Nongeometric Calabi-Yau compactifications and fractional mirror symmetry

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We construct a wide class of nongeometric compactifications of type II superstring theories preserving N = 1 space-time supersymmetry in four dimensions, starting from Calabi-Yau compactifications at Gepner points. Particular examples of this construction provide quantum equivalences between Calabi-Yau compactifications and non-Calabi-Yau ones, generalizing mirror symmetry. The associated Landau-Ginzburg models involve both chiral and twisted chiral multiplets and hence cannot be lifted to ordinary Calabi-Yau gauged linear sigma models.

I. INTRODUCTION

It is widely acknowledged that Calabi-Yau (CY) manifolds form only a small subset of supersymmetric string compactifications. Understanding more general compactifications is an important goal, both for probing the quantum geometry of string theory and for obtaining four-dimensional models with fewer moduli and fewer supersymmetries. Besides compactifications with Ramond-Ramond (RR) fluxes, which are quite successful in this respect but lack a usable world sheet formulation, it is desirable to find models with a better grip on α' corrections beyond the supergravity regime. Unlike in heterotic strings, it is not possible to consider type II compactifications with Neveu-Schwarz–Neveu-Schwarz (NSNS) three-form flux only, because of the tadpole condition $\int e^{-2\Phi} H \wedge \star H = 0$ coming from the equations of motion.

It leaves the possibility of using nongeometric fluxes; compactifications of this type have been described as asymmetric orbifolds of rational tori [1–3], using free-fermion models [4,5] or as T-folds that are locally geometric and their (generalized) T-duals [6–8]. Studying such nongeometric fluxes in interacting rather than free world sheet conformal field theories (CFTs) would allow us to understand how nongeometric compactifications can be defined in nontrivial backgrounds. A large class of supersymmetric compactifications on Calabi-Yau manifolds, in the stringy regime of negative Kähler moduli, is described by superconformal field theories constructed by Gepner using $\mathcal{N} = (2,2)$ minimal models as building blocks [9,10]. Some asymmetric (0, 2) Gepner models have been considered in the past, as heterotic compactifications with nonstandard gauge bundles [11–15]. In contrast type IIA/IIB asymmetric Gepner models have not been explored in detail; as we shall see they provide a good starting point for constructing large classes of nongeometric backgrounds.

Discriminating between abstract world sheet theories with and without a geometrical target-space interpretation is quite difficult. We use in this work a simple sufficient (but not necessary) criterion. Let us consider a compactification of type IIA or type IIB superstrings (without orientifolds, D-branes or RR fluxes) such that all space-time supersymmetry (SUSY) comes from the left-moving world sheet degrees of freedom. A geometric compactification of this sort would exist if the two connections with torsion $\nabla(\omega \pm H/2)$ appearing in the supersymmetry variations of the gravitini gave different G-structures, requiring nonzero three-form flux; for compact models this is forbidden by the tadpole condition quoted above.

The inspiration for this article originates from a recent work [16] where we described fibrations of $K3$ Gepner models over a two torus in type II, breaking space-time supersymmetry from the right movers only. Following our general argument it implies that, while going around a one cycle of the base, the $K3$ fiber undergoes a nongeometric symmetry twist. The symmetries of the $K3$ fiber appearing in the monodromies are actually neither geometric symmetries nor mirror symmetry. These new nongeometric symmetries of CY quantum sigma models are the focus of the present work. We embed them in a larger framework of nongeometric models based on solvable Calabi-Yau compactifications.

We construct a wide class of asymmetric Gepner models in type II, using the simple currents formalism [17], preserving space-time supersymmetry from the left movers, while the other half is generically broken. This is made possible by a specific choice of discrete torsion, which changes in particular the orbifold action on the Kähler moduli. This leads to many nongeometric compactifications with $\mathcal{N} = 1$ supersymmetry in four dimensions, and a reduced moduli space of vacua. We present some examples based on the quintic to illustrate these features.

In some cases, including those underlying the $K3$ fibrations over T^2 studied in [16], the nongeometric fluxes

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lead to superconformal field theories isomorphic to the original ones, albeit of a different nature. These are generalizations of mirror symmetry of (2,2) models [in heterotic strings, (0,2) extensions of mirror symmetry have been considered in [18] and subsequent works] that take the sigma models out of the realm of CY compactifications.

Mirror symmetry plays a major role in our understanding of CY manifolds, both in their physical and mathematical aspects [19,20]. It generalizes T-duality to CY sigma models as one exchanges the axial and vector R-symmetries of the superconformal algebra [21]. The first concrete realization was obtained by Greene and Plesser [22] using Gepner models; they have shown that an orbifold by the largest subgroup of discrete symmetries preserving space-time supersymmetry (bar permutations) gives an isomorphic conformal field theory with reversed right-moving R-charges. It provides an equivalence between type IIA compactified on some CY and type IIB on a topologically distinct one, such that their Hodge diamonds are mirror to each other. Using the gauged linear sigma model (GLSM) description [23] that includes the Gepner points, Hori and Vafa gave a proof of mirror symmetry [24]; in this context the dual models appear naturally as orbifolds of Landau-Ginzburg (LG) models.

As a special case of the general construction of asymmetric Gepner models that we present in this work, there exists a subclass of models such that the axial and vector R-symmetries for a single minimal model are exchanged; they are isomorphic as CFTs to the original theory. They correspond to "hybrid" LG orbifolds with both chiral and twisted chiral superfields, and hence cannot be lifted to ordinary Calabi-Yau GLSMs; this is a sign of the non-CY nature of these new dual models. Following the ideas of [24] we propose a hybrid GLSM that provides their UV completion. Given that the map can be applied stepwise to each and every minimal model until we reach the usual mirror theory, we give this symmetry the name of fractional mirror symmetry.

This work is organized as follows. In Sec. II we present a short overview of simple currents and Gepner models orbifolds. In Sec. III we provide the general construction of $\mathcal{N} = 1$ nongeometric compactifications, and study some explicit examples based on the quintic. In Sec. IV we define and study fractional mirror symmetry. Finally in Sec. V we give the conclusions and explain the relation between the $K3$ fibrations over tori of [16] and these new constructions. Useful facts about $\mathcal{N} = 2$ characters and representations are given in the appendix.

II. SIMPLE CURRENTS AND GEPNER MODELS

Let us first review briefly the simple current formalism [25,26], its relation with Gepner models and orbifolds thereof.

A. Simple currents and discrete torsion

In a conformal field theory a simple current J is a primary of the chiral algebra whose fusion with a generic primary gives a single primary: $J \star \phi_\mu = \phi_\nu$. This action defines the monodromy charge of the primary with respect to the current, $Q_\iota(\mu) = \Delta(\phi_\mu) + \Delta(J_\iota) - \Delta(J_\iota \star \phi_\mu)$ mod 1; two currents are mutually local if $Q_\iota(J_J) = 0$. We consider the extension of a rational CFT by a set of M simple currents J_ι. Provided that the simple currents action has no fixed points, the associated modular-invariant partition function is

$$Z = \sum_\mu \prod_{\iota=1}^{M} \sum_{b^\iota \in \mathbb{Z}_{n_\iota}} \chi_\mu(q) \chi_{\mu+\beta_J b^J}(\bar{q}) \delta^{(1)}(Q_\iota(\mu) + X_{\iota J} b^J), \quad (1)$$

with $J_\iota \star \phi_\mu = \phi_{\mu+\beta_\iota}$ and n_ι being the length of J_ι. The symmetric part of the matrix X is determined by the relative monodromies as $X_{\iota J} + X_{J\iota} = Q_\iota(J_J)$, while the antisymmetric part, discrete torsion, should be such that

$$\gcd(n_\iota, n_J) X_{\iota J} \in \mathbb{Z}. \quad (2)$$

If the left and right kernels of X are different, the simple-current-extended modular invariant is asymmetric.

B. Gepner models

A Gepner model for type II superstrings compactified on a CY threefold is obtained from a tensor product of r $\mathcal{N} = (2, 2)$ minimal models, whose central charges satisfy $\sum_{n=1}^{r} c_n = \sum_{n=1}^{r}(3 - 6/k_n) = 9$, tensored with a free $\mathbb{R}^2$ superconformal theory that represents the space-time part in the light-cone gauge. One needs to project the theory onto states with odd-integer left and right R-charges; this can be rephrased in the simple currents formalism. The simple currents of the minimal models are primaries with quantum numbers ($j = 0$, m, s). These simple currents can be grouped together with the current for a free fermion into a simple current J with labels

$$\beta_J = (s_0|m_1, \ldots, m_r|s_1, \ldots, s_r), \quad (3)$$

where s_0 is the fermionic $\mathbb{Z}_4$ charge of the $\mathbb{R}^2$ factor.

The Gepner modular invariant is obtained as a simple current extension, using first the sets of currents $\{J_n, n = 1, \ldots, r\}$, with

$$\beta_n = (2|0, \ldots, 0|0, \ldots, 0, \underbrace{2}_{n\text{-th position}}, 0, \ldots, 0) \quad (4)$$

enforcing world sheet supersymmetry, and second the current J_0, with

$$\beta_0 = (1|1, \ldots, 1|1, \ldots, 1), \quad (5)$$

ensuring the projection onto odd-integer R-charges, hence space-time supersymmetry. All these simple currents are mutually local.

In order to write the Gepner model partition function in a compact way we gather the free-fermion character $\theta_{s_0,2}/\eta$ and minimal model characters $\chi^j_{m,s}$ as

$$\chi^\lambda_\mu(q) = \frac{\theta_{s_0,2}(q)}{\eta(q)} \times \prod_{n=1}^{r} \chi^{j_n}_{m_n,s_n}(q), \qquad (6)$$

where we have grouped the associated quantum numbers as follows:

$$\lambda = (j_1, ..., j_r) \quad \text{and} \quad \mu = (s_0|m_1, ..., m_r|s_1, ..., s_r). \qquad (7)$$

The diagonal modular-invariant partition function of a CY_3 compactification at a Gepner point is then given by

$$Z = \frac{1}{2^r} \frac{1}{\tau_2^2 |\eta|^4} \sum_{\lambda,\mu} \sum_{b_0 \in \mathbb{Z}_K} (-1)^{b_0} \delta^{(1)}\left(\frac{Q_R - 1}{2}\right) \prod_{n=1}^{r} \sum_{b^n \in \mathbb{Z}_2} \delta^{(1)}\left(\frac{s_0 - s_n}{2}\right) \chi^\lambda_\mu(q) \chi^\lambda_{\mu + \beta_0 b_0 + \beta_l b^l}(\bar{q}), \qquad (8)$$

where Q_R is the left-moving world sheet R-charge and $K = \text{lcm}(4, 2k_1, ..., 2k_r)$. One can check that the right-moving R-charge $\bar{Q}_R$ also takes odd-integer values.

C. Supersymmetric orbifolds and mirror symmetry

Simple currents preserving world sheet and space-time supersymmetry should be mutually local with respect to the Gepner model currents $\{J_0, J_1, ..., J_r\}$; see [11]. Let us consider a generic simple current $\mathfrak{F}$ with

$$\beta_\mathfrak{F} = (0|2\rho_1, ..., 2\rho_r|0, ..., 0), \qquad \rho_n \in \mathbb{Z}. \qquad (9)$$

Any such current is mutually local with respect to the set of currents $\{J_n\}$; hence the corresponding extended partition function always preserves world sheet supersymmetry. Mutual locality with respect to the current J_0 (which ensures odd integrality of the R-charges) requires that

$$\sum_{n=1}^{r} \frac{\rho_n}{k_n} \in \mathbb{Z}. \qquad (10)$$

If this condition is satisfied one obtains an $\mathcal{N} = 2$ compactification, corresponding to a Calabi-Yau orbifold at a Gepner point.

Extending a Gepner model with all such supersymmetry-preserving simple currents (without discrete torsion) gives the mirror Gepner model, which is such that the right R-charge $\bar{Q}_R$ has opposite sign compared to the original model; it exchanges the chiral and twisted chiral rings of the theory, hence the complex structure and Kähler moduli spaces. This is the basis of the construction of mirror manifolds by Greene and Plesser [22].

III. NONGEOMETRIC CY COMPACTIFICATIONS

In this section we describe a way to obtain many nongeometric models starting from a Calabi-Yau compactification at a Gepner point.

A. General method

In order to construct new nongeometric compactifications we consider extensions of the Gepner model partition function by simple currents that are not mutually local with respect to the Gepner model currents. A generic current $\mathfrak{F}$ as in Eq. (9) is actually nonlocal with respect to J_0; hence space-time supersymmetry is completely broken (while world sheet supersymmetry is preserved). Indeed

$$Q_0(\mathfrak{F}) = \sum_{n=1}^{r} \frac{\rho_n}{k_n} \mod 1. \qquad (11)$$

Now comes the key step; there is a choice of discrete torsion, consistent with Eq. (2) for any $\{\rho_n \in \mathbb{Z}\}$, given by

$$X_{0\mathfrak{F}}^{\text{antisym}} = -\frac{1}{2} \sum_{n=1}^{r} \frac{\rho_n}{k_n}, \qquad (12)$$

bringing down the X matrix to a lower-triangular form. Its only nonzero entries are

$$X_{\mathfrak{F}\mathfrak{F}} = \sum_{n=1}^{r} \frac{\rho_n^2}{k_n}, \qquad X_{\mathfrak{F}0} = \sum_{n=1}^{r} \frac{\rho_n}{k_n}. \qquad (13)$$

This choice allows us to bring back the projection onto odd-integer left-moving R-charges Q_R into its original form.

The modular-invariant partition function of the $\mathfrak{F}$-extended Gepner model with this choice of discrete torsion is given by

$$Z = \frac{1}{2^r}\frac{1}{\tau_2^2|\eta|^4}\sum_{\lambda,\mu}\sum_{b_0\in\mathbb{Z}_K}(-1)^{b_0}\delta^{(1)}\left(\frac{Q_R-1}{2}\right)$$
$$\times\sum_{B\in\mathbb{Z}_N}\delta^{(1)}\left(\sum_{n=1}^r\frac{\rho_n(m_n+b_0+\rho_n B)}{k_n}\right)\prod_{n=1}^r\sum_{b^n\in\mathbb{Z}_2}\delta^{(1)}\left(\frac{s_0-s_n}{2}\right)\chi_\mu^\lambda(q)\chi_{\mu+\beta_0 b_0+\beta_l b^l+\beta_{\mathfrak{F}}B}^\lambda(\bar{q}), \tag{14}$$

where Q_R is the left-moving world sheet R-charge and the length of the simple current is given by $N = \mathrm{lcm}\,(\mathrm{lcm}(\rho_1,k_1)/\rho_1,\ldots,\mathrm{lcm}(\rho_r,k_r)/\rho_r))$.[1]

If some levels k_n are even, there may be fixed points under the simple current action, and multiplicity factors need to be added accordingly to the partition function. For simplicity of presentation we assume that we do not encounter this situation, which does not change the salient features of the construction; for instance one can take all the levels k_n to be odd.

Thanks to the discrete torsion the projection onto odd-integer world sheet R-charges, given by the discrete delta function in the first line, has been restored in the left-moving sector; hence space-time supersymmetry from the left movers is preserved. This supersymmetry is generated by spectral flow of the left-moving $\mathcal{N} = 2$ superconformal algebra as usual.

Twisted sectors associated with the $\mathfrak{F}$ extension (i.e. states with $B \neq 0$) can have fractional values of the right-moving world sheet R-charge $\bar{Q}_R$. Indeed,

$$\bar{Q}_R \equiv 1 + 2B\sum_{n=1}^r\frac{\rho_n}{k_n} \bmod 2\mathbb{Z}; \tag{15}$$

hence space-time supersymmetry from the right movers is generically broken; we end up with $\mathcal{N} = 1$ four-dimensional supersymmetry. Following our general argument given in the introduction, this construction provides a whole class of nongeometric quotients of CY sigma models at Gepner points.

Naturally it is possible to consider a simple current extension by several such currents, with a discrete torsion of the form (12) for each of them; these currents may or may not be mutually local. Discrete torsion with no components along the "Gepner currents" J_0 and J_n can be added without breaking further space-time supersymmetry.

B. Some quintic-based examples

We have given a method that allows us to obtain type IIA or type IIB $\mathcal{N} = 1$ compactifications to four dimensions, with neither orientifolds nor RR fluxes, starting from quite generic nonsupersymmetric geometric orbifolds of Calabi-Yau compactifications at Gepner points (this can be extended to a wider class of Landau-Ginzburg orbifolds; see Sec. V). The moduli spaces of such models are significantly reduced compared to the original $\mathcal{N} = 2$ CY compactifications.

To illustrate this general construction, let us consider several examples based on the quintic Calabi-Yau. The quintic is given by the hypersurface

$$Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 + Z_5^5 = 0 \tag{16}$$

in $\mathbb{P}^4$, the complex variables Z_n being homogeneous coordinates on the complex projective space. This Calabi-Yau has a unique complexified Kähler modulus t, whose real part is the volume modulus inherited from the ambient $\mathbb{P}^4$. The complex structure moduli correspond to deformations of Eq. (16) by monomials of degree five; there are 101 inequivalent of them.

In the regime $\Re(t) \to -\infty$, a quantum (2,2) nonlinear sigma model (NLSM) on the quintic is described by the Gepner model with $k_1 = \cdots = k_5 = 5$. Complex structure deformations correspond to marginal chiral operators of R-charges $Q_R = \bar{Q}_R = 1$. They are obtained from a tensor product of chiral operators in each minimal model, labeled by an $SU(2)$ spin j_n, with $j_n \in \{0, 1/2, \ldots, k_n/2 - 1\}$, such that $\sum_{n=1}^5 2j_n/k_n = 1$; they correspond to the monomials $Z_1^{2j_1}\cdots Z_5^{2j_5}$. Twisted chiral states (i.e. chiral with respect to the left-moving superconformal algebra and antichiral with respect to the right-moving superconformal algebra) appear in the twisted sectors of the Gepner model projection, i.e., with $b_0 \neq 0$ in the partition function (8). Explicitly, the complexified Kähler modulus has $2j_1 = \cdots = 2j_n = 1$ and $b_0 = 8$.

We consider simple current extensions of the form discussed in Sec. III, with discrete torsion leading generically to $\mathcal{N} = 1$ space-time supersymmetry. They are characterized by the integer-valued five-dimensional vector

$$\varrho = (\rho_1, \ldots, \rho_5), \tag{17}$$

giving the simple current action in each minimal model. We consider below the salient features of three representative cases.

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[1]If some of the ρ_ns vanish, the definition of N has to be modified accordingly; only nonzero entries enter the formula.

1. $\varrho = (1,2,3,2,1)$ *model*

On top of the projection onto odd-integer left R-charges, states in the partition function should satisfy the constraint

$$m_1 + 2m_2 + 3m_3 + 2m_4 + m_5 + 9b_0 + 19B \in 5\mathbb{Z}, \quad (18)$$

see Eq. (14), where the label of the twisted sectors is $B = 0, \ldots, 4$. For each of the 101 left chiral operators of charge $Q_R = 1$, one can solve Eq. (18) in a given twisted sector B.

Furthermore, the right $\mathbb{Z}_{k_n}$ charge of the nth minimal model is shifted in the twisted sectors as $m_n + b_0 \to m_n + b_0 + 2\rho_n B$. For a given spin j_n, chiral and antichiral states minimize the conformal dimension with respect to m_n. Therefore if $B \neq 0$ a formerly massless state could only stay massless if, in each minimal model, either $2\rho_n B \equiv 0 \mod 2k_n$, i.e. if the shift is trivial (thanks to the periodicity $m_n \sim m_n + 2k_n$), or if a formerly chiral or antichiral state becomes another antichiral or chiral state.

In this particular example, we found that out of the 101 original chiral states of charge $Q_R = \bar{Q}_R = 1$, corresponding to the complex structure deformations, only 18 operators remain massless. They all belong to the untwisted sector $B = 0$ in this case, and hence are still chiral operators; this is not a generic feature of the models as we will see in the next example. The former unique Kähler modulus of the quintic threefold is lifted, acquiring a string-scale mass

$$M_K = \sqrt{\frac{2}{\alpha'}}. \quad (19)$$

In the original Gepner model, the gravitino corresponding to the space-time supersymmetry associated with the right movers was obtained from the Neveu-Schwarz-Ramond primary operator with $2j_1 = \cdots = 2j_5 = m_1 = \cdots = m_5 = 0$ and $b_0 = 1$. For these quantum numbers the constraint (18) singles out the twisted sector $B = 4$. Accordingly the right gravitino is now massive, with

$$M_{\psi_\mu} = 2\sqrt{\frac{2}{\alpha'}}. \quad (20)$$

This mass scale is of the same order as massive string states. It indicates that one cannot reliably study this construction as a spontaneous $\mathcal{N} = 2 \to \mathcal{N} = 1$ SUSY breaking from an effective $\mathcal{N} = 2$ supergravity perspective.

2. $\varrho = (0,0,1,2,3)$ *model*

Only the last three minimal models are affected by the simple current extension. The untwisted sector ($B = 0$) contains the subset of marginal chiral operators, i.e. complex structure deformations, that are not projected out. Some of these, as $Z_3^2 Z_4 Z_5^2$, involve only the last three minimal models; others as $Z_1^2 Z_2^3$ involve only the first two ones; finally operators as $Z_1 Z_4^2 Z_5^2$ contain both. Overall there are 20 such marginal chiral operators.

In this model, the twisted sectors ($B \neq 0$) also contain marginal operators, of a peculiar nature. While they are chiral with respect to the left-moving superconformal algebra, they are neither chiral (c) nor antichiral (a) with respect to the right-moving one. In terms of the right-moving chiral or antichiral nature in the individual minimal model factors, one gets

(i) $B = 1$ sector: one operator from $Z_3 Z_4^2 Z_5^2$, of (c/a,c/a, a,c,a) chirality on the right;

(ii) $B = 2$ sector: two operators, e.g. from $Z_2 Z_3^2 Z_4 Z_5$ of (c/a,c,a,a,a) right chirality;

(iii) $B = 3$ sector: two operators, e.g. from $Z_2 Z_3 Z_4^2 Z_5$ of (c/a,c,c,a,a) right chirality;

(iv) $B = 4$ sector: two operators, e.g. from $Z_2 Z_3 Z_4 Z_5^2$ of (c/a,c,a,c,c) right chirality.

Hence all these twisted sector marginal operators are semichiral operators. Anticipating the discussion of the next section, there is no "duality frame" with respect to fractional mirror symmetry such that all massless operators are either chiral or twisted chiral.

3. $\varrho = (0,0,0,2,2)$ *model*

While analyzing the amount of space-time supersymmetry, one observes that some models preserve more supersymmetry than one can naively think; this example is one of them.

As written previously, in the original Gepner model the gravitino from the NSR sector is characterized by $2j_1 = \cdots = 2j_5 = m_1 = \cdots = m_5 = 0$ and $b_0 = 1$. In the present case, the corresponding state belongs to the twisted sector $B = 2$. While the first three minimal models are in the right Ramond ground state with $(j, m, s) = (0, 1, 1)$, the last two minimal models have to be in the Ramond ground state of opposite R-charge, i.e. with $(j, m, s) = (0, -1, -1)$, in order to get a massless operator.

In ordinary constructions of space-time supersymmetric compactifications, one imposes that the diagonal R-current has a spectrum of odd-integer charges, such that it can be exponentiated to a spin field mutually local with the physical states. As Eq. (15) indicates, the model that we consider does not satisfy this property on the right. Nevertheless there exists a different realization of space-time supersymmetry for the right-moving degrees of freedom. One can check that the supersymmetry operator that we have just constructed is mutually local with the other operators, owing both to the left-moving Gliozzi-Scherk-Olive projection $Q_R \in 2\mathbb{Z} + 1$ and to the projection coming from the simple current extension, namely $2(m_4 + m_5 + 2b_0) + 8B \in 5\mathbb{Z}$.

IV. FRACTIONAL MIRROR SYMMETRY

The last example of asymmetric Gepner model that we studied in the previous section was quite intriguing, as space-time supersymmetry among the right-moving

degrees of freedom was realized even though the right R-charge $\bar{Q}_R$ was not integer valued. In some cases the similarity between the geometric compactifications and the nongeometric ones goes beyond the amount of preserved supersymmetry.

A. Elementary simple current extensions and fractional mirror symmetry

A particular type of $\mathfrak{J}$ extension with discrete torsion has indeed remarkable properties, as not only the space-time supersymmetry is the same as the original Calabi-Yau compactification at a Gepner point, but the whole super-conformal field theories are isomorphic to each other. Extending a Gepner model partition function with an elementary simple current of labels

$$\beta_{\mathfrak{J}} = (0|2, 0, \ldots, 0|0, \ldots, 0), \qquad (21)$$

amounts, while taking into account the twisted sectors and discrete torsion, to replacing in the original partition function the antiholomor-phic character for the first minimal model with the character of opposite $\mathbb{Z}_{k_1}$ charge, namely

$$\chi^{j_1}_{m_1+b_0,s_1+b_0+2b_1}(\bar{q}) \overset{\mathfrak{J}\text{-ext}}{\to} \chi^{j_1}_{-m_1-b_0,s_1+b_0+2b_1}(\bar{q}). \qquad (22)$$

In the right Neveu-Schwarz sector, there is an equivalence between the map (22) and changing the sign of the right R-charge associated with the first minimal model. In the right Ramond sector it is also true if one changes the right-moving space-time chirality at the same time. As super-conformal field theories the original model and the new one are therefore isomorphic.

Starting from a type IIA Calabi-Yau compactification at a Gepner point, we obtain a type IIB theory on a Gepner model whose right-moving R-charge associated with the first minimal model has been reversed; with respect to the original right-moving diagonal R-current the spectrum of R-charges is not integer valued; hence the model is not associated with a Calabi-Yau. To put it differently the quotient does not preserve the holomorphic three form. These two models are isomorphic, and hence describe the same physics. This fractional mirror symmetry can be applied stepwise until one obtains the mirror description in the usual sense.

B. Hybrid Landau-Ginzburg models

A minimal model is the IR fixed point of a LG model with superpotential $W = Z^k$ [27]. Its mirror, obtained by a $\mathbb{Z}_k$ quotient, is a LG model for a twisted chiral superfield $\tilde{Z}$ with a twisted superpotential $\tilde{W} = \tilde{Z}^k$. In the present context we are considering a similar quotient acting inside a LG orbifold, with a discrete torsion that disentangles partly the two orbifolds—the diagonal one ensuring R-charge integrality and the $\mathbb{Z}_{k_1}$ quotient giving the fractional mirror. We end up with a hybrid

Landau-Ginzburg orbifold containing both a twisted chiral superfield $\tilde{Z}_1$ and chiral superfields $Z_{2,\ldots,r}$; hence it cannot be related to a Calabi-Yau GLSM.

This quantum equivalence need not be restricted to the Gepner points in the Calabi-Yau moduli space. To illustrate this point let us consider again the quintic. Away from the Gepner point, we expect that for every hypersurface in $\mathbb{P}^4$ of the form

$$z_1^5 + \sum \alpha_{abc} z_2^a z_3^b z_4^c z_5^{5-a-b-c} = 0 \qquad (23)$$

a fractional mirror with respect to the chiral superfield Z_1 exists. In other words the complex structure deformations that preserve the $\mathbb{Z}_5$ symmetry $z_1 \to e^{2i\pi/5} z_1$ are compatible with this duality. Kähler deformations are not compatible with it, as can be seen explicitly at the Gepner point; hence this symmetry is not manifest in the large-volume limit.

When these conditions are met the $\mathcal{N} = 2$ superconformal algebra can be split into the algebra coming from the LG model $W = Z_1^5$ and from the LG model for the other multiplets. This allows us to dualize Z_1 into a twisted chiral multiplet, giving a more general hybrid LG orbifold with superpotential $W = \sum \alpha_{abc} Z_2^a Z_3^b Z_4^c Z_5^{5-a-b-c}$ and twisted superpotential $\tilde{W} = \tilde{Z}_1^5$. One expects also that other accidental splittings of the superconformal algebra, corresponding to orbifolds of tensor products of Landau-Ginzburg models of more generic form [for instance if the LG potential splits as $W = G(Z_1, Z_2) + H(Z_3, Z_4, Z_5)$], should give rise to different fractional mirror symmetries.

On the Calabi-Yau side, Landau-Ginzburg orbifolds and Calabi-Yau NLSMs describe the infrared dynamics of (2,2) gauged linear sigma models in different regimes, continuously connected by varying the Fayet-Iliopoulos parameters, i.e. giving vacuum expectation values to marginal twisted chiral operators in the infrared description. It would be very helpful to have then a UV completion of the dual theory, which can be taken out of the hybrid Landau-Ginzburg regime where both formulations become overtly equivalent. We propose such a description below.

C. Fractional mirror symmetry and GLSMs

For concreteness we consider again the (2,2) gauged linear sigma model for the quintic threefold. This two-dimensional gauge theory contains a $U(1)$ vector superfield, five chiral superfields $Z_{1,\ldots,5}$ of charge one and a chiral superfield P of charge -5. They interact through the superpotential $W = PG(Z_n)$, where $G = \sum Z_n^5$ is the degree five homogeneous polynomial defining the CY hypersurface; furthermore the theory contains a twisted superpotential $\tilde{W} = -t\Sigma$, $t = r - i\theta$ being the complexified Fayet-Iliopoulos parameter and Σ the field-strength super-field. In the regime $\Re(t) \ll 0$, or equivalently when $p \neq 0$,

if flows to the diagonal $\mathbb{Z}_5$ orbifold of the Landau-Ginzburg model with superpotential $W = G(Z_n)$.

Using the approach of Hori and Vafa to mirror symmetry [24], one can dualize only the chiral superfield Z_1 to a twisted chiral superfield $\tilde{Y}$. One expects to get a hybrid GLSM with superpotential and twisted superpotential (with $\hat{G} = Z_2^5 + \cdots Z_5^5$):

$$W = P\hat{G}(Z_n), \qquad \tilde{W} = \Sigma(\tilde{Y} - t) + e^{-\tilde{Y}}, \qquad (24)$$

where the second term in the twisted superpotential comes from world sheet instantons.

Following for instance the discussion in [28], a geometrical NLSM regime of this model, if it existed, would be characterized by nonzero three-form flux H; this seems at odds with the tadpole condition recalled in the introduction. To settle this potential issue let us analyze classically what the predictions of the GLSM corresponding to (24) are. The vacua are determined by the scalar potential:

$$V(z_n, \tilde{y}, p, \sigma) = |\hat{G}(z_n)|^2 + |p|^2 \sum_{n=2}^{5} |\partial_n \hat{G}(z_n)|^2 + |\sigma - e^{-\tilde{y}}|^2$$

$$+ \frac{e^2}{2}(|z_2|^2 + \cdots + |z_5|^2 - 5|p|^2 + \Re(\tilde{y}) - \Re(t))^2 + |\sigma|^2(|z_2|^2 + \cdots + |z_5|^2 + 5^2|p|^2). \qquad (25)$$

A geometrical "phase" would be characterized by $|z_i| \neq 0$. It implies that $p = 0$ by transversality of $\hat{G}$, and also that $\sigma = 0$ in the vacuum as the superfields Z_n are minimally coupled to the vector superfield; see the last term in (25). The twisted F-term (third term) shows a runaway behavior as the vacuum is obtained for $\Re(\tilde{y}) \rightarrow +\infty$. As a consequence the D-term condition (last but one term) cannot be satisfied. Hence this two-dimensional theory has no regime with a NLSM description; this result, which is in accordance with the supergravity tadpole condition, relies crucially on the world sheet instanton contribution in $e^{-\tilde{Y}}$ to the twisted superpotential.

On the contrary there exists a hybrid Landau-Ginzburg phase. Setting $p \neq 0$, which breaks spontaneously the gauge group to $\mathbb{Z}_5$, implies that $z_n = 0$ by transversality of $\hat{G}$ and that $\sigma = 0$. Then the twisted F-term enforces $\Re(\tilde{y}) \rightarrow +\infty$ as above. Finally, the D-term condition shows that $|p|$ is driven to very large values. The effective superpotential for the chiral superfields Z_n is then of the form $W \sim \hat{G}(Z_n)$. Regarding the twisted chiral superfield, it was argued in [24] that the fundamental field after the duality in such a compact model is given by $e^{-\tilde{Y}} = \tilde{X}^5$. This conjecture was tested in the context of ordinary mirror symmetry by computing the masses of A-type or B-type boundary states saturating the Bogomol'nyi-Prasad-Sommerfeld bound; the same argument cannot be used in the hybrid models, but we expect that the same field redefinition can be carried over. It is not single valued, being invariant under $\tilde{X} \rightarrow e^{2i\pi/5}\tilde{X}$, hinting towards the orbifold structure that we obtained in the hybrid Landau-Ginzburg description. Clearly, a better understanding of the low-energy dynamics of this theory would be helpful in making the correspondence between the hybrid GLSMs and the hybrid LG models more precise.

V. CONCLUSIONS AND DISCUSSION

In this work we have constructed a wide class of compactifications of type IIA and type IIB superstring theories, starting from Calabi-Yau compactifications at Gepner points, whose generic features are $\mathcal{N} = 1$ space-time supersymmetry (with neither orientifolds nor fluxes) and a reduced moduli space of vacua.

As was explained in the introduction, such compactifications preserving only $\mathcal{N} = 1$ four-dimensional supersymmetry are necessarily nongeometric, as the ten-dimensional supercharges, related respectively to the left-moving and right-moving world sheet degrees of freedom, are not on the same footing. Technically, the origin of this nongeometrical nature was the introduction of a very specific discrete torsion, whose role was to turn a non-supersymmetric Gepner model orbifold into an $\mathcal{N} = 1$ theory.

One may argue that, after all, these models are almost geometric as the discrete torsion only plays a role in the twisted sectors. This is not actually correct, as the tensor product of minimal models becomes a CY sigma model at a Gepner point only after the extension by the Gepner currents J_0 and J_n has been implemented. The discrete torsion has an effect in the twisted sectors of the J_0 extension, giving the compactification a nongeometric nature. In particular the quotient has a different action on twisted chirals, i.e. on Kähler moduli, compared to the corresponding geometric orbifold.

As a special case of this construction, we have obtained new quantum symmetries associated with superconformal field theories lying in the moduli space of Calabi-Yau compactifications that we have called fractional mirror symmetry. Unlike the usual mirror symmetry which is understood everywhere in the CY moduli space, these new dualities are visible only when accidental discrete symmetries become manifest, in the Landau-Ginzburg regime. We have proposed a gauged linear sigma model description of the dual theory that provides a UV completion and can be taken out of this regime but does not exhibit a geometrical phase, as expected.

The asymmetric $K3$ fibrations over T^2 that we have given in [16] can be rephrased in light of the construction exposed in this article. These models, which we obtained considering some modular properties of $\mathcal{N} = 2$ characters, can be interpreted as fibrations of $K3$ at Gepner points over a two torus, with a nongeometric monodromy twist around each one cycle of the base. For this purpose one considers two elementary $\mathfrak{F}$ extensions as (21), acting respectively in the first and second factors of a $K3$ Gepner model, and as $\mathbb{Z}_{k_1}$ and $\mathbb{Z}_{k_2}$ shifts along the two torus. These models are close relatives of T-folds [6] and interpolate between the $K3$ sigma model in the large torus limit, and a "half-mirror" $K3$ in the opposite small-volume limit. As the world sheet realization of space-time supersymmetry on the right-moving side is different in the theories appearing in these two limits, the interpolating model naturally breaks this half of space-time supersymmetry. Furthermore, as shown in [6], at the minimum of the four-dimensional supergravity potential, which is where the on-shell world sheet description is defined, only fields invariant under the symmetry used in the twisting stay massless. It explains why, in many cases, all the $K3$ moduli are lifted in this construction. As there are no massless Ramond-Ramond fields in these models, it is possible to add D-branes alone to these compactifications without running into a problematic RR tadpole. One expects that the associated open string spectra are nonsupersymmetric, yet the potential phenomenological implications of such models are worth exploring in detail.

Finally it would be interesting to find whether these symmetries are related to the Mathieu moonshine, which suggests that $K3$ compactifications have an underlying M_{24} symmetry whose origin is not fully understood [29].

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APPENDIX: $\mathcal{N} = 2$ CHARACTERS

The characters of the $\mathcal{N} = (2,2)$ minimal model with $c = \bar{c} = 3 - 6/k$, i.e. the supersymmetric $SU(2)_k/U(1)$ gauged Wess-Zumino-Witten model, are conveniently defined through the characters $\chi^j_{m,s}$ of the $[SU(2)_{k-2} \times U(1)_2]/U(1)_k$ bosonic coset, obtained by splitting the Ramond and Neveu-Schwarz sectors according to the fermion number mod 2. Defining $q = e^{2\pi i \tau}$ and $z = e^{2\pi i \nu}$, these characters are determined implicitly through the identity

$$\chi^j_{k-2}(\nu|\tau)\theta_{s,2}(\nu-\nu'|\tau) = \sum_{m\in\mathbb{Z}_{2k}}\chi^j_{m,s}(\nu'|\tau)\theta_{m,k}\left(\nu - \frac{2\nu'}{k}\Big|\tau\right),$$

(A1)

in terms of the theta functions of $\widehat{\mathfrak{su}(2)}_k$:

$$\theta_{m,k}(\tau,\nu) = \sum_n q^{k(n+\frac{m}{2k})^2} z^{k(n+\frac{m}{2k})}, \qquad m \in \mathbb{Z}_{2k} \quad \text{(A2)}$$

and χ^j_{k-2} the characters of the affine algebra $\widehat{\mathfrak{su}(2)}_{k-2}$:

$$\chi^j_{k-2}(\nu|\tau) = \frac{\theta_{2j+1,k}(\nu|\tau) - \theta_{-(2j+1),k}(\nu|\tau)}{i\vartheta_1(\nu|\tau)}. \quad \text{(A3)}$$

Highest-weight representations are labeled by (j,m,s), corresponding to primaries of $SU(2)_{k-2} \times U(1)_k \times U(1)_2$. The following identifications apply:

$$(j,m,s) \sim (j,m+2k,s) \sim (j,m,s+4)$$
$$\sim \left(\frac{k}{2}-j-1, m+k, s+2\right) \quad \text{(A4)}$$

as the selection rule $2j + m + s = 0$ mod 2. The half-integer modded spin j is restricted to $0 \leq j \leq \frac{k}{2} - 1$. The conformal weights of the superconformal primary states are

$$\Delta = \frac{j(j+1)}{k} - \frac{m^2}{4k} + \frac{s^2}{8} \quad \text{for } -2j \leq m - s \leq 2j \tag{A5a}$$

$$\Delta = \frac{j(j+1)}{k} - \frac{m^2}{4k} + \frac{s^2}{8} + \frac{m-s-2j}{2} \quad \text{for } 2j \leq m - s \leq 2k - 2j - 4 \tag{A5b}$$

and their R-charge reads

$$Q_R = -\frac{s}{2} + \frac{m}{k} \mod 2. \tag{A6}$$

(i) Chiral primary states are obtained for $m = 2j$ and $s = 0$ (thus an even fermion number). Their conformal dimension reads

$$\Delta = \frac{Q_R}{2} = \frac{j}{k}. \qquad (A7)$$

Equivalently they are of the form $(j, m, s) = (j, -2(j+1), 2)$.

(ii) Antichiral primary states are obtained for $m = 2(j + 1)$ and $s = 2$ (thus an odd fermion number). Their conformal dimension reads

$$\Delta = -\frac{Q_R}{2} = \frac{1}{2} - \frac{j+1}{k}. \qquad (A8)$$

Equivalently they are of the form $(j, m, s) = (j, -2j, 0)$.

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