

Decay constants of the pion and its excitations in holographic QCD

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We investigate the leptonic decay constants of the pion and its excitations with a 5-d holographic model for quantum chromodynamics. We prove numerically that the leptonic decay constants of the excited states of the pion vanish in the chiral limit when chiral symmetry is dynamically broken. This nontrivial result is in agreement with a solid prediction of quantum chromodynamics and is based on a generalized Gell-Mann–Oakes–Renner relationship involving the decay constants and masses of the excited states of the pion. We also obtain an extended partially conserved axial-vector current relation that includes the fields of the excited states of the pion, a relation that was proposed long ago in the context of current algebra.

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I. INTRODUCTION

There is a solid prediction of quantum chromodynamics (QCD) that the leptonic decay constant of the excited states of the pion vanish in the chiral limit when chiral symmetry is dynamically broken [1]. The real world is not chirally symmetric, as the masses of the u and d quarks are not zero. But these masses are much smaller than the strong-interaction scale Λ_{QCD} and it is therefore natural to expect that the leptonic decay constants of the excited states of the pion are dramatically suppressed in nature. At first sight this prediction might seem surprising. Within a quark model perspective a suppression of the leptonic decay constants for excited states is expected; the leptonic decay constant for an S -wave state is proportional to the configuration-space wave function at the origin and, compared to the ground state, excited states have suppressed wave functions at the origin. However, within this perspective there is no obvious physical mechanism that suggests a dramatic reduction of the decay constants for the excited states. The key point behind the suppression of the decay constants, as we shall elaborate shortly ahead, is the dynamical breaking of chiral symmetry in QCD and the (pseudo) Goldstone boson nature of the ground-state pion.

The suppression of the leptonic decay constants of the pion's excited states is an interesting feature of nonperturbative QCD. Lattice QCD and models of nonperturbative QCD can benefit from this feature by using it as a gauge to validate techniques and truncation schemes in approximate calculations. A first lattice result from 2006 [2] for the

pion's first radial excitation, extrapolated to the chiral limit, gives $f_{\pi^1}/f_{\pi^0} \sim 0.08$ MeV; experimentally [3], $f_{\pi^1}/f_{\pi^0} < 0.064$ —the decay constant of the n th excited state is denoted in the present paper by f_{π^n} and that of the ground state by f_{π^0} . At about the same time, another lattice collaboration reports [4] a very small value for f_{π^1} , with an extrapolated value to the chiral limit consistent with zero. Finally, a very recent publication [5] reports lattice results for the three lowest excited states: f_{π^1} is modestly suppressed, f_{π^2} is significantly suppressed, and $f_{\pi^3} \simeq f_{\pi^1}$. Calculations based on sum rules [6–8], effective chiral Lagrangians [9], and a chiral quark model [10] also find strongly suppressed values for f_{π^1} .

In recent years a new class of models for tackling nonperturbative problems in QCD has received great attention in the literature. These are holographic models inspired on the gauge-gravity duality, in that a strongly coupled gauge theory in d dimensions is assumed to be described equivalently in terms of a gravitational theory in $d + 1$ dimensions. The assumed duality is based on the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [11–13], a conjectured relationship between conformal field theories and gravity theories in anti-de Sitter spaces—for recent reviews, see Refs. [14,15]. Although the holographic dual of QCD remains unknown, there exist several models attempting to construct the five-dimensional holographic dual of QCD by incorporating known nonperturbative features of QCD. Confinement, for example, can be modeled [16] by truncating the AdS space with the introduction of an infrared cutoff $z_0 \sim 1/\Lambda_{\text{QCD}}$ in the fifth dimension (the other four coordinates belong to the flat Minkowski spacetime). In such a “hard-wall” model, one considers a slice $0 \leq z \leq z_0$ of AdS space, and imposes boundary conditions on the fields at the infrared border z_0 .

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Dynamical chiral symmetry breaking (DCSB) can be incorporated [17,18] in the hard-wall model with the use of scalar and vector fields in the AdS space which are in correspondence, respectively, to the chiral order parameter and left- and right-handed currents of the $SU(2)_L \times SU(2)_R$ chiral flavor symmetry of QCD in Minkowski space-time. In this model the pion corresponds to the zero mode in the Kaluza-Klein expansion of the scalar field, it has a finite leptonic decay constant, and its mass satisfies the Gell-Mann–Oakes–Renner (GOR) relationship.

A particularly interesting approach in the holographic description of QCD is light-front holography (LFH), introduced in Refs. [19,20]. In LFH, hadronic amplitudes in AdS space are mapped to frame-independent light-front wave functions in Minkowski space. This is made possible with the identification of the coordinate z in AdS space with a Lorentz-invariant coordinate that measures the separation of the constituents within a hadron at equal light-front time. The way chiral symmetry is treated in LFH is nonstandard, as the vanishing of the pion mass in the chiral limit is not a result of the dynamical breaking of the symmetry; rather, it follows from the precise cancellation of the light-front kinetic energy and light-front potential energy terms for the quadratic confinement potential in a Schrödinger-like equation [21]. The same feature of obtaining a massless pion without DCSB is possible in a soft-wall LFH approach [22], in which confinement is modeled with a soft cutoff provided by a background dilaton field in the AdS space. The experimental values of the masses of the lowest radially and orbitally excited states of the pion are well reproduced, but the leptonic decay constants of the excited states do not vanish in the chiral limit.

Motivated by these results in LFH, in the present paper we obtain the leptonic decay constants of the pion and its excitations in a five-dimensional holographic hard-wall model for QCD. The decay constants are obtained directly from the Kaluza-Klein expansion of the holographic currents, without resorting to LFH. We use the model of Ref. [17] and prove numerically that the leptonic decay constants of the excited states of the pion vanish in the chiral limit. In particular, we show that these results follow from a generalized GOR relationship whose counterpart in QCD is [1]

$$f_{\pi^n} m_{\pi^n}^2 = 2m_q \rho_{\pi^n}, \quad (1.1)$$

where m_{π^n} is the mass of the pion's n th excited state, $m_q = m_u = m_d$ (we work in the approximation of isospin symmetry), and ρ_{π^n} is the gauge-invariant residue at the pole $P^2 = -m_{\pi^n}^2$ in the pseudoscalar vertex function; it is related to the matrix-valued Bethe-Salpeter wave function $\chi_{\pi^n}^a(P, q)$ via

$$i\rho_{\pi^n} \delta^{ab} := \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[t^a \gamma_5 \chi_{\pi^n}^b(q, P)], \quad (1.2)$$

with the $SU(2)$ generators t^a , $a = 1, 2, 3$, normalized as $2\text{Tr}(t^a t^b) = \delta^{ab}$. Although m_q and ρ_{π^n} are scale dependent, the product $m_q \rho_{\pi^n}$ is renormalization group invariant. For the ground-state pion, DCSB implies [23]

$$\rho_{\pi^0} = -\frac{1}{f_{\pi^0}} \langle \bar{q}q \rangle, \quad (1.3)$$

where $\langle \bar{q}q \rangle = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$ is the vacuum quark condensate; when this is used in Eq. (1.1), the GOR relationship is obtained

$$f_{\pi^0}^2 m_{\pi^0}^2 = 2m_q |\langle \bar{q}q \rangle|. \quad (1.4)$$

As mentioned above, this key result of DCSB is obtained in holographic QCD in a rather straightforward way [17]. On the other hand, the vanishing of the leptonic decay constants of the pion's excited states is more subtle in holographic QCD, as we discuss in the present paper. In QCD, this key result follows via the following chain of arguments [1]: (1) the existence of excited states entails finite matrix-valued $\chi_{\pi^n}(P, q)$ wave functions; (2) the integral in Eq. (1.2) is finite (this follows from the ultraviolet behavior of the QCD quark-antiquark scattering kernel); (3) then

$$\rho_{\pi^n}^0 := \lim_{m_q \rightarrow 0} \rho_{\pi^n} = \text{finite}, \quad (1.5)$$

and (4) since, by hypothesis, $m_{\pi^n}^2 \neq 0$ in the chiral limit, Eq. (1.1) implies $f_{\pi^n} = 0$ for $m_q = 0$.

The paper is organized as follows. In the next section we present the hard-wall model we use; we present the action and discuss how DCSB is implemented in the model, derive the equations of motion and holographic currents, and present the Kaluza-Klein expansion of the bulk fields. In Sec. III we derive the expressions for the leptonic decay constants via the Kaluza-Klein expansions for the holographic currents; we obtain an extended partially conserved axial-vector current relation that includes the fields of the excited states of the pion and derive the generalized GOR relationship (1.1). Numerical results are presented in Sec. IV. We discuss how the field equations are solved numerically and present results for the masses of the pion's ground and excited states, for the normalization of the field equations, and for the function ρ_{π^n} which appears in the generalized GOR relationship (1.1). Finally, we present the results for the leptonic decay constants and discuss the consistency of the results with the generalized GOR relationship. Section V presents our conclusions and perspectives for future work.

II. THE HARD-WALL MODEL

The AdS/QCD approach deals with the construction of 5-d holographic models for QCD-like theories by considering deformations of the AdS₅/CFT₄ correspondence.

This bottom-up approach has proved to be very useful in describing many nonperturbative aspects of QCD and is complementary to the top-down approach, where 5-d holographic models arise as gravitational solutions of critical or noncritical string theories.

In the AdS/QCD approach, the simplest way to implement confinement is the so called hard-wall model, which consists of a slice of 5-d anti-de Sitter spacetime [16]:

$$ds^2 = \frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2), \quad 0 < z \leq z_0 \quad (2.1)$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the metric of 4-d flat spacetime. We are working in units where the AdS radius is unity.

The hard-wall cutoff z_0 in the 5-d geometry corresponds to an infrared mass gap in the 4-d gauge theory, $z_0 = 1/\Lambda_{\text{QCD}}$. As a consequence, conformal symmetry is broken and the theory is confining, as can be shown through the holographic calculation of the quark-antiquark potential [24].

A. DCSB implementation in holographic QCD

DCSB was first implemented in holographic QCD in Refs. [17,18]. A first application of the model to study ground-state properties of the pion in the chiral limit, in particular the form factor and charge density distribution, was performed in Ref. [25]. Here we will follow the conventions and notation of Ref. [17]. The AdS/CFT correspondence maps 4-d field theory operators $\mathcal{O}(x)$ to 5-d fields $\phi(x, z)$. In the case of QCD the relevant operators for describing DCSB are the left- and right-handed currents $J_{L\mu}^a = \bar{q}_L \gamma_\mu t^a q_L$, $J_{R\mu}^a = \bar{q}_R \gamma_\mu t^a q_R$, corresponding to the $SU(N_f)_L \times SU(N_f)_R$ chiral flavor symmetry and the quark bilinear operator $\bar{q}_R q_L$ related to DCSB. In the dual theory, these 4-d operators correspond to 5-d gauge fields $L_m^a(x, z)$, $R_m^a(x, z)$ and a 5-d bifundamental scalar field $X(x, z)$, both living in an AdS slice described by Eq. (2.1).

Since the mass of a p-form in 5-d AdS spacetime is related to the dimension Δ of the dual 4-d operator via the relation $m^2 = (\Delta - p)(\Delta + p - 4)$, one has that the gauge fields $L_m^a(x, z)$, $R_m^a(x, z)$ are massless whereas the scalar field $X(x, z)$ has a negative mass squared $m^2 = -3$.

The action in Ref. [17] can be written as

$$S = \int d^5x \sqrt{|g|} \text{Tr} \left[(D^m X)^\dagger (D_m X) + 3|X|^2 - \frac{1}{4g_5^2} (L^{mn} L_{mn} + R^{mn} R_{mn}) \right], \quad (2.2)$$

where

$$D_m X := \partial_m X - iL_m X + iX R_m, \quad (2.3)$$

$$L_{mn} := \partial_m L_n - \partial_n L_m - i[L_m, L_n], \quad (2.4)$$

$$R_{mn} := \partial_m R_n - \partial_n R_m - i[R_m, R_n]. \quad (2.5)$$

The action includes the N_f gauge fields L_m and R_m , corresponding to the left and right flavor currents in QCD, and the bifundamental scalar X dual to the quark bilinear operator $\bar{q}_R q_L$. In this paper we restrict the discussion to the case $N_f = 2$, corresponding to the quark flavors u and d . The dynamics of the 5-d fields L_m , R_m , and X is described by the action in Eq. (2.2) and the classical solution that describes chiral symmetry breaking is given by

$$L_m^0 = R_m^0 = 0, \quad 2X_0 = \zeta M z + \frac{\Sigma}{\zeta} z^3, \quad (2.6)$$

with

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_u & 0 \\ 0 & \sigma_d \end{pmatrix}. \quad (2.7)$$

The parameter $\zeta = \sqrt{N_c}/(2\pi)$ in Eq. (2.6) is introduced to be consistent with the counting rules of large- N_c QCD [26]. The AdS/CFT dictionary leads to identification of the coefficients M and Σ with the 4-d quark mass and chiral condensate terms responsible for the explicit and dynamical breaking of chiral symmetry.

To investigate the consequences of DCSB on the mesons we consider perturbations around the background fields in Eq. (2.6). First of all it is convenient to rewrite the gauge field fluctuations in terms of vectorial and axial fields

$$L_m = V_m + A_m, \quad R_m = V_m - A_m. \quad (2.8)$$

The bifundamental field X is decomposed into the classical part X_0 and a pseudoscalar fluctuation π in the following form [27]:

$$X = e^{i\pi^a t^a} X_0 e^{i\pi^a t^a}. \quad (2.9)$$

We will work in the isospin symmetrical limit, $m_u = m_d =: m_q$ and $\sigma_u = \sigma_d =: \sigma_d$; in this limit the matrix X_0 is proportional to the unit matrix and X becomes

$$X = X_0 e^{2i\pi^a t^a}. \quad (2.10)$$

The meson spectrum is obtained from the kinetic terms; we expand the original action (2.2) up to quadratic order in $V_m = V_m^a t^a$, $A_m = A^a t^a$ and the fluctuation π^a :

$$S^{\text{Kin}} = \int d^5x \sqrt{|g|} \left[\frac{v^2}{2} (\partial_m \pi^a - A_m^a)^2 - \frac{1}{4g_5^2} (v_a^{mn} v_{mn}^a + a_a^{mn} a_{mn}^a) \right], \quad (2.11)$$

with

$$v_{mn}^a := \partial_m V_n^a - \partial_n V_m^a, \quad a_{mn}^a := \partial_m A_n^a - \partial_n A_m^a, \quad (2.12)$$

and

$$v(z) := \zeta m_q z + \frac{\sigma_q}{\zeta} z^3. \quad (2.13)$$

B. Equations of motion and holographic currents

Writing S^{Kin} as

$$S^{\text{Kin}} = \int d^5x \mathcal{L}^{\text{Kin}}, \quad (2.14)$$

its variation takes the Euler-Lagrange form

$$\begin{aligned} \delta S^{\text{Kin}} &= \int d^5x \left[\left(\frac{\partial \mathcal{L}^{\text{Kin}}}{\partial V_\ell^a} - \partial_m P_{V,a}^{m\ell} \right) \delta V_\ell^a \right. \\ &\quad \left. + \left(\frac{\partial \mathcal{L}^{\text{Kin}}}{\partial A_\ell^a} - \partial_m P_{A,a}^{m\ell} \right) \delta A_\ell^a + \left(\frac{\partial \mathcal{L}^{\text{Kin}}}{\partial \pi^a} - \partial_m P_{\pi,a}^m \right) \delta \pi^a \right] \\ &\quad + \int d^5x \partial_m (P_{V,a}^{m\ell} \delta V_\ell^a + P_{A,a}^{m\ell} \delta A_\ell^a + P_{\pi,a}^m \delta \pi^a), \quad (2.15) \end{aligned}$$

where $P_{V,a}^{m\ell}$ and $P_{A,a}^{m\ell}$ are the conjugate momenta to the vector fields V_ℓ^a and A_ℓ^a

$$P_{V,a}^{m\ell} := \frac{\partial \mathcal{L}^{\text{Kin}}}{\partial (\partial_m V_\ell^a)}, \quad P_{A,a}^{m\ell} := \frac{\partial \mathcal{L}^{\text{Kin}}}{\partial (\partial_m A_\ell^a)}, \quad (2.16)$$

and $P_{\pi,a}^m$ the conjugate momentum to the field π^a

$$P_{\pi,a}^m := \frac{\partial \mathcal{L}^{\text{Kin}}}{\partial (\partial_m \pi^a)}. \quad (2.17)$$

From Eq. (2.11) we find for the derivatives of \mathcal{L}^{Kin} with respect to the fields

$$\frac{\partial \mathcal{L}^{\text{Kin}}}{\partial V_\ell^a} = \frac{\partial \mathcal{L}^{\text{Kin}}}{\partial \pi^a} = 0, \quad (2.18)$$

$$\frac{\partial \mathcal{L}^{\text{Kin}}}{\partial A_\ell^a} = -v^2 \sqrt{|g|} (\partial^\ell \pi_a - A_\ell^a), \quad (2.19)$$

and for the conjugate momenta

$$P_{V,a}^{m\ell} = -\frac{1}{g_5^2} \sqrt{|g|} v_a^{m\ell}, \quad (2.20)$$

$$P_{A,a}^{m\ell} = -\frac{1}{g_5^2} \sqrt{|g|} a_a^{m\ell}, \quad (2.21)$$

$$P_{\pi,a}^m = v^2 \sqrt{|g|} (\partial^m \pi_a - A_a^m). \quad (2.22)$$

Imposing the stationarity condition $\delta S^{\text{Kin}} = 0$, we find from the first three terms in Eq. (2.15) the field equations

$$\partial_m (\sqrt{|g|} v_a^{mn}) = 0, \quad (2.23)$$

$$\partial_m (\sqrt{|g|} a_a^{mn}) - g_5^2 v^2 \sqrt{|g|} (\partial^n \pi^a - A_n^a) = 0, \quad (2.24)$$

$$\partial_m [v^2 \sqrt{|g|} (\partial^m \pi^a - A_a^m)] = 0. \quad (2.25)$$

The mass spectrum of the vector and axial-vector mesons and pions can be found by solving these equations of motion in momentum space under appropriate boundary conditions. We will choose, however, a different method based on the Kaluza-Klein expansion. As explained in the next subsection, the advantage of the Kaluza-Klein method is the extraction of a 4-d off-shell action for the mesons.

The last three terms in (2.15) form a surface term, whose nonvanishing contribution can be written as

$$\delta S_{\text{Bdy}}^{\text{Kin}} = \int d^4x (P_{V,a}^{z\mu} \delta V_\mu^a + P_{A,a}^{z\mu} \delta A_\mu^a + P_{\pi,a}^z \delta \pi^a)_{z=\epsilon}^{z=z_0}. \quad (2.26)$$

The terms at $z = z_0$ vanish under Neumann boundary conditions

$$\partial_z V_\mu^a|_{z=z_0} = \partial_z A_\mu^a|_{z=z_0} = \partial_z \pi^a|_{z=z_0} = 0, \quad (2.27)$$

and for the gauge choice $V_z^a = A_z^a = 0$. The boundary terms at $z = \epsilon$ can be written as (we distinguish vectorial Minkowski indices $\hat{\mu}$ and vectorial AdS indices μ)

$$\begin{aligned} \delta S_{\text{Bdy}}^{\text{Kin}} &= - \int d^4x [\langle J_{V,a}^{\hat{\mu}} \rangle (\delta V_{\hat{\mu}}^a)_{z=\epsilon} + \langle J_{A,a}^{\hat{\mu}} \rangle (\delta A_{\hat{\mu}}^a)_{z=\epsilon} \\ &\quad + \langle J_{\pi,a} \rangle (\delta \pi^a)_{z=\epsilon}], \quad (2.28) \end{aligned}$$

where we find the holographic currents

$$\langle J_{V,a}^{\hat{\mu}}(x) \rangle = P_{V,a}^{z\mu}|_{z=\epsilon} = -\frac{1}{g_5^2} (\sqrt{|g|} v^{z\mu})_{z=\epsilon}, \quad (2.29)$$

$$\langle J_{A,a}^{\hat{\mu}}(x) \rangle = P_{A,a}^{z\mu}|_{z=\epsilon} = -\frac{1}{g_5^2} (\sqrt{|g|} a^{z\mu})_{z=\epsilon}, \quad (2.30)$$

$$\begin{aligned} \langle J_{\pi,a}(x) \rangle &= P_{\pi,a}^z|_{z=\epsilon} = [\sqrt{|g|}v^2(\partial^z\pi_a - A_z^a)]_{z=\epsilon} \\ &= \partial_\mu \langle J_{A,a}^\mu(x) \rangle. \end{aligned} \quad (2.31)$$

These holographic currents are identified with the vacuum expectation values of the 4-d vectorial, axial, and pion current operators in Minkowski spacetime.

C. The Kaluza-Klein expansion

First of all we evaluate the metric (2.1) to write the action of Eq. (2.11) in the form

$$\begin{aligned} S^{\text{Kin}} &= \int d^4x \int \frac{dz}{z} \left\{ \frac{v^2}{2z^2} [-(\partial_z\pi^a - A_z^a)^2 + (\partial_{\hat{\mu}}\pi^a - A_{\hat{\mu}}^a)^2] \right. \\ &\quad \left. - \frac{1}{4g_5^2} [-2(v_{z\hat{\mu}}^a)^2 + (v_{\hat{\mu}\hat{\nu}}^a)^2 - 2(a_{z\hat{\mu}}^a)^2 + (a_{\hat{\mu}\hat{\nu}}^a)^2] \right\}. \end{aligned} \quad (2.32)$$

The axial-vector field $A_{\hat{\mu}}^a$ can be decomposed into transverse and longitudinal parts:

$$A_{\hat{\mu}}^a = A_{\hat{\mu}}^{\perp,a} + \partial_{\hat{\mu}}\phi^a, \quad (2.33)$$

where $\partial_{\hat{\mu}}A_{\perp,a}^{\hat{\mu}} = 0$. The transverse part will be associated with the axial-vector mesons whereas the longitudinal part will be associated with the pions. The action (2.32) is invariant under the gauge transformations

$$V_m^a \rightarrow V_m^a - \partial_m\lambda_V^a, \quad (2.34)$$

$$A_m^a \rightarrow A_m^a - \partial_m\lambda_A^a, \quad (2.35)$$

$$\pi^a \rightarrow \pi^a - \lambda_A^a. \quad (2.36)$$

Because of this, we can fix the gauge as $V_z^a = A_z^a = 0$ and then Eq. (2.32) reduces to

$$\begin{aligned} S^{\text{Kin}} &= \int d^4x \int \frac{dz}{z} \left\{ \frac{v^2}{2z^2} [-(\partial_z\pi^a)^2 + (\partial_{\hat{\mu}}\pi^a - \partial_{\hat{\mu}}\phi^a)^2] \right. \\ &\quad + (A_{\hat{\mu}}^{\perp,a})^2 + \partial_{\hat{\mu}}(\dots) - \frac{1}{4g_5^2} [-2(\partial_z V_{\hat{\mu}}^a)^2 + (v_{\hat{\mu}\hat{\nu}}^a)^2 \\ &\quad \left. - 2(\partial_z A_{\hat{\mu}}^a)^2 - 2(\partial_z \partial_{\hat{\mu}}\phi^a)^2 + (a_{\hat{\mu}\hat{\nu}}^{\perp,a})^2] \right\}, \end{aligned} \quad (2.37)$$

where the terms in (...) vanish by choosing appropriate boundary conditions.

The action (2.37) is in a suitable form for the Kaluza-Klein expansion. This consists in expanding the 5-d fields in an infinite discrete set of modes. Each mode will be a product of a pure wave function in the radial coordinate z and a meson field depending on the Minkowski coordinates x . For the present case, the Kaluza-Klein expansion for the bulk fields $V_{\hat{\mu}}^a$, $A_{\hat{\mu}}^{\perp,a}$, π^a , and ϕ^a take the form

$$V_{\hat{\mu}}^a(x, z) = g_5 \sum_{n=0}^{\infty} v^{a,n}(z) \hat{V}_{\hat{\mu}}^{a,n}(x), \quad (2.38)$$

$$A_{\hat{\mu}}^{\perp,a}(x, z) = g_5 \sum_{n=0}^{\infty} a^{a,n}(z) \hat{A}_{\hat{\mu}}^{a,n}(x), \quad (2.39)$$

$$\pi^a(x, z) = g_5 \sum_{n=0}^{\infty} \pi^{a,n}(z) \hat{\pi}^{a,n}(x), \quad (2.40)$$

$$\phi^a(x, z) = g_5 \sum_{n=0}^{\infty} \phi^{a,n}(z) \hat{\phi}^{a,n}(x). \quad (2.41)$$

The wave functions $\phi^{a,n}(z)$ and $\pi^{a,n}(z)$ are not independent. The relation between them can be obtained from the 5-d field equations in Eqs. (2.23)–(2.25) or via an off-shell integration, as described below.

Using the Kaluza-Klein expansions (2.38)–(2.41) in the action (2.37), one can separate the z and x integrals and write the action as

$$\begin{aligned} S^{\text{Kin}} &= \sum_{n,m=0}^{\infty} \int d^4x \left\{ \frac{1}{2} \Delta_{\pi}^{a,nm} [\partial_{\hat{\mu}}\pi^{a,n}(x)] \partial^{\hat{\mu}}\pi^{a,m}(x) \right. \\ &\quad - \frac{1}{2} M_{\pi}^{a,nm} \hat{\pi}^{a,n}(x) \hat{\pi}^{a,m}(x) - \frac{1}{4} \Delta_V^{a,nm} \hat{v}_{\hat{\mu}\hat{\nu}}^{a,n}(x) \hat{v}_{a,m}^{\hat{\mu}\hat{\nu}}(x) \\ &\quad + \frac{1}{2} M_V^{a,nm} \hat{V}_{\hat{\mu}}^{a,n}(x) \hat{V}_{a,m}^{\hat{\mu}}(x) - \frac{1}{4} \Delta_A^{a,nm} \hat{a}_{\hat{\mu}\hat{\nu}}^{a,n}(x) \hat{a}_{a,m}^{\hat{\mu}\hat{\nu}}(x) \\ &\quad \left. + \frac{1}{2} M_A^{a,nm} \hat{A}_{\hat{\mu}}^{a,n}(x) \hat{A}_{a,m}^{\hat{\mu}}(x) \right\}, \end{aligned} \quad (2.42)$$

where the coefficients for the 4-d fields are given by the following z integrals

$$\begin{aligned} \Delta_{\pi}^{a,nm} &= \int \frac{dz}{z} \{ [\partial_z \phi^{a,n}(z)] \partial_z \phi^{a,m}(z) \\ &\quad + \beta(z) [\pi^{a,n}(z) - \phi^{a,n}(z)] [\pi^{a,m}(z) - \phi^{a,m}(z)] \}, \end{aligned} \quad (2.43)$$

$$\begin{aligned} \Delta_V^{a,nm} &= \int \frac{dz}{z} v^{a,n}(z) v^{a,m}(z), \\ \Delta_A^{a,nm} &= \int \frac{dz}{z} a^{a,n}(z) a^{a,m}(z), \end{aligned} \quad (2.44)$$

$$\begin{aligned} M_{\pi}^{a,nm} &= \int \frac{dz}{z} \beta(z) [\partial_z \pi^{a,n}(z)] \partial_z \pi^{a,m}(z), \\ M_V^{a,nm} &= \int \frac{dz}{z} [\partial_z v^{a,n}(z)] \partial_z v^{a,m}(z), \\ M_A^{a,nm} &= \int \frac{dz}{z} \{ [\partial_z a^{a,n}(z)] \partial_z a^{a,m}(z) \\ &\quad + \beta(z) a^{a,n}(z) a^{a,m}(z) \}, \end{aligned} \quad (2.45)$$

and we have defined $\beta(z)$ as

$$\beta(z) := \frac{g_5^2}{z^2} v(z)^2 = g_5^2 \left(\zeta m_q + \frac{\sigma_q}{\zeta} z^2 \right)^2. \quad (2.46)$$

Now the goal is to obtain a 4-d action with standard kinetic terms for the vector $\hat{V}_{\hat{\mu}}^{a,n}$ and axial-vector $\hat{A}_{\hat{\mu}}^{a,n}$ mesons and pions $\pi^{a,n}$. This can be achieved imposing the conditions

$$\Delta_{\pi}^{a,nm} = \Delta_V^{a,nm} = \Delta_A^{a,nm} = \delta^{nm}, \quad (2.47)$$

$$M_{\pi}^{a,nm} = m_{\pi^{a,n}}^2 \delta^{nm}, \quad M_V^{a,nm} = m_{V^{a,n}}^2 \delta^{nm}, \quad (2.48)$$

$$M_A^{a,nm} = m_{A^{a,n}}^2 \delta^{nm}. \quad (2.49)$$

This way we arrive at the following 4-d action:

$$\begin{aligned} S^{\text{Kin}} = \sum_{n=0}^{\infty} \int d^4x \left\{ \frac{1}{2} [\partial_{\hat{\mu}} \hat{\pi}^{a,n}(x)]^2 - \frac{1}{2} m_{\pi^{a,n}}^2 [\hat{\pi}^{a,n}(x)]^2 \right. \\ \left. - \frac{1}{4} [\hat{v}_{\hat{\mu}\hat{\nu}}^{a,n}(x)]^2 + \frac{1}{2} m_{V^{a,n}}^2 [\hat{V}_{\hat{\mu}}^{a,n}(x)]^2 \right. \\ \left. - \frac{1}{4} [\hat{a}_{\hat{\mu}\hat{\nu}}^{a,n}(x)]^2 + \frac{1}{2} m_{A^{a,n}}^2 [\hat{A}_{\hat{\mu}}^{a,n}(x)]^2 \right\}. \quad (2.50) \end{aligned}$$

The conditions (2.47) are precisely the normalization rules for the wave functions $v^{a,n}(z)$, $a^{a,n}(z)$, $\pi^{a,n}(z)$, and $\phi^{a,n}(z)$. Moreover, the conditions on the masses, Eqs. (2.48) and (2.49), can be obtained from the normalization conditions (2.47) as long as we impose the following equations

$$\frac{\beta(z)}{z} [\pi^{a,n}(z) - \phi^{a,n}(z)] = -\partial_z \left[\frac{1}{z} \partial_z \phi^{a,n}(z) \right], \quad (2.51)$$

$$\beta(z) \partial_z \pi^{a,n}(z) = m_{\pi^{a,n}}^2 \partial_z \phi^{a,n}(z), \quad (2.52)$$

$$-\partial_z \left[\frac{1}{z} \partial_z v^{a,n}(z) \right] = \frac{m_{V^{a,n}}^2}{z} v^{a,n}(z), \quad (2.53)$$

$$\left[-\partial_z \left(\frac{1}{z} \partial_z \right) + \frac{1}{z} \beta(z) \right] a^{a,n}(z) = \frac{m_{A^{a,n}}^2}{z} a^{a,n}(z). \quad (2.54)$$

These equations can be interpreted as the on-shell conditions for the wave functions $v^{a,n}(z)$, $a^{a,n}(z)$, $\pi^{a,n}(z)$, and $\phi^{a,n}(z)$. It should be noted that these equations can simply be obtained from the 5-d field equations in Eqs. (2.23)–(2.25). However, the convenience of our method is the fact that the final 4-d action (2.50) remains off shell, a property that can be very useful when considering scattering amplitudes, which require the knowledge of Feynman rules.

To conclude, we mention the meson masses $m_{\pi^{a,n}}$, $m_{V^{a,n}}$, and $m_{A^{a,n}}$ are completely determined once we find solutions for Eqs. (2.51)–(2.54). In the following section, solutions to

these equations are found imposing Dirichlet boundary conditions at $z = \epsilon$:

$$\pi^{a,n}|_{z=\epsilon} = v^{a,n}|_{z=\epsilon} = a^{a,n}|_{z=\epsilon} = 0, \quad (2.55)$$

and Neumann boundary conditions at $z = z_0$:

$$\partial_z \pi^{a,n}|_{z=z_0} = \partial_z v^{a,n}|_{z=z_0} = \partial_z a^{a,n}|_{z=z_0} = 0. \quad (2.56)$$

Since there is no flavor mixing, the equations of motion and boundary conditions have the same form for every flavor index a and from now on we will omit that index. The latter is relevant for the calculation of pion decay constants.

III. LEPTONIC DECAY CONSTANTS OF THE EXCITED STATES OF THE PION

In this section we present the main result of our paper: the behavior of the leptonic decay constants of the excited states of the pion near the chiral limit. First we show how to calculate the decay constants f_{π^n} from holography and arrive at the equivalent to Eq. (1.1) in QCD. After describing some technical details on the calculation of the decay constants, we present our numerical results for the f_{π^n} .

A. Holographic calculation of the decay constants

The simplest way to extract the leptonic decay constant is to use the Kaluza-Klein expansions (2.38)–(2.41) for the holographic currents (2.29)–(2.31):

$$\langle J_V^{\hat{\mu}}(x) \rangle = \sum_{n=0}^{\infty} \left[\frac{1}{g_5 z} \partial_z v^n(z) \right]_{z=\epsilon} \hat{V}_n^{\hat{\mu}}(x), \quad (3.1)$$

$$\begin{aligned} \langle J_A^{\hat{\mu}}(x) \rangle = \sum_{n=0}^{\infty} \left[\frac{1}{g_5 z} \partial_z a^n(z) \right]_{z=\epsilon} \hat{A}_n^{\hat{\mu}}(x) \\ + \sum_{n=0}^{\infty} \left[\frac{1}{g_5 z} \partial_z \phi^n(z) \right]_{z=\epsilon} \partial^{\hat{\mu}} \hat{\pi}^n(x), \quad (3.2) \end{aligned}$$

$$\begin{aligned} \partial_{\hat{\mu}} \langle J_A^{\hat{\mu}}(x) \rangle = \langle J_{\Pi}(x) \rangle \\ = - \sum_{n=0}^{\infty} \left[\frac{\beta(z)}{g_5 z} \partial_z \pi^n(z) \right]_{z=\epsilon} \hat{\pi}_n(x). \quad (3.3) \end{aligned}$$

As explained in the previous section, we are omitting the flavor index a . Here the 4-d fields $\hat{V}_n^{\hat{\mu}}(x)$, $\hat{A}_n^{\hat{\mu}}(x)$, and $\hat{\pi}^n(x)$ are on shell. From the current expansions (3.1) and (3.2) we are able to extract the decay constants for the vector mesons (g_{V^n}), the axial-vector mesons (g_{A^n}), and the pions (f_{π^n}):

$$g_{V^n} = \left[\frac{1}{g_5 z} \partial_z v^n(z) \right]_{z=\epsilon}, \quad (3.4)$$

$$g_{A^n} = \left[\frac{1}{g_5 z} \partial_z a^n(z) \right]_{z=\epsilon}, \quad (3.5)$$

$$f_{\pi^n} = \left[-\frac{1}{g_5 z} \partial_z \phi^n(z) \right]_{z=\epsilon}. \quad (3.6)$$

These results are consistent with the standard definition of meson decay constants:

$$\langle 0 | J_V^{\hat{\mu}}(0) | V_n(p, \lambda) \rangle = \epsilon^{\hat{\mu}}(p, \lambda) g_{V^n}, \quad (3.7)$$

$$\langle 0 | J_A^{\hat{\mu}}(0) | A_n(p, \lambda) \rangle = \epsilon^{\hat{\mu}}(p, \lambda) g_{A^n}, \quad (3.8)$$

$$\langle 0 | J_A^{\hat{\mu}}(0) | \pi_n(p) \rangle = i p^{\hat{\mu}} f_{\pi^n}. \quad (3.9)$$

Taking the divergence of (3.2) and using (3.6), one obtains the interesting relation

$$f_{\pi^n} m_{\pi^n}^2 = -\frac{1}{g_5} \left[\frac{\beta(z)}{z} \partial_z \pi^n(z) \right]_{z=\epsilon}, \quad (3.10)$$

where we made use of the on-shell equation for the pion field, $\partial^2 \hat{\pi}^n(x) = -m_{\pi^n}^2 \hat{\pi}^n(x)$. Moreover, using this result into Eq. (3.3), the divergence of the axial current takes the form of an extended partial conservation of the axial current (PCAC) relation

$$\partial_\mu \langle J_A^{\hat{\mu}}(x) \rangle = \sum_{n=0}^{\infty} f_{\pi^n} m_{\pi^n}^2 \hat{\pi}_n(x). \quad (3.11)$$

Interestingly, such a relation was proposed long ago [28] in the context of current algebra studies. In particular, such an extended PCAC relation leads naturally to the vanishing of the leptonic decay constants of the pion's excited states when the ground-state pion is the Goldstone boson of DCSB and $m_{\pi^n} \neq 0$, $n \geq 1$, in the chiral limit. Moreover, from Eq. (3.10) one obtains a generalized GOR relationship in the form of Eq. (1.1) if one makes the identification

$$2m_q \rho_{\pi^n} := -\frac{1}{g_5} \left[\frac{\beta(z)}{z} \partial_z \pi^n(z) \right]_{z=\epsilon}. \quad (3.12)$$

As we will show, independently of the mode number n , the function ρ_{π^n} is finite for $m_q \rightarrow 0$. This, like in QCD, allows us to predict the behavior of f_{π^n} close to the chiral limit.

The vanishing of the leptonic decay constants of the pion's excited states in the chiral limit could also be inferred from the analysis of the spectral representation of the two-point function of axial-vector currents. The residues of the pole terms in the two-point function are proportional to matrix elements related to the leptonic decay constants and, in the chiral limit, the pole terms associated with the pion's excited states would be absent—see e.g. Eq. (51) of Ref. [29].

B. Normalization and asymptotic expansion

It is possible to decouple the system of equations (2.51) and (2.52) and find an independent equation for the function $\Pi^n(z) := \partial_z \pi^n(z)$

$$(z\partial_z)^2 \Pi^n(z) + A(z) z \partial_z \Pi^n(z) + B^n(z) \Pi^n(z) = 0, \quad (3.13)$$

with $A(z)$ and $B_n(z)$ given by

$$A(z) = z \partial_z \ln \beta(z) - 2, \quad (3.14)$$

$$B_n(z) = 1 + z^2 [\partial_z^2 \ln \beta(z) + m_{\pi^n}^2 - \beta(z)]. \quad (3.15)$$

From (3.10) and (3.12) we find that the function ρ_{π^n} is determined by Π^n through the relation

$$\rho_{\pi^n} = -\frac{1}{2m_q g_5} \left[\frac{\beta(z)}{z} \Pi^n(z) \right]_{z=\epsilon}. \quad (3.16)$$

Expanding $\Pi^n(z)$ in powers of z near the boundary we find the asymptotic solution

$$\begin{aligned} \Pi^n(z) &= C_n \left[-z + \frac{1}{4} \left(m_{\pi^n}^2 + \frac{32\pi^2 \sigma}{3m_q} - 3m_q^2 \right) z^3 + \dots \right] \\ &=: C_n \Pi_U^n(z), \end{aligned} \quad (3.17)$$

where the dots represent higher powers in z and we have defined $\Pi_U^n(z)$ as the un-normalized function associated with $\Pi^n(z)$. In the numerical procedure we will focus on the function $\Pi_U^n(z)$. Note that the solution (3.17) naturally satisfies the Dirichlet boundary condition (2.55).

The constant C_n in Eq. (3.17) is determined from the normalization condition $\Delta_{\pi^n}^{nm} = \delta^{nm}$. From Eqs. (2.51) and (2.52) we find that $\Pi^n(z)$ obeys the normalization condition

$$\int \frac{dz}{z} \beta(z) \Pi^n(z) \Pi^m(z) = m_{\pi^n}^2 \delta^{mn}, \quad (3.18)$$

which, in turn, leads to

$$C_n = \frac{m_{\pi^n}}{N_{\pi^n}}, \quad (3.19)$$

with

$$N_{\pi^n}^2 = \int \frac{dz}{z} \beta(z) [\Pi_U^n(z)]^2. \quad (3.20)$$

We end this subsection noticing that, by using the asymptotic expansion (3.17) and the definition (2.46), the holographic prescription (3.16) takes the simple form

$$\rho_{\pi^n} = \frac{g_5 \zeta^2 m_q m_{\pi^n}}{2 N_{\pi^n}}. \quad (3.21)$$

This formula will be very useful in the following section.

IV. NUMERICAL RESULTS

In this section we present our numerical results. We start with the mass spectrum of the pions. The spectrum is obtained by solving Eq. (3.13) for the auxiliary wave function $\Pi^n(z) = \partial_z \pi^n(z)$ and imposing the boundary conditions $\pi^n(\epsilon) = 0$ and $\partial_z \pi^n(z_0) = 0$. We integrate numerically Eq. (3.13) from $z = \epsilon$ to $z = z_0$, using the asymptotic solution (3.17) to extract the value for the derivative of $\Pi^n(z)$ at $z = \epsilon$. We use the shooting method, which consists in shooting values in the parameter plane m_{π^n} vs m_q until we find a solution that satisfies the IR condition $\Pi^n(z_0) = \partial_z \pi^n(z_0) = 0$. Due to the linearity property of Eq. (3.13), to get the pion spectrum it is sufficient to work with the un-normalized wave function $\Pi_U^n(z)$, defined in Eq. (3.17). This is equivalent to setting C_n to 1.

The model we use has three free parameters, z_0 , m_q , and σ_q . We use the same values used in Refs. [17,27]. The hard-wall cutoff z_0 is the dual of the QCD infrared mass gap and is completely fixed (in the limit of exact isospin symmetry) by the mass of the ρ meson, namely $z_0 = (322.5 \text{ MeV})^{-1}$. Therefore, z_0 essentially sets the scale of all hadronic masses. The remaining parameters are fixed fitting the experimental values of the mass $m_{\pi^0} = 139.6 \text{ MeV}$ and leptonic decay constant $f_{\pi^0} = 92.4 \text{ MeV}$ of the ground-state pion¹; their values are $m_q = 8.31 \text{ MeV}$ and $\sigma_q = (213.7 \text{ MeV})^3$. Figure 1 displays the results for the m_q dependence of the pion masses, for the ground state and first three excited states. We have also obtained solutions for $n > 3$ up to $n = 6$; the m_q dependence of those solutions is similar to that shown in Fig. 1 for the three lowest excited states. The mass of the ground-state pion can be fitted as $m_{\pi^0} \sim m_q^{1/2}$ near the chiral limit, which is consistent with the GOR (1.4). On the other hand, the masses of the excited states can be fitted with the linear form $m_{\pi^n} = m_{\pi^n}^0 + a_n m_q$, where $m_{\pi^n}^0$ are the corresponding masses in the chiral limit.

For establishing the finiteness of ρ_{π^n} in the chiral limit, we consider the m_q dependence of the normalization constant N_{π^n} , defined in Eq. (3.20). N_{π^n} is completely determined from the knowledge of the un-normalized wave function $\Pi_U^n(z)$, which enters in the determination of the mass spectrum. The results are displayed in Fig. 2. The curve for the ground-state pion can be fitted as $N_{\pi^0} \sim m_q^{3/2}$, while those for the excited states can be fitted with a linear

¹Although the experimental value of the leptonic decay constant of the pion has been updated recently by the Particle Data Group [30], to the central value $f_\pi = 92.2 \text{ MeV}$, we prefer to use the old value to compare results with Refs. [17,27].

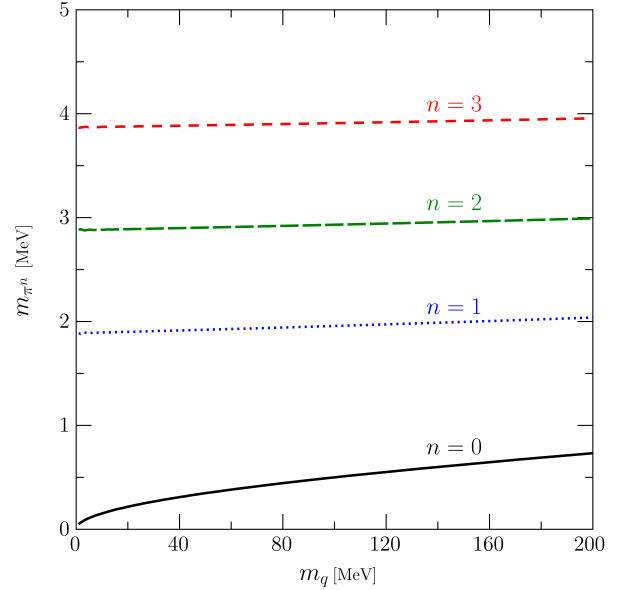


FIG. 1 (color online). Quark mass dependence of the pion masses.

function $N_{\pi^n} \sim m_q$, $n \geq 1$. As we show next, these different m_q dependences of N_{π^n} for the ground state and the excited states, when combined with the different m_q dependence of m_{π^n} , are responsible for the finiteness of f_{π^0} and the vanishing of f_{π^n} for $n \geq 1$ in the chiral limit.

As the spectrum and the normalization constant N_{π^n} are known, the function ρ_{π^n} can be readily determined using the formula in Eq. (3.21). Figure 3 displays the results. The curves in this figure show that ρ_{π^n} is finite as $m_q \rightarrow 0$ and remarkably independent of m_q for $0 \leq m_q \leq 30 \text{ MeV}$ for

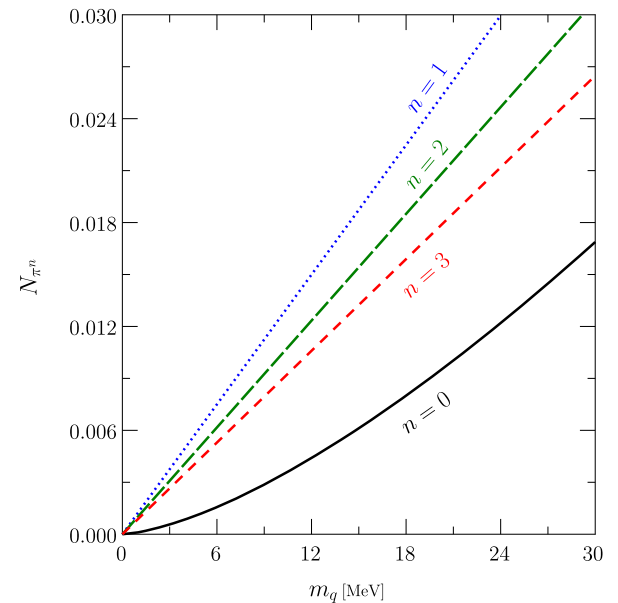
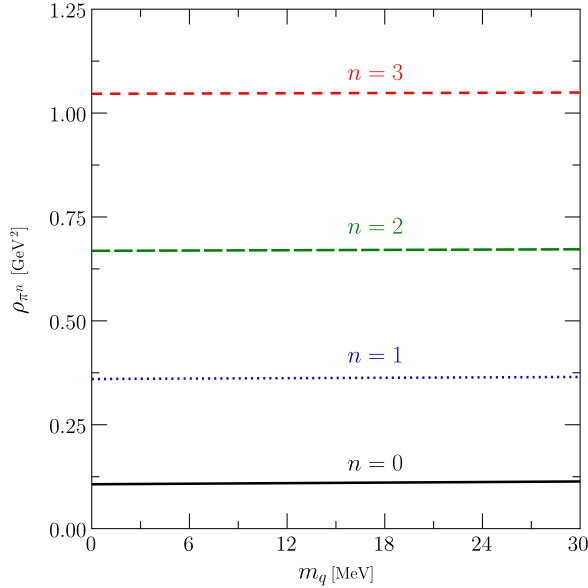


FIG. 2 (color online). Quark mass dependence of the normalization constants.


 FIG. 3 (color online). Quark mass dependence of ρ_{π^n} .

all the values of n investigated. This clearly establishes the finiteness of ρ_{π^n} at small m_q and, as we discuss next, leads to the conclusion that in the chiral limit, $f_{\pi^n} = 0$ for $n \geq 1$.

At this stage we are able to present the main result of the present paper, namely the behavior of the pion decay constants f_{π^n} near the chiral limit. The numerical results are displayed in Fig. 4. As one can see from the figure, while the ground-state pion possesses a finite leptonic decay constant f_{π^0} the excited states have leptonic decay constants f_{π^n} that vanish in the chiral limit. As mentioned at the beginning of the present section, for $m_q = 8.31$ MeV, the ground-state pion decay constant is $f_{\pi^0} \approx 92.4$ MeV. Table I presents results for f_{π^n} .

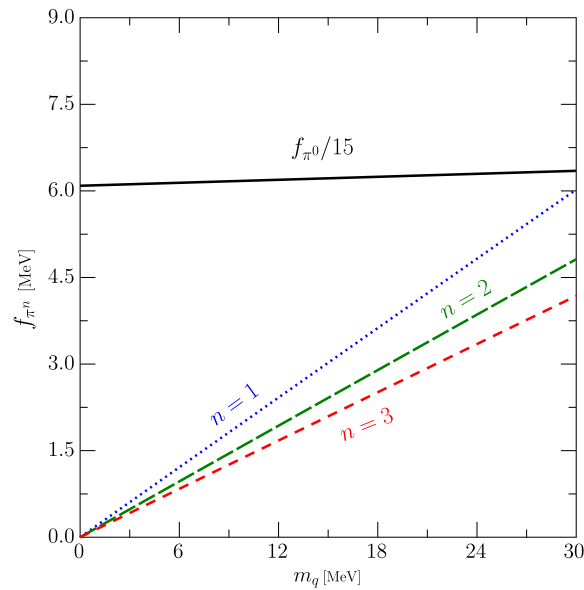

 FIG. 4 (color online). Quark mass dependence of f_{π^n} .

TABLE I. Leptonic decay constants for the ground state and the first three excited states of the pion.

n	0	1	2	3
f_{π^n} (MeV)	92.4	1.68	1.34	1.16

Experimental extraction of the leptonic decay constants of the excited states is very difficult because the decays of such states are dominated by strong decays. There is a bound on f_{π^1} extracted from B decays [3], which is $f_{\pi^1} < 0.064 f_{\pi^0}$. Our result in Table I is perfectly compatible with this bound, but is about five times smaller than the lattice result of Ref. [2].

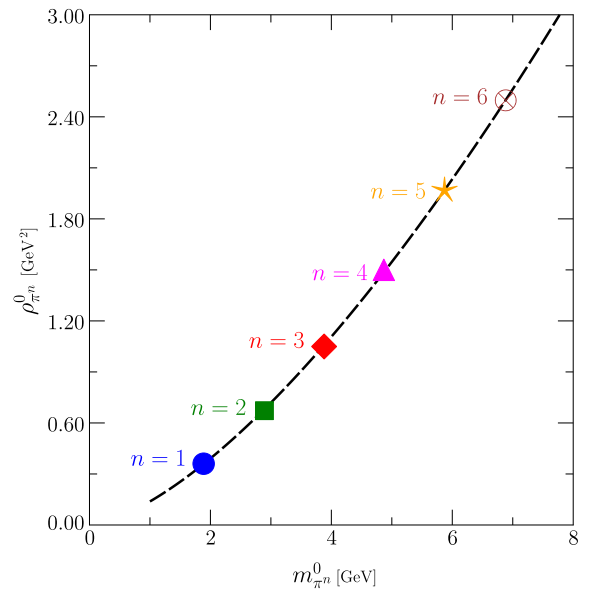
We also note that the curves for the excited states can be fitted with a linear quark mass dependence. Such a linear m_q scaling of f_{π^n} for $n \geq 1$ is precisely the one predicted in QCD [1] through the generalized GOR relationship (1.1):

$$f_{\pi^n} = \frac{2m_q \rho_{\pi^n}}{m_{\pi^n}^2} \sim m_q, \quad n \geq 1, \quad (4.1)$$

as $m_{\pi^n}^2 \sim (m_q)^0$ and $\rho_{\pi^n} \sim (m_q)^0$.

Finally, for completeness we examine $\rho_{\pi^n}^0$, the chiral limit of ρ_{π^n} defined in Eq. (1.5), as a function of the pion masses of excited states in the chiral limit, $m_{\pi^n}^0$. Note that although the ρ_{π^n} are, to a good precision, m_q independent, the masses m_{π^n} are slightly dependent on m_q and this makes the m_{π^n} dependence of ρ_{π^n} nontrivial. The results are shown in Fig. 5. We found that the six lowest discrete eigenvalues can be fitted as

$$\rho_{\pi^n}^0 = \gamma (m_{\pi^n}^0)^{3/2}, \quad n \geq 1, \quad (4.2)$$


 FIG. 5 (color online). The function $\rho_{\pi^n}^0$, defined in Eq. (1.5), for the first six excited states. The dashed line is a fit to the discrete eigenvalues.

with $\gamma = 4.375 \text{ MeV}^{1/2}$. With this, Eq. (3.10) then implies that the generalized GOR relationship takes the form

$$f_{\pi^n}^0 := \lim_{m_q \rightarrow 0} f_{\pi^n} = \gamma \frac{2m_q}{\sqrt{m_{\pi^n}^0}}, \quad n \geq 1. \quad (4.3)$$

It is important to remark that the hard-wall model fails to reproduce the Regge behavior of mesons. This is a typical feature of holographic models based on the supergravity approximation. A more realistic holographic model, the soft-wall model of Karch *et al.* [31], which contains an additional background dilaton field, is able to obtain the correct Regge behavior of mesons and it is expected to arise from string theory beyond the supergravity approximation. In addition, as discussed in Ref. [31], a chiral condensate consistent with the dynamical breaking of chiral symmetry in QCD requires the addition of higher order terms in the background scalar potential $U(X)$, beyond the $3|X|^2$ term in the action (2.2)—Ref. [32], for example, adds a quartic term. Although going from a hard-wall background to a soft-wall background definitely changes the meson wave functions, the holographic prescriptions for the currents, obtained in Sec. III A, will not be modified. This is because the soft-wall dilaton goes to zero in the ultraviolet. Moreover, the ρ function given in Eqs. (3.12) and (3.21) should remain valid since the background scalar X and the (un-normalized) pion wave function should have the same ultraviolet behavior as in the hard-wall model. This means the way the derived GOR relation is modified by the soft-wall dilaton and potential $U(X)$ will be dictated by the normalization constants of the pion wave functions. Therefore, it seems that the general qualitative conclusion of the dramatic suppression of the leptonic decay constants of the pion's excited states will therefore not be modified in the soft-wall model. A detailed study of this will be the subject of a future work.

V. CONCLUSIONS AND PERSPECTIVES

We have investigated the leptonic decay constants of the pion and its excitations in a five-dimensional holographic hard-wall model for QCD. We have used the model proposed in Refs. [17,18] for implementing dynamical chiral symmetry breaking and introduced a direct way of calculating the decay constants via the definition of holographic currents. We have proved numerically that the leptonic decay constants of the excited states of the pion vanish in the chiral limit. In addition, we have shown that these results follow from the generalized GOR relationship Eq. (1.1), whose counterpart in QCD was first derived in QCD in Ref. [1].

Our results for the vanishing of the leptonic decay constants of the pion's excited states in the chiral limit, besides being in agreement with QCD, might shed light on the failure of LFH in reproducing such results. A key feature of the approach we followed in the present paper is

the generalized GOR relationship of Eq. (1.1). The generalized GOR relationship is a direct consequence of the dynamical breaking of chiral symmetry implemented by the action of Eq. (2.2) via a scalar field $X(x, z)$ that has a negative mass squared, which leads to the extended PCAC relation given in Eq. (3.11). On the other hand, as mentioned in the Introduction, the way chiral symmetry is treated in LFH is nonstandard, as the vanishing of the pion mass in the chiral limit is not a result of the dynamical breaking of the symmetry [21]. In LFH the vanishing of the pion mass when $m_q = 0$ follows from the precise cancellation of the light-front kinetic energy and light-front potential energy terms for the quadratic confinement potential in a Schrödinger-like equation [21]. Although similar cancellations occur in chiral models of QCD in Coulomb gauge [33–41], there is, however, one crucial aspect in the calculation of the pion leptonic decay constants in LFH [22] that might not capture the full chiral dynamics of the pion bound state, namely the truncation of the Fock-space decomposition of the pion's light-front wave function to its lowest quark-antiquark valence component. In this sense, it would be interesting to recalculate the leptonic decay constants including higher Fock-space components in the pion's wave function in the LFH approach.

Notwithstanding our results are derived in a hard-wall model of holographic QCD, we believe that the vanishing of the leptonic decay constants of the pion's excited states in the chiral limit will happen in any holographic model that implements dynamical chiral symmetry breaking and reproduces the generalized GOR relationship. This and other aspects of the problem investigated here should be investigated with soft-wall models. Besides improving on the mass spectrum, such a model makes closer contact with LFH.

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