the action; on the other hand, $\omega(t')$ is expected to be negative to have a positive R^2 term. In the above expressions, $\beta_{2,3,4}$ and γ correspond to [7]

$$\beta_2 = \frac{133}{10}, \qquad \beta_3 = \frac{10}{3}\omega^2 + 5\omega + \frac{5}{12},$$
$$\beta_4 = \frac{\lambda^2}{2}\left(5 + \frac{1}{4\omega^2}\right) + \frac{\lambda}{3}(\kappa^2)^2\Lambda\left(20\omega + 15 - \frac{1}{2\omega}\right),$$
$$\gamma = \lambda\left(\frac{10}{3}\omega - \frac{13}{6} - \frac{1}{4\omega}\right). \tag{2.9}$$

[1]Note that higher-derivative theory of the type in (2.1) as well as other higher-derivative modified gravities may even pass solar system tests, for instance, due to the chameleon scenario [10] and so on.

The RG parameter t' is given by

$$t' = \frac{t'_0}{2}\log\left[\frac{R}{R_0}\right]^2, \tag{2.10}$$

where $t'_0 > 0$ is dimensionless constant introduced for the sake of completeness and R_0 is the mass scale for the Ricci scalar. We set R_0 as the value of the Ricci scalar in the current nearly de Sitter universe ($R_0 = 4\Lambda$, Λ being the cosmological constant), such that $t'(R = R_0) = 0$ today, while in the past $0 < t'(R_0 < R)$. Note that the de Sitter solution of the current accelerated expansion is a final attractor of Friedmann universe.

For Eq. (2.5) we also have the explicit solution

$$\lambda(t') = \frac{\lambda(0)}{1 + \lambda(0)\beta_2 t'}, \tag{2.11}$$

where $\lambda(0)$ is the integration constant corresponding to the value of λ at $t' = 0$, namely $\lambda(t = t_0) \equiv \lambda(R = R_0) = \lambda(0)$.

One important remark is in order: when we introduce the effective running constants in (2.1), we also get a contribution from the Gauss-Bonnet and $\Box R$ in the RG-improved effective action, since it is not more possible to write the Gauss-Bonnet term like a total derivative and $\Box R$ in terms of a flux in three dimensions. This fact will be discussed in below, but for the moment we work with the simplified action.

Let us consider the flat Friedmann-Robertson-Walker (FRW) spacetime, whose general form is given by

$$ds^2 = -N(t)^2 dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \tag{2.12}$$

where $a \equiv a(t)$ is the scale factor depending on the cosmological time t and $N \equiv N(t)$ is an arbitrary lapse function, which describes the gauge freedom associated with the reparametrization invariance of the action. For the above metric, the Ricci scalar and the square of the Weyl tensor read

$$R = \frac{1}{N^2}\left[6\left(\frac{\dot{a}}{a}\right)^2 + 6\left(\frac{\ddot{a}}{a}\right) - 6\left(\frac{\dot{N}}{N}\right)\left(\frac{\dot{a}}{a}\right)\right], \qquad C^2 = 0, \tag{2.13}$$

where the dot denotes the derivative with respect to the cosmological time t. The fact that the Weyl tensor is zero on the general form of the metric indicates that its contribution to the action and, therefore, to the derivation of the field equations of the theory is null. In fact one can write on FRW background

$$\delta I_{C^2} = \frac{1}{\lambda(t')}\delta(\sqrt{-g}C^2) + (\sqrt{-g}C^2)\delta\left(\frac{1}{\lambda(t')}\right) = \frac{1}{\lambda(t')}\delta(\sqrt{-g}C^2), \tag{2.14}$$

but

$$\frac{1}{\lambda(t')}\delta(\sqrt{-g}C^2) = 0, \tag{2.15}$$

and it is well known that the square of the Weyl tensor does not enter in the Friedmann-like equations.

To derive the equations of motions, we will use a method based on the Lagrangian multiplayer [13–16]. If we plug the expression for the Ricci scalar (2.13) into the action (2.4), we get the higher-derivative Lagrangian theory. In order to derive a standard (first order) Lagrangian theory, we introduce a Lagrangian multiplier ξ as [13,14],

$$I = \int_{\mathcal{M}} d^4\sqrt{-g}\left[\frac{R}{\kappa^2(t')} - \frac{\omega(t')}{3\lambda(t')}R^2 - \Lambda(t) - \xi\left[R - \frac{1}{N^2}\left[6\left(\frac{\dot{a}}{a}\right)^2 + 6\left(\frac{\ddot{a}}{a}\right) - 6\left(\frac{\dot{N}}{N}\right)\left(\frac{\dot{a}}{a}\right)\right]\right]\right], \tag{2.16}$$

where we have taken into account (2.13). By making the derivation with respect to R, one finds

$$\xi = -2R\frac{\omega(t')}{3\lambda(t')} + \frac{1}{\kappa^2} - \Delta(t')\frac{dt'}{dR}, \tag{2.17}$$

where

$$\Delta(t') = \left[\frac{R}{(\kappa^2(t'))^2}\frac{d\kappa^2(t')}{dt'} + R^2\frac{d}{dt'}\left(\frac{\omega(t')}{3\lambda(t')}\right) + \frac{d\Lambda(t')}{dt'}\right], \tag{2.18}$$

since it is understood that the functions $\kappa^2(t'), \Lambda(t'), \lambda(t')$ and $\omega(t')$ depend on R throught t' as in Eq. (2.10).

Therefore, by substituting (2.17) and making an integration by parts one obtains the (standard) Lagrangian

$$\mathcal{L}(a, \dot{a}, N, R, \dot{R}) = -Na^3\Lambda(t') - \frac{6\dot{a}^2 a}{\kappa^2(t')N} + \frac{6\dot{a}a^2(\dot{\overline{\kappa^2(t')}})}{N(\kappa^2(t'))^2} + \frac{\omega(t')}{3\lambda(t')}a^3 N\left[R^2 + \frac{12R}{N^2}\frac{\dot{a}^2}{a^2} + \frac{12\dot{a}\dot{R}}{aN^2}\right] + \frac{d}{dt'}\left[\frac{\omega(t')}{3\lambda(t')}\right]\left(\frac{dt'}{dR}\dot{R}\right)\frac{12Ra^2\dot{a}}{N} + 6a^3N\left(\frac{R}{6} + \frac{\dot{a}^2}{a^2N^2}\right)\Delta(t')\frac{dt'}{dR} + 6\dot{a}\left(\frac{a^2}{N}\right)\left[\frac{d\Delta(t')}{dt'}\left(\frac{dt'}{dR}\right)^2 + \Delta(t')\frac{d^2t'}{dR^2}\right]\dot{R}. \tag{2.19}$$

If we derive this Lagrangian with respect to $N(t)$ and, therefore, we choose the gauge $N(t)=1$, we get

$$0=-a^3\Lambda(t')+\frac{6\dot{a}^2a}{\kappa^2(t')}-\frac{6\dot{a}a^2(\dot{\kappa^2(t')})}{(\kappa^2(t'))^2}+\frac{\omega(t')}{3\lambda(t')}a^3\left[R^2-12R\frac{\dot{a}^2}{a^2}-\frac{12\dot{a}\,\dot{R}}{a}\right]-12Ra^2\dot{a}\frac{d}{dt'}\left(\frac{\omega(t')}{3\lambda(t')}\right)\left(\frac{dt'}{dR}\dot{R}\right)$$
$$+6a^3\left(\frac{R}{6}-\frac{\dot{a}^2}{a^2}\right)\Delta(t')\frac{dt'}{dR}-6\dot{a}a^2\left[\frac{d\Delta(t')}{dt'}\left(\frac{dt'}{dR}\right)^2+\Delta(t')\frac{d^2t'}{dR^2}\right]\dot{R}. \tag{2.20}$$

The variation with respect to $a(t)$ leads to

$$0=-3a^2\Lambda(t')+\frac{6}{\kappa^2(t')}(\dot{a}^2+2\ddot{a}a)+\frac{6}{\kappa^2(t')}\left(\frac{2a^2(\dot{\kappa^2(t')})^2}{(\kappa^2(t'))^2}-\frac{2\dot{a}a(\dot{\kappa^2(t')})}{\kappa^2(t')}-\frac{a^2(\ddot{\kappa^2(t')})}{\kappa^2(t')}\right)$$
$$+\frac{\omega(t')}{\lambda(t')}(R^2a^2-4R\dot{a}^2-8\dot{R}\,\dot{a}\,a-8R\ddot{a}a-4\ddot{R}a^2)-24\frac{d}{dt}\left(\frac{\omega(t')}{\lambda(t')}\right)[\dot{R}a^2+Ra\dot{a}]-12\frac{d^2}{dt^2}\left(\frac{\omega(t')}{\lambda(t')}\right)Ra^2$$
$$+(3a^2R-6\dot{a}^2-12a\ddot{a})\Delta(t')\frac{dt'}{dR}-(12a\dot{a}\,\dot{R}+6a^2\ddot{R})\left[\frac{d\Delta(t')}{dt'}\left(\frac{dt'}{dR}\right)^2+\Delta(t')\frac{d^2t'}{dR^2}\right]$$
$$-6a^2\dot{R}^2\left[\frac{d^2\Delta(t')}{dt'^2}\left(\frac{dt'}{dR}\right)^3+3\frac{d\Delta(t')}{dt'}\left(\frac{dt'}{dR}\right)\frac{d^2t'}{dR^2}+\Delta(t')\frac{d^3t'}{dR^3}\right], \tag{2.21}$$

where we have set $N(t)=1$ again and $d/dt\equiv\dot{R}(dt'/dR)d/dt'$. Finally, the variation of the Lagrangian with respect to R, remembering that t' is a function of R, returns to be the expression in (2.13), and by putting $N(t)=1$, we have

$$R=6\left(\frac{\dot{a}}{a}\right)^2+6\left(\frac{\ddot{a}}{a}\right). \tag{2.22}$$

We obtained a system of three second order equations (2.20)–(2.22), where one is redundant (in the absence of matter contributions), namely it can be derived from the other two.

Equations (2.20) and (2.22) can be rewritten as

$$0=-\Lambda(t')+\frac{6H^2}{\kappa^2(t')}-\frac{6H}{(\kappa^2(t'))^2}\frac{d\kappa^2(t')}{dt'}\left(\frac{t_0'\dot{R}}{R}\right)+\frac{\omega(t')}{3\lambda(t')}[6R\dot{H}-12H\dot{R}]-12H\frac{d}{dt'}\left(\frac{\omega(t')}{3\lambda(t')}\right)(\dot{R}t_0')$$
$$+6(H^2+\dot{H})\Delta(t')\frac{t_0'}{R}-6H\left[\frac{d\Delta(t')}{dt'}\left(\frac{t_0'}{R}\right)^2-\Delta(t')\frac{t_0'}{R^2}\right]\dot{R}, \tag{2.23}$$

$$R=12H^2+6\dot{H}, \tag{2.24}$$

where we have introduced the Hubble parameter $H=\dot{a}/a$ and we have used (2.10) to write $dt'/dR=t_0'/R$. In the following expression, we explicitly develop Eq. (2.23) in terms of the functions $\lambda(t')$, $\omega(t')$, $\kappa^2(t')$, and $\Lambda(t')$ by using the set of equations (2.5)–(2.8) and Eq. (2.24) for the Ricci scalar,

$$0=\frac{12\omega(-6H^2\dot{H}-2H\ddot{H}+\dot{H}^2)}{\lambda}-\frac{H\lambda t_0'(40\omega^2-26\omega-3)(4H\dot{H}+\ddot{H})}{2\kappa^2\omega(2H^2+\dot{H})}+\frac{6H^2}{\kappa^2}$$
$$-\frac{t_0'}{360\kappa^4\omega^3(2H^2+\dot{H})^2}(H(4H\dot{H}+\ddot{H})(\lambda t_0'(120\kappa^4\omega^3(4\omega+3)(2\omega(100\omega+549)+25)(2H^2+\dot{H})^2$$
$$-2\kappa^2\lambda\omega(24H^2(\omega(\omega(20\omega(100\omega+409)-2121)+210)+15)+12\dot{H}(\omega(\omega(20\omega(100\omega+409)-2121)+210)+15)$$
$$+\kappa^2\Lambda(4\omega(1616\omega-355)-45))-15\lambda^2(\omega(2\omega(4\omega(50\omega+97)-25)-71)-5))-180\kappa^2\omega^2(2H^2+\dot{H})$$
$$\times(20\kappa^2\omega(4\omega(2\omega+3)+1)(2H^2+\dot{H})+\lambda(-40\omega^2+26\omega+3)))+15\omega(2H^4+7H^2\dot{H}+H\ddot{H}+\dot{H}^2)$$
$$\times(120\kappa^4\omega^2(4\omega(2\omega+3)+1)(2H^2+\dot{H})^2+4\kappa^2\lambda\omega(6H^2(-40\omega^2+26\omega+3)+\dot{H}(6(13-20\omega)\omega+9)$$
$$-2\kappa^2\Lambda(28\omega+1))-3\lambda^2(20\omega^2+1)))+10Ht_0(8\omega^2+12\omega+1)(4H\dot{H}+\ddot{H})-\Lambda. \tag{2.25}$$

Here, $\lambda \equiv \lambda(t')$, $\omega \equiv \omega(t')$, $\kappa^2 \equiv \kappa^2(t')$ and $\Lambda \equiv \Lambda(t')$. One should remember that t' is related to R as in Eq. (2.10), and only $\lambda(t')$ is given by (2.11). Note that the above approach suggests the consistent way to account for quantum effects of higher-derivative gravity. Note also that different approach to take into account such quantum effects at the inflationary universe was developed in Ref. [8].

On the de Sitter solution $R_{\mathrm{dS}} = 12H_{\mathrm{dS}}^2$, where H_{dS} is a constant, the system is simplified as

$$\begin{aligned}0 = \frac{6H^2}{\kappa^2} - \frac{t_0'}{48(\kappa^2)^2\omega^2}(480H^4(\kappa^2)^2\omega^2(4\omega(2\omega+3)+1) \\ +4\kappa^2\lambda\omega(6H^2(-40\omega^2+26\omega+3) - 2\kappa^2\Lambda(28\omega+1)) \\ -3\lambda^2(20\omega^2+1)) - \Lambda, \end{aligned} \tag{2.26}$$

where the functions $\lambda, \omega, \kappa^2$ and Λ are assumed to be constant and $H \equiv H_{\mathrm{dS}}$.

Hence, we obtained consistent system of quantum-corrected FRW equations from the RG-improved effective action corresponding to higher-derivative quantum gravity.

III. THE ASYMPTOTIC BEHAVIOR OF THE EFFECTIVE COUPLING CONSTANTS AND THE DE SITTER SOLUTION FOR INFLATION

In order to solve the system (2.25), we need to investigate the asymptotic behavior of the implicitly given effective coupling constants $\omega(t'), \kappa^2(t'), \Lambda(t')$, when $t' \to \infty$, namely, at the high curvature limit ($R \to \infty$) describing inflation [see (2.10)]. Equation (2.6) has two fixed points at

$$\omega_1 \simeq -0.02, \qquad \omega_2 \simeq -5.47, \tag{3.1}$$

and the analysis of the solution around the fixed points $\omega(t') = \omega_{1,2} + \delta\omega(t')$, with $|\delta\omega(t')| \ll 1$, leads to

$$\begin{aligned}\frac{d\omega(t')}{dt'} &\simeq -\lambda(t')\left(\frac{20}{3}\omega + \frac{183}{10}\right)\Big|_{\omega_{1,2}} \delta\omega(t') - \lambda(t')^2\beta_2 \\ &\times \left(\frac{dt'}{d\omega(t')}\right)\left(\frac{10}{3}\omega^2 + \frac{183}{10}\omega + \frac{5}{12}\right)\Big|_{\omega_{1,2}} \delta\omega(t') \\ &= -\lambda(t')\left(\frac{20}{3}\omega + \frac{158}{5}\right)\Big|_{\omega_{1,2}} \delta\omega(t'),\end{aligned} \tag{3.2}$$

such that,

$$\begin{aligned}\omega(t') &= \omega_{1,2} + \frac{c_0}{(1+\lambda(0)\beta_2 t')^q}, \\ q &= \frac{1}{\beta_2}\left(\frac{20}{3}\omega + \frac{158}{5}\right)\Big|_{\omega_{1,2}}, \\ |c_0| &\ll 1,\end{aligned} \tag{3.3}$$

where c_0 is a constant and we have introduced $\lambda(t')$ as in (2.11). We immediatly see that $q \simeq 2.37$ for ω_1 rendering the solution stable when $t' \to \infty$, but for ω_2 one gets $q \simeq -0.37$ and the solution is unstable when $t' \to \infty$. Thus, we expect that for large values of t' the function $\omega(t')$ tends to the attractor ω_1. Since between ω_1 and ω_2 the derivative $d\omega(t')/dt'$ with $0 < \lambda(t')$ is positive, $\omega(t')$ grows up with t' and approaches to ω_1 being $\omega(t') < \omega_1$. When $\omega_2 < \omega(t') < \omega_1$ we may estimate from (3.2),

$$\frac{d\omega(t')}{dt'} = -\frac{\lambda(t')}{2}\left(\frac{20}{3}\right)(\omega_1 - \omega_2)\delta\omega(t'). \tag{3.4}$$

Therefore, the solution (3.3) is rewritten as (see third Ref. in [12]),

$$\begin{aligned}\omega(t') &= \omega_1 + \frac{c_0}{(1+\lambda(0)\beta_2 t)^p}, \\ p &= \left(\frac{10}{3}\right)\frac{(\omega_1-\omega_2)}{\beta_2} \simeq 1.36, \qquad |c_0| \ll 1.\end{aligned} \tag{3.5}$$

Note that related study for the behavior of above dimensionless coupling constants in relation with dimensional transmutation is given in Ref. [17].

In order to study the behavior of $\kappa^2(t')$ and $\Lambda(t')$, we introduce

$$\tilde{\Lambda}(t') = (\kappa^2(t'))^2\Lambda(t'), \tag{3.6}$$

and Eq. (2.8) with Eq. (2.7) lead to

$$\begin{aligned}\frac{d\tilde{\Lambda}(t')}{dt'} = \beta_4 \equiv \frac{\lambda(t')^2}{2}\left(5 + \frac{1}{4\omega(t')^2}\right) \\ + \lambda(t')\tilde{\Lambda}(t')\left(\frac{20}{3}\omega(t') + 5 - \frac{1}{6\omega(t')}\right).\end{aligned} \tag{3.7}$$

In the asymptotic limit $\omega(t') \simeq \omega_1$ we get

$$\begin{aligned}\tilde{\Lambda} &= -\frac{3\lambda(0)(1+20\omega_1^2)}{4\omega_1(1+\lambda(0)\beta_2 t')(-1+30\omega_1+6\beta_2\omega_1+40\omega_1^2)} \\ &\quad + \tilde{\Lambda}_0(1+\lambda(0)\beta_2 t')^{W/\beta_2}, \\ W &= \frac{20}{3}\omega_1 + 5 - \frac{1}{6\omega_1} = 13.2.\end{aligned} \tag{3.8}$$

As a consequence,

$$\tilde{\Lambda}(t') \simeq \tilde{\Lambda}_0(1+\lambda(0)\beta_2 t')^{W/\beta_2}, \tag{3.9}$$

where the constant $\tilde{\Lambda}_0$ is assumed to be positive. On the other side, from Eq. (2.7) we have at $\omega(t') \simeq \omega_1$,

$$\kappa^2(t') \simeq \kappa_0^2(1+\lambda(0)\beta_2 t')^{Z/\beta_2},$$
$$Z = \left(\frac{10}{3}\omega_1 - \frac{13}{6} - \frac{1}{4\omega_1}\right) \simeq 10.27, \tag{3.10}$$

such that finally

$$\Lambda(t') \simeq \frac{\tilde{\Lambda}_0}{(\kappa_0^2)^2}(1+\lambda(0)\beta_2 t')^{X/\beta_2},$$
$$X = (W - 2Z) \simeq -7.34. \tag{3.11}$$

Let us summarize the results. From the investigation of the asymptotic region, we can derive the effective running coupling constants of the model (2.4) as

$$\lambda(t') = \frac{\lambda(0)}{(1+\lambda(0)\beta_2 t')}, \qquad \omega \simeq \omega_1 + \frac{c_0}{(1+\lambda(0)\beta_2 t')^{1.36}},$$
$$\kappa^2(t') \simeq \kappa_0^2(1+\lambda(0)\beta_2 t')^{0.77},$$
$$\Lambda(t') \simeq \Lambda_0 \frac{1}{(1+\lambda(0)\beta_2 t')^{0.55}}. \tag{3.12}$$

Here, $\Lambda_0 = \tilde{\Lambda}_0/(\kappa_0^2)^2$ and $|c_0| \ll |\omega_1|$, and we will omit its contribution at large t'. One remark is in order. In principle these expressions correspond to the behavior of the coupling constants in the high energy limit, when $t' \to \infty$ and $R_0 \ll R$, R_0 being the Ricci scalar at the present time, and they are valid as soon as $\omega(t')$ is close to ω_1. However, we may assume that the structure of the coupling constants keeps the same form at every epoch, since in fact out of inflation the curvature of the universe drastically decreases, $t' \to 1$, and the coupling constants are expected to be constant: in fact, we can consider $\omega(t')$ sufficiently close to $-\omega_1$ at every time; namely, we will not consider the additional corrections at small curvature. In particular, at the present de Sitter epoch with $R = R_0$ and $t_0' = 0$ (see Eq. (2.10) and the comment below), we must find

$$\kappa^2(t_0') \equiv \kappa_0^2 = \frac{16\pi}{M_{Pl}^2}, \qquad \Lambda(t_0') \equiv \Lambda_0 = 2\Lambda, \tag{3.13}$$

where $M_{\rm Pl}$ is the Planck mass and Λ is the cosmological constant, which is much smaller than the curvature at the inflation scale. By considering $\lambda(0)$ of the order of the unit to avoid the R^2-correction at the present epoch, at the time of inflation one can put $\Lambda(t') = 0$.

Let us assume that $R = R_{\rm dS}$ describes the curvature of (de Sitter) inflation. Since it must be $R_0 \ll R_{\rm dS} \equiv 12H_{\rm dS}^2$, where $R_0 = 4\Lambda$, one has

$$\log\left[\frac{R_{\rm dS}}{R_0}\right] = \log[H_{\rm dS}^2\kappa_0^2] - \log\left[\frac{\Lambda}{3}\kappa_0^2\right] \simeq -\log\left[\frac{\Lambda}{3}\kappa_0^2\right]. \tag{3.14}$$

Thus, from (2.10) we get

$$t' \simeq -t_0' \log\left[\frac{\Lambda}{3}\kappa_0^2\right], \qquad 1 \ll t', \tag{3.15}$$

namely t' expresses the rate of the curvature of the current Universe with respect to the Planck mass on logaritm scale: this approximation is valid as soon as $R_{\rm dS}$ is near to $M_{\rm Pl}^2$ during inflation, where "near" is understood as "with respect to the cosmological constant scale." In fact, the solution of Eq. (2.26) depends on the value of today $\lambda(0)$, which fixes the bound of inflation. From (2.26), we derive the following solution,

$$H_{\rm dS}^2\kappa_0^2 \simeq \frac{0.0146}{t_0'(\lambda(0)t')^{0.77}} \equiv \frac{0.0146}{t_0'^{1.77}(\lambda(0))^{0.77}}\frac{1}{[-\log[\frac{\Lambda}{3}\kappa_0^2]]^{0.77}}, \tag{3.16}$$

where we have taken into account that $1 \ll t'$. If we use the recent cosmological data [1] for the evaluation of Λ in Planck units (see also Ref. [18]),

$$\Lambda\kappa_0^2 \simeq 1.7 \times 10^{-121}, \tag{3.17}$$

and we set for simplicity $t_0' = 1$, we finally obtain

$$H_{\rm dS}^2\kappa_0^2 \simeq \frac{19 \times 10^{-5}}{\lambda(0)^{0.77}}. \tag{3.18}$$

For example, for $\lambda(0) = 1$, we have

$$-\frac{\omega_2}{3\lambda(0)}(4\Lambda\kappa_0^2)R \simeq 4.53 \times 10^{-123}R \ll R,$$
$$\frac{1.7 \times 10^{-121}}{3}M_{\rm Pl}^2 \simeq \left(\frac{\Lambda}{3}\right) \ll H_{\rm dS}^2 \simeq 3.8 \times 10^{-6}M_{\rm Pl}^2. \tag{3.19}$$

The first condition guarantees that at the present epoch the R_0^2-contribution to the action (2.4) is negligible with respect to the Hilbert-Einstein term R_0/κ_0^2, where $R_0 = 4\Lambda$. The second condition shows that de Sitter solution of inflation takes place at very high curvature near to the Planck scale, such that the approximation (3.14) is well satisfied. We also note that during inflation,

$$\frac{R}{\kappa^2(t')} \simeq 1.6 \times 10^{-9}M_{\rm Pl}^4 \ll -\frac{\omega(t')}{3\lambda(t')}R^2 \simeq 5.1 \times 10^{-8}M_{\rm Pl}^4, \tag{3.20}$$

and the second term in (2.4) is dominant with respect to the Hilbert-Einstein contribution at the early Universe, thanks to the fact that the running constant $\kappa^2(t')$ increases back into the past.

IV. DYNAMICS OF INFLATION

In this section, we would like to analyze the behavior of the model (2.4) at high curvature, when the de Sitter solution describing inflation (3.16) takes place. First of all, in order to have the exit from inflation, one must show that the solution is unstable. Hence, we can try to describe the inflation in terms of the e-folds number and slow-roll parameters.

A. Instability of the de Sitter universe

Let us consider the following form of Hubble parameter which is used in Eq. (2.25),

$$H = H_{\mathrm{dS}} + \delta H(t), \qquad |\delta H(t)| \ll 1, \tag{4.1}$$

where $\delta H(t)$ is the perturbation with respect to de Sitter inflation. By making use of Eq. (2.26) and (3.12)–(3.13) with $c_0, \Lambda = 0$ in Eq. (2.25), and by multiplying it by κ_0^3, one has at the first order in $\delta H(t) \equiv \delta H$,

$$\begin{aligned} 0 = {}& (\kappa_0 \dot{\delta} H)\left[t'_0\left((H_{\mathrm{dS}}\kappa_0)^2\left(34.344 - \frac{0.913 t'_0}{t'}\right) + \frac{0.001 t' + 0.003 t'_0}{t'^3 (H_{\mathrm{dS}}\kappa_0)^2 (\lambda(0)t')^{1.54}} + \frac{0.346 t'_0 - 0.086 t'}{t'^2 (\lambda(0)t')^{0.77}}\right) + 19.152 t' (H_{\mathrm{dS}}\kappa_0)^2\right] \\ &+ \frac{(\kappa_0^2 \ddot{\delta} H)}{t'^3 (H_{\mathrm{dS}}\kappa_0)^3}\left[t'^2 (H_{\mathrm{dS}}\kappa_0)^4 (6.384 t'^2 + t'_0(11.448 t' - 0.228 t'_0)) \right. \\ &\left. - \frac{0.043 t'^2 t'_0 (H_{\mathrm{dS}}\kappa_0)^2}{(\lambda(0)t')^{0.77}} + \frac{0.001 t'^2_0}{(\lambda(0)t')^{1.54}} + \frac{0.087 t' t'^2_0 (H_{\mathrm{dS}}\kappa_0)^2}{(\lambda(0)t')^{0.77}} + \frac{2\times 10^{-4} t' t'_0}{(\lambda(0)t')^{1.54}}\right] \\ &+ (H_{\mathrm{dS}}\kappa_0)\delta H\left[\frac{0.223}{(\lambda(0)t')^{0.77}} + \frac{0.172 \lambda(0) t'_0}{(\lambda(0)t')^{1.77}} - 30.528 t'_0 (H_{\mathrm{dS}}\kappa_0)^2\right]. \end{aligned} \tag{4.2}$$

If we assume

$$1 \ll (H_{\mathrm{dS}}\kappa_0)^2 t'^{2.27}, \tag{4.3}$$

the above expression is simplified as

$$D_0 \delta H + t'[19.152 (H_{\mathrm{dS}}\kappa_0)(\kappa_0 \dot{\delta} H) + 6.384 (\kappa_0^2 \ddot{\delta} H)] \simeq 0, \tag{4.4}$$

where

$$D_0 = \left(\frac{0.223}{(\lambda(0)t')^{0.77}} - 30.528 t'_0 (H_{\mathrm{dS}}\kappa_0)^2\right). \tag{4.5}$$

Thus, the solution of the equation reads

$$\begin{aligned} &\delta H = h_\pm \exp[A_\pm t], \\ &A_\pm = \left[\frac{H_{\mathrm{dS}}}{2}\left(-3 \pm \sqrt{9 - \frac{0.627 D_0}{(H_{\mathrm{dS}}\kappa_0)^2 t'}}\right)\right], \\ &|h_\pm| \ll 1, \end{aligned} \tag{4.6}$$

where $h_\pm$ are the integration constants corresponding to plus and minus signs inside $A_\pm$. By choosing the sign plus in (4.6), the solution is unstable under the condition

$$D_0 < 0. \tag{4.7}$$

We would like to note that if we ignore the contribution from δH in (4.4), we get

$$-\frac{\omega}{3\lambda}[(-216 H_{\mathrm{dS}}^2)\dot{\delta} H(t) - 72 \ddot{\delta} H(t)] \simeq 0, \tag{4.8}$$

which is the equation for perturbation around the de Sitter solution in pure R^2 theory with Lagrangian $\mathcal{L} = -(\omega/(3\lambda))R^2$, $\omega/3\lambda$ being constant. From this equation it is not possible to know if the solution is stable or not, since δH mainly goes like $\delta H \sim$ const in the time and even a small contribution from the coefficient in front of $\delta H(t)$ could make the solution unstable so that a further analysis is required. In particular, the fact that the coefficient in front to R^2 is not a constant contributes to the instability of the solution, since for the Lagrangian $\mathcal{L} = -(\omega(t')/(3\lambda(t')))R^2$ we get the equation

$$\begin{aligned} &-\frac{\omega}{3\lambda}[(-216 H_{\mathrm{dS}}^2)\dot{\delta} H - 72 \ddot{\delta} H] \\ &\quad + (24 H_{\mathrm{dS}})^2 (6 H_{\mathrm{dS}})^3 \frac{d}{dR}\left(\frac{\omega(t')}{3\lambda(t')}\right)\delta H \simeq 0, \end{aligned} \tag{4.9}$$

where we have omitted the additional contributions to $\dot{\delta} H, \ddot{\delta} H$. The term related to δH corresponds to the last term of D_0 in (4.5), and, if it is dominant, it makes the solution (4.6) unstable.

Let us discuss the conditions (4.3) and (4.7). If

$$\frac{0.007}{t_0(\lambda(0)t')^{0.77}} < (H_{\rm dS}\kappa_0)^2, \tag{4.10}$$

both of the conditions are well satisfied and by taking into account the de Sitter solution (3.16), we see that this formula holds always true and it is independent of the bound on inflation encoded in $\lambda(0)$. It means that the de Sitter solution is unstable with

$$D_0 \simeq -\frac{0.223}{(\lambda(0)t')^{0.77}}, \tag{4.11}$$

where we have used (3.16). Moreover,

$$A_+ \simeq 0.796\frac{H_{\rm dS}t'_0}{t'}, \qquad A_- \simeq -3H_{\rm dS}, \tag{4.12}$$

where D_0 has been considered very small. For example, by setting $H_{\rm dS}\kappa_0$ with (3.16)–(3.17) and by putting $t'_0 = 1$ and $\lambda(0) = 1$, one derives

$$\delta H = h_- e^{-5833\times10^{-6}M_{\rm Pl}t} + h_+ e^{5.54\times10^{-6}M_{\rm Pl}t}. \tag{4.13}$$

During inflation, as soon as $t \ll 1/A_+$, avoiding the contribution of h_- which quickly disappears, one may estimate

$$\delta H \simeq h_+, \qquad \dot{\delta}H \simeq h_+A_+, \qquad \ddot{\delta}H \simeq h_+A_+^2, \tag{4.14}$$

where A_+ is the instability parameter. The duration of inflation Δt is of the order of magnitude

$$\Delta t \sim \frac{1}{A_+}, \tag{4.15}$$

but may continue after the linear approximation of the perturbation. In the case of (4.13), one has

$$\Delta t \sim \frac{18\times10^4}{M_{\rm Pl}}. \tag{4.16}$$

The inflation solves the problems of initial conditions of the Friedmann universe (horizon and velocities problems), if $\dot{a}_{\rm i}/\dot{a}_0 < 10^{-5}$, where $\dot{a}_{\rm i}, \dot{a}_0$ are the time derivatives of the scale factor at the big bang and today, respectively, and 10^{-5} is the estimated value of the inhomogeneity (anisotropy) in our Universe. Since in the decelerating universe, $\dot{a}(t)$ decreases by a factor 10^{28}, it is required that $\dot{a}_{\rm i}/\dot{a}_{\rm f} < 10^{-33}$, with $a_{\rm i}$ the scale factor at the beginning of inflation and $a_{\rm f}$ the scale factor at the end of inflation. If inflation is governed by a (quasi–) de Sitter solution where $a(t) = \exp(H_{\rm dS}t)$, we introduce the number of e-folds N as

$$N = \ln\left(\frac{a_{\rm f}}{a_{\rm i}}\right) \equiv \int_{t_i}^{t_f} H(t)dt, \tag{4.17}$$

and inflation is viable if $N > 76$, but the spectrum of fluctuations of CMB say that it is enough $\mathcal{N} \simeq 55$ to have thermalization of the observable Universe. In our case,

$$N \simeq H_{\rm dS}\Delta t \sim \frac{H_{\rm dS}}{A_+} \simeq 1.26\left(\frac{t'}{t'_0}\right), \tag{4.18}$$

due to the fact that the Hubble parameter is almost a constant during inflation. In order to obtain a viable inflation, it must be

$$61 < \left(\frac{t'}{t'_0}\right). \tag{4.19}$$

It means that, from (2.10) and (3.17),

$$3.1R_0 \times 10^{26} < R, \tag{4.20}$$

and this condition is always satisfied for realistic inflation. For the case of (3.18), where the Hubble parameter during inflation is 117 times larger than today and the duration of inflation is given by (4.16), we get

$$N \sim 339, \tag{4.21}$$

and it is guaranteed that the thermalization of a portion of the Universe is much larger with respect to the observed one.

It is clear that a large e-folds number, which corresponds to a huge amount of inflation, may be related to the fact that the Universe remains extremely close to the de Sitter spacetime during inflation. In fact, even if we cannot impose any upper limit to the e-folds number without additional data about the decay of the primordial accelerated expansion (the so-called "false vacuum"), and we expect that the homogeneity and isotropy continue for some distance beyond our observable Universe, the primordial perturbations at the end of inflation depend on the e-folds. As a consequence, as we will see in the next subsection, a large e-folds number could generate the wrong predictions for the spectral index. In the last part of the work, we will find how it is possible to make inflation shorter, according to a correct prediction of such an index.

B. Slow-roll parameters and the spectral index

During inflation the Hubble parameter must slowly decrease, and the following approximations must be met:

$$\left|\frac{\dot{H}}{H^2}\right| \ll 1, \qquad \left|\frac{\ddot{H}}{H\dot{H}}\right| \ll 1. \tag{4.22}$$

Thus, one introduces the slow-roll parameters,

$$\epsilon = -\frac{\dot{H}}{H^2}, \qquad \eta = -\frac{\dot{H}}{H^2} - \frac{\ddot{H}}{2H\dot{H}} \equiv 2\epsilon - \frac{1}{2\epsilon H}\dot{\epsilon}, \tag{4.23}$$

whose magnitude must be small during inflation, and $\dot{H}$ is assumed to be negative. In particular, since the acceleration is expressed as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2, \tag{4.24}$$

we see that the Universe expands in an accelerated way as soon as $\epsilon < 1$. By integrating the formula for the (positive and almost constant) ϵ parameter in (4.23), we also get

$$H(t) = \frac{1}{\epsilon(t_{\rm dS} + t)}, \qquad t_{\rm dS} \simeq \frac{1}{\epsilon H_{\rm dS}}, \tag{4.25}$$

where $t_{\rm dS}$ is a positive time parameter, and when the time increases, the Hubble parameter decreases. In the limit $t/t_{\rm dS} \ll 1$, one has

$$H(t) \simeq H_{\rm dS} - H_{\rm dS}^2 \epsilon t, \tag{4.26}$$

and by taking into account (4.14), we get

$$\epsilon \simeq \frac{(-h_+)A_+}{(H_{\rm dS})^2} = 0.796272\left(\frac{t_0'}{t'}\right)\frac{(-h_+)}{H_{\rm dS}}, \tag{4.27}$$

where $h_+ < 0$ and A_+ is given by (5.19). This relation is consistent with a direct evaluation of the slow-roll parameter ϵ (4.23) in the slow-roll limit (4.22) of the equation of motion (2.25),

$$\begin{aligned} 0 = {}& 2\lambda^2\epsilon[480H^4\kappa^4\omega^3(4\omega+3)(2\omega(100\omega+549)+25) + 2\kappa^4\lambda\Lambda\omega(4(355-1616\omega)\omega+45) \\ & -15\lambda^2(\omega(2\omega(4\omega(50\omega+97)-25)-71)-5)] + 720\kappa^2\omega^3(72H^4\kappa^2\omega\epsilon + 6H^2\lambda - \kappa^2\lambda\Lambda) \\ & + 15\lambda\omega(-480H^4\kappa^4\omega^2(4\omega(2\omega+3)+1)(8\epsilon+1) + 4\kappa^2\lambda\omega(3H^2(4\omega-3)(10\omega+1)(7\epsilon+2) \\ & +2\kappa^2\Lambda(28\omega+1)) + \lambda^2(60\omega^2+3)). \end{aligned} \tag{4.28}$$

By using (3.12)–(3.13) with $c_0 = \Lambda = 0$, one obtains the solution

$$\epsilon \simeq \frac{-\frac{3\times10^{-4}t_0'}{t'^2(\lambda(0)t')^{1.54}} - \frac{0.086\lambda(0)t_0'(H\kappa_0)^2}{(\lambda(0)t')^{1.77}} - \frac{0.112(H\kappa_0)^2}{(\lambda(0)t')^{0.77}} + 7.632t_0(H\kappa_0)^4}{-\frac{0.003(t_0')^2}{t'^3(\lambda(0)t')^{1.54}} + \frac{0.913(t_0')^2(H\kappa_0)^4}{t'} + t_0'(H\kappa_0)^2\left(\frac{0.301\lambda(0)}{(\lambda(0)t')^{1.77}} - 61.056(H\kappa_0)^2\right) - 19.152t'(H\kappa_0)^4}, \tag{4.29}$$

and under the condition (4.3), we derive

$$\epsilon \simeq \frac{0.006}{t'(\lambda(0)t')^{0.77}(H\kappa_0)^2} - \frac{0.398}{t'}. \tag{4.30}$$

By expanding $H(t)$ around the de Sitter solution (3.16), we finally get

$$\begin{aligned} \epsilon &\simeq \frac{-2(0.006)}{t'(\lambda(0)t')^{0.77}(H_{\rm dS}\kappa_0)^3}\kappa_0\delta H \\ &= \frac{0.012}{t'(\lambda(0)t')^{0.77}(H_{\rm dS}\kappa_0)}\frac{\epsilon}{\kappa_0 A_+}, \end{aligned} \tag{4.31}$$

where Eqs. (4.14) and (4.27) are considered: the equation is well satisfied by using (3.16) again and (5.19). Thus, the ϵ slow-roll parameter is related to the (initial) amplitude of perturbation and by using (4.18), one may estimate

$$\epsilon \simeq \frac{(-h_+)A_+}{(H_{\rm dS})^2} \sim \frac{(-h_+)}{(H_{\rm dS})N}. \tag{4.32}$$

Moreover, for the η slow-roll parameter in (4.23) with (4.14), one has

$$\eta \simeq -\frac{A_+}{2H_{\rm dS}} \simeq -\frac{0.398t_0'}{t'} \sim \frac{1}{2N}. \tag{4.33}$$

Both of the paramters $\epsilon, |\eta|$ (4.32)–(4.33) are very small during inflation, and the slow-roll approximations (4.22) hold true. We also note that, since $|h_+| \ll H_{\rm dS}$,

$$\epsilon \ll |\eta|, \tag{4.34}$$

like in other scalar-tensor theories for inflation, where usually $\epsilon \sim 1/N^2$, as in (4.32) if we consider $(-h_+)/H_{\rm dS} \sim 1/N$.

Given the slow-roll parameters, one can evaluate the Universe's anisotropy coming from inflation by introducing the spectral indexes. To be specific, the amplitude of the primordial scalar power spectrum reads

$$\Delta^2_{\mathcal{R}} = \frac{\kappa^2 H^2}{8\pi^2 \epsilon}, \tag{4.35}$$

and for slow-roll inflation the spectral index n_s and the tensor-to-scalar ratio are given by

$$n_s = 1 - 4\eta, \qquad r = 48\epsilon^2, \tag{4.36}$$

where we use the results for modified gravity [19]. The last Planck data [1] constrain these quantities as

$$n_s = 0.9603 \pm 0.0073, \qquad r < 0.11. \tag{4.37}$$

For our model, one has the scalar power spectrum

$$\Delta_{\mathcal{R}} \simeq 1.25585(H_{\mathrm{dS}}\kappa_0)^3 \left(\frac{t'}{t'_0}\right)(-\kappa_0^2 h_+)^{-1}, \tag{4.38}$$

and the spectral index and the tensor-to-scalar ratio,

$$n_s = 1 - \frac{2A_+}{H_{\mathrm{dS}}} \sim 1 - \frac{2}{N},$$
$$r = \frac{48A_+^2}{H_{\mathrm{dS}}^2}\frac{(-h_+)^2}{H_{\mathrm{dS}}^2} \ll \frac{1}{N}, \tag{4.39}$$

where we have used (4.34). We see that the tensor-to-scalar ratio can satisfy the Planck results, with the e-folds of realistic inflation being quite large. On the other side, in order to find the spectral index n_s in agreement with the Planck data (4.37), we must require

$$21 < \frac{2A_+}{H_{\mathrm{dS}}}\left(= 2.5117\left(\frac{t'}{t'_0}\right)\right) < 31. \tag{4.40}$$

Since A_+/H_{dS} depends on the ratio between the curvature of the Universe at the time of inflation and the curvature of today's Universe, it results particulary high and does not satisfy this condition, contributing to render near to one the spectral index n_s of the model. For example, in the case of (3.18) where the Hubble parameter during inflation is 117 times larger than today and the e-folds $N \sim H_{\mathrm{dS}}/(A_+) \simeq 339$ as in (4.21),

$$n_s \simeq 0.994, \qquad r \simeq 0.0004\frac{(-h_+)^2}{H_{\mathrm{dS}}^2}. \tag{4.41}$$

Since $(-h_+/H_{\mathrm{dS}}) \ll 1$, the tensor-to-scalar ratio is much smaller than 0.11, but the spectral index does not satisfy the Planck data. This should be compared with analysis of inflationary parameters for general $F(R)$ theory in fluidlike presentation [20] which maybe consistent with Planck data.

The large e-folds number and the n_s spectral index too close to one are consequences of the small value of A_+ (5.19), which depends on $d(\omega(t')/3\lambda(t'))/dt'$, as we explained under (4.9). In particular, the fact that $d(\omega(t')/3\lambda(t'))/dt' = -\beta_3/3$, where β_3 is given in (2.9), such that $\beta_3 \ll 1$, makes this term too small compared with the coefficients in front of $\ddot{\delta}H(t), \dot{\delta}H(t)$ in the equation for perturbation (4.8). In the next section, we suggest a possible solution of the problem returning to the general action (2.4) with the Gauss-Bonnet and $\Box R$ terms which have been omitted in the above study.

V. THE ACCOUNT OF GAUSS-BONNET AND $\Box R$ TERMS AND THE SPECTRAL INDEX

As was mentioned in Sec. II, to construct the Lagrangian of higher-derivative gravity, the Gauss-Bonnet and $\Box R$ terms must also be taken into account. They may give a nonzero contribution to the dynamical equations if the coefficients in front of them are not constant but depend on the curvature. This is precisely what happens when one solves the RG equation and gets the RG-improved effective action. In the first part of this work we did not consider such contributions. Let us analyze their role on the dynamics of the inflation induced by higher-derivative quantum gravity. Let us consider the following additional piece to the action (2.4),

$$I_{G,\Box R} = -\int_{\mathcal{M}} d^4x\sqrt{-g}[\gamma(t')G - \zeta(t')\Box R], \tag{5.1}$$

where G is given by (2.3) and $\gamma(t'), \zeta(t')$ are effective coupling constants depending on t' (2.10) and, therefore, on R. We assume

$$\gamma(t') = \gamma_0(1 + c_1 t'),$$
$$\zeta(t') = \zeta_0(1 + c_2 t'), \tag{5.2}$$

where γ_0, ζ_0 are generic constants and $c_{1,2}$ are numerical coefficients whose explicit values are not necessary in the below analysis. As is explained in Ref. [7], this is a result of the one-loop quantum calculation of these terms (vacuum polarization). For a recent discussion of the contribution of the GB term in higher-derivative gravity, see Ref. [21]. Actually, the calculation of surface terms may be done in less or more than four dimensions, with subsequent dimensional continuation.

Hence, when $t' \ll 1$, at the low curvature limit, $\gamma(t'), \zeta(t')$ tends to constants, the derivatives do not diverge, and (5.1) turns out to be zero; on the other side, when $1 \ll t'$, at the high curvature limit, they give a significant contribution to the dynamical equations of motion. The Gauss-Bonnet represents a new curvature invariant. On the FRW metric, it (2.12) reads

$$G = \frac{24\dot{a}^2}{a^3 N^5}(\ddot{a}N - \dot{a}\dot{N}). \tag{5.3}$$

Adding to the Lagrangian (2.16) the piece (5.1), we make an integration by parts with respect to $\Box R$, where $\Box R = (\sqrt{-g})^{-1}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu R) \equiv -(\sqrt{-g})^{-1}\partial_t(\sqrt{-g}\partial_t R)$, and introduce a new Lagrangian multiplier σ for the Gauss-Bonnet term [15], such that

$$I_{G,\Box R} = -\int_{\mathcal{M}} d^4x\sqrt{-g}\left[\gamma(t')G + \sigma\left[G - \frac{24\dot{a}^2}{a^3N^5}(\ddot{a}N - \dot{a}\dot{N})\right] - \left(\frac{d\zeta}{dt'}\frac{dt'}{dA}\dot{A}^2\right)\right], \qquad \sigma = -\gamma(t'). \tag{5.4}$$

Here the second expression has been derived from the variation with respect to G, and $A \equiv A(N, \dot{N}, a, \dot{a})$ is the explicit form of the Ricci scalar as a function of the metric (2.13),

$$A(N, \dot{N}, a, \dot{a}) = \frac{1}{N^2}\left[6\left(\frac{\dot{a}}{a}\right)^2 + 6\left(\frac{\ddot{a}}{a}\right) - 6\left(\frac{\dot{N}}{N}\right)\left(\frac{\dot{a}}{a}\right)\right]. \tag{5.5}$$

Thus, $\Delta(t')$ in (2.17) reads

$$\Delta(t') = \left[\frac{R}{(\kappa^2(t'))^2}\frac{d\kappa^2(t')}{dt'} + R^2\frac{d}{dt'}\left(\frac{\omega(t')}{3\lambda(t')}\right) + \frac{d\Lambda(t')}{dt'} + \frac{d\gamma(t')}{dt'}G\right], \tag{5.6}$$

and the additional piece of the Lagrangian (2.19) turns out to be

$$\mathcal{L}_{G,\Box R}(N, \dot{N}, \ddot{N}, a, \dot{a}, \ddot{a}, R, \dot{R}) = 6\dot{a}\left(\frac{a^2}{N}\right)\left[\frac{d\gamma(t')}{dt'}\frac{dt'}{dR}\dot{G}\right] + \frac{8\dot{a}^3}{N^3}\frac{d\gamma(t')}{dt'}\frac{dt'}{dR}\dot{R} + (Na^3)\left(\frac{d\zeta}{dt'}\frac{dt'}{dA}\dot{A}^2\right), \tag{5.7}$$

where the first piece comes from the integration by parts of the second derivative metric functions of the Ricci scalar, the second term comes from the ones of the Gauss-Bonnet, and the last piece corresponds to the $\Box R$ term. Note that now the Lagrangian depends on the higher derivatives of the metric due to the introduction of $\dot{A}^2$. Equation (2.23), in the gauge $N = 1$, is derived as

$$\begin{aligned} 0 = {} & -\Lambda(t') + \frac{6H^2}{\kappa^2(t')} - \frac{6H}{(\kappa^2(t'))^2}\frac{d\kappa^2(t')}{dt'}\left(\frac{t_0'\dot{R}}{R}\right) + \frac{\omega(t')}{3\lambda(t')}[6R\dot{H} - 12H\dot{R}] - 12H\frac{d}{dt'}\left(\frac{\omega(t')}{3\lambda(t')}\right)(\dot{R}t_0') + 6(H^2 + \dot{H})\Delta(t')\frac{t_0'}{R} \\ & - 6H\left[\frac{d\Delta(t')}{dt'}\left(\frac{t_0'}{R}\right)^2 - \Delta(t')\frac{t_0'}{R^2}\right]\dot{R} - 24H^3\frac{d\gamma(t')}{dt'}\frac{t_0'\dot{R}}{R} - 6H\left[\frac{d\gamma(t')}{dt'}\frac{t_0'\dot{G}}{R}\right] - 3\mathcal{A}\dot{R}^2 - 2\mathcal{B}\dot{R}^2 R \\ & + 6\frac{d}{dt}[2\mathcal{A}(4H^2 + 3\dot{H})\dot{R} + \mathcal{B}H\dot{R}^2] + 18H[2\mathcal{A}(4H^2 + 3\dot{H})\dot{R} + \mathcal{B}H\dot{R}^2] - 36(3H^2 + \dot{H})\mathcal{A}H\dot{R} - 72H\frac{d}{dt}(\mathcal{A}H\dot{R}) \\ & - 12\frac{d^2}{dt^2}(\mathcal{A}H\dot{R}), \end{aligned} \tag{5.8}$$

where

$$\mathcal{A} = \left(\frac{d\zeta(t')}{dt'}\frac{t_0'}{R}\right), \qquad \mathcal{B} = \left[\frac{d^2\zeta(t')}{dt'^2}\left(\frac{t_0'}{R}\right)^2 - \frac{d\zeta(t')}{dt'}\frac{t_0'}{R^2}\right], \tag{5.9}$$

and the Ricci scalar R is given by (2.24). The derivative of the Lagrangian with respect to the Gauss-Bonnet leads to the Ricci scalar in (2.13), and the derivative with respect to the Ricci scalar leads to the Gauss-Bonnet one in (5.3), which reads in the gauge $N = 1$,

$$G = 24H^2(H^2 + \dot{H}). \tag{5.10}$$

In the de Sitter solution $R_{\mathrm{dS}} = 12H_{\mathrm{dS}}^2$, $G_{\mathrm{dS}} = 24H_{\mathrm{dS}}^4$, with H_{dS} being constant, Eq. (2.26) is corrected as

$$\begin{aligned} 0 = {} & \frac{6H^2}{\kappa^2} - \frac{t_0'}{48(\kappa^2)^2\omega^2}(480H^4(\kappa^2)^2\omega^2(4\omega(2\omega + 3) + 1) \\ & + 4\kappa^2\lambda\omega(6H^2(-40\omega^2 + 26\omega + 3) - 2\kappa^2\Lambda(28\omega + 1)) \\ & - 3\lambda^2(20\omega^2 + 1)) - \Lambda + 12H^4\frac{d\gamma}{dt'}t_0', \end{aligned} \tag{5.11}$$

where the functions $\lambda, \omega, \kappa^2, \Lambda$, and $\gamma, d\gamma/dt'$ are constants in the time. By using (3.12)–(3.13) with $c_0 = \Lambda = 0$, and $1 \ll t'$, we obtain the solution

$$H_{\mathrm{dS}}^2\kappa_0^2 \simeq \frac{322.762}{(22085.2 - 34725.2(d\gamma/dt'))t_0'(\lambda(0)t')^{0.77}}, \qquad \frac{d\gamma}{dt'} < 0, \tag{5.12}$$

where $|d\gamma/dt'| \ll t'^2$ is used and we require that such a derivative is negative [$\gamma_0 c_1 < 0$ in (5.2)]. Thus, given the form of $\gamma_{t'}(t')$, the de Sitter solution depends on the current value of $\lambda(t'=0) = \lambda(0)$. Obviously, the $\Box R$-term does not give any contribution to the de Sitter solution. By using again the parametrization (3.12)–(3.13) with $c_0 = \Lambda = 0$, and, therefore, by multiplying (5.8) by κ_0^3, and by perturbating it with respect to the de Sitter solution (5.12) as in (4.1), we get

$$\frac{\kappa_0}{t'^3(H_{\rm dS}\kappa_0)^3}[\kappa_0\ddot{\delta H}(t'^2(H_{\rm dS}\kappa_0)^4(6.384t'^2 + t't'_0(-18\gamma_{t'}(t') + 18\zeta_{t'}(t') - 6\gamma_{t't'}(t')t'_0 + 11.448) - 0.228t_0'^2)$$
$$- 0.043t'^2t'_0(H_{\rm dS}\kappa_0)^2(\lambda(0)t')^{-0.77} + 0.001t_0'^2(\lambda(0)t')^{-1.54} + 0.087t't_0'^2(H_{\rm dS}\kappa_0)^2(\lambda(0)t')^{-0.77}$$
$$+2\times 10^{-4}t't'_0(\lambda(0)t')^{-1.54}) + (H_{\rm dS}\kappa_0)\dot{\delta H}(t'^2(H_{\rm dS}\kappa_0)^4(19.152t'^2 + t't'_0(-54\gamma_{t'}(t') + 72\zeta_{t'}(t')$$
$$-24\gamma_{t't'}(t')t'_0 + 34.344) - 0.913t_0'^2) - 0.086t'^2t'_0(H_{\rm dS}\kappa_0)^2(\lambda(0)t')^{-0.77} + 0.003t_0'^2(\lambda(0)t')^{-1.54}$$
$$+0.346t't_0'^2(H_{\rm dS}\kappa_0)^2(\lambda(0)t')^{-0.77} + 0.001t't'_0(\lambda(0)t')^{-1.54})] + (H_{\rm dS}\kappa_0)\delta H\left[\frac{0.223}{(\lambda(0)t')^{0.77}}\right.$$
$$\left.+t'_0\left(\frac{0.172\lambda(0)}{(\lambda(0)t')^{1.77}} + (H_{\rm dS}\kappa_0)^2(48\gamma_{t'}(t') - 30.528)\right)\right] = 0, \qquad H(t) = H_{\rm dS} + \delta H(t), \qquad |\delta H(t)| \ll 1, \qquad (5.13)$$

where we introduced the notation

$$\gamma_{t'}(t') \equiv \frac{d\gamma(t')}{dt'}, \qquad \gamma_{t't'}(t') \equiv \frac{d^2\gamma(t')}{dt'^2},$$
$$\zeta_{t'}(t') \equiv \frac{d\zeta(t')}{dt'}. \qquad (5.14)$$

If one assumes (4.3) and takes into account that $|\gamma_{t'}(t')|, |\zeta_{t'}(t')| \ll t'$, and $|\gamma_{t't'}(t')| \ll 1$, this expression is simplified as

$$\tilde{D}_0\delta H + t'[19.152(H_{\rm dS}\kappa_0)(\kappa_0\dot{\delta H}) + 6.384(\kappa_0^2\ddot{\delta H})] \simeq 0, \qquad (5.15)$$

where

$$\tilde{D}_0 = \left[\frac{0.223}{(\lambda(0)t')^{0.77}} - (30.528 - 48\gamma_{t'}(t'))t'_0(H_{\rm dS}\kappa_0)^2\right]. \qquad (5.16)$$

Thus, the solution of the above differential equation reads

$$\delta H = h_\pm \exp[\tilde{A}_\pm t],$$
$$\tilde{A}_\pm = \left[\frac{H_{\rm dS}}{2}\left(-3 \pm \sqrt{9 - \frac{0.627\tilde{D}_0}{(H_{\rm dS}\kappa_0)^2 t'}}\right)\right],$$
$$|h_\pm| \ll 1, \qquad (5.17)$$

where $h_\pm$ are the integration constants corresponding to the signs: plus and minus inside $\tilde{A}_\pm$. The solution is unstable if $\tilde{D}_0 < 0$, namely,

$$\frac{0.223074}{(\lambda(0)t')^{0.77}} < [30.528 - 48\gamma_{t'}(t')]t'_0(H_{\rm dS}\kappa_0)^2, \qquad (5.18)$$

and, by using (5.12), one sees that this inequality is always satisfied independently on the value of $\gamma_{t'}(t')$. As a consequence, (4.3) which we have used to derive (5.15) is verified, and it is interesting to note that $\tilde{D}_0$ evaluated with respect to the de Sitter solution (5.12) is equal to D_0 in (4.11) evaluated with respect to the de Sitter solution (3.16), from which we can understand that the Gauss-Bonnet term contribution to the stability of the de Sitter solution behaves like the one of the R^2 term (see (4.8)–(4.9) and the related comment). By using (5.12), one gets

$$\tilde{A}_+ \simeq 36019\times 10^{-9}\frac{H_{\rm dS}t'_0}{t'}(22085.2 - 34725.2\gamma_{t'}(t')),$$
$$\tilde{A}_- \simeq -3H_{\rm dS}, \qquad (5.19)$$

where $\tilde{D}_0$ is taken to be small. Thanks to the presence of the Gauss-Bonnet term in the action, the instability parameter $\tilde{A}_+$ can be increased with respect to the case considered before. Let us introduce our ansatz (5.2). We obtain

$$H_{\rm dS}^2\kappa_0^2 \simeq \frac{322.762}{[22085.2 - 34725.2\gamma_0 c_1]t'_0(\lambda(0)t')^{0.77}},$$
$$\gamma_0 c_1 < 0, \qquad (5.20)$$

$$\tilde{A}_+ \simeq 36019\times 10^{-9}\frac{H_{\rm dS}t'_0}{t'}(22085.2 - 34725.2\gamma_0 c_1),$$
$$\tilde{A}_- \simeq -3H_{\rm dS}. \qquad (5.21)$$

As a consequence, the instability parameter $\tilde{A}_+$ is larger than A_+ in the absence of Gauss-Bonnet correction if $\gamma_0 c_1$ is negative, namely, by taking $0 < c_1$ and $\gamma_0 < 0$, the Gauss-Bonnet contribution to the action is positive [see (5.1)]: the analysis of inflation is similar to the previous case, but the e-folds and, therefore, the spectral index n_s are smaller.

To be specific, the η slow-roll parameter (4.33) and the spectral index n_s in (4.36) read

$$\eta \simeq -\frac{18\times 10^{-6} t'_0 (22085.2 - 34725.2\gamma_0 c_1)}{t'},$$
$$n_s \simeq 1 - \frac{72038\times 10^{-9} t'_0 (22085.2 - 34725.2\gamma_0 c_1)}{t'}, \tag{5.22}$$

since we can still use (4.34). The spectral index n_s is consistent with Planck data (4.37) if

$$450 < \frac{t'_0}{t'}(22085.2 - 34725.2\gamma_0 c_1) < 653. \tag{5.23}$$

If we set $\lambda(0) = t_0 = 1$ and take (3.15) together with (3.17), we get from (5.23),

$$-4.61 < \gamma_0 c_1 < -2.98. \tag{5.24}$$

For example, for $c_1 = 1$ and $\gamma_0 = -3$ we find

$$n_s \simeq 0.96740, \tag{5.25}$$

which is in agreement with the Planck data (4.37). The de Sitter solution results to be $H^2_{\rm dS} \simeq 3.17\times 10^{-7} M^2_{\rm Pl}$, and inflation takes place near to the Planck scale, such that (3.15) is valid. In this kind of model, as we noted in Sec. IV B, the e-folds $N \sim 2/(1-n_s)$, and in the present case we have $N \sim 60$: this is an order of magnitude/lower bound of the e-folds which permits the thermalization of the observable Universe [the acceleration finishes when $\epsilon = 1$, and, therefore, the exact amount of inflation depends on the initial amplitude $|h_+|$ as in (4.32)]. Thanks to the Gauss-Bonnet contribution in the action, we can see that the value of the e-folds has considerably decreased (see for example [(4.21)], rendering correct the prediction of the spectral index. In the present example, a viable inflation is obtained for $1 \ll t'/t'_0$, which is always true due to the large curvature scale of inflation.

We have demonstrated that the contribution from the RG-improved Gauss-Bonnet term can modify the instability of de Sitter solution describing inflation given a viable spectral index. In our derivation, we have taken into account also the $\Box R$ contribution, but, due to the ansatz (5.2), it disappears. However, we furnished the formalism to treat the Lagrangian (5.1) with generic coefficients: if they grow up in the early-time Universe, they modify the dynamics of inflation and can lead to a model compatible with the Planck data.

As a final result of the work, we are able to present the very general quantum-corrected Lagrangian constructed with second-degree corrections to the Einstein gravity,

$$I = \int_{\mathcal{M}} d^4\sqrt{-g}\left[\frac{R}{\kappa^2(t')} - \frac{\omega(t')}{3\lambda(t')}R^2 + \frac{1}{\lambda(t')}C^2 - \gamma(t')G + \zeta(t')\Box R - \Lambda(t')\right], \qquad t' = \frac{t'_0}{2}\log\left[\frac{R}{R_0}\right]^2, \tag{5.26}$$

where t_0 is a number and $R_0 = 4\Lambda$ is the curvature of today's Universe, with Λ being the cosmological constant. The one-loop running coupling constants $\lambda(t'), \omega(t')$, $\kappa^2(t'), \Lambda(t'), \gamma(t')$ and $\zeta(t')$ are found from higher-derivative quantum gravity. They can be written as

$$\lambda(t') = \frac{\lambda(0)}{(1+\lambda(0)(133/10)t')},$$
$$\omega(t') = \omega_1,$$
$$\kappa^2(t') = \kappa_0^2(1+\lambda(0)(133/10)t')^{0.77},$$
$$\Lambda(t') = \frac{\Lambda_0}{(1+\lambda(0)(133/10)t')^{0.55}}, \tag{5.27}$$

with $\omega_1 = -0.02$, $\kappa_0^2 = 16\pi/M^2_{\rm Pl}$, $\Lambda_0 = 2\Lambda$. The expressions for $\omega(t'), \kappa^2(t')$ and $\Lambda(t')$ are derived by investigating the asymptotic behavior of the running constants at high curvature. However, the derivatives of the coupling constants obey a set of RG equations that we have taken into account in our analysis. The form of $\gamma(t')$ and $\zeta(t')$ is given by

$$\gamma(t') = \gamma_0(1 + c_1 t'),$$
$$\zeta(t') = \zeta_0(1 + c_2 t'),$$
$$c_1\gamma_0 < 0, \tag{5.28}$$

γ_0, ζ_0 and $c_{1,2}$ constants. Finally, $\lambda(0)$ is a number related to the bound of inflation. At small curvature ($t' \ll 1$), the action (5.26) reads

$$I = \int_{\mathcal{M}} d^4\sqrt{-g}\left[\frac{R}{\kappa_0} + \frac{0.02}{\lambda(0)}R^2 + \frac{1}{\lambda(0)}C^2 - 2\Lambda\right],$$
$$t' = \frac{t'_0}{2}\log\left[\frac{R}{R_0}\right]^2, \tag{5.29}$$

and the contributions of the Gauss-Bonnet and $\Box R$ terms disappear when the coefficients become constant.

Inflation is described at high curvature for $1 \ll t'$, near to the Planck mass. The model possesses a de Sitter solution which depends on $\lambda(0)$. This solution is always unstable and the model exits from inflation. It is possible to calculate the behavior of perturbations and show that the slow-roll conditions of inflation are satisfied with the ϵ slow-roll parameter much smaller than the η slow-roll parameter.

The amount of inflation (e-folds) is sufficiently large, the tensor-to-scalar ratio r is very close to zero, and due to the contribution of the RG-improved Gauss-Bonnet term in the action, the spectral index n_s satisfies the Planck data. The RG-improved $\Box R$ term does not play any important role in the dynamics of inflation.

After inflation, the reheating process with the particle production must take place to recover the FRW universe. These processes occur when the curvature (Ricci scalar) oscillates and eventually in the presence of the interaction between the gravity and matter quantum fields. At the end of inflation $t' \to 0$ and the model turns out to be a quadratic correction R^2 of Einstein's gravity (on FRW metric the square of Weyl tensor gives a zero contribution): this model has been well investigated in the literature, and it has been demonstrated that it is compatible with the reheating scenario.

VI. DISCUSSION

In this work we investigated the inflationary universe, taking into account quantum gravity effects in frames of the RG-improved effective action of higher-derivative quantum gravity. The effective coupling constants in higher-derivative quantum gravity obey a set of one-loop RG equations found in Refs. [12] and may show the asymptotically free behavior. These one-loop RG equations which define the effective coupling constants are used to derive quantum-corrected dynamical FRW equations. In order to find the explicit form of the running coupling constants, their (asymptotically free) behavior at the high-energy scale is used.

The model possesses a de Sitter solution at high curvature to describe the expanding inflationary universe. The bound of the de Sitter solution depends on the value of the running constant of the R^2 term today. We have demonstrated that the de Sitter solution is always unstable and takes place near the Planck scale. Thus, it is possible to evaluate the instability parameter of the model and the amplitude of perturbations. The slow-roll conditions are well satisfied, and the η slow-roll parameter is much larger than the ϵ slow-roll parameter: their behavior with respect to the e-folds N seems to be the same as the ones in scalar-tensor theories (see review [22]) for inflation ($\epsilon \sim 1/N^2$ and $|\eta| \sim 1/N$). The amount of inflation of the model is sufficiently large, and the tensor-to-scalar ratio r is very close to zero. However, in order to have the correct spectral index n_s compatible with the Planck data, it is necessary to take into account the contribution of the RG-improved Gauss-Bonnet term in the action. Note that the other RG-improved surface term ($\Box R$) does not play any important role during inflation. At low energy, the effective running constants become constant, and we recover the Friedmann universe.

It would be very interesting to compare the inflationary predictions (including the exit and reheating) of higher-derivative quantum gravity with those of Einstein quantum gravity in more detail. This will be considered elsewhere.

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