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Son-Yamamoto relation and holographic renormalization group flows

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Motivated by the Son-Yamamoto (SY) relation, which connects the three-point and two-point correlators, we consider the holographic renormalization group (RG) flows in the bottom-up approach to holographic QCD via the Hamilton-Jacobi equation with respect to the radial coordinate. It is shown that the SY relation is diagonal with respect to the RG flow in the 5D Yang-Mills-Chern-Simons model, while the RG equation acquires an inhomogeneous term in the model with an additional scalar field, which encodes the chiral condensate.

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I. INTRODUCTION

The derivation of the renormalization group (RG) flows in effective field theories is a subtle issue; in particular, in the chiral perturbation theory, only one-loop calculations are well justified. Moreover, there is no simple way to incorporate the nonperturbative effects into RG dynamics. Holography provides a new tool to consider this problem; one could investigate the dependence on the radial coordinate, which is related to a RG scale. It was argued in [1] (see [2] for a review) that the RG equation can be identified with the Hamilton-Jacobi (HJ) equation when the radial anti–de Sitter (AdS)–like holographic coordinate is treated as the time variable. This identification is consistent with the standard holographic recipe, where the classical action in the bulk theory serves as the generating functional for the correlators in the boundary theory. The HJ equations in the bulk are supplemented by the Hamiltonian constraints—Gauss law for the bulk gauge theory and the Arnowitt-Deser-Misner constraints for the bulk gravity. The recent discussion on the relation between the holographic RG and the conventional Wilsonian RG can be found in [3].

It is important, generally, to derive the properties of the different objects under a renormalization. The simplest objects to study are the β function and the anomalous dimensions of the local operators. Some of them are not renormalized due to the conservation laws behind them. A more general situation concerns the RG properties of the correlators of the different local operators. If some operator corresponds to the special symmetry, e.g., the trace of the energy stress tensor that corresponds to the dilatation, one can derive low-energy theorems as in [4]. These low-energy theorems are the simplest examples of when the nonperturbative effects can be accounted for in the RG dynamics. It was shown that QCD low-energy theorems are fulfilled in the holographic approach as well [5].

A separate question concerns the mixing of the operators and correlators under the RG flows. This mixing can be quite complicated, and the matrix of anomalous dimensions of the local operators can have a huge dimension. Sometimes, say in $N = 4$ super Yang-Mills, the diagonalization of the matrix of the one-loop anomalous dimensions turns out to be equivalent to the evaluation of the spectrum of some integrable system that follows from the special symmetries of the dilatation operator. When we consider the RG properties of the multipoint correlators, the situation is more involved. Roughly speaking, one can use the operator product expansion of the local operator first, then consider the RG behavior of the emerging sum of the local operators and coefficient functions and, finally, try to collect them back into the form of the initial correlators. There is no guarantee that it will acquire the form of the initial correlator, because the anomalous dimensions of the local operators generally do not know about each other.

In this paper we consider the behavior of the simplest correlators under the nonperturbative RG motivated by the Son-Yamamoto (SY) relation, derived in the holographic setting in the 5D Yang-Mills-Chern-Simons (YM-CS) model [6]. This relation connects three-point and two-point correlators and, crucially, involves the axial anomaly. If true, this relation would provide the highly nontrivial anomaly-matching conditions for the resonances. It is puzzling how this relation could be obtained purely in the field theory framework without appealing to holography. Also, the potential Ward identity behind it has not been identified yet. Moreover, even its status is quite controversial [7]. It reproduces the Vainshtein relation for the magnetic susceptibility of the chiral condensate [8] and is fulfilled in the chiral perturbation theory for two flavors in the leading chiral logs [9]; however, there are explicit

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examples when it does not work [6,10]. Since its derivation in [6] is a bit tricky, it is important to recognize its role in a more general setting. It is interesting to understand the principle behind this relation when it works, and what is wrong with it in the models where it is not true.

To this end, we shall focus at the RG behavior of the SY relation in the framework of the holographic HJ equation. Because of the 5D Chern-Simons (CS) term, the canonical momentum of the gauge field in the Hamiltonian picture is modified, and we shall take the anomalous shift into account. We shall consider the HJ equation supplemented by the Gauss law constraint in the simplest holographic models for QCD, which can be thought of as a generalization of the chiral Lagrangian when the whole tower of mesons is taken into account. We shall demonstrate that in the simplest model with the gauge fields in the bulk, the SY relation is diagonal with respect to nonperturbative RG flow generated by the HJ equation. However, when condensate is taken into account, the inhomogeneous term arises in the RG equation, which means that the SY relation cannot be true at all scales. We shall also find the two-point correlator diagonal under the nonperturbative RG flow.

The paper is organized as follows. First, we briefly discuss the models under consideration and the HJ approach to nonperturbative RG evolution. Then we will demonstrate, by explicit calculation, that the SY relation is diagonal under RG flow in the 5D YM-CS model, but gets mixed with other correlators in the model with an additional scalar field. Some directions for the future research are summarized in the Conclusion.

II. BOTTOM-UP MODELS OF HOLOGRAPHIC QCD

In this paper, following [6], we will consider two holographic QCD models. The first model [11] deals with the bulk Yang-Mills-Chern-Simons theory, where chiral symmetry breaking is incorporated by a boundary condition on the additional scalar field. The model involves vector fields $A_L = A_L^a t^a$, $A_R = A_R^a t^a$, where t^a are generators of the algebra $u(N_f)$, which are dual to left and right quark currents $j_L = j_V - j_A$, $j_R = j_V + j_A$, and scalar field X, whose boundary value is related to chiral condensate $\langle \bar{q} q \rangle$. The action reads as

$$S_{\mathrm{YMX}} = \int d^5x \sqrt{g} \mathrm{Tr}\left(|D_m X|^2 + 3|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2) \right), \tag{1}$$

where $D_m X = \partial_m X + i(A_{Rm} - A_{Lm})X$ and $F_{L,Rmn} = \partial_m A_{L,Rn} - \partial_n A_{L,Rm} - i[A_{L,Rm}, A_{L,Rn}]$. In order to reproduce the chiral anomaly we add the Chern-Simons term,

$$S = S_{\mathrm{YM}}(A_L, A_R) + S_{\mathrm{CS}}(A_L) - S_{\mathrm{CS}}(A_R), \tag{2}$$

with

$$S_{\mathrm{CS}}(A) = \kappa \mathrm{Tr}\left(AF^2 - \frac{i}{2} A^3 F - \frac{1}{10} A^5 \right), \quad \kappa = -\frac{N_c}{24\pi^2}. \tag{3}$$

The expectation value of the scalar field is fixed in the chiral limit by the solution to the classical equation of motion

$$X(z) = \frac{\sigma z^3}{2}, \tag{4}$$

where σ is proportional to the chiral condensate. We assume the AdS5 metric in the bulk theory,

$$ds^2 = \frac{1}{z^2}(-dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu). \tag{5}$$

In our notations roman letters denote five-dimensional coordinates, which we raise and lower using the AdS5 metric, whereas greek letters are used for four-dimensional objects, which we manipulate using the Minkowski metric $\eta_{\mu\nu}$. The physical 4D world is located at the "UV" boundary of AdS5 space, $z = \epsilon \to 0$. The theory also needs an "IR" boundary, located at $z = z_m \approx 1/\Lambda_{\mathrm{QCD}}$. Below we will consider the following IR boundary conditions [11]:

$$\partial_z A_{A\mu} = \partial_z A_{V\mu} = 0. \tag{6}$$

In the second model there is no scalar field, and the chiral symmetry breaking occurs due to different boundary conditions for A_L and A_R. The IR brane is located at $z = 0$ and UV brane is located at $z = z_0$. The action reads as [6]

$$S = \frac{1}{2} \int d^4x \int_{z_0} dz \left\{ f^2(z) \mathrm{Tr}(F_{Lz\mu}^2 + F_{Rz\mu}^2) - \frac{1}{2g^2(z)} \mathrm{Tr}(F_{L\mu\nu}^2 + F_{R\mu\nu}^2) \right\} + S_{\mathrm{CS}}. \tag{7}$$

Following Son and Yamamoto we assume $f(z)$ and $g(z)$ to satisfy the following conditions: $f(-z) = f(z)$ and $g(-z) = g(z)$. It is more convenient to work with vector and axial gauge connections,

$$A_{L\mu} = V_\mu + A_\mu, \qquad A_{R\mu} = V_\mu - A_\mu, \tag{8}$$

which obey the Neumann and the Dirichlet boundary conditions, respectively,

$$\partial_z V_\mu(z = 0) = 0, \qquad A_\mu(z = 0) = 0. \tag{9}$$

This model suffers from the problem that three-point correlation $\langle VVA \rangle$ does not vanish when one of the momenta of vector fields tends to zero [12]. We can add a surface term to the action to resolve this problem, which leads us to the expression for the CS term [12] (in the gauge $A_z = V_z = 0$)

$$S_{\mathrm{CS}} = \int d^4xdz\mathrm{Tr}\left(4\kappa\epsilon^{z\alpha\beta\gamma\lambda\eta}\left(3A_\alpha\frac{F_{V\beta\gamma}}{2}F_{Vz\lambda} + A_\alpha\frac{F_{A\beta\gamma}}{2}F_{Az\lambda}\right)\right). \tag{10}$$

III. HAMILTON-JACOBI EQUATION

The standard way to evaluate correlation functions using holography is to solve equations of motion in the five-dimensional bulk and vary the on-shell action with respect to boundary conditions. However, from classical mechanics and field theory we know that there is an alternative approach, namely the Hamilton-Jacobi equation, which sometimes works more effectively.

Suppose we have a 5D holographic model. This means that we deal with a five-dimensional geometry and five-dimensional bulk action S_{5D}. Let us denote the fifth coordinate by z. The physical 4D world with coordinates x^μ lies at the UV boundary, whose fifth coordinate we will denote by ϵ. The value of ϵ can be thought of as a UV cutoff. If we are interested in 4D correlators of fields j_α (where α just enumerates fields), then, according to the holographic picture, we have to insert corresponding sources O_α in the 4D action,

$$S_{4D}[O] = \int d^4x\left(\mathcal{L}_{4D} + \sum_\alpha j_\alpha(x)O_\alpha(x)\right). \tag{11}$$

Then we have to promote $O_\alpha(x)$ to 5D dynamical fields $O_\alpha(z,x)$, and the five-dimensional bulk action S_{5D} mentioned above is the action for these fields. Below we will keep track of the fields' arguments: $O_\alpha(x)$ means boundary value, whereas $O_\alpha(z,x)$ is 5D dynamical field such that $O_\alpha(z=\epsilon,x) = O_\alpha(x)$. In the proper limit we have to solve classical equations of motion in the bulk with the fixed values of $O_\alpha(z,x)$ at the physical UV boundary. Then, the 4D quantum generating function equals to

$$Z^{\mathrm{quantum}}_{4D}[O_\alpha(x)] = \exp(iS^{\text{on-shell}}_{5D}[\epsilon, O_\alpha(x)]). \tag{12}$$

Note that the on-shell action depends only on the boundary value $O_\alpha(z=\epsilon,x) = O_\alpha(x)$ *and* the value of ϵ. In this approach, $O_\alpha(x)$ plays the role of a classical background chemical potential for $j_\alpha(x)$. In 5D classical field theory one can switch to the Hamiltonian description in which we will trade ϵ to be the "time." We introduce canonical momenta

$$\pi_\alpha(z,x) = \frac{\partial\mathcal{L}_{5D}}{\partial(\partial_\epsilon O_\alpha(z,x))}. \tag{13}$$

On-shell value is given by

$$\pi^{\text{on-shell}}_\alpha(z,x) = \frac{\delta S^{\text{on-shell}}_{5D}}{\delta O_\alpha(x)}. \tag{14}$$

The Hamiltonian is given by the Legendre transform,

$$H(\pi_\alpha(z,x), O_\alpha(z,x), \epsilon) = \int d^4x\sum_\alpha \pi_\alpha(z,x)\partial_\epsilon O_\alpha(z,x) - \mathcal{L}_{5D}, \tag{15}$$

and the general form of the Hamilton-Jacobi equation reads as

$$\frac{\partial S^{\text{on-shell}}_{5D}}{\partial\epsilon} + H\left(\frac{\delta S^{\text{on-shell}}_{5D}}{\delta O_\alpha(x)}, O_\alpha(x), \epsilon\right) = 0. \tag{16}$$

The advantage of the HJ equation is that we can obtain a hierarchy of equations for correlators if we vary the HJ equations with respect to O_α, since $\langle j_\alpha(x)\rangle^{\mathrm{quantum}}_{4D} = \frac{\delta S^{\text{on-shell}}_{5D}}{\delta O_\alpha(x)}$.

It is instructive to obtain two-point functions for models from the previous section using this method. For simplicity, let us calculate $\langle VV\rangle$ for the first model. Neglecting the axial part, we arrive at the Hamiltonian (we drop indices 5D and on-shell for S)

$$H = -\frac{1}{2}g_5^2 z\int d^4x\left(\frac{\delta S}{\delta A_{V\mu}(z,x)}\right) + \frac{1}{4g_5^2 z}\mathrm{Tr}\int d^4xF_{\mu\nu}(z,x)^2. \tag{17}$$

Hence, the HJ equation reads as

$$\frac{\partial S}{\partial\epsilon} - \frac{1}{2}g_5^2\epsilon\int d^4x\left(\frac{\delta S}{\delta A_{V\mu}(\epsilon,x)}\right) + \frac{1}{g_5^2\epsilon}\mathrm{Tr}\int d^4xF_{\mu\nu}(\epsilon,x)^2 = 0. \tag{18}$$

If we assume that the correlation function is not too singular at the limit $\epsilon\to 0$, so that $\epsilon\langle j_V j_V\rangle \to 0$, then, after varying twice with respect to the V_A, we have near the UV boundary

$$\frac{\partial\langle j^a_{V\mu}(-p)j^b_{V\nu}(p)\rangle}{\partial\epsilon} = -\frac{1}{2g_5^2\epsilon}(p^2\eta_{\mu\nu} - p_\mu p_\nu)\delta^{ab}, \tag{19}$$

therefore,

$$\langle j^a_{V\mu}(-p)j^b_{V\nu}(p)\rangle = -\frac{1}{2g_5^2}\log(p\epsilon)(p^2\eta_{\mu\nu} - p_\mu p_\nu)\delta^{ab}, \tag{20}$$

which exactly coincides with the result found in [11].

Now let us discuss boundary conditions for the Hamilton-Jacobi equation. Because we deal with the first-order differential equation, to specify boundary conditions we need to know the values of correlators at the particular z. To this end, consider the bulk action

$$S = \int_{z_{ir}}^{z_{uv}} dz d^4 x \mathcal{L}. \tag{21}$$

If we take the limit $z_{uv} \to z_{ir}$, then naively we have

$$S \approx (z_{ir} - z_{uv}) \int d^4 x \mathcal{L}(z = z_{uv}), \tag{22}$$

and taking variations with respect to the boundary UV values is exceptionally simple. So far everything was applicable for both models, so we did not specify z_{ir}, z_{uv} and $\mathcal{L}$. However, we should be careful with terms like $\partial_z A_{V\mu}$.

In the first model we have Neumann boundary conditions (6); therefore, in the leading order in $z_{ir} - z_{uv} = z_m - \epsilon$ we can neglect $\partial_z A_{V\mu}$ and $\partial_z A_{A\mu}$. So, we are left with

$$S = (z_m - \epsilon)\mathrm{Tr} \int d^4 x \bigg\{ -\frac{1}{4g_5^2 \epsilon} (F^2_{L\mu\nu}(z_{uv}, x) + F^2_{R\mu\nu}(z_{uv}, x)) + \frac{3}{\epsilon^3} A^2_{A\mu}(z_{uv}, x) |X|^2 (z_{uv}, x) \bigg\}. \tag{23}$$

In the second model we consider the limit $z_0 \to 0$. We have different boundary conditions for A_μ and V_μ [see Eq. (9)]. Again we can neglect $\partial_z V_\mu$. However, we can no longer neglect $\partial_z A_\mu$: at the UV boundary the value of A_μ can be arbitrary but at the IR boundary it must be zero, so we have a very sharp transition $\partial_z A_\mu = \frac{1}{z_0} A_\mu + O(1)$. Now it is straightforward to write down the leading terms in the Lagrangian,

$$S = z_0 \int d^4 x \mathrm{Tr} \left(\frac{f^2(z_0)}{z_0^2} A^2_\mu(z_0) - \frac{1}{2g^2(z_0)} F^2_{V\mu\nu}(z_0) \right), \tag{24}$$

where the additional non-Abelian terms from the CS term are omitted.

IV. SON-YAMAMOTO RELATION

The Son-Yamamoto relation [6] connects three-point functions and two-point functions in the models described above. Let us introduce the usual notations as follows:

$$\begin{aligned}
\langle V^{a\perp}_\mu(q) V^{b\perp}_\nu(-q) \rangle &= \delta^{ab} \Pi^\perp_{\mu\nu}(q) q^2 \Pi_V(q) \\
\langle A^{a\perp}_\mu(q) A^{b\perp}_\nu(-q) \rangle &= \delta^{ab} \Pi^\perp_{\mu\nu}(q) q^2 \Pi_A(q) \\
\langle V^a_\mu(k) V^{\perp b}_\nu(-k-Q) A^{\perp c}_\alpha(Q) \rangle &= \frac{Q^2}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} k^\beta w_T(Q) \\
&\quad \times \mathrm{Tr}(t^a t^b t^c), \\
k &\to 0 \\
\Pi^\perp_{\mu\nu}(q) &= \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \\
\Pi^\parallel_{\mu\nu}(q) &= \frac{q_\mu q_\nu}{q^2}.
\end{aligned} \tag{25}$$

Slightly abusing notation, we use the same letter for quark currents and their holographic duals. It is easy to distinguish them: if V_μ or A_μ are located within a correlator $\langle \ldots \rangle$ then they should be understood as a quark current, for example, $\langle V_\mu A_\nu \rangle$ means $\langle j_{V\mu} j_{A\nu} \rangle$. Otherwise, the letters V_μ or A_μ refer to holographic duals of corresponding currents.

We start from the 5D Yang-Mills action for the second model,

$$S = \int d^4 x dz \mathrm{Tr} \bigg(f^2(z)((\partial_z A_\mu)^2 + (\partial_z V_\mu)^2) + \frac{1}{2g^2(z)} (F^2_{A\mu\nu} + F^2_{V\mu\nu}) + 12\kappa \epsilon^{z\alpha\beta\gamma\lambda\eta} A_\alpha \frac{F_{V\beta\gamma}}{2} F_{Vz\lambda} \bigg), \tag{26}$$

and omit the term AF_AF_A, which makes no contribution to the three-point correlator $\langle VVA \rangle$. Introducing the ansatz for the bulk fields (superscript 0 indicates boundary value),

$$\begin{aligned}
V_\mu(z,q) &= V^{0\perp}_\mu(q) V(z,q) + V^{0\parallel}_\mu(q) \psi_V(z,q) \\
A_\mu(z,q) &= A^{0\perp}_\mu(q) A(z,q) + A^{0\parallel}_\mu(q) \psi_A(z,q),
\end{aligned} \tag{27}$$

we recover the results found in [6],

$$\begin{aligned}
\Pi_V &= \frac{2}{q^2} f(z_0)^2 V'(z_0, q) \\
w_T &= \frac{48\kappa}{q^2} \int_0^{z_0} A(z,q) V'(z,q) dz.
\end{aligned} \tag{28}$$

Note that although we use greek letters for four-dimensional indices, objects $V_\mu(z,q)$ are five-dimensional, because we have adopted the gauge $V_0(z,q) = 0$. On the other hand, objects with superscript 0, such as $V^0_\mu(q)$, are boundary values of corresponding fields and, therefore, are four-dimensional objects. In what follows we will keep arguments of our fields explicit, in order to prevent any confusion.

Our aim is now to reproduce the Son-Yamamoto relation for the transversal part of a triangle anomalies, taking variational derivatives of the Hamilton-Jacobi equation. The Hamiltonian is given by

$$H = \mathrm{Tr}\int d^4x\partial_z A_\mu(z,x)\pi^\mu_A(z,x) + \partial_z V_\mu(z,x)\pi^\mu_V(z,x) - \mathcal{L}, \tag{29}$$

where $\pi_{A\mu}(z,x) = \frac{\partial L}{\partial(\partial_z A_\mu(z,x))}$ and $\pi_{V\mu}$ is defined in the similar fashion. Let us rewrite the last expression for H in terms of $A_\mu(z,x)$, $V_\mu(z,x)$, $\pi_{A\mu}(z,x)$, and $\pi_{V\mu}(z,x)$. First, we need to obtain an exact expression for canonical momenta,

$$\pi^\mu_A(z,x) = \frac{\partial L}{\partial(\partial_z A_\mu(z,x))} = 2f^2(z)\partial^z A^\mu(z,x), \tag{30}$$

$$\begin{aligned}\pi^\mu_V(z,x) &= \frac{\partial L}{\partial(\partial_z V_\mu(z,x))} \\ &= 2f^2(z)\partial^z V^\mu(z,x) \\ &\quad + 6\kappa\epsilon^{\alpha\beta\gamma z\mu}A_\alpha(z,x)F_{V\beta\gamma}(z,x),\end{aligned} \tag{31}$$

which yields the expressions for $\partial_z V_\mu$ and $\partial_z A_\mu$,

$$\partial_z A_\mu(z,x) = \frac{\pi_{A\mu}(z,x)}{2f^2(z)}, \tag{32}$$

$$\partial_z V^a_\mu(z,x) = \frac{\pi^a_{V\mu}(z,x)}{2f^2(z)} - 3\frac{\kappa}{f^2(z)}\epsilon^{z\alpha\beta\gamma\mu}A^b_\alpha(z,x)F^c_{V\beta\gamma}(z,x)\mathrm{Tr}(t^a t^b t^c). \tag{33}$$

We are now able to write down the resulting Hamilton-Jacobi equation,

$$\begin{aligned}&\frac{\partial S}{\partial z_0} + \mathrm{Tr}\int d^4x\bigg\{\frac{1}{2f(z)^2}\pi^\mu_A\pi_{A\mu} + \frac{1}{2f(z)^2}\pi^\mu_V(\pi_{V\mu} - 6\kappa\epsilon^{\alpha\beta\gamma}_{z\mu}A_\alpha F_{V\beta\gamma}) - f^2(z)((\partial_z A_\mu)^2 + (\partial_z V_\mu)^2) \\ &\quad + \frac{1}{2g^2(z)}(F^2_{A\mu\nu} + F^2_{V\mu\nu}) + 6\kappa\epsilon^{z\alpha\beta\gamma\lambda}A_\alpha F_{V\beta\gamma}F_{Vz\lambda}\bigg\} = 0,\end{aligned} \tag{34}$$

which can be presented in the following form:

$$\begin{aligned}&\frac{\partial S}{\partial z_0} + \int d^4x\bigg\{\frac{1}{4f(z)^2}\pi^2_{A\mu} + \frac{1}{4f(z)^2}(\pi_{V\mu} - \phi_{V\mu})^2 - \frac{1}{2g(z)^2}(F^2_{A\mu\nu} + F^2_{V\mu\nu})\bigg\} = 0 \\ &\phi_{V\mu} = 6\kappa\epsilon^{z\alpha\beta\gamma\mu}A_\alpha F_{V\beta\gamma}.\end{aligned} \tag{35}$$

We omitted $u(N_f)$ indices for brevity; the notation should be self-evident.

We turn now to the discussion of the RG properties of the multipoint correlators. First, let us derive the two-point correlator diagonal with respect to the RG flow. Taking the second variational derivative of the HJ equation after the simple algebra, we obtain

$$\frac{\partial}{\partial z_0}(\Pi_A - \Pi_V) = -\frac{q^2}{2f^2}(\Pi^2_A - \Pi^2_V). \tag{36}$$

We see that the difference between the axial and vector correlators is diagonal, and the sum of these correlators defines the q^2-dependent "anomalous dimension."

We consider now the three-point functions involving one axial and two vector currents. Taking the Fourier transform of the HJ equation and applying the variation $\frac{\delta^3}{\delta V^0_\eta(k)\delta V^0_\chi(-q-k)\delta A^0_\lambda(q)}$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial z_0}\frac{\delta^3 S}{\delta V^0_\eta(k)\delta V^0_\chi(-q-k)\delta A^0_\lambda(q)} + \frac{1}{2f^2}\int d^4p\bigg\{\langle A_\lambda A_\mu\rangle(q,p)\langle V_\eta V_\chi A_\mu\rangle(k,-q-k,-p) \\ &\quad + \langle V_\eta A_\lambda V_\mu\rangle(k,q,p)\langle V_\chi V_\mu\rangle(-q-k,-p) - \frac{\delta^2\phi_{V\mu}(z_0,p)}{\delta V_\eta(z_0,k)\delta A_\lambda(z_0,q)}(k,q,p)\langle V_\chi V_\mu\rangle(-q-k,-p)\bigg\} = 0.\end{aligned} \tag{37}$$

Note that we set boundary values $A^0_\mu(x)$ and $V^0_\mu(x)$ to zero, because we consider the situation without background fields.

Let us write down every term in some detail, as follows:

$$\langle A_\lambda A_\mu\rangle(q,p) = \delta(p+q)q^2\Pi^\perp_{\lambda\mu}\Pi_A,$$
$$\langle V_\chi V_\mu\rangle(-q-k,-p) = \delta(k+q+p)p^2\Pi^\perp_{\chi\mu}\Pi_V, \quad (38)$$

$$\langle V_\eta V_\chi A_\mu\rangle(k,-q-k,-p) = \frac{p^2}{4\pi^2}\Pi^{\alpha\perp}_\mu(\Pi^{\beta\perp}_\chi w_T + \Pi^{\beta\|}_\chi w_L)\epsilon_{\alpha\beta\gamma\eta}k^\gamma\delta(p+q), \quad (39)$$

$$\langle V_\eta A_\lambda V_\mu\rangle(k,q,p) = \frac{p^2}{4\pi^2}\Pi^{\alpha\perp}_\lambda(\Pi^{\beta\perp}_\mu w_T + \Pi^{\beta\|}_\mu w_L)\epsilon_{\alpha\beta\gamma\eta}k^\gamma\delta(p+q+k). \quad (40)$$

For the Fourier transform of ϕ_V we get

$$\phi_V(z_0,p) = 6\kappa\epsilon^{z\alpha\beta\gamma\mu}\int dq' A_\alpha(z_0,q')F_{V\beta\gamma}(z_0,-q'-p), \quad (41)$$

$$\frac{\delta^2\phi_{V\mu}}{\delta V_\eta(z_0,k)\delta A_\lambda(z_0,q)}(z_0,k,q,p) = 12\kappa\epsilon^{z\alpha\beta\gamma\mu}k_\beta\delta(k+p+q). \quad (42)$$

We now contract the Hamilton-Jacobi equation with q^η to get rid of the longitudinal part of the three-point correlator, and arrive at the equation for w_T,

$$q^\eta\frac{\partial}{\partial z_0}\langle V_\eta V_\chi A_\lambda\rangle(k,-q-k,q) = \frac{1}{2f^2(z_0)}\left[\frac{q^4}{4\pi^4}(\Pi_A+\Pi_V)w_T\epsilon_{\lambda\chi\beta\eta}k^\beta q^\eta + 12\kappa q^2\Pi_V\epsilon^{\lambda\chi\beta\eta}k_\beta q^\eta\right]. \quad (43)$$

This expression can be presented in the following form, suitable for the discussion of the SY relation:

$$\frac{q^2}{4\pi^2}\frac{\partial w_T}{\partial z_0} = \frac{1}{2f^2(z_0)}\left[\frac{q^4}{4\pi^2}(\Pi_A+\Pi_V)w_T + 6\kappa q^2(\Pi_A+\Pi_V) + 6\kappa q^2(\Pi_V-\Pi_A)\right]. \quad (44)$$

Recall that the original Son-Yamamoto relation reads as [6]

$$(S-Y) = w_T - \frac{N_c}{Q^2} + \frac{N_c}{f_\pi^2}(\Pi_A-\Pi_V) = 0, \quad (45)$$

where

$$\frac{1}{f_\pi(z_0)^2} = \frac{1}{2}\int_0^{z_0} dz\frac{1}{f(z)^2}. \quad (46)$$

If we take into account the diagonal two-point correlator and substitute it into the variation of the HJ equation, we obtain

$$\frac{\partial}{\partial z_0}(S-Y) = -\frac{(\Pi_A+\Pi_V)q^2}{2f^2}(S-Y). \quad (47)$$

Therefore, in this model, the SY relation is diagonal under the renormalization group. In order to prove that the $(S-Y)$ is zero for all z_0, let us take the limit $z_0\to 0$; that is, we consider the SY relation near the UV boundary. In the model under consideration, the chiral symmetry is broken because of the boundary conditions; therefore, we should be careful at this point. The boundary conditions do not play any role for the HJ equation itself, but they are important for the boundary conditions for the HJ equation. Recalling (24), we see that $w_T\to 0$ and

$$\frac{N_c}{f_\pi^2}(\Pi_A-\Pi_V)\to\frac{N_c}{q^2}. \quad (48)$$

Hence, the SY relation indeed holds at the UV boundary and, therefore, it holds for all z_0.

In the first model with the additional scalar field we have similar HJ equations,

$$\frac{\partial}{\partial\epsilon}(\Pi_A-\Pi_V) = \frac{g_5^2\epsilon q^2}{2}(\Pi_A^2-\Pi_V^2) + \frac{3}{\epsilon^3}|X|^2, \quad (49)$$

$$\frac{q^2}{4\pi^2}\frac{\partial w_T}{\partial z_0} - \frac{g_5^2\epsilon}{2}\left[\frac{q^4}{4\pi^2}(\Pi_A+\Pi_V)w_T + 6\kappa q^2(\Pi_A+\Pi_V) + 6\kappa q^2(\Pi_V-\Pi_A)\right] = 0, \quad (50)$$

and

$$\frac{\partial(S-Y)}{\partial\epsilon} = (\Pi_A+\Pi_V)\frac{q^2g_5^2\epsilon}{2}(S-Y) + \frac{3N_c}{\epsilon^3 f_\pi^2}|X|^2. \quad (51)$$

Note that in this equation f_π is defined as in the second model (see Eq. (46)].

Therefore, we see that in the model with an additional scalar field, the RG equation for the SY relation acquires the inhomogeneous term that corresponds to its failure. In order to make this expression diagonal and look for the modified diagonal correlator, one could potentially add a term θ to the SY relation that would have to satisfy the following equation:

$$\frac{\partial\theta}{\partial\epsilon} = \theta(\Pi_A+\Pi_V)\frac{q^2g_5^2\epsilon}{2} - \frac{3N_c}{\epsilon^3 f_\pi^2}|X|^2. \quad (52)$$

However, we have not found its proper operator realization.

It is interesting to note that Eq. (36) tells us that the flow for $\Pi_A-\Pi_V$ is diagonal, but we do not expect $\Pi_A=\Pi_V$. In the second model, the problem is due to the boundary conditions: $\Pi_A = \Pi_V$ does not hold for $z_0 = 0$. In the first model, the chiral symmetry is broken only by the quark condensate X [see Eq. (49)]. If it vanishes, then we indeed have $\Pi_A = \Pi_V$.

V. CONCLUSION

In this paper we have examined the behavior of the SY relation under the nonperturbative RG flow. We have found that when the SY relation is fulfilled, it is diagonal under the action of the RG flow generated by the HJ equation, while when it is not true, the inhomogeneous term in the RG equation is present. We expect that the diagonal evolution under RG flow should be one of the guiding principles in searches for more complicated anomaly-matching conditions. It would be interesting to obtain the higher correlators that are diagonal under RG flow in holographic models of QCD. However, we have seen that the issue of the diagonalization is model dependent in holographic models of QCD; hence, it is important to perform similar consideration for the models when the bulk dual theory is uniquely defined, as in supersymmetry gauge theories. It would be interesting to formulate the diagonalization of the correlators in terms of the string world sheet theory. The diagonalization of the matrix of the anomalous dimensions corresponds to the diagonalization of the spin chain Hamiltonian that arises upon the discretization of the world sheet sigma model; it would be interesting to formulate the problem of the diagonalization of correlators in a similar manner.

It would be also interesting to investigate a few related problems. First, the similar diagonalization problem can be discussed in the gravity sector of the bulk theory, when the Wheeler-deWitt equation plays the role of the HJ equation. The second question concerns the baryonic sector of the theory. The baryons correspond to the instantons in the bulk theory [13]; hence, an interesting question concerns the solution to the HJ equations in the sector with nonvanishing topological charge. Finally, it would be interesting to find similar diagonal correlators from the HJ equation in the holographic bulk descriptions of condensed matter models.

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