Model independent method for a quantitative estimation of $SU(3)$ flavor symmetry breaking using Dalitz plot distributions

Dibyakrupa Sahoo,[1] Rahul Sinha,[1] and N. G. Deshpande[2]

[1]*The Institute of Mathematical Sciences, Taramani, Chennai 600113, India*
[2]*Institute of Theoretical Science, University of Oregon, Eugene, Oregon 94703, USA*

(Received 9 March 2015; published 29 April 2015)

The light hadron states are satisfactorily described in the quark model using $SU(3)$ flavor symmetry. If the $SU(3)$ flavor symmetry relating the light hadrons were exact, one would have an exchange symmetry between these hadrons arising out of the exchange of the up, down and strange quarks. This aspect of $SU(3)$ symmetry is used extensively to relate many decay modes of heavy quarks. However, the nature of the effects of $SU(3)$ breaking in such decays is not well understood, and hence a reliable estimate of $SU(3)$ breaking effects is missing. In this work we propose a new method to quantitatively estimate the extent of flavor symmetry breaking and better understand the nature of such breaking using the Dalitz plot. We study the three noncommuting $SU(2)$ symmetries [subsumed in $SU(3)$ flavor symmetry], isospin (or T-spin), U-spin and V-spin, using the Dalitz plots of some three-body meson decays. We look at the Dalitz plot distributions of decays in which pairs of the final three particles are related by two distinct $SU(2)$ symmetries. We show that such decay modes have characteristic distributions that enable the measurement of violation of each of the three $SU(2)$ symmetries via Dalitz plot asymmetries in a single decay mode. Experimental estimates of these easily measurable asymmetries would help in better understanding the weak decays of heavy mesons into both two and three light mesons.

DOI: 10.1103/PhysRevD.91.076013 PACS numbers: 11.30.Hv, 13.25.Ft, 13.25.Hw

I. INTRODUCTION

A satisfactory understanding of the light hadronic states using $SU(3)$ flavor symmetry is one of the outstanding success stories of particle physics [1–5]. In its true essence, the $SU(3)$ flavor symmetry denotes the full exchange symmetry among the up (u), down (d) and strange (s) quarks. Another implication of $SU(3)$ flavor symmetry, if it were an exact symmetry, is that the mesons formed by combining the quarks u, d, s and the antiquarks $\bar{u}$, $\bar{d}$, $\bar{s}$ belonging to the same representation of $SU(3)$ would also be degenerate. One treats the three quarks on the same footing even though the quark masses differ by allowing for a breaking of the symmetry. The success of the Gell-Mann–Okubo mass formula in relating the hadron masses is that it takes the small $SU(3)$ breaking into account but does not depend on the details of $SU(3)$ breaking effects. Such $SU(3)$ breaking effects cannot be calculated and must be estimated using experimental inputs. Traditionally, the mass differences between these mesons have been used as a measure of the extent of the breaking of $SU(3)$ flavor symmetry. The masses of these mesons, which are bound states of quark-antiquark pairs, depend on their binding energies. It is not possible to estimate these binding energies from QCD calculations since these resonances lie in the nonrelativistic low-energy regime. Moreover, the electromagnetic interactions between the quark and the antiquark in the meson also contribute toward its binding energy. Thus, by measuring the mass differences among the mesons, one does not fully solicit the breaking of $SU(3)$ flavor symmetry. Another usual way to explore the breaking $SU(3)$ flavor symmetry is to look at specific loop diagrams where the down and strange quarks contribute. The loop effects affect the amplitude of the process under consideration, and its physical manifestations are then studied for a quantitative estimation of the breaking of $SU(3)$ flavor symmetry. Since the up quark has a different electric charge than down and strange, it cannot be treated in the same way in these studies of loop contributions. Therefore, such a method also fails to probe the full exchange symmetry of these three light quarks. Hence, all estimates of $SU(3)$ breaking are currently empirical.

Several studies exist in the literature that have used broken $SU(3)$ flavor symmetry (i) in various decay modes using the methods of amplitudes (usually isospin and U-spin amplitudes) and various quark diagrams [6–52] and (ii) in determinations of weak phases and CP violating phases [53–64]. These methods involve comparison of observables in distinct decay modes which are related by some underlying $SU(2)$ symmetries, such as isospin, U-spin or V-spin. However, the full exchange symmetry among the three light quarks has not yet been fully exploited, in a single decay mode. Hadronic weak decays involve several unknown parameters which can be reduced by the use of $SU(3)$ flavor symmetry. Since $SU(3)$ flavor symmetry is still extensively used to relate the few decay modes of heavy quarks, it is important to realize other ways to experimentally measure the breaking of $SU(3)$ flavor

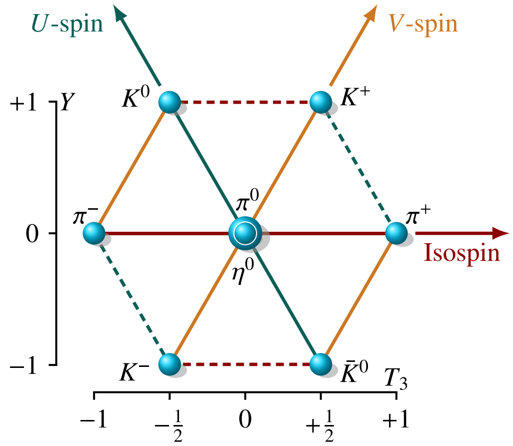

FIG. 1 (color online). The $SU(3)$ meson octet of light pseudoscalar mesons. Here the horizontal axis shows the eigenvalues of isospin (T_3), and the vertical axis shows the eigenvalues of hypercharge ($Y = B + S$, with B being baryon number and S being the strangeness number). The dotted lines parallel to the U-spin (or isospin) axis signify that in no two-body decays of B or D meson can the two connected mesons appear together in the final state as that would violate conservation of electric charge (or strangeness by two units).

symmetry and understand better the complete nature of $SU(3)$ breaking. In this paper we propose a method to achieve precisely this by looking at asymmetries in the Dalitz plot under exchange of the mesons in the final state. These asymmetries can be measured in different regions of the Dalitz plot. In particular these asymmetries can be measured both along resonances and in the nonresonant regions. A quantitative estimate of the variation of these asymmetries obtained experimentally would provide valuable understanding of $SU(3)$ breaking effects. It would also be interesting to find regions of the Dalitz plots where $SU(3)$ is a good symmetry. The $SU(3)$ flavor symmetry subsumes three important and noncommuting $SU(2)$ symmetries: isospin (or T-spin), U-spin and V-spin. All the members of a $SU(3)$ multiplet are related to one another by combined operations of the raising and lowering operators of these individual $SU(2)$ symmetries. In this paper we restrict ourselves to the breaking of these $SU(2)$ symmetries in combinations.

We shall work with the three-body decays of the type $P \to M_1 M_2 M_3$, where P can be either a B or a D meson and the final particles M_1, M_2 and M_3 are distinct members of the lightest pseudoscalar $SU(3)$ multiplet (see Fig. 1). Our approach toward experimental estimation of breaking of $SU(3)$ flavor symmetry primarily looks for violations of two constituent $SU(2)$ symmetries. Therefore, our final state would have a pair of particles in one $SU(2)$ multiplet and another pair belonging to a different $SU(2)$ multiplet. If the $SU(2)$ symmetry is assumed to be exact, the pairs of final state particles that are members of the $SU(2)$ multiplet are identical bosons in the symmetry limit and must be totally symmetric under exchange. This implies that if the wave function is symmetric under $SU(2)$ exchange it must be even under space exchange, whereas if it is antisymmetric in $SU(2)$, it must be odd under space exchange, too. We shall explicitly explore this exchange symmetry to deduce some simple relations that predict a pattern in the distribution of events in the concerned Dalitz plot. Any deviation from this predicted Dalitz plot distribution would, therefore, constitute a test of the breaking of $SU(3)$ flavor symmetry. Dalitz plots have previously been used in Refs. [65–67] to extract weak decay amplitudes and to study CP, CPT and Bose symmetry violations. Here we use the Dalitz plot to look for breaking of $SU(3)$ flavor symmetry in a single decay mode.

We start Sec. II by explaining briefly in Sec. II A the kind of Dalitz plot we shall use to elucidate our method and also set up the notation to be followed thenceforth. We shall then illustrate the method in full detail in Sec. II B by considering the decay mode $B^+ \to K^0\pi^0\pi^+$ which tests both isospin and U-spin simultaneously. We show in detail how the exchange $\pi^0 \leftrightarrow \pi^+$ under isospin and $K^0 \leftrightarrow \pi^0$ under U-spin results in a characteristic distribution of events in the Dalitz plot if both isospin and U-spin are exact symmetries. The method can equally well be applied to the decay mode $D_s^+ \to K^0\pi^0\pi^+$. We then show how G-parity generalized to the V-spin further influences the distribution of events in the Dalitz plot. The definition of G-parity and its generalization to the U-spin and V-spin are discussed in Appendix A for ready reference. We provide Dalitz plot asymmetries which can then be easily used to make a quantitative estimate of the breaking of $SU(3)$ flavor symmetry. Then, we sketch out the necessary steps for handling cases of both isospin and V-spin violation (in Sec. II C) as well as both U-spin and V-spin violation (in Sec. II D) by considering the decay modes B_d^0 or $\bar{B}_s^0 \to K^+\pi^0\pi^-$ and B^+ or $D^+ \to K^+\pi^0\bar{K}^0$, respectively. Finally in Sec. II E, we sketch out how our method can be applied to a decay mode $D^+ \to \pi^+\pi^0\bar{K}^0$ where each pair of particles in the final state can be directly related by one of the three $SU(2)$ symmetries, namely isospin, U-spin and V-spin. We point out how the Dalitz plot distribution for this mode differs from the ones considered in the earlier subsections. Finally, we conclude in Sec. III, emphasizing the salient features of our method.

II. METHOD

A. General considerations

The method described in this paper relies on the simultaneous application of two of the $SU(2)$ symmetries subsumed in $SU(3)$, i.e. isospin (or T-spin), U-spin or V-spin, to a three body decay $P \to M_1 M_2 M_3$, where M_1, M_2 and M_3 are chosen such that M_1 and M_2 belong to the triplet of one of the $SU(2)$ subgroups and M_2 and M_3 belong to another. To be definite M_2 is always chosen to be

TABLE I. We look at decays with the final states $M_1M_2M_3$ given as in the table here. The particle M_2, which is always π^0, being at the center of the pseudoscalar meson octet belongs to all the three $SU(2)$ symmetries under consideration. The states are denoted with subscripts for clarity; e.g. the state $|U=1, U_3=+1\rangle$ is denoted as $|1,+1\rangle_U$. Modes with conjugate final states can as well be studied in a similar manner. The primed states such as $|1',\pm1\rangle$ arise from the $|0,0\rangle$ component of π^0 under U-spin and V-spin considerations as discussed in the text. The last mode in the table with final state $\pi^+\pi^0\bar{K}^0$ has another exchange symmetry, namely exchange of π^+ and $\bar{K}^0$ under V-spin. Thus, $|\pi^+\bar{K}^0\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle_V + |0,0\rangle_V)$ under V-spin.

Final state			Kind of $SU(2)$ exchange		Expression for the state	
M_1	M_2	M_3	$M_1 \leftrightarrow M_2$	$M_2 \leftrightarrow M_3$	$\|M_1M_2\rangle$	$\|M_2M_3\rangle$
K^0	π^0	π^+	U-spin	Isospin	$\frac{1}{2\sqrt{2}}(\|2,+1\rangle_U + \|1,+1\rangle_U) - \frac{\sqrt{3}}{2}\|1',+1\rangle_U$	$-\frac{1}{\sqrt{2}}(\|2,+1\rangle_I - \|1,+1\rangle_I)$
K^+	π^0	π^-	V-spin	Isospin	$-\frac{1}{2\sqrt{2}}(\|2,+1\rangle_V + \|1,+1\rangle_V) + \frac{\sqrt{3}}{2}\|1',+1\rangle_V$	$\frac{1}{\sqrt{2}}(\|2,-1\rangle_I + \|1,-1\rangle_I)$
K^+	π^0	$\bar{K}^0$	V-spin	U-spin	$-\frac{1}{2\sqrt{2}}(\|2,+1\rangle_V + \|1,+1\rangle_V) + \frac{\sqrt{3}}{2}\|1',+1\rangle_V$	$\frac{1}{2\sqrt{2}}(\|2,-1\rangle_U + \|1,-1\rangle_U) - \frac{\sqrt{3}}{2}\|1',-1\rangle_U$
π^+	π^0	$\bar{K}^0$	Isospin	U-spin	$-\frac{1}{\sqrt{2}}(\|2,+1\rangle_I + \|1,+1\rangle_I)$	$-\frac{1}{2\sqrt{2}}(\|2,-1\rangle_U + \|1,-1\rangle_U) + \frac{\sqrt{3}}{2}\|1',-1\rangle_U$

the π^0, and the modes we consider are listed in Table I. Under the limit of exact $SU(2)$, all the mesons belonging to the triplet are identical bosons and must exhibit an overall Bose symmetry under exchange. This behavior must also be reflected in the Dalitz plot for the decay. We can construct a Dalitz plot out of the Mandelstam-like variables s, t and u. Let us denote the 4-momenta of particles P and M_i (where $i \in \{1,2,3\}$) by p and p_i and their masses by m and m_i, respectively. The variables s, t, u are defined in terms of the 4-momenta as follows:

$$\begin{aligned} s &= (p-p_1)^2 = (p_2+p_3)^2, \\ t &= (p-p_2)^2 = (p_1+p_3)^2, \\ u &= (p-p_3)^2 = (p_1+p_2)^2. \end{aligned} \tag{1}$$

It is easy to observe that $(m_2+m_3)^2 \le s \le (m-m_1)^2$, $(m_1+m_3)^2 \le t \le (m-m_2)^2$, $(m_1+m_2)^2 \le u \le (m-m_3)^2$ and $s+t+u = m^2+m_1^2+m_2^2+m_3^2 = M^2$ (say).

To give equal weight to s, t and u, we shall work with a ternary plot of which s, t, u form the three axes. This leads to an equilateral triangle as shown in Fig. 2. When the final particles are ultrarelativistic, the full interior of the equilateral triangle tends to get occupied. In any case the Dalitz plot under our consideration would always lie inside the equilateral triangle. The physically allowed region is schematically shown in Fig. 2 by the yellow colored region inside the equilateral triangle. The boundary of the Dalitz plot for a three-body decay process under consideration would not look symmetric under the exchanges $s \leftrightarrow t \leftrightarrow u$ due to the breaking of flavor $SU(3)$ symmetry on account of masses m_1, m_2 and m_3 being different. Any event inside the Dalitz plot, as illustrated in Fig. 2, can be specified by its radial distance (r) from the center of the equilateral triangle and the angle subtended by its position vector with any of the three axes s, t or u. The angle subtended by the position vector with the s axis is denoted by θ, the one with the u axis is denoted by θ', and the one with the t-axis is denoted by θ''. An event described by some values of s, t and u corresponds to some values of r and θ as calculable from the relations given below:

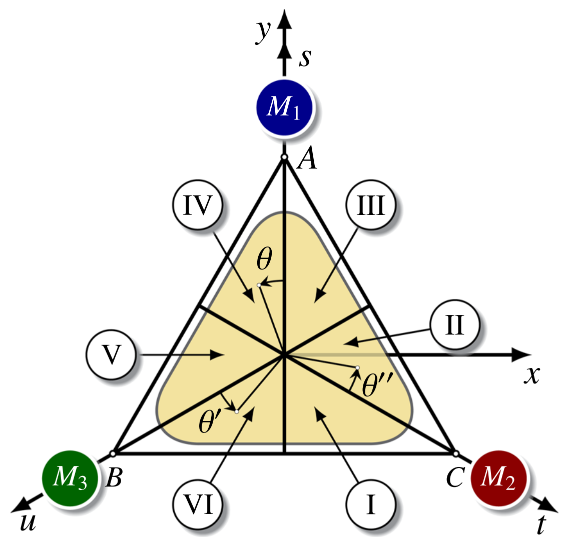

FIG. 2 (color online). A hypothetical Dalitz plot for the decay $P \to M_1M_2M_3$, where the variables s, t, u are defined in Eq. (1). The three sides of the equilateral triangle are given by $s=0$, $t=0$ and $u=0$. The three vertices A, B and C correspond to $s=M^2$, $u=M^2$ and $t=M^2$, respectively. The three medians divide the interior of the equilateral triangle into six regions of equal area. These six sextants are denoted by I, II, III, IV, V and VI. The three vertices A, B and C of the equilateral triangle have rectangular coordinates $A=(0,2)$, $B=(-\sqrt{3},-1)$ and $C=(\sqrt{3},-1)$. This rectangular coordinate system has its origin at the center of the equilateral triangle, and the y axis is along the s axis as shown here. The angles θ, θ' and θ'' are defined in the text. The blobs with M_1, M_2 and M_3 serve as mnemonic to suggest that the exchanges $s \leftrightarrow t \leftrightarrow u$ are equivalent to the particle exchanges $M_1 \leftrightarrow M_2 \leftrightarrow M_3$, respectively. The physically allowed region is always inside the equilateral triangle as shown, schematically, by the yellow colored region.

$$s = \frac{M^2}{3}(1 + r\cos\theta), \tag{2}$$

$$t = \frac{M^2}{3}\left(1 + r\cos\left(\frac{2\pi}{3} + \theta\right)\right), \tag{3}$$

$$u = \frac{M^2}{3}\left(1 + r\cos\left(\frac{2\pi}{3} - \theta\right)\right). \tag{4}$$

One can easily change the basis from (r, θ) to either (r, θ') or (r, θ'') by noting the fact that $\theta = \theta' + \frac{2\pi}{3}$ and $\theta = \theta'' + \frac{4\pi}{3}$ (see Fig. 2).

Before we analyze the specific decay modes, we would like to point out a few simple facts about the neutral pion, which plays a pivotal role in all our decays. The neutral pion is a pure isotriplet state $|1,0\rangle_I \equiv \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$:

$$|\pi^0\rangle = -|1,0\rangle_I. \tag{5}$$

But in case of U-spin, it is a linear combination of the U-spin triplet state $|1,0\rangle_U \equiv \frac{1}{\sqrt{2}}(s\bar{s} - d\bar{d})$ and the U-spin singlet but $SU(3)$ octet state $|0,0\rangle_{U,8} \equiv \frac{1}{\sqrt{6}}(d\bar{d} + s\bar{s} - 2u\bar{u})$:

$$|\pi^0\rangle = \frac{1}{2}|1,0\rangle_U - \frac{\sqrt{3}}{2}|0,0\rangle_{U,8}. \tag{6}$$

Similarly in case of V-spin, π^0 is given by a linear combination of the V-spin triplet state $|1,0\rangle_V \equiv \frac{1}{\sqrt{2}}(s\bar{s} - u\bar{u})$ and the V-spin singlet but $SU(3)$ octet state $|0,0\rangle_{V,8} \equiv \frac{1}{\sqrt{6}}(u\bar{u} + s\bar{s} - 2d\bar{d})$:

$$|\pi^0\rangle = -\frac{1}{2}|1,0\rangle_V + \frac{\sqrt{3}}{2}|0,0\rangle_{V,8}. \tag{7}$$

We have put subscripts I, U and V in the states to indicate that they are written in isospin, U-spin and V-spin bases, respectively.

B. Decay mode with final state $K^0\pi^0\pi^+$

We begin by considering as an example the decay mode $B^+ \to K^0\pi^0\pi^+$. We will see that the application of both the isospin and U-spin results in unique tests of the validity of both these constituent symmetries of $SU(3)$. The π^0 and π^+ in the final state are identical under isospin, and the final state must be totally symmetric under exchange. Under U-spin (see Fig. 1), the K^0 and π^0 can be considered as identical bosons and must similarly be totally symmetric under exchange. This ensures the following exchanges in the Dalitz plot:

$$U\text{-spin exchange} \equiv K^0 \leftrightarrow \pi^0 \Rightarrow s \leftrightarrow t,$$
$$\text{isospin exchange} \equiv \pi^0 \leftrightarrow \pi^+ \Rightarrow t \leftrightarrow u.$$

Under exact U-spin and isospin, the final state $K^0\pi^0\pi^+$ has, therefore, the following two possibilities:

(1) $K^0\pi^0$ would exist in either a symmetrical or antisymmetrical state with respect to their exchange in space.
(2) $\pi^0\pi^+$ would exist in either a symmetrical or antisymmetrical state with respect to their exchange in space.

The amplitude for this decay would then be described by four independent functions defined by their symmetry and antisymmetry properties as enunciated below:

(1) $\mathcal{A}_{SS}(s,t,u)$ which is symmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$, or
(2) $\mathcal{A}_{AA}(s,t,u)$ which is antisymmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$, or
(3) $\mathcal{A}_{SA}(s,t,u)$ which is symmetric under $s \leftrightarrow t$ and anti-symmetric under $t \leftrightarrow u$, or
(4) $\mathcal{A}_{AS}(s,t,u)$ which is antisymmetric under $s \leftrightarrow t$ and symmetric under $t \leftrightarrow u$.

We now analyze each of the possible amplitude functions in the most general manner. We start by $\mathcal{A}_{SS}(s,t,u)$, which is a function symmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$, to show that $\mathcal{A}_{SS}(s,t,u)$ must also be symmetric under $s \leftrightarrow u$:

$$\begin{aligned}\mathcal{A}_{SS}(s,t,u) &\overset{s\leftrightarrow t}{=} \mathcal{A}_{SS}(t,s,u) \overset{t\leftrightarrow u}{=} \mathcal{A}_{SS}(u,s,t)\\ &\overset{s\leftrightarrow t}{=} \mathcal{A}_{SS}(u,t,s).\end{aligned}$$

Since we have shown that $\mathcal{A}_{SS}(s,t,u) = \mathcal{A}_{SS}(u,t,s)$, we have demonstrated that $\mathcal{A}_{SS}(s,t,u)$ is also symmetric under $s \leftrightarrow u$. Hence, we conclude that $\mathcal{A}_{SS}(s,t,u)$ is a fully symmetric amplitude function. Let us next consider $\mathcal{A}_{AA}(s,t,u)$ which is a function antisymmetric under both $s \leftrightarrow t$ and $t \leftrightarrow u$, to show that it is also antisymmetric under $s \leftrightarrow u$:

$$\begin{aligned}\mathcal{A}_{AA}(s,t,u) &\overset{s\leftrightarrow t}{=} -\mathcal{A}_{AA}(t,s,u) \overset{t\leftrightarrow u}{=} +\mathcal{A}_{AA}(u,s,t)\\ &\overset{s\leftrightarrow t}{=} -\mathcal{A}_{AA}(u,t,s).\end{aligned}$$

Since $\mathcal{A}_{AA}(s,t,u) = -\mathcal{A}_{AA}(u,t,s)$ we require that $\mathcal{A}_{AA}(s,t,u)$ must also be antisymmetric under $s \leftrightarrow u$. Hence, we conclude that $\mathcal{A}_{AA}(s,t,u)$ is a fully antisymmetric amplitude function. Following the same arguments as above, it is easy to conclude that both $\mathcal{A}_{SA}(s,t,u)$ and $\mathcal{A}_{AS}(s,t,u)$ must be identically zero. The details are as follows. The function $\mathcal{A}_{SA}(s,t,u)$ which is symmetric under $s \leftrightarrow t$ and anti-symmetric under $t \leftrightarrow u$ must satisfy

$$\begin{aligned}\mathcal{A}_{SA}(s,t,u) &\overset{s\leftrightarrow t}{=} \mathcal{A}_{SA}(t,s,u) \overset{t\leftrightarrow u}{=} -\mathcal{A}_{SA}(u,s,t)\\ &\overset{s\leftrightarrow t}{=} -\mathcal{A}_{SA}(u,t,s) \overset{t\leftrightarrow u}{=} +\mathcal{A}_{SA}(t,u,s)\\ &\overset{s\leftrightarrow t}{=} +\mathcal{A}_{SA}(s,u,t) \overset{t\leftrightarrow u}{=} -\mathcal{A}_{SA}(s,t,u) = 0.\end{aligned}$$

Similarly, $\mathcal{A}_{AS}(s,t,u)$ being a function antisymmetric under $s \leftrightarrow t$ and symmetric under $t \leftrightarrow u$ satisfies

$$\begin{aligned}\mathcal{A}_{AS}(s,t,u) &\overset{s\leftrightarrow t}{=} -\mathcal{A}_{AS}(t,s,u) \overset{t\leftrightarrow u}{=} -\mathcal{A}_{AS}(u,s,t)\\ &\overset{s\leftrightarrow t}{=} +\mathcal{A}_{AS}(u,t,s) \overset{t\leftrightarrow u}{=} +\mathcal{A}_{AS}(t,u,s)\\ &\overset{s\leftrightarrow t}{=} -\mathcal{A}_{AS}(s,u,t) \overset{t\leftrightarrow u}{=} -\mathcal{A}_{AS}(s,t,u) = 0.\end{aligned}$$

We have shown that both $\mathcal{A}_{SA}(s,t,u) = 0$ and $\mathcal{A}_{SA}(s,t,u) = 0$, which implies that these amplitudes never contribute to the distribution of events on the Dalitz plot. Since the function describing the distribution of events in the Dalitz plot is proportional to the amplitude mod-square, it also has only two parts, one which is fully symmetric under $s \leftrightarrow t \leftrightarrow u$ and another which is fully antisymmetric under $s \leftrightarrow t \leftrightarrow u$.

We now examine the decay mode $B^+ \to K^0\pi^0\pi^+$ in detail, by writing down the decay amplitude in terms of isospin and U-spin amplitudes, eventually obtaining the same conclusion as above about the distribution of events in the Dalitz plot under consideration. The $\pi^0\pi^+$ combination can exist in isospin states $|2,+1\rangle_I$ and $|1,+1\rangle_I$ (see Table I). If isospin were an exact symmetry, the state $|\pi^0\pi^+\rangle$ would remain unchanged under $\pi^0 \leftrightarrow \pi^+$ exchange. This puts the $|2,+1\rangle_I$ state in a space symmetric (even partial wave) state and the $|1,+1\rangle_I$ state in a space antisymmetric (odd partial wave) state. The isospin decomposition of the final state $|K^0\pi^0\pi^+\rangle$ is given by

$$\begin{aligned}|K^0\pi^0\pi^+\rangle = &-\frac{1}{\sqrt{5}}\left|\frac{5}{2},+\frac{1}{2}\right\rangle_I^e + \frac{\sqrt{3}}{\sqrt{10}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_I^e\\ &+\frac{1}{\sqrt{6}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_I^o - \frac{1}{\sqrt{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle_I^o,\end{aligned} \tag{8}$$

where the superscripts e and o denote the even and odd natures of the state under the exchange $\pi^0 \leftrightarrow \pi^+$. The sign change in the odd states above is due to the odd $|1,+1\rangle_I$ isospin component of the $|\pi^0\pi^+\rangle$ state switching sign under $\pi^0 \leftrightarrow \pi^+$ exchange, whereas the $|2,+1\rangle_I$ is even under the same exchange. Since B^+ has isospin state $|\frac{1}{2},+\frac{1}{2}\rangle_I$, and only $\Delta I = 0, 1$ currents are allowed by the Hamiltonian in the standard model, we would have no contributions from the $|\frac{5}{2},+\frac{1}{2}\rangle_I$ state. The $|\frac{3}{2},+\frac{1}{2}\rangle_I$ state can arise from both $|\frac{1}{2},-\frac{1}{2}\rangle_I \otimes |2,+1\rangle_I$ and $|\frac{1}{2},-\frac{1}{2}\rangle_I \otimes |1,+1\rangle_I$, with the first contribution being symmetric and the later being antisymmetric. The state $|\frac{1}{2},+\frac{1}{2}\rangle_I$ on the other hand is purely antisymmetric. Even though we shall work with the standard model Hamiltonian, our conclusions are general and are valid even when more general Hamiltonians exist.

The isospin $I = \frac{1}{2}$ initial state decays to a final state that can be decomposed into either $I = \frac{1}{2}$ or $I = \frac{3}{2}$ states via a Hamiltonian that allows $\Delta I = 0$ or $\Delta I = 1$ transitions. The transition with $\Delta I = 1$ results in two amplitudes with $I = \frac{1}{2}$ or $I = \frac{3}{2}$ represented as $T_{1,\frac{1}{2}}$ and $T_{1,\frac{3}{2}}$, respectively, whereas the $\Delta I = 0$ transition results only in a single amplitude with final state $I = \frac{1}{2}$ labeled as $T_{0,\frac{1}{2}}$. The isospin amplitudes $T_{1,\frac{1}{2}}$, $T_{1,\frac{3}{2}}$ and $T_{0,\frac{1}{2}}$ are defined [16] in terms of the Hamiltonian to be

$$\begin{aligned}T_{1,\frac{3}{2}} &= \sqrt{\frac{1}{3}}\left\langle\frac{3}{2},\pm\frac{1}{2}\right|\mathcal{H}_{\Delta I=1}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle,\\ T_{1,\frac{1}{2}} &= \pm\sqrt{\frac{2}{3}}\left\langle\frac{1}{2},\pm\frac{1}{2}\right|\mathcal{H}_{\Delta I=1}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle,\\ T_{0,\frac{1}{2}} &= \sqrt{\frac{2}{3}}\left\langle\frac{1}{2},\pm\frac{1}{2}\right|\mathcal{H}_{\Delta I=0}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle.\end{aligned} \tag{9}$$

The amplitude for the decay $B^+ \to K^0\pi^0\pi^+$ can then be written in terms of the isospin amplitudes as

$$\begin{aligned}A(B^+ \to K^0\pi^0\pi^+) = &\frac{3}{\sqrt{10}}T^{\mathrm{e}}_{1,\frac{3}{2}}X\\ &-\frac{1}{\sqrt{2}}\left(T^{\mathrm{o}}_{1,\frac{3}{2}} + T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}}\right)Y\sin\theta,\end{aligned} \tag{10}$$

where X and $Y\sin\theta$ are introduced to take care of the spatial and kinematic contributions as is seen from the discussion above [see Eqs. (3) and (4)]. In general, X and Y can be arbitrary functions of r and $\cos\theta$. The functions X and Y are in general mode dependent; however, they are the same for modes related by isospin symmetry. We retain the subscripts e and o merely to track the even or odd isospin state of the two pions in the three-body final state.

On the other hand, if U-spin were an exact symmetry, the state $K^0\pi^0$ must remain unchanged under $K^0 \leftrightarrow \pi^0$ exchange. Under U-spin the $K^0\pi^0$ state can exist in $|2,+1\rangle_U$ and $|1,+1\rangle_U$ (see Table I), out of which $|1,+1\rangle_U$ has a contribution from the $|0,0\rangle_{U,8}$ admixture in π^0 which is denoted by $|1',+1\rangle_U$. Both $|2,+1\rangle_U$ and the $|1,+1\rangle_U$ coming from the $|0,0\rangle_{U,8}$ contribution of π^0 exist in space symmetric (even partial wave) states, and that part of $|1,+1\rangle_U$ arising out of the $|1,0\rangle_U$ part of π^0 exists in a space antisymmetric (odd partial wave) state. The U-spin decomposition of the final state $|K^0\pi^0\pi^+\rangle$ is given by

$$\begin{aligned}|K^0\pi^0\pi^+\rangle = &-\frac{1}{2\sqrt{5}}\left|\frac{5}{2},+\frac{1}{2}\right\rangle_U^e - \frac{\sqrt{3}}{2\sqrt{10}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_U^e\\ &-\frac{1}{2\sqrt{6}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_U^o - \frac{1}{2\sqrt{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle_U^o\\ &+\frac{1}{2}\left|\frac{3'}{2},+\frac{1}{2}\right\rangle_U^e + \frac{1}{\sqrt{2}}\left|\frac{1'}{2},+\frac{1}{2}\right\rangle_U^e,\end{aligned} \tag{11}$$

where the superscripts e and o denote that the state is even or odd under the exchange $K^0 \leftrightarrow \pi^0$. The origin of the sign change in the odd terms above is easy to understand from the U-spin decomposition of the $|K^0\pi^0\rangle$ state,

$$|K^0\pi^0\rangle = \frac{1}{2\sqrt{2}}(|2,+1\rangle_U + |1,+1\rangle_U) - \frac{\sqrt{3}}{2}|1',+1\rangle_U,$$

which transforms as follows under the $K^0 \leftrightarrow \pi^0$ exchange:

$$|\pi^0K^0\rangle = \frac{1}{2\sqrt{2}}(|2,+1\rangle_U - |1,+1\rangle_U) - \frac{\sqrt{3}}{2}|1',+1\rangle_U.$$

We recollect that $|1,+1\rangle_U$ is an odd state under $K^0 \leftrightarrow \pi^0$ exchange, whereas $|2,+1\rangle_U$ and $|1',+1\rangle_U$ are even states under the same exchange. It is easy to see that $|\frac{5}{2},+\frac{1}{2}\rangle_U$ and $|\frac{3}{2},+\frac{1}{2}\rangle_U$ states do not contribute since the parent particle B^+ is a U-spin singlet, and only the $\Delta U = \frac{1}{2}$ current contributes to the decay. This unique feature follows from the fact that the electroweak penguin does not violate U-spin as d and s quarks carry the same electric charge (see Ref. [34]). Hence, only the $|\frac{1}{2},\frac{1}{2}\rangle_U$ and $|\frac{1'}{2},\frac{1}{2}\rangle_U$ can contribute to the decay amplitude, and they correspond to antisymmetric and symmetric contributions under $K^0 \leftrightarrow \pi^0$, respectively. The U-spin amplitudes

$$U_{\frac{1}{2},\frac{1}{2}} = \pm\sqrt{\frac{2}{3}}\left\langle\frac{1}{2},\pm\frac{1}{2}\middle|\mathcal{H}_{\Delta U=\frac{1}{2}}\middle|0,0\right\rangle,$$
$$U'_{\frac{1}{2},\frac{1}{2}} = \sqrt{\frac{1}{3}}\left\langle\frac{1'}{2},\pm\frac{1}{2}\middle|\mathcal{H}_{\Delta U=\frac{1}{2}}\middle|0,0\right\rangle. \quad (12)$$

Hence, the amplitude for the decay $B^+ \to K^0\pi^0\pi^+$ can then be written in terms of the U-spin amplitudes as

$$A(B^+ \to K^0\pi^0\pi^+) = \frac{3}{\sqrt{10}}U'^{\mathrm{e}}_{\frac{1}{2},\frac{1}{2}}X' + U^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}}Y'\sin\theta', \quad (13)$$

where X' and Y' are functions that are, in general, arbitrary functions of r and $\cos\theta'$ and are introduced to take care of spatial and kinematic contributions to the decay amplitude. The subscripts e and o are again retained to merely track the even or odd U-spin state of K^0 and π^0 in the three-body final state. As argued earlier the amplitude for the decay has two parts, one fully symmetric under the exchanges $s \leftrightarrow t \leftrightarrow u$ [i.e. $\mathcal{A}_{SS}(s,t,u)$] and another fully antisymmetric under the same exchanges [i.e. $\mathcal{A}_{AA}(s,t,u)$]. Comparing Eqs. (10) and (13), we obtain

$$\mathcal{A}_{SS} = \frac{3}{\sqrt{10}}T^{\mathrm{e}}_{1,\frac{3}{2}}X = \frac{3}{\sqrt{10}}U'^{\mathrm{e}}_{\frac{1}{2},\frac{1}{2}}X' \quad (14)$$

$$\mathcal{A}_{AA} = -\frac{1}{\sqrt{2}}(T^{\mathrm{o}}_{1,\frac{3}{2}} + T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}})Y\sin\theta$$
$$= U^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}}Y'\sin\theta'. \quad (15)$$

The exchange $s \leftrightarrow t \leftrightarrow u$ being equivalent to $\theta \leftrightarrow \theta' \leftrightarrow \theta''$ implies that the fully antisymmetric amplitude $\mathcal{A}_{AA}(s,t,u)$ must be proportional to $\sin 3\theta$ because $\sin 3\theta = \sin 3\theta' = \sin 3\theta''$ as $\theta = \theta' + \frac{2\pi}{3} = \theta'' + \frac{4\pi}{3}$. From elementary trigonometry we know that $\sin 3\theta = \sin\theta(4\cos^2\theta - 1)$. This implies that the factor $(4\cos^2\theta - 1)$ is an even function of $\cos\theta$ and is a part of both Y and Y' in Eq. (15); i.e. $Y = y(4\cos^2\theta - 1)$ and $Y' = y'(4\cos^2\theta' - 1)$ for some y and y' such that

$$\mathcal{A}_{AA} = -\frac{1}{\sqrt{2}}(T^{\mathrm{o}}_{1,\frac{3}{2}} + T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}})y\sin 3\theta$$
$$= U^{\mathrm{o}}_{\frac{1}{2},\frac{1}{2}}y'\sin 3\theta'. \quad (16)$$

The Dalitz plot can be divided into six sextants by means of the s, t and u axes which go along the medians of an equilateral triangle as shown in Figs. 2 and 3. Since the Dalitz plot distribution function is proportional to the amplitude mod-square, it would also have a part which is fully symmetric under $s \leftrightarrow t \leftrightarrow u$ [denoted by $f_{SS}(s,t,u)$] and another part which is fully antisymmetric under the same exchanges [denoted by $f_{AA}(s,t,u)$]:

$$f_{SS}(s,t,u) \propto |\mathcal{A}_{SS}(s,t,u)|^2 + |\mathcal{A}_{AA}(s,t,u)|^2, \quad (17)$$

$$f_{AA}(s,t,u) \propto 2\mathrm{Re}(\mathcal{A}_{SS}(s,t,u)\cdot\mathcal{A}^*_{AA}(s,t,u)). \quad (18)$$

Let us denote the function describing distribution of events in any sextant, say the ith one, of the Dalitz plot by $f_i(r,\theta)$, where the coordinates (r,θ) lie in the sextant i and we could have as well used the other equivalent choices θ' or θ'' instead of θ, the choice of which is subject to the underlying symmetry being considered (see Fig. 3). Henceforth, we shall drop (r,θ) from the distribution functions, except when necessary, as we implicitly assume the r and θ dependence in them. The distribution function must have only two parts as said above, the fully symmetric and the fully antisymmetric parts. Let us assume that in sextant I the Dalitz plot distribution is given by the function

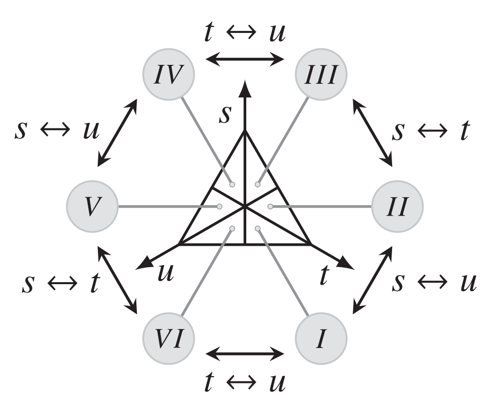

FIG. 3. Exchanges that take us from one sextant to another in the Dalitz plot. It must be noted that the following exchanges are also equivalent: $s \leftrightarrow t \leftrightarrow u \equiv \theta \leftrightarrow \theta' \leftrightarrow \theta''$ as well as $t \leftrightarrow u \equiv \theta \leftrightarrow -\theta$, $s \leftrightarrow t \equiv \theta' \leftrightarrow -\theta'$ and $u \leftrightarrow s \equiv \theta'' \leftrightarrow -\theta''$.

$$f_I = f_{SS}(s,t,u) + f_{AA}(s,t,u). \tag{19}$$

It is then trivial to see that the Dalitz plot distributions in the even numbered sextants should be identical to one another, and the odd numbered sextants would also be identically populated, because

$$f_I = f_{III} = f_V = f_{SS}(s,t,u) + f_{AA}(s,t,u), \tag{20}$$

$$f_{II} = f_{IV} = f_{VI} = f_{SS}(s,t,u) - f_{AA}(s,t,u). \tag{21}$$

This is the signature of exact $SU(3)$ flavor symmetry in the Dalitz plots under our consideration. Any deviation from this conclusion would constitute an observable evidence for violation of the $SU(3)$ flavor symmetry.

Until now the exchange properties of $K^0 \leftrightarrow \pi^0$ under U-spin and $\pi^0 \leftrightarrow \pi^+$ under isospin have been used to obtain the even and odd amplitudes contributing to $B^+ \to K^0\pi^0\pi^+$. Since K^0 and π^+ belong to different multiplets of V-spin, in order to consider the symmetry properties under $K^0 \leftrightarrow \pi^+$, one needs to define the G-parity analog of the V-spin, denoted by G_V and defined in Appendix A. Since charge conjugation is a good symmetry in strong interaction, G_V is as good as the V-spin itself. The state $|K^0\pi^+\rangle$ is composed of states which are even and odd under G_V-parity,

$$|K^0\pi^+\rangle = \frac{1}{2}(|K^0\pi^+\rangle_\mathrm{e} + |K^0\pi^+\rangle_\mathrm{o}),$$

where

$$|K^0\pi^+\rangle_\mathrm{e} = |K^0\pi^+\rangle - |\pi^+K^0\rangle,$$
$$|K^0\pi^+\rangle_\mathrm{o} = |K^0\pi^+\rangle + |\pi^+K^0\rangle,$$

and

$$G_V|K^0\pi^+\rangle_\mathrm{e} = +|K^0\pi^+\rangle_\mathrm{e},$$
$$G_V|K^0\pi^+\rangle_\mathrm{o} = -|K^0\pi^+\rangle_\mathrm{o}.$$

We have already proven that the amplitudes for the decay $B^+ \to K^0\pi^0\pi^+$ have two parts, one even and the other odd under the exchange of any two particles in the final state. Hence, $\mathcal{A}_{SS}$ is odd under G_V, and $\mathcal{A}_{AA}$ is even under G_V. Since the two G_V-parity amplitudes do not interfere, the two amplitudes $\mathcal{A}_{SS}$ and $\mathcal{A}_{AA}$ do not interfere in the Dalitz plot distribution resulting in f_{AA} being zero [Eq. (18)]. Therefore, if G_V is a good symmetry of nature, it is interesting to conclude that the Dalitz plot is completely symmetric under $s \leftrightarrow t \leftrightarrow u$. This implies that

$$f_I = f_{II} = f_{III} = f_{IV} = f_V = f_{VI} \equiv f_{SS}(s,t,u). \tag{22}$$

This expression holds only if isospin, U-spin and V-spin are all exact symmetries. However, if G_V is broken, the Dalitz plot distribution will still follow Eqs. (20) and (21) when the isospin and U-spin are exact symmetries. In the case when G_V is exact, the exchange properties of the distribution functions f_I to f_{VI} imply that, (a) if U-spin is an exact symmetry, then $f_{II} = f_{III}, f_I = f_{IV}$ and $f_V = f_{VI}$ irrespective of the validity of isospin symmetry and, (b) if isospin is an exact symmetry, then $f_{II} = f_V, f_I = f_{VI}$ and $f_{III} = f_{IV}$ irrespective of the validity of the U-spin symmetry. However, when both G_V and either the isospin or U-spin is broken, then Eqs. (20) and (21) are no longer valid. In such a case, we have the following possibilities:

(i) Test for isospin symmetry: By isospin symmetry, the sextants I, II, III get mapped to the sextants VI, V, IV, respectively. We note that, when isospin is *not* broken, then

$$f_I + f_{VI} = f_{III} + f_{IV} = f_V + f_{II} = 2f_{SS}(s,t,u), \tag{23}$$

$$f_I - f_{VI} = f_{III} - f_{IV} = f_V - f_{II} = 2f_{AA}(s,t,u). \tag{24}$$

However, when isospin is broken, the values of f_{SS} and f_{AA} extracted from sextants I and VI need not be same as those extracted from either II and V or III and IV. For further clarification of this statement, we introduce two quantities Σ^i_j and Δ^i_j defined as

$$\Sigma^i_j(r,\theta) = f_i + f_j, \tag{25}$$

$$\Delta^i_j(r,\theta) = f_i - f_j, \tag{26}$$

where i and j are two sextants and $i \neq j$. For conciseness of expressions, we shall also drop the explicit (r,θ) dependence of Σ^i_j and Δ^i_j. In terms of these quantities, the signature of isospin breaking can be succinctly summarized by the inequalities

$$\Sigma^I_{VI} \neq \Sigma^{III}_{IV} \neq \Sigma^V_{II}, \tag{27}$$

$$\Delta^I_{VI} \neq \Delta^{III}_{IV} \neq \Delta^V_{II}. \tag{28}$$

An asymmetry can now be constructed to measure the isospin breaking as follows:

$$\mathbb{A}_{\text{Isospin}} = \left|\frac{\Sigma^I_{VI} - \Sigma^{III}_{IV}}{\Sigma^I_{VI} + \Sigma^{III}_{IV}}\right| + \left|\frac{\Sigma^{III}_{IV} - \Sigma^V_{II}}{\Sigma^{III}_{IV} + \Sigma^V_{II}}\right| + \left|\frac{\Sigma^V_{II} - \Sigma^I_{VI}}{\Sigma^V_{II} + \Sigma^I_{VI}}\right| + \left|\frac{\Delta^I_{VI} - \Delta^{III}_{IV}}{\Delta^I_{VI} + \Delta^{III}_{IV}}\right| + \left|\frac{\Delta^{III}_{IV} - \Delta^V_{II}}{\Delta^{III}_{IV} + \Delta^V_{II}}\right| + \left|\frac{\Delta^V_{II} - \Delta^I_{VI}}{\Delta^V_{II} + \Delta^I_{VI}}\right|. \tag{29}$$

(ii) Test for U-spin symmetry: By U-spin symmetry, the sextants VI, I, II get mapped to the sextants

V, IV, III, respectively. We note that, when U-spin is not broken, then

$$\Sigma^{I}_{IV} = \Sigma^{III}_{II} = \Sigma^{V}_{VI} = 2f_{SS}(s,t,u), \tag{30}$$

$$\Delta^{I}_{IV} = \Delta^{III}_{II} = \Delta^{V}_{VI} = 2f_{AA}(s,t,u). \tag{31}$$

Here it is profitable to consider the Σ's and Δ's being functions of (r, θ') as we are considering $s \leftrightarrow t$ exchange which is equivalent to $\theta' \leftrightarrow -\theta'$. When the U-spin is broken,

$$\Sigma^{I}_{IV} \neq \Sigma^{III}_{II} \neq \Sigma^{V}_{VI}, \tag{32}$$

$$\Delta^{I}_{IV} \neq \Delta^{III}_{II} \neq \Delta^{V}_{VI}. \tag{33}$$

The asymmetry for U-spin breaking is, therefore, given by

$$\mathbb{A}_{U\text{-spin}} = \left|\frac{\Sigma^{I}_{IV} - \Sigma^{III}_{II}}{\Sigma^{I}_{IV} + \Sigma^{III}_{II}}\right| + \left|\frac{\Sigma^{III}_{II} - \Sigma^{V}_{VI}}{\Sigma^{III}_{II} + \Sigma^{V}_{VI}}\right| + \left|\frac{\Sigma^{V}_{VI} - \Sigma^{I}_{IV}}{\Sigma^{V}_{VI} + \Sigma^{I}_{IV}}\right| + \left|\frac{\Delta^{I}_{IV} - \Delta^{III}_{II}}{\Delta^{I}_{IV} + \Delta^{III}_{II}}\right| + \left|\frac{\Delta^{III}_{II} - \Delta^{V}_{VI}}{\Delta^{III}_{II} + \Delta^{V}_{VI}}\right| + \left|\frac{\Delta^{V}_{VI} - \Delta^{I}_{IV}}{\Delta^{V}_{VI} + \Delta^{I}_{IV}}\right|. \tag{34}$$

(iii) Test for V-spin symmetry: As said before, G_V-parity is as badly broken as the V-spin because charge conjugation is a good symmetry. When V-spin symmetry is broken, then G_V is also broken, and the distribution of events in the Dalitz plot sextants would follow Eqs. (20) and (21). In addition to that, when V-spin is broken, K^0 and π^+ need not be even under exchange. This leads to

$$\Sigma^{V}_{IV} \neq \Sigma^{III}_{VI} \neq \Sigma^{I}_{II}, \tag{35}$$

$$\Delta^{V}_{IV} \neq \Delta^{III}_{VI} \neq \Delta^{I}_{II}. \tag{36}$$

The asymmetry for V-spin breaking is, therefore, given by

$$\mathbb{A}_{V\text{-spin}} = \left|\frac{\Sigma^{V}_{IV} - \Sigma^{I}_{II}}{\Sigma^{V}_{IV} + \Sigma^{I}_{II}}\right| + \left|\frac{\Sigma^{I}_{II} - \Sigma^{III}_{VI}}{\Sigma^{I}_{II} + \Sigma^{III}_{VI}}\right| + \left|\frac{\Sigma^{III}_{VI} - \Sigma^{V}_{IV}}{\Sigma^{III}_{VI} + \Sigma^{V}_{IV}}\right| + \left|\frac{\Delta^{V}_{IV} - \Delta^{I}_{II}}{\Delta^{V}_{IV} + \Delta^{I}_{II}}\right| + \left|\frac{\Delta^{I}_{II} - \Delta^{III}_{VI}}{\Delta^{I}_{II} + \Delta^{III}_{VI}}\right| + \left|\frac{\Delta^{III}_{VI} - \Delta^{V}_{IV}}{\Delta^{III}_{VI} + \Delta^{V}_{IV}}\right|. \tag{37}$$

Hence, the extent of the breaking of isospin, U-spin and V-spin can easily be measured from the Dalitz plot distribution. The asymmetries measuring isospin, U-spin and V-spin are functions of r and $3\theta \equiv 3\theta' \equiv 3\theta''$ [see the discussion leading to Eq. (16)]. These asymmetries are, thus, valid in the full Dalitz plot, including the resonant contributions, and can be measured in different regions of the Dalitz plot. In particular these asymmetries can be measured both along resonances and in the nonresonant

TABLE II. Comparison of decays of B^+ and D_s^+ to the final state $K^0\pi^0\pi^+$.

$B^+ \to K^0\pi^0\pi^+$							
Isospin (initial state $\lvert\frac{1}{2},+\frac{1}{2}\rangle$)				U-spin (initial state $\lvert 0,0\rangle$)			
Transition	Final state	Symmetry	Amplitude	Transition	Final state	Symmetry	Amplitude
$\Delta I = 1$	$\lvert\frac{3}{2},+\frac{1}{2}\rangle$	Mixed	$\frac{3}{\sqrt{10}}T^{e}_{1,\frac{3}{2}}X + \frac{1}{\sqrt{2}}T^{o}_{1,\frac{3}{2}}Y\sin\theta$	$\Delta U = \frac{1}{2}$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{2\sqrt{2}}U^{o}_{\frac{1}{2},\frac{3}{2}}Y'\sin\theta'$
$\Delta I = 1$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{\sqrt{2}}T^{o}_{1,\frac{1}{2}}Y\sin\theta$	$\Delta U = \frac{1'}{2}$	$\lvert\frac{1}{2},+\frac{1'}{2}\rangle$	Even	$\frac{\sqrt{3}}{\sqrt{2}}U'^{e}_{\frac{1}{2},\frac{1'}{2}}X'$
$\Delta I = 0$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{\sqrt{2}}T^{o}_{0,\frac{1}{2}}Y\sin\theta$				

$D^+{}_s \to K^0\pi^0\pi^+$							
Isospin (initial state $\lvert 0,0\rangle$)				U-spin (initial state $\lvert\frac{1}{2},+\frac{1}{2}\rangle$)			
Transition	Final state	Symmetry	Amplitude	Transition	Final state	Symmetry	Amplitude
$\Delta I = \frac{3}{2}$	$\lvert\frac{3}{2},+\frac{1}{2}\rangle$	Mixed	$\frac{\sqrt{3}}{\sqrt{10}}T^{e}_{\frac{3}{2},\frac{3}{2}}X + \frac{1}{\sqrt{6}}T^{o}_{\frac{3}{2},\frac{3}{2}}Y\sin\theta$	$\Delta U = 1$	$\lvert\frac{3}{2},+\frac{1}{2}\rangle$	Mixed	$-\frac{3}{2\sqrt{10}}U^{e}_{1,\frac{3}{2}}X' - \frac{1}{2\sqrt{2}}U^{o}_{1,\frac{3}{2}}Y'\sin\theta'$
$\Delta I = \frac{1}{2}$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{\sqrt{2}}T^{o}_{\frac{1}{2},\frac{1}{2}}Y\sin\theta$	$\Delta U = 1$	$\lvert\frac{3'}{2},+\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{2}U'^{e}_{1,\frac{3'}{2}}X'$
				$\Delta U = 1$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{2\sqrt{2}}U^{o}_{1,\frac{1}{2}}Y'\sin\theta'$
				$\Delta U = 1$	$\lvert\frac{1'}{2},+\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{2}U'^{e}_{1,\frac{1'}{2}}X'$
				$\Delta U = 0$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{2\sqrt{2}}U^{o}_{0,\frac{1}{2}}Y'\sin\theta'$
				$\Delta U = 0$	$\lvert\frac{1'}{2},+\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{2}U'^{e}_{0,\frac{1'}{2}}X'$

regions. A quantitative estimate of the variation of these asymmetries obtained experimentally would provide valuable understanding of $SU(3)$ breaking effects. It would also be interesting to find regions of the Dalitz plots where $SU(3)$ is a good symmetry. The procedure discussed above can also be applied to other decay modes with the same final state. In particular one can study the Dalitz plot distribution for the decay $D_s^+ \to K^0\pi^0\pi^+$ in a similar manner. The amplitudes for this mode are tabulated in Table. II.

C. Decay mode with final state $K^+\pi^0\pi^-$

Let us now consider the decay B_d^0 or $\bar{B}_s^0 \to K^+\pi^0\pi^-$ in which isospin symmetry allows the exchange of π^0 and π^- and V-spin symmetry allows exchange of K^+ and π^0. This leads to the following exchanges in the Dalitz plot:

$$V\text{-spin} \equiv K^+ \leftrightarrow \pi^0 \Rightarrow s \leftrightarrow t,$$

$$\text{Isospin} \equiv \pi^0 \leftrightarrow \pi^- \Rightarrow t \leftrightarrow u.$$

Under exact isospin and V-spin, the final state $K^+\pi^0\pi^-$ has the following two possibilities:

(i) $K^+\pi^0$ would exist in either a symmetrical or antisymmetrical state with respect to their exchange in space.

(ii) $\pi^0\pi^-$ would exist in either symmetrical or antisymmetrical state with respect to their exchange in space.

Following the steps as enunciated in Sec. II B, the amplitude for the decay can be shown to have two components, one which is fully symmetric under exchange of any pair of final particles and the other which is fully antisymmetric under the same exchange.

The final state can be expanded in terms of isospin and V-spin states as follows:

(i) Isospin:

$$|K^+\pi^0\pi^-\rangle = \frac{1}{\sqrt{5}}\left|\frac{5}{2},-\frac{1}{2}\right\rangle_I^e + \sqrt{\frac{3}{10}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle_I^e + \frac{1}{\sqrt{6}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle_I^o + \frac{1}{\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle_I^o, \tag{38}$$

where the superscripts e and o denote even and odd behavior of the state under the exchange $\pi^0 \leftrightarrow \pi^-$.

(ii) V-spin:

$$|K^+\pi^0\pi^-\rangle = \frac{1}{2\sqrt{5}}\left|\frac{5}{2},+\frac{1}{2}\right\rangle_V^e + \frac{\sqrt{3}}{2\sqrt{10}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_V^e + \frac{1}{2\sqrt{6}}\left|\frac{3}{2},+\frac{1}{2}\right\rangle_V^o + \frac{1}{2\sqrt{3}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle_V^o - \frac{1}{2}\left|\frac{3'}{2},+\frac{1}{2}\right\rangle_V^e - \frac{1}{\sqrt{2}}\left|\frac{1'}{2},+\frac{1}{2}\right\rangle_V^e,$$

where the superscripts e and o denote even and odd behavior of the state under the exchange $K^+ \leftrightarrow \pi^0$.

The sign changes as can be noticed in the above states arise from the exchange of particles in the two particle states given below (as also noted in Table I):

(i) Isospin:

$$|\pi^0\pi^-\rangle = \frac{1}{\sqrt{2}}(|2,-1\rangle_I + |1,-1\rangle_I), \tag{39}$$

$$|\pi^-\pi^0\rangle = \frac{1}{\sqrt{2}}(|2,-1\rangle_I - |1,-1\rangle_I). \tag{40}$$

(ii) V-spin:

$$|K^+\pi^0\rangle = -\frac{1}{2\sqrt{2}}(|2,+1\rangle_V + |1,+1\rangle_V) + \frac{\sqrt{3}}{2}|1',+1\rangle_V, \tag{41}$$

$$|\pi^0K^+\rangle = -\frac{1}{2\sqrt{2}}(|2,+1\rangle_V - |1,+1\rangle_V) + \frac{\sqrt{3}}{2}|1',+1\rangle_V. \tag{42}$$

It would be clear from the expressions above that if isospin were an exact symmetry the $|2,-1\rangle_I$ and $|1,-1\rangle_I$ states of $|\pi^-\pi^0\rangle$ would exist in even and odd partial wave states, respectively, as was the case in Sec. II B also. On the other hand, if the V-spin were an exact symmetry, the state $|K^+\pi^0\rangle$ must remain unchanged under $K^+ \leftrightarrow \pi^0$ exchange. Under V-spin the $|K^+\pi^0\rangle$ state can exist in $|2,+1\rangle_V$ and $|1,+1\rangle_V$, out of which $|1,+1\rangle_V$ has a contribution from the $|0,0\rangle_{V,8}$ admixture in π^0, denoted above by $|1',+1\rangle_V$. Both the state $|2,+1\rangle_V$ and the state $|1',+1\rangle_V$ exist in space symmetric (even partial wave) states, and that part of $|1,+1\rangle_V$ arising out of the $|1,0\rangle_V$ part of π^0 exists in a space antisymmetric (odd partial wave) state.

If we consider the initial state to be B_d^0 which is the isospin $|\frac{1}{2},+\frac{1}{2}\rangle_I$ state but V-spin singlet $|0,0\rangle_V$ state, the standard model Hamiltonian allows only $\Delta I = 0, 1$ and $\Delta V = \frac{1}{2},\frac{3}{2}$ transitions. Therefore, in addition to the isospin amplitudes of Eq. (9), we can define the following V-spin amplitudes:

$$V_{\frac{3}{2},\frac{3}{2}} = \left\langle\frac{3}{2},\pm\frac{1}{2}\right|\mathcal{H}_{\Delta V=\frac{3}{2}}\left|0,0\right\rangle, \tag{43}$$

$$V'_{\frac{3}{2},\frac{3}{2}} = \left\langle\frac{3'}{2},\pm\frac{1}{2}\right|\mathcal{H}_{\Delta V=\frac{3}{2}}\left|0,0\right\rangle, \tag{44}$$

$$V_{\frac{1}{2},\frac{1}{2}} = \pm\sqrt{\frac{2}{3}}\left\langle\frac{1}{2},\pm\frac{1}{2}\right|\mathcal{H}_{\Delta V=\frac{1}{2}}\left|0,0\right\rangle, \tag{45}$$

$$V'_{\frac{1}{2},\frac{1}{2}} = \sqrt{\frac{1}{3}}\left\langle\frac{1'}{2},\pm\frac{1}{2}\right|\mathcal{H}_{\Delta V=\frac{1}{2}}\left|0,0\right\rangle. \tag{46}$$

The amplitude for the process $B_d^0 \to K^+\pi^0\pi^-$ can, therefore, be written as

$$A(B_d^0 \to K^+\pi^0\pi^-) = -\frac{3}{\sqrt{10}} T^e_{1,\frac{3}{2}} X + \frac{1}{\sqrt{2}}\left(-T^o_{1,\frac{1}{2}} + T^o_{0,\frac{1}{2}}\right) Y \sin\theta, \quad (47)$$

$$A(B_d^0 \to K^+\pi^0\pi^-) = \sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{20}} V^e_{\frac{3}{2},\frac{3}{2}} - \frac{1}{\sqrt{6}} V'^e_{\frac{3}{2},\frac{3}{2}'} - V'^e_{\frac{1}{2},\frac{1}{2}'}\right) X'' + \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{3}} V^o_{\frac{3}{2},\frac{3}{2}} + V^o_{\frac{1}{2},\frac{1}{2}}\right) Y'' \sin\theta'', \quad (48)$$

where X'' and Y'' are functions that are, in general, arbitrary functions of r and $\cos\theta''$ and are introduced to take care of spatial and kinematic contributions to the decay amplitude. As argued before, the part of the amplitude even under isospin must also be even under the V-spin, and the part odd under isospin must again be odd under the V-spin:

$$\mathcal{A}_{SS} = \frac{3}{\sqrt{10}} T^{\mathrm{e}}_{1,\frac{3}{2}} X = \sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{20}} V^e_{\frac{3}{2},\frac{3}{2}} - \frac{1}{\sqrt{6}} V'^e_{\frac{3}{2},\frac{3}{2}'} - V'^e_{\frac{1}{2},\frac{1}{2}'}\right) X'', \quad (49)$$

$$\mathcal{A}_{AA} = \frac{1}{\sqrt{2}}\left(-T^{\mathrm{o}}_{1,\frac{1}{2}} + T^{\mathrm{o}}_{0,\frac{1}{2}}\right) Y \sin\theta = \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{3}} V^o_{\frac{3}{2},\frac{3}{2}} + V^o_{\frac{1}{2},\frac{1}{2}}\right) Y'' \sin\theta''. \quad (50)$$

We can conclude that the Dalitz plot distribution in the even numbered sextants would be identical to one another, and those of odd numbered sextants would also be similar. Any deviation from this would constitute a signature of simultaneous violations of the isospin and V-spin.

Since K^+ and π^- belong to different multiplets of the U-spin, in order to consider the symmetry properties under $K^+ \leftrightarrow \pi^-$, one needs to define the G-parity analog of the U-spin, denoted by G_U and defined in Appendix A. Since charge conjugation is a good symmetry in strong interaction, G_U is as good as the U-spin itself. The state $|K^+\pi^-\rangle$ is composed of states which are even and odd under G_U-parity,

$$|K^+\pi^-\rangle = \frac{1}{2}(|K^+\pi^-\rangle_{\mathrm{e}} + |K^+\pi^-\rangle_{\mathrm{o}}),$$

where

$$|K^+\pi^-\rangle_e = |K^+\pi^-\rangle - |\pi^- K^+\rangle,$$
$$|K^+\pi^-\rangle_o = |K^+\pi^-\rangle + |\pi^- K^+\rangle,$$

and

$$G_U|K^+\pi^-\rangle_e = |K^+\pi^-\rangle_e,$$
$$G_U|K^+\pi^-\rangle_o = -|K^+\pi^-\rangle_o.$$

We have already proven that the amplitudes for the decay $B_d^0 \to K^+\pi^0\pi^-$ have two parts, one even and the other odd, under the exchange of any two particles in the final state. Hence, $\mathcal{A}_{SS}$ is odd under G_U, and $\mathcal{A}_{AA}$ is even under G_U. Since the two G_U-parity amplitudes do not interfere the two amplitudes, $\mathcal{A}_{SS}$ and $\mathcal{A}_{AA}$ do not interfere in the Dalitz plot distribution resulting in f_{AA} being zero [Eq. (18)]. Therefore, if G_U is a good symmetry of nature, it is interesting to conclude that the Dalitz plot is completely symmetric under $s \leftrightarrow t \leftrightarrow u$. The Dalitz plot asymmetries which would be a measure of the extent of the breaking of the $SU(3)$ flavor symmetry are, therefore, given by

$$\mathbb{A}_{\text{Isospin}} = \left|\frac{\Sigma^I_{VI} - \Sigma^{III}_{IV}}{\Sigma^I_{VI} + \Sigma^{III}_{IV}}\right| + \left|\frac{\Sigma^{III}_{IV} - \Sigma^V_{II}}{\Sigma^{III}_{IV} + \Sigma^V_{II}}\right| + \left|\frac{\Sigma^V_{II} - \Sigma^I_{VI}}{\Sigma^V_{II} + \Sigma^I_{VI}}\right| + \left|\frac{\Delta^I_{VI} - \Delta^{III}_{IV}}{\Delta^I_{VI} + \Delta^{III}_{IV}}\right| + \left|\frac{\Delta^{III}_{IV} - \Delta^V_{II}}{\Delta^{III}_{IV} + \Delta^V_{II}}\right| + \left|\frac{\Delta^V_{II} - \Delta^I_{VI}}{\Delta^V_{II} + \Delta^I_{VI}}\right|, \quad (51)$$

$$\mathbb{A}_{U\text{-spin}} = \left|\frac{\Sigma^V_{IV} - \Sigma^I_{II}}{\Sigma^V_{IV} + \Sigma^I_{II}}\right| + \left|\frac{\Sigma^I_{II} - \Sigma^{III}_{VI}}{\Sigma^I_{II} + \Sigma^{III}_{VI}}\right| + \left|\frac{\Sigma^{III}_{VI} - \Sigma^V_{IV}}{\Sigma^{III}_{VI} + \Sigma^V_{IV}}\right| + \left|\frac{\Delta^V_{IV} - \Delta^I_{II}}{\Delta^V_{IV} + \Delta^I_{II}}\right| + \left|\frac{\Delta^I_{II} - \Delta^{III}_{VI}}{\Delta^I_{II} + \Delta^{III}_{VI}}\right| + \left|\frac{\Delta^{III}_{VI} - \Delta^V_{IV}}{\Delta^{III}_{VI} + \Delta^V_{IV}}\right|, \quad (52)$$

$$\mathbb{A}_{V\text{-spin}} = \left|\frac{\Sigma^I_{IV} - \Sigma^{III}_{II}}{\Sigma^I_{IV} + \Sigma^{III}_{II}}\right| + \left|\frac{\Sigma^{III}_{II} - \Sigma^V_{VI}}{\Sigma^{III}_{II} + \Sigma^V_{VI}}\right| + \left|\frac{\Sigma^V_{VI} - \Sigma^I_{IV}}{\Sigma^V_{VI} + \Sigma^I_{IV}}\right| + \left|\frac{\Delta^I_{IV} - \Delta^{III}_{II}}{\Delta^I_{IV} + \Delta^{III}_{II}}\right| + \left|\frac{\Delta^{III}_{II} - \Delta^V_{VI}}{\Delta^{III}_{II} + \Delta^V_{VI}}\right| + \left|\frac{\Delta^V_{VI} - \Delta^I_{IV}}{\Delta^V_{VI} + \Delta^I_{IV}}\right|, \quad (53)$$

where the Σ's and Δ's are as defined in Eqs. (25) and (26), respectively. It must again be noted that these asymmetries are in general functions of r and θ (or θ' or θ'') and are defined throughout the Dalitz plot region, including resonant regions. It would again be interesting to look for patterns in the variations of these asymmetries inside the Dalitz plot. Observation of these asymmetries would quantify the extent of the breaking of $SU(3)$ flavor symmetry in the concerned decay mode. One can also look for such asymmetries in the Dalitz plot for $\bar{B}_s^0 \to K^+\pi^0\pi^-$. The amplitudes for this process are given in Table III.

D. Decay mode with final state $K^+\pi^0\bar{K}^0$

For the study of simultaneous violations of both the U-spin and V-spin, we look at decays such as B^+ or $D^+ \to K^+\pi^0\bar{K}^0$ and their conjugate modes. In this state, K^+ and π^0 are exchangeable under V-spin, and π^0, $\bar{K}^0$ are exchangeable under U-spin. Under V-spin, the $K^+\pi^0$ state

TABLE III. Comparison of decays of B_d^0 and $\bar{B}_s^0$ to the final state $K^+\pi^0\pi^-$.

$B_d^0 \to K^+\pi^0\pi^-$							
Isospin (initial state $\vert\frac{1}{2},-\frac{1}{2}\rangle$)				V-spin (initial state $\vert 0,0\rangle$)			
Transition	Final state	Symmetry	Amplitude	Transition	Final state	Symmetry	Amplitude
$\Delta I = 1$	$\vert\frac{3}{2},-\frac{1}{2}\rangle$	Mixed	$\frac{3}{\sqrt{10}} T^e_{1,\frac{3}{2}} X + \frac{1}{\sqrt{2}} T^o_{1,\frac{3}{2}} Y \sin\theta$	$\Delta V = \frac{3}{2}$	$\vert\frac{3}{2},+\frac{1}{2}\rangle$	Mixed	$\frac{\sqrt{3}}{2\sqrt{10}} V^e_{\frac{3}{2},\frac{3}{2}} X'' + \frac{1}{2\sqrt{6}} V^o_{\frac{3}{2},\frac{3}{2}} Y'' \sin\theta''$
$\Delta I = 1$	$\vert\frac{1}{2},-\frac{1}{2}\rangle$	Odd	$-\frac{1}{\sqrt{2}} T^o_{1,\frac{1}{2}} Y \sin\theta$	$\Delta V = \frac{3}{2}$	$\vert\frac{3'}{2},+\frac{1}{2}\rangle$	Even	$-\frac{1}{2} V'^e_{\frac{3}{2},\frac{3'}{2}} X''$
$\Delta I = 0$	$\vert\frac{1}{2},-\frac{1}{2}\rangle$	Odd	$\frac{1}{\sqrt{2}} T^o_{0,\frac{1}{2}} Y \sin\theta$	$\Delta V = \frac{1}{2}$	$\vert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$\frac{1}{2\sqrt{2}} V^o_{\frac{1}{2},\frac{1}{2}} Y'' \sin\theta''$
				$\Delta V = \frac{1}{2}$	$\vert\frac{1'}{2},+\frac{1}{2}\rangle$	Even	$-\frac{\sqrt{3}}{\sqrt{2}} V'^e_{\frac{1}{2},\frac{1'}{2}} X''$

$\bar{B}_s^0 \to K^+\pi^0\pi^-$							
Isospin (initial state $\vert 0,0\rangle$)				V-spin (initial state $\vert\frac{1}{2},+\frac{1}{2}\rangle$)			
Transition	Final state	Summetry	Amplitude	Transition	Final state	Symmetry	Amplitude
$\Delta I = \frac{3}{2}$	$\vert\frac{3}{2},-\frac{1}{2}\rangle$	Mixed	$\frac{\sqrt{3}}{\sqrt{10}} T^e_{\frac{3}{2},\frac{3}{2}} X + \frac{1}{\sqrt{6}} T^o_{\frac{3}{2},\frac{3}{2}} Y \sin\theta$	$\Delta V = 1$	$\vert\frac{3}{2},+\frac{1}{2}\rangle$	Mixed	$\frac{3}{2\sqrt{10}} V'^e_{1,\frac{3}{2}} X'' + \frac{1}{2\sqrt{2}} V^o_{1,\frac{3}{2}} Y'' \sin\theta''$
$\Delta I = \frac{1}{2}$	$\vert\frac{1}{2},-\frac{1}{2}\rangle$	Odd	$-\frac{1}{\sqrt{2}} T^o_{\frac{1}{2},\frac{1}{2}} Y \sin\theta$	$\Delta V = 1$	$\vert\frac{3'}{2},+\frac{1}{2}\rangle$	Even	$-\frac{\sqrt{3}}{2} V'^e_{1,\frac{3'}{2}} X''$
				$\Delta V = 1$	$\vert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$\frac{1}{2\sqrt{2}} V^o_{1,\frac{1}{2}} Y'' \sin\theta''$
				$\Delta V = 1$	$\vert\frac{1'}{2},+\frac{1}{2}\rangle$	Even	$-\frac{\sqrt{3}}{2} V^e_{1,\frac{1'}{2}} X''$
				$\Delta V = 0$	$\vert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$\frac{1}{2\sqrt{2}} V^o_{0,\frac{1}{2}} Y'' \sin\theta''$
				$\Delta V = 0$	$\vert\frac{1'}{2},+\frac{1}{2}\rangle$	Even	$-\frac{\sqrt{3}}{2} V'^e_{0,\frac{1'}{2}} X''$

can exist in $|2,+1\rangle_V$ and $|1,+1\rangle_V$, out of which the state $|1,+1\rangle_V$ has a contribution from the $|0,0\rangle_{V,8}$ admixture in π^0. Thus, assuming V-spin to be an exact symmetry would put the state $|2,+1\rangle_V$ and that part of the $|1,+1\rangle_V$ state coming from the $|0,0\rangle_{V,8}$ contribution of π^0 in a space symmetric (even partial wave) state. The remaining part of the $|1,+1\rangle_V$ state would be in a space antisymmetric (odd partial wave) state. Similarly, the $\pi^0\bar{K}^0$ state would exist in $|2,-1\rangle_U$ and $|1,-1\rangle_U$, out of which the state $|1,-1\rangle_U$ has a contribution from the $|0,0\rangle_{U,8}$ admixture in π^0. Thus, if U-spin were assumed to be an exact symmetry, the state $|2,-1\rangle_U$ and the $|1,-1\rangle_U$ state coming from $|0,0\rangle_{U,8}$ part of π^0 would exist in space symmetric (even partial wave) states, and the other part of $|1,-1\rangle_U$ would exist in a space antisymmetric (odd partial wave) state.

Therefore, under exact U-spin and V-spin, the final state $K^+\pi^0\bar{K}^0$ has the following two possibilities:

(i) $K^+\pi^0$ would exist in either a symmetrical or antisymmetrical state with respect to their exchange in space.

(ii) $\pi^0\bar{K}^0$ would exist in either a symmetrical or antisymmetrical state with respect to their exchange in space.

Again, following the steps as enunciated in Sec. II B, we can conclude that the Dalitz plot distribution in the even numbered sextants would be identical to one another, and those of odd numbered sextants would also be similar, as given in Eqs. (20) and (21). Any deviation from this would constitute a signature of simultaneous violations of U-spin and V-spin. We can once again reaffirm the same logic as given in Secs. II B and II C, by invoking the G_I-parity operator (see Appendix A) to connect K^+ and $\bar{K}^0$ belonging to two different isospin doublets. This would lead to a fully symmetric Dalitz plot which would be broken when G_I is broken. The amplitudes for the two decay modes under consideration are given in Table IV. The Dalitz plot asymmetries that can be useful in this case are given by

$$\mathbb{A}_{\text{Isospin}} = \left|\frac{\Sigma^V_{IV} - \Sigma^I_{II}}{\Sigma^V_{IV} + \Sigma^I_{II}}\right| + \left|\frac{\Sigma^I_{II} - \Sigma^{III}_{VI}}{\Sigma^I_{II} + \Sigma^{III}_{VI}}\right| + \left|\frac{\Sigma^{III}_{VI} - \Sigma^V_{IV}}{\Sigma^{III}_{VI} + \Sigma^V_{IV}}\right| + \left|\frac{\Delta^V_{IV} - \Delta^I_{II}}{\Delta^V_{IV} + \Delta^I_{II}}\right| + \left|\frac{\Delta^I_{II} - \Delta^{III}_{VI}}{\Delta^I_{II} + \Delta^{III}_{VI}}\right| + \left|\frac{\Delta^{III}_{VI} - \Delta^V_{IV}}{\Delta^{III}_{VI} + \Delta^V_{IV}}\right|, \tag{54}$$

$$\mathbb{A}_{U\text{-spin}} = \left|\frac{\Sigma^I_{VI} - \Sigma^{III}_{IV}}{\Sigma^I_{VI} + \Sigma^{III}_{IV}}\right| + \left|\frac{\Sigma^{III}_{IV} - \Sigma^V_{II}}{\Sigma^{III}_{IV} + \Sigma^V_{II}}\right| + \left|\frac{\Sigma^V_{II} - \Sigma^I_{VI}}{\Sigma^V_{II} + \Sigma^I_{VI}}\right| + \left|\frac{\Delta^I_{VI} - \Delta^{III}_{IV}}{\Delta^I_{VI} + \Delta^{III}_{IV}}\right| + \left|\frac{\Delta^{III}_{IV} - \Delta^V_{II}}{\Delta^{III}_{IV} + \Delta^V_{II}}\right| + \left|\frac{\Delta^V_{II} - \Delta^I_{VI}}{\Delta^V_{II} + \Delta^I_{VI}}\right|, \tag{55}$$

$$\mathbb{A}_{V\text{-spin}} = \left|\frac{\Sigma^I_{IV} - \Sigma^{III}_{II}}{\Sigma^I_{IV} + \Sigma^{III}_{II}}\right| + \left|\frac{\Sigma^{III}_{II} - \Sigma^V_{VI}}{\Sigma^{III}_{II} + \Sigma^V_{VI}}\right| + \left|\frac{\Sigma^V_{VI} - \Sigma^I_{IV}}{\Sigma^V_{VI} + \Sigma^I_{IV}}\right| + \left|\frac{\Delta^I_{IV} - \Delta^{III}_{II}}{\Delta^I_{IV} + \Delta^{III}_{II}}\right| + \left|\frac{\Delta^{III}_{II} - \Delta^V_{VI}}{\Delta^{III}_{II} + \Delta^V_{VI}}\right| + \left|\frac{\Delta^V_{VI} - \Delta^I_{IV}}{\Delta^V_{VI} + \Delta^I_{IV}}\right|. \tag{56}$$

TABLE IV. Comparison of amplitudes for the decays of B^+ and D^+ to the final state $K^+\pi^0\bar{K}^0$.

$B^+ \to K^+\pi^0\bar{K}^0$							
U-spin (initial state $\lvert 0,0\rangle$)				V-spin (initial state $\lvert\frac{1}{2},+\frac{1}{2}\rangle$)			
Transition	Final state	Symmetry	Amplitude	Transition	Final state	Symmetry	Amplitude
$\Delta U=\frac{1}{2}$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle$	Odd	$\frac{1}{2\sqrt{2}}U^{o}_{\frac{1}{2},\frac{1}{2}}Y'\sin\theta'$	$\Delta V=1$	$\lvert\frac{3}{2},+\frac{1}{2}\rangle$	Mixed	$-\frac{3}{2\sqrt{10}}V^{e}_{1,\frac{3}{2}}X''-\frac{1}{2\sqrt{2}}V^{o}_{1,\frac{3}{2}}Y''\sin\theta''$
$\Delta U=\frac{1}{2}$	$\lvert\frac{1'}{2},-\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{\sqrt{2}}U'^{e}_{\frac{1}{2},\frac{1'}{2}}X'$	$\Delta V=1$	$\lvert\frac{3'}{2},+\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{2}V'^{e}_{1,\frac{3'}{2}}X''$
$\Delta U=\frac{3}{2}$	$\lvert\frac{3}{2},-\frac{1}{2}\rangle$	Mixed	$\frac{\sqrt{3}}{2\sqrt{10}}U^{e}_{\frac{3}{2},\frac{3}{2}}X'-\frac{1}{2\sqrt{6}}U^{o}_{\frac{3}{2},\frac{3}{2}}Y'\sin\theta'$	$\Delta V=1$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{2\sqrt{2}}V^{o}_{1,\frac{1}{2}}Y''\sin\theta''$
$\Delta U=\frac{3}{2}$	$\lvert\frac{3'}{2},-\frac{1}{2}\rangle$	Even	$\frac{1}{2}U'^{e}_{\frac{1}{2},\frac{1'}{2}}X'$	$\Delta V=1$	$\lvert\frac{1'}{2},+\frac{1}{2}\rangle$	Enen	$\frac{\sqrt{3}}{2}V'^{e}_{1,\frac{1'}{2}}X''$
				$\Delta V=0$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{2\sqrt{2}}V^{o}_{0,\frac{1}{2}}Y''\sin\theta''$
				$\Delta V=0$	$\lvert\frac{1'}{2},+\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{2}V'^{e}_{0,\frac{1}{2}}X''$

$D^+ \to K^+\pi^0\bar{K}^0$							
U-spin (initial state $-\lvert\frac{1}{2},-\frac{1}{2}\rangle$)				V-spin (initial state $\lvert 0,0\rangle$)			
Transition	Final state	Symmetry	Amplitude	Transition	Final state	Symmetry	Amplitude
$\Delta U=1$	$\lvert\frac{3}{2},-\frac{1}{2}\rangle$	Mixed	$-\frac{3}{2\sqrt{10}}U^{e}_{1,\frac{3}{2}}X'-\frac{1}{2\sqrt{2}}U^{o}_{1,\frac{3}{2}}Y'\sin\theta'$	$\Delta V=\frac{3}{2}$	$\lvert\frac{3}{2},+\frac{1}{2}\rangle$	Mixed	$-\frac{\sqrt{3}}{2\sqrt{10}}V^{e}_{\frac{3}{2},\frac{3}{2}}X''-\frac{1}{2\sqrt{6}}V^{o}_{\frac{3}{2},\frac{3}{2}}Y''\sin\theta''$
$\Delta U=1$	$\lvert\frac{3'}{2},-\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{2}U'^{e}_{1,\frac{3'}{2}}X'$	$\Delta V=\frac{3}{2}$	$\lvert\frac{3'}{2},+\frac{1}{2}\rangle$	Even	$\frac{1}{2}V'^{e}_{\frac{3}{2},\frac{3'}{2}}X''$
$\Delta U=1$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle$	Odd	$\frac{1}{2\sqrt{2}}U^{o}_{1,\frac{1}{2}}Y'\sin\theta'$	$\Delta V=\frac{1}{2}$	$\lvert\frac{1}{2},+\frac{1}{2}\rangle$	Odd	$-\frac{1}{2}V^{o}_{\frac{1}{2},\frac{1}{2}}Y''\sin\theta''$
$\Delta U=1$	$\lvert\frac{1'}{2},-\frac{1}{2}\rangle$	Even	$-\frac{\sqrt{3}}{2}U'^{e}_{1,\frac{1'}{2}}X'$	$\Delta V=\frac{1}{2}$	$\lvert\frac{1'}{2},+\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{\sqrt{2}}V'^{e}_{\frac{1}{2},\frac{1'}{2}}X''$
$\Delta U=0$	$\lvert\frac{1}{2},-\frac{1}{2}\rangle$	Odd	$-\frac{1}{2\sqrt{2}}U^{o}_{0,\frac{1}{2}}Y'\sin\theta'$				
$\Delta U=0$	$\lvert\frac{1'}{2},-\frac{1}{2}\rangle$	Even	$\frac{\sqrt{3}}{2}U'^{e}_{0,\frac{1'}{2}}X'$				

Once again the asymmetries being, in general, functions of r and θ (or θ' or θ''), it would be quite interesting to look for their variation across the Dalitz plot. These would be the visible signatures of the breaking of $SU(3)$ flavor symmetry.

E. Decay mode with final state $\pi^+\pi^0\bar{K}^0$

Finally, we consider a mode where each pair of particles in the final states can be directly related by one of the three $SU(2)$ symmetries, namely isospin, U-spin and V-spin. Here we do not need G_I, G_V or G_U to relate the final states. We consider as an example decays with final state $\pi^+\pi^0\bar{K}^0$ such as $D^+\to\pi^+\pi^0\bar{K}^0$ and the conjugate mode. In the final state considered here, isospin exchange implies $\pi^0\leftrightarrow\pi^+$, U-spin exchange implies $\pi^0\leftrightarrow\bar{K}^0$, and V-spin exchange implies $\pi^+\leftrightarrow\bar{K}^0$. The $SU(2)$ decompositions of all the pairs of particles under their respective $SU(2)$ symmetries have already been considered in Secs. II B, II C and II D. Once again, the steps elaborated on in Sec. II B are applicable to this case also. The amplitudes for this decay mode can be easily read off from Table V. However, in this mode the even and odd contributions to the decay amplitude can interfere as they are not eigenstates of G_V, resulting in even and odd numbered sextants to have distinctly different density of events as depicted in Eqs. (20) and (21). Note that the Dalitz plot distributions for the even (odd) numbered sextants of the Dalitz plot would still be identical if isospin and U-spin are exact symmetries. The breakdown of isospin, U-spin and V-spin could be quantitatively measured using the following asymmetries:

TABLE V. Amplitudes for the decay $D^+\to\pi^+\pi^0\bar{K}^0$. The V-spin amplitudes can be written in a similar manner. For brevity we have not written them explicitly.

$D^+\to\pi^+\pi^0\bar{K}^0$							
Isospin (initial state $\lvert\frac{1}{2},+\frac{1}{2}\rangle$)				U-spin (initial state $-\lvert\frac{1}{2},-\frac{1}{2}\rangle$)			
Transition	Final state	Symmetry	Amplitude	Transition	Final state	Symmetry	Amplitude
$\Delta I=1$	$\lvert\frac{3}{2},+\frac{3}{2}\rangle$	Mixed	$\frac{\sqrt{3}}{\sqrt{10}}T^{e}_{1,\frac{3}{2}}X+\frac{\sqrt{3}}{\sqrt{2}}T^{o}_{1,\frac{3}{2}}Y\sin\theta$	$\Delta U=1$	$\lvert\frac{3}{2},-\frac{3}{2}\rangle$	Mixed	$\frac{\sqrt{3}}{2\sqrt{5}}U^{e}_{1,\frac{3}{2}}X'-\frac{\sqrt{3}}{2}U^{o}_{1,\frac{3}{2}}Y'\sin\theta'$
					$\lvert\frac{3'}{2},-\frac{3}{2}\rangle$	Even	$\frac{3}{2}U'^{e}_{1,\frac{3'}{2}}X'$

$$\mathbb{A}_{\text{Isospin}} = \left|\frac{\Sigma_{IV}^{I} - \Sigma_{II}^{III}}{\Sigma_{IV}^{I} + \Sigma_{II}^{III}}\right| + \left|\frac{\Sigma_{II}^{III} - \Sigma_{VI}^{V}}{\Sigma_{II}^{III} + \Sigma_{VI}^{V}}\right| + \left|\frac{\Sigma_{VI}^{V} - \Sigma_{IV}^{I}}{\Sigma_{VI}^{V} + \Sigma_{IV}^{I}}\right| + \left|\frac{\Delta_{IV}^{I} - \Delta_{II}^{III}}{\Delta_{IV}^{I} + \Delta_{II}^{III}}\right| + \left|\frac{\Delta_{II}^{III} - \Delta_{VI}^{V}}{\Delta_{II}^{III} + \Delta_{VI}^{V}}\right| + \left|\frac{\Delta_{VI}^{V} - \Delta_{IV}^{I}}{\Delta_{VI}^{V} + \Delta_{IV}^{I}}\right|, \tag{57}$$

$$\mathbb{A}_{U\text{-spin}} = \left|\frac{\Sigma_{VI}^{I} - \Sigma_{IV}^{III}}{\Sigma_{VI}^{I} + \Sigma_{IV}^{III}}\right| + \left|\frac{\Sigma_{IV}^{III} - \Sigma_{II}^{V}}{\Sigma_{IV}^{III} + \Sigma_{II}^{V}}\right| + \left|\frac{\Sigma_{II}^{V} - \Sigma_{VI}^{I}}{\Sigma_{II}^{V} + \Sigma_{VI}^{I}}\right| + \left|\frac{\Delta_{VI}^{I} - \Delta_{IV}^{III}}{\Delta_{VI}^{I} + \Delta_{IV}^{III}}\right| + \left|\frac{\Delta_{IV}^{III} - \Delta_{II}^{V}}{\Delta_{IV}^{III} + \Delta_{II}^{V}}\right| + \left|\frac{\Delta_{II}^{V} - \Delta_{VI}^{I}}{\Delta_{II}^{V} + \Delta_{VI}^{I}}\right|, \tag{58}$$

$$\mathbb{A}_{V\text{-spin}} = \left|\frac{\Sigma_{IV}^{V} - \Sigma_{II}^{I}}{\Sigma_{IV}^{V} + \Sigma_{II}^{I}}\right| + \left|\frac{\Sigma_{II}^{I} - \Sigma_{VI}^{III}}{\Sigma_{II}^{I} + \Sigma_{VI}^{III}}\right| + \left|\frac{\Sigma_{VI}^{III} - \Sigma_{IV}^{V}}{\Sigma_{VI}^{III} + \Sigma_{IV}^{V}}\right| + \left|\frac{\Delta_{IV}^{V} - \Delta_{II}^{I}}{\Delta_{IV}^{V} + \Delta_{II}^{I}}\right| + \left|\frac{\Delta_{II}^{I} - \Delta_{VI}^{III}}{\Delta_{II}^{I} + \Delta_{VI}^{III}}\right| + \left|\frac{\Delta_{VI}^{III} - \Delta_{IV}^{V}}{\Delta_{VI}^{III} + \Delta_{IV}^{V}}\right|. \tag{59}$$

Once again these asymmetries being, in general, functions of r and θ (or θ' or θ''), it would be very interesting to look for their variation across the Dalitz plot. These would constitute the visible signatures of the breaking of $SU(3)$ flavor symmetry.

III. CONCLUSION

In this paper we have elucidated a new model independent method to look for the breaking of the $SU(3)$ flavor symmetry in many three-body decay modes, namely B^+ or $D_s^+ \to K^0\pi^0\pi^+$, B_d^0 or $\bar{B}_s^0 \to K^+\pi^0\pi^-$, B^+ or $D^+ \to K^+\pi^0\bar{K}^0$ and $D^+ \to \pi^+\pi^0\bar{K}^0$. The novelty in choosing these decay modes is that pairs of the final state do belong to at least two different $SU(2)$ triplets, and hence under the assumption of exact $SU(3)$ flavor symmetry, the amplitude for the process has two parts: one fully symmetric and another fully antisymmetric under the exchanges $s \leftrightarrow t \leftrightarrow u$. This gives rise to a characteristic pattern in the Dalitz plot distribution: the alternate sextants must have identical distribution of events. Any deviation from this behavior would constitute an evidence for the breaking of $SU(3)$ flavor symmetry, which indeed is broken in nature. We have provided mode specific Dalitz plot asymmetries which can be used to quantify the extent of $SU(3)$ symmetry breaking in each of the decay modes under our consideration. These asymmetries are defined in the full region of the Dalitz plot and can be measured both along resonances and in the nonresonant regions. A quantitative estimate of the variation of these asymmetries obtained experimentally would provide a valuable understanding of $SU(3)$ breaking effects. It would also be interesting to find regions of the Dalitz plots where $SU(3)$ is a good symmetry. A better understanding and measured estimate of $SU(3)$ breaking would help in reliably estimating hadronic uncertainties and hence result in effectively using it to measure weak phases and search for new physics effects beyond the standard model.

ACKNOWLEDGMENTS

N. G. D. thanks The Institute of Mathematical Sciences, Chennai, for hospitality, where part of the work was done.

APPENDIX: G-PARITY AND FINAL STATES

The G-parity operator G_I (or G_U or G_V) is defined as a rotation through π radian (180°) around the second axis of isospin (or U-spin or V-spin) space, followed by charge conjugation ($\mathcal{C}$), $G_I = \mathcal{C}e^{i\pi I_2} = \mathcal{C}e^{i\pi\tau_2/2}$, where I_2 is the second generator of $SU(2)$ isospin (or U-spin or V-spin) group and τ_2 is the second Pauli matrix. G-parity as defined here transforms the various $SU(2)$ multiplets as follows:

$$G_I\begin{pmatrix}\pi^+\\ \pi^0\\ \pi^-\end{pmatrix} = -\begin{pmatrix}\pi^+\\ \pi^0\\ \pi^-\end{pmatrix}, \qquad G_I\begin{pmatrix}K^+\\ K^0\end{pmatrix} = \begin{pmatrix}\bar{K}^0\\ -K^-\end{pmatrix}, \qquad G_I\begin{pmatrix}\bar{K}^0\\ -K^-\end{pmatrix} = -\begin{pmatrix}K^+\\ K^0\end{pmatrix},$$

$$G_U\begin{pmatrix}K^0\\ \pi^0\\ \bar{K}^0\end{pmatrix} = -\begin{pmatrix}K^0\\ \pi^0\\ \bar{K}^0\end{pmatrix}, \qquad G_U\begin{pmatrix}K^+\\ \pi^+\end{pmatrix} = \begin{pmatrix}\pi^-\\ -K^-\end{pmatrix}, \qquad G_U\begin{pmatrix}\pi^-\\ -K^-\end{pmatrix} = -\begin{pmatrix}K^+\\ \pi^+\end{pmatrix},$$

$$G_V\begin{pmatrix}K^+\\ \pi^0\\ K^-\end{pmatrix} = -\begin{pmatrix}K^+\\ \pi^0\\ K^-\end{pmatrix}, \qquad G_V\begin{pmatrix}\pi^+\\ \bar{K}^0\end{pmatrix} = \begin{pmatrix}K^0\\ -\pi^-\end{pmatrix}, \qquad G_V\begin{pmatrix}K^0\\ -\pi^-\end{pmatrix} = -\begin{pmatrix}\pi^+\\ \bar{K}^0\end{pmatrix}.$$

[1] M. Gell-Mann, California Institute of Technology Synchrotron Laboratory Report No. CTSL–20, 1961 (unpublished).
[2] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[3] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).
[4] S. Okubo, Prog. Theor. Phys. **27**, 949 (1962).
[5] S. Okubo and C. Ryan, Nuovo Cimento **34**, 776 (1964).
[6] R. L. Kingsley, S. B. Treiman, F. Wilczek, and A. Zee, Phys. Rev. D **11**, 1919 (1975).
[7] M. B. Voloshin, V. I. Zakharov, and L. B. Okun, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 403 (1975) [JETP Lett.**21**, 183 (1975)].
[8] L. L. Wang and F. Wilczek, Phys. Rev. Lett. **43**, 816 (1979).
[9] C. Quigg, Z. Phys. C **4**, 55 (1980).
[10] D. Zeppenfeld, Z. Phys. C **8**, 77 (1981).
[11] L. L. Chau and H. Y. Cheng, Phys. Rev. Lett. **56**, 1655 (1986).
[12] L. L. Chau and H. Y. Cheng, Phys. Rev. D **36**, 137 (1987).
[13] M. J. Savage and M. B. Wise, Phys. Rev. D **39**, 3346 (1989); **40**, 3127(E) (1989).
[14] L. L. Chau, H. Y. Cheng, W. K. Sze, H. Yao, and B. Tseng, Phys. Rev. D **43**, 2176 (1991); **58**, 019902(E) (1998).
[15] M. J. Savage, Phys. Lett. B **257**, 414 (1991).
[16] H. J. Lipkin, Y. Nir, H. R. Quinn, and A. Snyder, Phys. Rev. D **44**, 1454 (1991).
[17] L. L. Chau and H. Y. Cheng, Phys. Lett. B **280**, 281 (1992).
[18] W. Kwong and S. P. Rosen, Phys. Lett. B **298**, 413 (1993).
[19] I. Hinchliffe and T. A. Kaeding, Phys. Rev. D **54**, 914 (1996).
[20] M. Gronau, O. F. Hernandez, D. London, and J. L. Rosner, Phys. Rev. D **52**, 6356 (1995).
[21] S. Oh, Phys. Rev. D **60**, 034006 (1999).
[22] J. L. Rosner, Phys. Rev. D **60**, 114026 (1999).
[23] M. Gronau and J. L. Rosner, Phys. Lett. B **500**, 247 (2001).
[24] N. G. Deshpande, X. G. He, and J. Q. Shi, Phys. Rev. D **62**, 034018 (2000).
[25] Y. L. Wu and Y. F. Zhou, Eur. Phys. J. C **32**, s179 (2004).
[26] A. Khodjamirian, T. Mannel, and M. Melcher, Phys. Rev. D **68**, 114007 (2003).
[27] C. W. Chiang, M. Gronau, Z. Luo, J. L. Rosner, and D. A. Suprun, Phys. Rev. D **69**, 034001 (2004).
[28] M. Zhong, Y. L. Wu, and W. Y. Wang, Eur. Phys. J. C **32**, s191 (2004).
[29] C. W. Chiang, M. Gronau, J. L. Rosner, and D. A. Suprun, Phys. Rev. D **70**, 034020 (2004).
[30] Y. L. Wu, M. Zhong, and Y. F. Zhou, Eur. Phys. J. C **42**, 391 (2005).
[31] Y. L. Wu and Y. F. Zhou, Phys. Rev. D **72**, 034037 (2005).
[32] M. Gronau and J. L. Rosner, Phys. Rev. D **72**, 094031 (2005).
[33] C. W. Chiang and Y. F. Zhou, J. High Energy Phys. 12 (2006) 027.
[34] A. Soni and D. A. Suprun, Phys. Rev. D **75**, 054006 (2007).
[35] C. W. Chiang and E. Senaha, Phys. Rev. D **75**, 074021 (2007).
[36] C. W. Chiang and Y. F. Zhou, J. Phys. Conf. Ser. **110**, 052056 (2008).
[37] B. Bhattacharya and J. L. Rosner, Phys. Rev. D **77**, 114020 (2008).
[38] C. W. Chiang and Y. F. Zhou, J. High Energy Phys. 03 (2009) 055.
[39] D. H. Wei, J. Phys. G **36**, 115006 (2009).
[40] M. Jung and T. Mannel, Phys. Rev. D **80**, 116002 (2009).
[41] M. Imbeault and D. London, Phys. Rev. D **84**, 056002 (2011).
[42] H. Y. Cheng and S. Oh, J. High Energy Phys. 09 (2011) 024.
[43] D. Pirtskhalava and P. Uttayarat, Phys. Lett. B **712**, 81 (2012).
[44] H. Y. Cheng and C. W. Chiang, Phys. Rev. D **86**, 014014 (2012).
[45] T. N. Pham, Phys. Rev. D **87**, 016002 (2013).
[46] G. Hiller, M. Jung, and S. Schacht, Phys. Rev. D **87**, 014024 (2013).
[47] B. Bhattacharya, M. Gronau, M. Imbeault, D. London, and J. L. Rosner, Phys. Rev. D **89**, 074043 (2014).
[48] H. Y. Cheng, C. W. Chiang, and A. L. Kuo, Phys. Rev. D **91**, 014011 (2015).
[49] T. N. Pham, arXiv:1409.6160.
[50] X. G. He, G. N. Li, and D. Xu, Phys. Rev. D **91**, 014029 (2015).
[51] T. Ledwig, J. Martin Camalich, L. S. Geng, and M. J. Vicente Vacas, Phys. Rev. D **90**, 054502 (2014).
[52] M. Gronau, Phys. Rev. D **91**, 076007 (2015).
[53] C. S. Kim, D. London, and T. Yoshikawa, Phys. Rev. D **57**, 4010 (1998).
[54] M. Gronau and D. Pirjol, Phys. Lett. B **449**, 321 (1999).
[55] A. Datta and D. London, Phys. Lett. B **584**, 81 (2004).
[56] M. Gronau and J. Zupan, Phys. Rev. D **71**, 074017 (2005).
[57] M. Gronau, Nucl. Phys. B, Proc. Suppl. **163**, 16 (2007).
[58] M. Gronau and J. L. Rosner, Phys. Lett. B **651**, 166 (2007).
[59] L. Calibbi, J. Jones-Perez, A. Masiero, J. h. Park, W. Porod, and O. Vives, *Proc. Sci.*, EPS-HEP2009 (2009) 167.
[60] N. Rey-Le Lorier and D. London, Phys. Rev. D **85**, 016010 (2012).
[61] X. G. He, S. F. Li, and H. H. Lin, J. High Energy Phys. 08 (2013) 065.
[62] B. Bhattacharya, M. Imbeault, and D. London, Phys. Lett. B **728**, 206 (2014).
[63] Y. Grossman, Z. Ligeti, and D. J. Robinson, J. High Energy Phys. 01 (2014) 066.
[64] C. S. Fong and E. Nardi, Phys. Rev. D **89**, 036008 (2014).
[65] R. Sinha, N. G. Deshpande, S. Pakvasa, and C. Sharma, Phys. Rev. Lett. **107**, 271801 (2011).
[66] D. Sahoo, R. Sinha, N. G. Deshpande, and S. Pakvasa, Phys. Rev. D **89**, 071903 (2014).
[67] D. Sahoo, R. Sinha, and N. G. Deshpande, Phys. Rev. D **91**, 051901 (2015).