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Brown-York mass and the hoop conjecture in nonspherical massive systems

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We discuss the relation between the concentration of the Brown-York mass and the formation of trapped surfaces in nonspherical massive systems. In particular, we formulate and prove a precise version of the Thorne hoop conjecture in conformally flat three-geometries sliced by equipotential foliation leaves. An intriguing relationship between the total rest mass and the Brown-York mass is shown. This is a further investigation of the previous work on the Brown-York mass hoop conjecture in spherical symmetry.

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I. INTRODUCTION

The Brown-York quasilocal mass [1] is one of many mass concepts developed in last decades. Its potential has been revealed in [2], where the Brown-York mass has been used in order to prove the trapped surface conjecture [3] in spherically symmetric geometries. The trapped surface conjecture states that large mass enclosed in a small volume has to be trapped, and it constitutes an attempt to concretize a loose idea expressed by Thorne in his hoop conjecture [4].

In this paper we demonstrate that the Brown-York mass is useful in proving the hoop conjecture in certain classes of nonspherical geometries. They include systems having an equipotential surface foliation, that is convex in a certain sense. We present sufficient conditions for the existence of trapped surfaces. It appears convenient to split the consideration into two stages. In the first step one deals with 2-surfaces that satisfy an integral condition—that they are averaged trapped surfaces [5]. In the second step one finds additional conditions that ensure the pointwise trapping of averaged trapped surfaces.

The paper is organized as follows. Section II gives a concise historical account on the hoop conjecture. Section III contains the description of the formalism and needed definitions. We propose various sufficient and necessary conditions for the existence of averaged trapped surfaces in general settings. This is done in terms of the reference geometry. Section IV defines an equipotential foliation, assuming conformally flat geometry. We provide therein necessary and sufficient conditions for averaged trapped surfaces in terms of the original physical geometry. As an aside, but important result, we show that the Brown-York mass is not larger than the total rest mass. The question whether an averaged trapped surface is indeed pointwise trapped is examined in Sec. V. There is an extra mass term required to balance the nonsphericality, that can be written down explicitly. The last section summarizes obtained results.

II. HOOP CONJECTURE

There is a folk belief in general relativity that if matter is sufficiently concentrated into a finite volume, the gravitational system ultimately has to collapse to a black hole. Thorne proposed a hoop conjecture (HC) [4] which states:

> *Horizons form when and only when a mass M gets compacted into a region whose circumference in EVERY direction satisfies $C \lesssim 4\pi M$.*

His conjecture deals with global event horizons and the "circumference" and the mass are deliberately left unspecified. Notice, however, that in the Schwarzschild spacetime we have the equality: $C = 4\pi M$. In this case M is the asymptotic mass [6]. Seifert [3] formulated the more concrete trapped surface conjecture (TSC), according to which massive singularities have to be surrounded by a closed trapped 2-surface. This is an easier concept, because trapped surfaces are local in time. Proving the HC would require the study of the full history of a spacetime, while in order to prove the TSC one needs only to consider a single Cauchy slice.

There have been many attempts to prove the HC/TSC. In the early period the concentration of matter was assumed in spherically symmetric spacetimes [7–15]. Early results have been reviewed in [16]. Recently Khuri [17] applied in this context the generalized Jang equation [18]. Schoen and Yau dealt with nonsymmetric spacetimes [19]. Their sufficient condition for the formation of trapped surfaces required a special spacelike foliation of a spacetime with large extrinsic curvature. Within a single Cauchy slice, assuming the matter density to be large on a "large region," trapped surfaces have to form. That was a consequence of the blow-up analysis of the Jang equation [20] for an asymptotically flat initial data set [21]. In later studies the TSC has been proved in special classes of systems, with

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matter [16,22,23] or in vacuum [9,24]. Some of the recent development has been reviewed in [25].

In spherical symmetric systems one can measure their “size” by the circumference. It is reasonable to take $C = 2\pi R$ where R is the Schwarzschild or the areal radius of the surface in question, i.e. $R = \sqrt{\mathrm{Area}/4\pi}$. Then one can prove a precise statement of the hoop conjecture using the Brown-York mass as the mass measure [2]. The theorem says that if $C < 2\pi \mathfrak{m}_{\mathrm{BY}}$, then the surface is trapped. There exists a discrepancy between the 4π in Thorne’s HC and the coefficient 2π in [2]. This can be traced back to the fact that at the horizon of the Schwarzschild spacetime the Brown-York mass is equal to R, the area radius of the horizon. That is $\mathfrak{m}_{\mathrm{BY}} = 2M$, where M is the asymptotic mass.

III. AVERAGED TRAPPED SURFACES

Let (Ω^3, g, K) be a subset of a Cauchy slice for the Einstein field equations. Here g is the 3-metric of a Cauchy hypersurface and K stands for its extrinsic curvature. We assume that Ω is time-symmetric, i.e. it lies in a totally geodesic Cauchy hypersurface, $K \equiv 0$.

In this paper, we concentrate on the case of Ω being a compact domain with boundary and since it is time symmetric, we use the Brown-York mass as our measure of the mass within Ω.

Assume further that the boundary $\Sigma = \partial\Omega$ is a topological 2-sphere. There exists a unit normal n (directed outward) to Σ; its divergence $\nabla_i n^i$ is equal to the mean curvature k. Here ∇_i denotes the covariant derivative with respect to the 3-metric g. The sign of k has an important physical meaning. Take a bundle of outgoing null rays, normal to Σ. If $k > 0$ along Σ, then the bundle is divergent; while if $k < 0$, then the null rays must converge. If $k < 0$ everywhere along Σ, then the two-surface is said to be *trapped*. Trapped surfaces (TS) do not exist in the Euclidean geometry and their presence is associated with strongly curved geometries.

On the other hand, the first derivative of the area of Σ with respect to the uniform normal deformation gives the total mean curvature,

$$H(\Sigma) = \int_\Sigma k \mathrm{d}\Sigma. \tag{1}$$

The concept of a trapped surface is purely local, but it appears useful to deal with surfaces that are trapped in the average:

Definition A surface Σ is called an *averaged trapped surface* (ATS) if $H(\Sigma)$ is negative.

Assume further that Σ has positive Gauss curvature and thus can be isometrically embedded into the Euclidean space $\mathbb{R}^3$, i.e. $i\colon \Sigma \hookrightarrow i(\Sigma) \subset \mathbb{R}^3$. This isometric embedding is called the Weyl embedding and it is unique up to a rigid motion in $\mathbb{R}^3$ [26].

Then the Brown-York mass [1] is defined as

$$\mathfrak{m}_{\mathrm{BY}}(\Sigma, g) = \frac{1}{8\pi}\int_\Sigma (k_0 - k)\mathrm{d}\Sigma \tag{2}$$

where k is the mean curvature of Σ with respect to the physical metric g and k_0 is that of $i(\Sigma)$ with respect to the Euclidean metric. Note that k_0 is completely determined by the intrinsic 2-metric on the surface Σ but does not depend on the extrinsic geometry how Σ bends in Ω.

The above definition implies in a straightforward way the *important* proposition.

Proposition 1: The surface Σ is an ATS if and only if

$$\mathfrak{m}_{\mathrm{BY}}(\Sigma, g) > \frac{1}{8\pi}\int_\Sigma k_0 \mathrm{d}\Sigma. \tag{3}$$

It is interesting that here the integral and the mean curvature k_0 in Proposition 1 are in the Euclidean space. One can employ well-known geometric estimates and reexpress the proposition in a number of ways. This is done in the remainder of this section. The total mean curvature $\int_\Sigma k_0 \mathrm{d}\Sigma$ represents an “averaged size” of a solid convex body in the Euclidean space [27]. Suppose that a compact oriented convex surface Σ lies in $\mathbb{R}^3$. Let x_0 be a fixed point enclosed by Σ. The Minkowski integral formula [[28], Lemma 6.2.9, Page 136] gives

$$\int_\Sigma \frac{k_0}{2}\mathrm{d}\Sigma = \int_\Sigma K(x) < n(x), \qquad X(x) - x_0 >_{\mathbb{R}^3} \mathrm{d}\Sigma. \tag{4}$$

Here $K(x)$ is the Gauss curvature and $X(x)$ is the position vector of Σ in $\mathbb{R}^3$, $n(x)$ is the unit normal at $X(x)$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

Recall that Σ is a topological sphere. By the Gauss-Bonnet theorem $\int_\Sigma K(x)\mathrm{d}\Sigma = 2\pi\chi(S^2) = 4\pi$, it gives an upper bound of the right-hand side of (4), $4\pi \sup_{x\in\Sigma}|X(x) - x_0|$.

If we measure the “size” of a surface by looking at the position vector of its image when embedded isometrically into $\mathbb{R}^3$, then we have

Theorem 1: (Sufficient Condition for an ATS.) If

$$\mathfrak{m}_{\mathrm{BY}}(\Sigma, g) > \sup_{x\in\Sigma}|X(x) - x_0|, \tag{5}$$

then Σ is an ATS.

Another upper bound of the total mean curvature is given by the Blaschke cap body inequality, cf. Page 387 in [29]. Let V be a compact convex body in $\mathbb{R}^3$. Then

$$\mathrm{Area}(\partial V) \geq \sqrt{3\mathrm{Vol}(V)\int_\Sigma \frac{k_0}{2}\mathrm{d}(\partial V)}$$

where k_0 is the mean curvature of the boundary ∂V. This leads to the following theorem.

Theorem 2: (Sufficient Condition for an ATS.) If

$$\mathfrak{m}_{\rm BY}(\Sigma, g) > \frac{1}{4\pi}\frac{(\mathrm{Area}(\Sigma))^2}{3\mathrm{Vol}(\Omega_0)}, \qquad (6)$$

then Σ is an ATS. Here Ω_0 is the body in $\mathbb{R}^3$ enclosed by the image of Σ via the (unique) Weyl embedding.

At this stage we have a hybrid picture. The Brown-York mass lives in a physical space while the upper bounds are given in the reference space. Again things are easy in the spherically symmetric case [2], when $k_0 = 2/R$, where R is the areal radius. In general Riemannian geometries, life becomes harder. It is difficult to define a workable concept of a "circumference" [30,31]. Fortunately, one finds a quantitative link between the total mean curvature in the reference space $\int_\Sigma k_0 \mathrm{d}\Sigma$ and the original physical geometric data in a class of foliations of conformally flat 3-manifolds. The details will be discussed in the next section.

To provide necessary conditions for an ATS, we need the lower bound estimates of the total mean curvature $\int_\Sigma k_0 \mathrm{d}\Sigma$. There are two candidates both of which are in terms of intrinsic 2-geometry of the surface. One is given by the classical geometric inequality [29] and the other one is given by the Birkhoff invariant of the intrinsic 2-metric [32]. Let (Σ, h) be a topological sphere with a 2-metric h and let $F\colon \Sigma \to \mathbb{R}$ be a function with just two critical points, a maximum and a minimum. Then for any $c \in \mathbb{R}$, each level set $F^{-1}(c)$ has a length $l(c)$. The Birkhoff invariant $\beta(\Sigma, h)$ is defined as $\beta(\Sigma, h) \coloneqq \inf_F \max_c l(c)$.

Theorem 3: (Necessary Condition for an ATS.) Assume that Σ is an ATS, then

$$(1)\ \mathfrak{m}_{\rm BY}(\Sigma, g) > \sqrt{\frac{\mathrm{Area}(\Sigma)}{4\pi}}; \qquad (7)$$

$$(2)\ \mathfrak{m}_{\rm BY}(\Sigma, g) > \frac{1}{8\pi}\cdot 4\beta = \frac{\beta}{2\pi}. \qquad (8)$$

Here β is the Birkhoff invariant of the surface Σ.

IV. CONFORMALLY FLAT GEOMETRIES

Herein we shall investigate the following concrete class of three-spaces. Assume that

(i) g is conformally flat, $g_{ab} = f^4 \hat{g}_{ab}$ where $\hat{g}_{ab}$ is the standard Euclidean metric.

(ii) There is an equipotential foliation on Ω,

$$g = f^4(\sigma)[\hat{g}_{\sigma\sigma}\mathrm{d}\sigma^2 + \hat{g}_{ij}\mathrm{d}x^i \mathrm{d}x^j] \quad (i, j = 2, 3) \quad (9)$$

where $\sigma \geq 0$ and σ foliates the level surfaces of f which are assumed to be convex, and x^2 and x^3 are quasi-angle variables.

(iii) $\Sigma = \{\sigma = \sigma_0\}$.

Thus, $\hat{n} = (\hat{n}_\sigma, 0, 0)$ and $\hat{n}_\sigma = \sqrt{\hat{g}_{\sigma\sigma}}$. The conformal factor f satisfies the elliptic equation $\hat{\Delta} f = -2\pi\rho f^5$, which is the Hamiltonian constraint for momentarily static initial data of the Einstein equations. The energy density ρ is nonnegative due to energy conditions.

Remark.—In order to detect whether a surface Σ is trapped or not, one only needs the geometry in a neighborhood of Σ in Ω, or within Σ. The dominant view nowadays is that trapped surfaces are of physical interest because of their roles in the proof of scenarios of the cosmic censorship. That demands that Ω constitutes a domain of an asymptotically flat Cauchy slice and hence one should assume that the equipotential surface foliation, that covers Ω, is extendible onto the entire slice with the asymptotic condition $f(\infty) = 1$. That in turn implies that $f|_\Sigma \geq 1$ by the maximum principle.

We emphasize that Σ refers to the $\sigma = \sigma_0$ surface with induced metric $f^4(\sigma_0)(\hat{g}_{ij}\mathrm{d}x^i \mathrm{d}x^j)$. Denote by $\hat{\Sigma}$ the σ-constant surface with induced metric $\hat{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j$. Let k be the mean curvature of Σ with respect to the physical metric g and let $\hat{k}$ be the mean curvature of $\hat{\Sigma}$ with respect to the Euclidean metric $\hat{g}_{\sigma\sigma}\mathrm{d}\sigma^2 + \hat{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

We isometrically embed Σ into the reference space $f^4(\sigma_0)[\hat{g}_{\sigma\sigma}\mathrm{d}\sigma^2 + \hat{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j]$ which is also Euclidean. Then it gives a relation between k_0 and $\hat{k}$, i.e. $k_0 = \hat{k}/f^2(\sigma_0)$ and that of the induced area forms is $\mathrm{d}\Sigma = f^4(\sigma_0)\mathrm{d}\hat{\Sigma}$.

Below we shall write down some criteria for ATS's, obtained in Sec. III, in terms of the geometry of the physical space (Ω, g).

(i) Proposition 1 states that Σ must be an ATS if $\mathfrak{m}_{\rm BY}(\Sigma, g) > \int_\Sigma k_0\mathrm{d}\Sigma$; but $\int_\Sigma k_0\mathrm{d}\Sigma = f^2(\sigma_0)\int_{\hat{\Sigma}} \hat{k}\mathrm{d}\hat{\Sigma}$ represents an "averaged areal size" R_{Av} of a body enclosed by the (convex) 2-surface Σ. The sufficiency condition states simply $\mathfrak{m}_{\rm BY}(\Sigma, g) > R_{Av}(\Sigma)$.

(ii) In the same way one may also rewrite Theorem 1 as: If $\mathfrak{m}_{\rm BY}(\Sigma, g) > R_{\sup}(\Sigma)$, where $R_{\sup}(\Sigma) \coloneqq f^2(\sigma_0)\sup_{x\in\hat{\Sigma}}|\hat{X}(x) - x_0|$, then Σ is an ATS. Here $\hat{X}$ is the position vector of the surface $\hat{\Sigma}$ in the Euclidean space $\hat{g}_{\sigma\sigma}\mathrm{d}\sigma^2 + \hat{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

As a consequence of the uniqueness of the Weyl embedding, the lower bounds given in Theorems 2 and 3 are completely determined by the intrinsic 2-geometry on the surface. They are the same no matter calculated either in the physical space (Ω, g) or in the reference Euclidean space. In particular,

(iii) If we define the areal radius of Σ as $R_S \coloneqq \sqrt{\mathrm{Area}(\Sigma)/4\pi}$, then the first condition in Theorem 3 becomes $\mathfrak{m}_{\rm BY}(\Sigma, g) > R_S$.

(iv) In the second condition of Theorem 3, the Birkhoff invariant β is the minimum length of a closed string being slipped over the 2-surface [33]. One defines the Birkhoff radius $R_B := \beta/2\pi$ and then the condition becomes $\mathfrak{m}_{\rm BY}(\Sigma, g) > R_B$.

We have introduced the above four size measures for Σ. Notice that in spherical geometries and for a round sphere Σ centered at the symmetry center, all these measures coincide, $R_{\rm Av} = R_{\rm sup} = R_S = R_B$.

The physical and "embedded" mean curvatures along Σ are related as

$$k|_\Sigma = \frac{\hat{k}|_{\hat{\Sigma}(\sigma=\sigma_0)}}{f^2(\sigma_0)} + \frac{4}{f^3(\sigma_0)}\hat{n}^a\hat{\nabla}_a f|_{\hat{\Sigma}(\sigma=\sigma_0)}. \tag{10}$$

There is a simple calculation that allows us to give an upper bound onto the Brown-York mass $\mathfrak{m}_{\rm BY}(\Sigma, g)$ by the total rest mass $M(\Sigma) = \int_\Omega \rho {\rm dvol}_g$ within Σ. Indeed,

$$\begin{aligned}\mathfrak{m}_{\rm BY}(\Sigma, g) &= \frac{1}{8\pi}\int_\Sigma \frac{\hat{k}}{f^2(\sigma_0)} f^4(\sigma_0){\rm d}\hat{\Sigma} \\ &\quad - \frac{1}{8\pi}\int_\Sigma \left(\frac{\hat{k}}{f^2(\sigma_0)} + \frac{4}{f^3(\sigma_0)}\hat{n}^a\hat{\nabla}_a f\right) f^4(\sigma_0){\rm d}\hat{\Sigma} \\ &= -\frac{4f(\sigma_0)}{8\pi}\int_\Sigma \hat{n}^a\hat{\nabla}_a f{\rm d}\hat{\Sigma} = -\frac{f(\sigma_0)}{2\pi}\int_\Omega \hat{\Delta} f{\rm dvol}_{\hat{g}} \\ &= \int_\Omega f(\sigma_0)\rho f^5 f^{-6}{\rm dvol}_g = \int_\Omega \rho\frac{f(\sigma_0)}{f}{\rm dvol}_g,\end{aligned} \tag{11}$$

where $\hat{\Delta} f = -2\pi\rho f^5$. If we assume the dominant (or weak) energy condition, i.e. $\rho \geq 0$, then f is a superharmonic function (with respect to the Euclidean metric) and the maximum principle yields that for any $x \in \Omega$, $f(x) \geq f|_\Sigma = f(\sigma_0)$. Therefore,

Theorem 4: One has $\mathfrak{m}_{\rm BY}(\Sigma, g) \leq M(\Sigma)$ where $M(\Sigma) = \int_\Omega \rho{\rm dvol}_g$.

Remark.—The total rest mass has been employed in [7,8,23,34] in the derivation of sufficient conditions for ATS's and further TS's under certain additional conditions. The conditions therein are of the form $M(\Sigma) > D(\Sigma)$ where $D(\Sigma)$ is a certain "size measure" coming from an upper bound of the geometric size of the domain enclosed by Σ. As a corollary of Theorem 4, if $\mathfrak{m}_{\rm BY}(\Sigma, g) > D(\Sigma)$, then Σ must be an ATS or TS. One is expecting to find a refined size measure $D'(\Sigma)$ (which is smaller than $D(\Sigma)$) for the Brown-York mass. We shall do it in the next section.

If Σ is a marginally trapped massive shell where the derivative of f has a discontinuity, then Eq. (11), as a consequence of integration by parts, is no longer valid. But in spherical symmetry, one shows that the total rest mass equals twice of the asymptotic mass [[35], Eq. (1.14)]. This value also agrees with the Brown-York mass.

V. FROM AVERAGED TRAPPED SURFACE TO TRAPPED SURFACE

In spherically symmetric geometries, if we take a spherical two-surface Σ centered at the symmetry center, then its mean curvature becomes a constant. That means that it is trapped if and only if it is an ATS. This is the situation considered in [2]. In nonspherical geometries, the existence of an ATS is *not* sufficient to make use of the Penrose singularity theorem [36] predicting incomplete null geodesics. In this section, we formulate certain additional conditions which guarantee that an "ellipsoidal" ATS is indeed trapped. We would ask how much Brown-York mass compacted into the system can produce a pointwise TS. The reasoning is analogous to that used in [16,22,23,34].

Suppose that Σ is not a TS, there must be at least a point on which the mean curvature k is non-negative. Then the maximal value of $n_\sigma k$ must be non-negative and hence

$$\int_\Sigma n^\sigma (n_\sigma k)_{\rm max}{\rm d}\Sigma \geq 0. \tag{12}$$

There is no summation for σ here. Instead, n^σ denotes the particular σ-component of the unit normal in the equipotential foliation (9). Then one must have

$$\frac{1}{8\pi}\int_\Sigma n^\sigma[(n_\sigma k)_{\rm max} - n_\sigma k]d\Sigma + \frac{1}{8\pi}\int_\Sigma k_0{\rm d}\Sigma \geq \mathfrak{m}_{\rm BY}(\Sigma, g). \tag{13}$$

Equivalently, we have

Proposition 2: If

$$\mathfrak{m}_{\rm BY}(\Sigma, g) > \frac{1}{8\pi}\int_\Sigma n^\sigma[(n_\sigma k)_{\rm max} - n_\sigma k]{\rm d}\Sigma + \frac{1}{8\pi}\int_\Sigma k_0{\rm d}\Sigma, \tag{14}$$

then Σ must be a pointwise TS.

Now we apply Eq. (12) in [23]:

$$\partial_\sigma\left(\int_{\hat{\Sigma}} \hat{k}{\rm d}\hat{\Sigma}\right) = 2\int_{\hat{\Sigma}} \hat{K}\hat{n}_\sigma{\rm d}\hat{\Sigma} := 8\pi C(\sigma). \tag{15}$$

Integrating from 0 to σ_0, we have

$$\begin{aligned}\frac{1}{8\pi}\int_\Sigma k_0 d\Sigma &= \frac{1}{8\pi}\int_{\hat{\Sigma}(\sigma=\sigma_0)} \frac{\hat{k}}{f^2(\sigma_0)} f^4(\sigma_0){\rm d}\hat{\Sigma} \\ &= \frac{f^2(\sigma_0)}{8\pi}\int_{\hat{\Sigma}(\sigma=0)} \hat{k}{\rm d}\hat{\Sigma} + f^2(\sigma_0)\int_0^{\sigma_0} C(s){\rm d}s.\end{aligned} \tag{16}$$

Note that the $\{\sigma = 0\}$ "surface" is the set of points for which the conformal factor f achieves its maximal value.

One may find an upper bound of the first term, Eq. (18) in [37],

$$\frac{f^2(\sigma_0)}{8\pi}\int_{\hat{\Sigma}(\sigma=0)}\hat{k}\mathrm{d}\hat{\Sigma} \le \frac{f^2(\sigma_0)\pi}{4}\sup l(S(0)). \qquad (17)$$

Here $\sup l(S(0))$ is the largest flat radius of the disk on which the conformal factor f achieves its maximal value.

Finally, we have arrived at

Theorem 5: (Sufficient condition for a pointwise TS.) If

$$\begin{aligned}\mathfrak{m}_{\mathrm{BY}}(\Sigma, g) > &\frac{1}{8\pi}\int_\Sigma n^\sigma[(n_\sigma k)_{\max} - n_\sigma k]\mathrm{d}\Sigma \\ &+\frac{f^2(\sigma_0)\pi}{4}\sup l(S(0)) \\ &+ f^2(\sigma_0)\int_0^{\sigma_0} C(s)\mathrm{d}s, \end{aligned} \qquad (18)$$

then Σ is a pointwise TS.

The physical significance of this theorem is as follows.

(i) The line integral term $f^2(\sigma_0)\int_0^{\sigma_0} C(s)\mathrm{d}s$ represents an appropriate "size" of the hoop for mass concentration.

(ii) The surface integral term $\int_\Sigma n^\sigma[(n_\sigma k)_{\max} - n_\sigma k]\mathrm{d}\Sigma/8\pi$ reflects the "boundary effect" when the surface is not spherical.

(iii) The radius term $f^2(\sigma_0)\pi \sup l(S(0))/4$ is influenced by the behavior of the conformal factor f within the entire foliation and thus can be interpreted as the "global deviation" the system from being spherical.

The sum

$$\frac{1}{8\pi}\int_\Sigma n^\sigma[(n_\sigma k)_{\max} - n_\sigma k]\mathrm{d}\Sigma + \frac{f^2(\sigma_0)\pi}{4}\sup l(S(0)) \qquad (19)$$

is the energy required to balance the nonsphericality when producing a trapped surface. By maximum principle, for $0 \le \sigma \le \sigma_0$, $f(\sigma) \ge f(\sigma_0)$. Then the radius term $f^2(\sigma_0)\pi \sup l(S(0))/4$ and the hoop term $f^2(\sigma_0)\int_0^{\sigma_0} C(s)\mathrm{d}s$ are both less than the terms in the sufficient condition for TS in terms of $M(\Sigma)$ [23], $f^2(0)\pi \sup l(S(0))/4 \coloneqq \pi \mathrm{rad}(0)/4$ and $\int_0^{\sigma_0} f^2(s)C(s)\mathrm{d}s$, respectively. However, the "boundary effect" energy compensating terms $\int_\Sigma n^\sigma[(n_\sigma k)_{\max} - n_\sigma k]\mathrm{d}\Sigma/8\pi$ are the same. We obtain a sufficient condition for TS's, employing the Brown-York mass, that is finer than that implied by Theorem 4 (cf. Remark beneath Theorem 4).

In spherical symmetry, both of the two terms in Eq. (19) vanish, and $C(s) \equiv 1$ and hence $f^2(\sigma_0)\int_0^{\sigma_0} C(s)\mathrm{d}s$ equals the areal radius of the surface. The inequality (18) is sharp and it reduces to the result in [2].

VI. CONCLUSIONS

We have shown, using conformally flat geometries and a suitable foliation, that the Brown-York mass is bounded from above by the total rest mass. The more nonspherical is a surface, the more Brown-York energy must be compacted within to make it trapped. We employ a number of geometric inequalities in Euclidean space, that yield several necessary and sufficient conditions for ATS's and pointwise trapped surfaces. These results hold true for a large class of nonspherical geometries whose metrics are conformal (with convex layer surfaces) to the flat metric, and for adapted foliations.

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