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Myers-Perry black hole in an external gravitational field

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We obtain a new exact solution of the 5D Einstein equations in vacuum describing a distorted Myers-Perry black hole with a single angular momentum. Locally, the solution is interpreted as a black hole distorted by a stationary $U(1) \times U(1)$ symmetric distribution of external matter. Technically, the solution is constructed by applying a twofold Bäcklund transformation on a 5D distorted Minkowski spacetime as a seed. The physical quantities of the solution are calculated, and a local Smarr-like relation on the black hole horizon is derived. It possesses the same form as the Smarr-like relation for the asymptotically flat Myers-Perry black hole. It is demonstrated that in contrast to the asymptotically flat Myers-Perry black hole, the ratio of the horizon angular momentum and the mass J^2/M^3 is unbounded, and can grow arbitrarily large. We study the properties of the ergoregion and the horizon surface. The external field does not influence the horizon topology. The horizon geometry however is distorted, and any regular axisymmetric geometry is possible.

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I. INTRODUCTION

Black holes are one of the most important predictions of general relativity. They are interesting from a purely theoretical viewpoint since they give insight into the properties of the gravitational theory. On the other hand they represent intriguing astrophysical objects that are involved in various physical processes, including accretion disks and jets. At this point there is vast observational evidence for their existence. Besides supermassive black holes located in the centers of galaxies there are stellar mass black holes, which occur as companions in binary systems.

Most of the theoretical investigations concerning black holes consider them as isolated objects. Various exact solutions describing such systems exist, and their properties are comparatively well studied. Such is the Kerr-Newman family of solutions, which represents the unique stationary charged black hole in asymptotically flat spacetime within the classical general relativity in four dimensions. Most astrophysical situations, however, would suggest that the black hole is not isolated but is interacting with some external distribution of matter. In general these systems are dynamical, and due to their complexity are subject only to numerical or perturbative treatment. Exact solutions exist only in idealized cases assuming stationarity, or even staticity, and a very special form of the external matter. A plausible way to describe a nonisolated black hole by an exact solution is to construct a local solution, which is physically relevant only in a close neighborhood of the black hole, but still incorporates into itself information about the external matter fields. A major advantage is that these solutions are valid for broad classes of external matter, the only restrictions coming from some regularity conditions. They are still stationary, since solution generation techniques rely heavily on spacetime symmetries. However, they can be considered as a possible approximation for dynamical black holes relaxing on a time scale much shorter than the external matter, or for equilibrium systems of black hole and matter moving in a quasistationary state. Such scenarios include for example a black hole surrounded by an accretion disk, or a galaxy with a central black hole.

The idea was originally developed in the work of Geroch and Hartle [1], where they considered general static black holes in four dimensions in the presence of external matter fields and investigated their properties, thermodynamical behavior and Hawking radiation. They discussed solutions to the static Einstein equations in vacuum which contain a regular horizon, which are free of singularities in the domain of outer communications, and are asymptotically nonflat if considered as global solutions. It was suggested that such solutions can be interesting if regarded as local solutions, which are valid only in some neighborhood of the black hole horizon. Provided they can be extended in some intermediate region in spacetime to some nonvacuum solutions to

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the Einstein equations which are asymptotically flat, a physically reasonable global solution can be constructed. Then, the vacuum solution can be interpreted as describing locally a black hole distorted by certain external matter. To be able to apply the described argument, we should assume moreover that the black hole and the external matter are separated in spacetime, so that there exists a certain vacuum neighborhood surrounding the black hole, and that the external matter is confined to a compact region, in order for the spacetime to be asymptotically flat.

Several local black hole solutions in the presence of external matter were constructed explicitly, which were called after the work of Geroch and Hartle distorted black holes. Static vacuum solutions were obtained in [2–6], and rotating generalizations describing Kerr black holes in arbitrary axisymmetric external gravitational fields were constructed in [7,8] using solitonic techniques. It was observed that distorted black holes in four dimensions can possess also a regular horizon with toroidal topology apart from the spherical one [9,10], in contrast to their nondistorted asymptotically flat counterparts. However, in order to be able to make an asymptotically flat nonvacuum extension of toroidal black hole solutions, there should exist a region outside the horizon containing external matter with negative energy density. Thus, the dominant energy condition is violated, and distorted black holes with a toroidal horizon are considered to be not of astrophysical importance, although theoretically interesting.

In analogy with the vacuum solutions distorted charged black holes were also constructed in the classical general relativity and dilaton gravity in four dimensions. Static solutions were obtained in [11,12], while a Kerr-Newman black hole in general external gravitational field was constructed in [13]. They are local solutions to the static Einstein-Maxwell(-dilaton) equations possessing only electric charge. Similar to the vacuum solutions, they should contain a regular horizon, possess no singularities in the domain of outer communications, and admit an extension to a solution with external matter fields which is asymptotically flat. In the same sense, two classes of distorted black hole solutions to the Einstein-Klein-Gordon equations, minimally coupled to gravity, were also obtained in [14].

With the development of string theory, brane-world scenarios, and holographic ideas black hole solutions in higher spacetime dimensions became relevant. In this relation distorted black hole solutions to the static axisymmetric Einstein-Maxwell equations in five dimensions were obtained, including a static vacuum solution with a horizon with spherical topology (distorted 5D Schwarzschild-Tangherlini black hole) [15], and a static electrically charged solution with spherical horizon topology (distorted 5D Reissner-Nordström black hole) [16]. In this framework the restrictions on the asymptotic structure of spacetime were also naturally relaxed, and spacetimes with one (or several) compactified spacelike dimensions were considered. Static black holes with a distorted horizon in such spacetimes were investigated [17], however in this case the distortion of the horizon does not result from the interaction with some external matter, but follows from the compactification.

The distorted black holes are not only interesting as more realistic solutions for potential astrophysical application, but also important from a purely theoretical viewpoint. They are more general stationary and axisymmetric solutions than the isolated black holes, and can provide deeper insights into the black hole properties. A series of works were devoted to investigating how the properties of the isolated black holes are influenced if they are distorted by external matter field, and which of them remain unaffected. It was established that the 4D static distorted black holes belong to the Petrov type D on the horizon, like their asymptotically flat counterparts, although in the rest of the spacetime they are algebraically general [18]. It was also demonstrated that the distorted Kerr-Newman black hole satisfies on the horizon the standard Smarr relation for the nondistorted case [13], and the same geometric inequalities between the electric charge, horizon area, and angular momentum [19,20]. These features were also observed in the case of the 5D distorted Reissner-Nordström black hole [16]. It was shown that for this solution, the spacetime singularities are located behind the black hole's inner (Cauchy) horizon, provided that the sources of the distortion satisfy the strong energy condition, and the inner horizon remains regular if the distortion fields are finite and smooth at the outer horizon. There exists a certain duality transformation between the inner and outer horizon surfaces which links surface gravity, electrostatic potential, and spacetime curvature invariants calculated at the black hole horizons. The product of the inner and outer horizon areas depends only on the black hole's electric charge and the geometric mean of the areas is the upper (lower) limit for the inner (outer) horizon area.

Within the framework of isolated horizons it was proven that a local first law of thermodynamics is valid on the distorted black holes horizon [21–23], which possesses the same form as the first law for the corresponding asymptotically flat black holes. On the other hand, the distorted black hole can exhibit a very different horizon geometry than the isolated black holes. Even in the four-dimensional static case their horizon surfaces are axisymmetric, rather than spherically symmetric, and they can be highly elongated, or flattened [24]. Only in the extremal limit, the horizon of the distorted Reissner-Nordström black hole is proven to be spherically symmetric [25]. The distorted black holes can also have different ratios between the mass and the angular momentum than the asymptotically flat ones. It is well known that the mass M and the angular momentum J of the Kerr black hole should satisfy the inequality $|J|/M^2 \leq 1$ in order for an event horizon to exist. However, if the Kerr black hole is situated in an external matter field this ratio not only can exceed one, but become arbitrary large. This effect is also observed in a numerical solution describing a similar astrophysical situation as the distorted black holes [26,27]. A stationary and axisymmetric configuration of a perfect fluid ring rotating around a central black hole was investigated, and it was found that the angular momentum/mass ratio of the black hole can reach $|J|/M^2 > 10^4$, meaning that it is practically unbounded.

The purpose of this work is to construct a five-dimensional Myers-Perry black hole in an external gravitational field and examine its properties. The solution is obtained by applying a twofold Bäcklund transformation, which was developed by Neugebauer in order to solve the four-dimensional stationary and axisymmetric problem. The transformation is applied on a 5D vacuum Weyl solution as a seed, describing a regular region in spacetime in the presence of a static external distribution of matter. As a result, a 5D stationary and "axisymmetric"[1] vacuum solution is obtained, which is rotating only with respect to one of the symmetry axes. It possesses a regular Killing horizon with spherical topology and is asymptotically nonflat. The asymptotically flat Myers-Perry black hole is contained in it as a limiting case. Therefore, the solution is interpreted in the spirit of Geroch and Hartle as describing locally a Myers-Perry black hole in the presence of an external matter field. The solution is also a generalization of the 5D distorted Schwarzschild-Tangherlini black hole obtained in [15], which is recovered in the static limit. The physical properties of the distorted Myers-Perry black hole are investigated. The local mass and angular momentum on the horizon are calculated, as well as its temperature and entropy.

The paper is organized as follows. In the next section we review the 5D asymptotically flat Myers-Perry black hole and some of its distinctive features, which are relevant for our work. We also provide a representation of the 5D Myers-Perry solution with single rotation in prolate spheroidal coordinates, since we will use them in the construction of the distorted solution. In Sec. III, we briefly describe the Bäcklund transformation, which we will apply as a solution generation technique for obtaining the distorted solution. Section IV is devoted to the actual construction of the distorted Myers-Perry black hole. First, we construct a suitable seed solution, and then we perform a twofold Bäcklund transformation on it in prolate spheroidal coordinates. The regularity of the solution is analyzed, and appropriate restrictions on the solution parameters are imposed, so that the distorted Myers-Perry black hole is completely regular in the domain of outer communications. In Sec. V, some physical properties of the solution are investigated. The local mass and angular momentum of the horizon are computed, as well as the temperature and entropy, and a Smarr-like relation is derived. It is demonstrated that the ratio of the horizon angular momentum and mass is not bounded, and can grow unlimitedly. In Sec. VI, the horizon geometry and the behavior of the ergoregion are analyzed.

[1]This solution is actually $U(1)\times U(1)$ symmetric. A d-dimensional, axisymmetric spacetime which admits the $SO(d-2)$ isometry group cannot be considered as an appropriate higher-dimensional generalization of the four-dimensional Weyl form. Instead one has to consider a d-dimensional spacetime which admits the $R^1\times U(1)\times U(1)$ isometry group.

II. THE MYERS-PERRY SOLUTION

The Myers-Perry solution [28] describes a family of black holes with spherical horizon topology in a spacetime with arbitrary dimension D, such that $D\geq 4$. It is a stationary solution to the Einstein equations in vacuum, meaning that it possesses an asymptotically timelike Killing vector. It is also axisymmetric, in the sense that there exist N spacelike Killing fields, where N is the integer part of $(D-1)/2$, and they correspond to N rotational axes. Therefore, the solution is characterized in general with N independent angular momenta. The Myers-Perry family is the higher-dimensional generalization of the Kerr black hole, which is included as the particular case for $D=4$.

In this work, we consider the Myers-Perry black hole in five-dimensional spacetime, which is represented in Boyer-Lindquist coordinates by the metric

$$ds^2 = -dt^2 + \Sigma\left(\frac{r^2}{\Delta}dr^2 + d\theta^2\right) + (r^2+a_1^2)\sin^2\theta d\phi^2 + (r^2+a_2^2)\cos^2\theta d\psi^2 + \frac{m}{\Sigma}(dt - a_1\sin^2\theta d\phi - a_2\cos^2\theta d\psi)^2, \quad (1)$$

where

$$\Sigma = r^2 + a_1^2\cos^2\theta + a_2^2\sin^2\theta,$$
$$\Delta = (r^2+a_1^2)(r^2+a_2^2) - mr^2. \quad (2)$$

The timelike Killing field is given by $\partial/\partial t$, while the spacelike Killing fields are $\partial/\partial\phi$ and $\partial/\partial\psi$. The solution is characterized by three parameters: m and a_i, $i=1,2$. The parameter m is related to the mass of the solution, while a_1 and a_2 are rotation parameters related to the angular momenta with respect to the two rotational axes J_ϕ and J_ψ. In general, an event horizon is present, located at the largest positive root $r=r_{\rm H}$ of the function

$$\Delta(r) = (r^2+a_1^2)(r^2+a_2^2) - mr^2, \quad (3)$$

which is given explicitly by

$$r_{\rm H}^2 = \frac{1}{2}\left(m - a_1^2 - a_2^2 + \sqrt{(m-a_1^2-a_2^2)^2 - 4a_1^2a_2^2}\right). \quad (4)$$

The horizon exists only if the solution parameters obey the condition

$$a_1^2 + a_2^2 + 2|a_1 a_2| \le m. \tag{5}$$

If the function $\Delta(r)$ possesses another positive root $r_I < r_H$, the solution contains an inner (Cauchy) horizon, and it can become extremal in the limit when the two horizon radii coincide. The event and the inner horizon are Killing horizons with respect to the Killing field

$$K = \frac{\partial}{\partial_t} + \Omega_\phi \frac{\partial}{\partial_\phi} + \Omega_\psi \frac{\partial}{\partial_\psi}, \tag{6}$$

where the constant coefficients Ω_ϕ and Ω_ψ represent the angular velocities, with which the black hole rotates with respect to the axes of the Killing vectors $\frac{\partial}{\partial_\phi}$ and $\frac{\partial}{\partial_\psi}$. Their explicit form is given by the expressions

$$\Omega_\phi = \frac{a_1}{(r_H^2 + a_1^2)}, \qquad \Omega_\psi = \frac{a_2}{(r_H^2 + a_2^2)}. \tag{7}$$

In the case when the rotation parameters a_1 and a_2 vanish, we obtain the five-dimensional Schwarzschild-Tangherlini black hole

$$ds^2 = -\left(1 - \frac{m}{r^2}\right)dt^2 + \left(1 - \frac{m}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\psi^2 + \cos^2\theta d\phi^2). \tag{8}$$

Physically, the five-dimensional Myers-Perry solution is characterized by its Arnowitt-Deser-Misner (ADM) mass M and the angular momenta with respect to the two rotational axes J_ϕ and J_ψ. They can be determined either by examining the asymptotic behavior of the metric functions g_{tt}, $g_{t\phi}$, and $g_{t\psi}$, or equivalently by calculating the corresponding Komar integrals [29]. Thus, the following quantities are obtained[2]:

$$M = \frac{3\pi}{8} m, \qquad J_\phi = \frac{\pi}{4} m a_1, \qquad J_\psi = \frac{\pi}{4} m a_2. \tag{9}$$

The condition restricting the existence of the horizon (5) can be expressed in terms of the physical quantities as

$$M^3 \ge \frac{27\pi}{32}(J_\phi^2 + J_\psi^2 + 2|J_\phi J_\psi|). \tag{10}$$

Another form of the Myers-Perry black hole can be obtained by introducing prolate spheroidal coordinates x and y on the two-dimensional surfaces which are orthogonal to the Killing fields. They are particularly convenient for representing the solution in the case where one of the angular momenta, e.g. J_ϕ, vanishes. Then, the associated rotational parameter a_1 vanishes as well, and the prolate spheroidal coordinates are related to the Boyer-Lindquist coordinates r and θ by the expressions

$$x = \frac{r^2}{2\sigma} - 1, \qquad y = \cos 2\theta. \tag{11}$$

The parameter σ is a real number, connected to the mass parameter m and the nonzero rotational parameter a_2 of the solution in Boyer-Lindquist coordinates as $4\sigma = m - a_2^2$. The Myers-Perry black hole with a single rotation acquires the form [30]

$$\begin{aligned} ds^2 = & -\frac{x - 1 - \alpha^2(1-y)}{x + 1 + \alpha^2(1+y)} \\ & \times \left(dt + 2\sigma^{1/2}\alpha \frac{(1+\alpha^2)(1-y)}{x - 1 - \alpha^2(1-y)} d\psi\right)^2 \\ & + \sigma \frac{(x-1)(1-y)(x+1+\alpha^2(1+y))}{x - 1 - \alpha^2(1-y)} d\psi^2 \\ & + \sigma(x+1)(1+y)d\phi^2 \\ & + \frac{\sigma}{2}(x+1+\alpha^2(1+y))\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right). \end{aligned} \tag{12}$$

The prolate spheroidal coordinates x and y take the ranges $x \ge 1$ and $-1 \le y \le 1$. The black hole horizon is located at $x = 1$, while the axes of the spacelike Killing fields $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\psi}$ are located at $y = -1$ and $y = 1$ respectively. The physical infinity corresponds to the limit $x \to \infty$. The solution is characterized by the parameters σ and α, as the former has the meaning of a mass parameter, while the later is interpreted as a rotational parameter. It is related to the parameter set of the solution (m, a_2) in Boyer-Lindquist coordinates as $\alpha^2 = a_2^2/(m - a_2^2)$. The mass M, the angular momentum J, and the angular velocity Ω of the solution are given in terms of the parameters $\{\sigma, \alpha\}$ as

$$\begin{aligned} M &= \frac{3\pi}{2}\sigma(1+\alpha^2), \\ J &= 2\pi\sigma^{\frac{3}{2}}\alpha(1+\alpha^2), \\ \Omega &= \frac{\alpha}{2\sqrt{\sigma}(1+\alpha^2)}. \end{aligned} \tag{13}$$

In the limit when the rotation parameter α vanishes, the solution reduces to the five-dimensional Schwarzschild-Tangherlini black hole in the prolate spheroidal coordinates

$$\begin{aligned} ds^2 = & -\frac{x-1}{x+1}dt^2 + \sigma(1-y)(x+1)d\psi^2 \\ & + \sigma(x+1)(1+y)d\phi^2 \\ & + \frac{\sigma}{2}(x+1)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right). \end{aligned} \tag{14}$$

[2] We use geometrical units, i.e. the gravitational constant is set to $G = 1$.

III. GENERATION OF STATIONARY AXISYMMETRIC SOLUTIONS BY MEANS OF BÄCKLUND TRANSFORMATIONS

The construction of analytic solutions to the 5D stationary and axisymmetric Einstein equations in vacuum is comparatively well studied. It is proven that the problem is completely integrable [31,32], an associated linear problem is constructed, and a solution generation technique by means of the inverse scattering method is developed [33–35]. In the particular case when the solution is rotating only with respect to a single axis, i.e. one of the spacelike Killing fields is hypersurface orthogonal, a Bäcklund transformation is also obtained which can be applied to generate a new solution. In this section we will briefly describe the method of construction of solutions by using the Bäcklund transformation found by Neugebauer [36–38]. The Bäcklund transformations relate different solutions of a partial differential equation in an algebraic way. Thus, applied on an already known solution, called a seed, they lead to a new solution with minimal analytic operations. Neugebauer's transformation was originally derived in order to solve the four-dimensional stationary and axisymmetric problem. However, it can be also applied to the five-dimensional stationary and axisymmetric Einstein equations in vacuum, when one of the spacelike Killing fields is hypersurface orthogonal [30,39–44], since the corresponding field equations closely resemble the four-dimensional case.

The general stationary and axisymmetric solution to the five-dimensional Einstein equations, which rotates only in a single plane, can be represented by the metric

$$ds^2 = -e^{2\chi-u}(dt-\omega d\psi)^2 + e^{-2\chi-u}\rho^2 d\psi^2 + e^{-2\chi-u}e^{2\Gamma}(d\rho^2+dz^2) + e^{2u}d\phi^2. \tag{15}$$

The asymptotically timelike Killing field is represented as $\partial/\partial t$, and the spacelike Killing fields are given by $\partial/\partial\phi$ and $\partial/\partial\psi$. The two-dimensional surfaces orthogonal to the Killing fields are parametrized by the Weyl coordinates ρ and z, and all the metric functions depend only on them. The Einstein equations in vacuum determining such a solution consist of a nonlinear system of equations for the metric function χ and the twist potential f defined as

$$\partial_\rho f = -\frac{e^{4\chi}}{\rho}\partial_z\omega, \qquad \partial_z f = \frac{e^{4\chi}}{\rho}\partial_\rho\omega, \tag{16}$$

a Laplace equation for the metric function u, and a decoupled linear system for the remaining metric function Γ. It is always integrable for a particular solution (χ, u, f). In analogy to the four-dimensional case we can introduce an Ernst potential $\mathcal{E}$ defined as

$$\mathcal{E} = e^{2\chi} + if, \tag{17}$$

and describe the problem by means of $\mathcal{E}$ and its complex conjugate $\bar{\mathcal{E}}$ instead of the couple of functions (χ, f). The field equations acquire the form

$$(\mathcal{E}+\bar{\mathcal{E}})(\partial_\rho^2\mathcal{E} + \rho^{-1}\partial_\rho\mathcal{E} + \partial_z^2\mathcal{E}) = 2(\partial_\rho\mathcal{E}\partial_\rho\mathcal{E} + \partial_z\mathcal{E}\partial_z\mathcal{E}),$$
$$\partial_\rho^2 u + \rho^{-1}\partial_\rho u + \partial_z^2 u = 0, \tag{18}$$

$$\rho^{-1}\partial_\rho\Gamma = \frac{1}{(\mathcal{E}+\bar{\mathcal{E}})^2}[\partial_\rho\mathcal{E}\partial_\rho\bar{\mathcal{E}} - \partial_z\mathcal{E}\partial_z\bar{\mathcal{E}}] + \frac{3}{4}[(\partial_\rho u)^2 - (\partial_z u)^2],$$
$$\rho^{-1}\partial_z\Gamma = \frac{2}{(\mathcal{E}+\bar{\mathcal{E}})^2}\partial_\rho\mathcal{E}\partial_z\bar{\mathcal{E}} + \frac{3}{2}\partial_\rho u\partial_z u.$$

Similar to the four-dimensional case we obtain a nonlinear equation for the Ernst potential, called the Ernst equation, and it represents the main difficulty for solving the system. The solutions to the Laplace equation are well studied, and provided the Ernst potential and the metric function u are known, the metric function Γ can be obtained in a straightforward (although technically cumbersome) way.

Solutions to the Ernst equation can be constructed by using the Bäcklund transformation developed by Neugebauer. The Bäcklund transformation relates a new solution for the Ernst potential $\mathcal{E}$ to an already known solution $\mathcal{E}_0$, called a seed, in an algebraic way, after performing minimal analytic operations. In some cases the solution which we try to obtain is related to a particular seed solution by a sequence of N subsequent Bäcklund transformations, referred to as an N-fold Bäcklund transformation. The Bäcklund transformation is determined by a couple of functions α and λ which are solutions to the following system of Riccati equations [37]:

$$d\lambda = \rho^{-1}(\lambda-1)[\lambda\rho_{,\zeta}d\zeta + \rho_{,\bar{\zeta}}d\bar{\zeta}],$$
$$d\alpha = (\mathcal{E}_0+\bar{\mathcal{E}}_0)^{-1}[(\alpha-\lambda^{1/2})\bar{\mathcal{E}}_{0,\zeta} + (\alpha^2\lambda^{1/2}-\alpha)\mathcal{E}_{0,\zeta}]d\zeta + (\mathcal{E}_0+\bar{\mathcal{E}}_0)^{-1}[(\alpha-\lambda^{-1/2})\bar{\mathcal{E}}_{0,\bar{\zeta}} + (\alpha^2\lambda^{-1/2}-\alpha)\mathcal{E}_{0,\bar{\zeta}}]d\bar{\zeta}, \tag{19}$$

where $\mathcal{E}_0$ is the Ernst potential for the seed solution, $\zeta = \rho + iz$, the bar denotes complex conjugation, and $(\ldots)$ denotes differentiation. The equation for λ is solved by the function

$$\lambda = \frac{k - i\bar{\zeta}}{k + i\zeta}, \tag{20}$$

where k is a real integration constant. The equation for α depends on the explicit form of the Ernst potential for the seed solution $\mathcal{E}_0$ and its complex conjugate $\bar{\mathcal{E}_0}$. An important class of seed solutions, which is convenient for the applications, is the Weyl class of static axisymmetic solutions to the 5D Einstein equations in vacuum [45]. They are described by the line element (15) in the particular case when we require that the twist potential f vanishes, and the Ernst potential corresponding to them is real $\mathcal{E}_0 = e^{2\chi_0}$. The second Riccati equation for such a seed is solved by the function

$$\alpha = \frac{\mu + ie^{2\Phi}}{\mu - ie^{2\Phi}}, \tag{21}$$

where μ is a real integration constant, and Φ obeys the equation

$$d\Phi = \frac{1}{2}\lambda^{1/2}\partial_\zeta(\ln\mathcal{E}_0)d\zeta + \frac{1}{2}\lambda^{-1/2}\partial_{\bar\zeta}(\ln\mathcal{E}_0)d\bar\zeta. \tag{22}$$

A single Bäcklund transformation performed on a seed solution $\mathcal{E}_0$ requires the integration of (19), which introduces a couple of integration constants (k_1, μ_1). Performing N subsequent Bäcklund transformations on a seed solution $\mathcal{E}_0$ includes solving the same Riccati equations, but each iteration introduces a new pair of integration constants (k_n, μ_n), $n = 1\ldots N$. The Ernst potential obtained by the application of $2N$ Bäcklund transformations on a seed potential $\mathcal{E}_0$ is constructed in the form [37,46]

$$\mathcal{E} = \mathcal{E}_0 \frac{\det\left(\frac{\alpha_p R_{k_p} - \alpha_q R_{k_q}}{k_p - k_q} - 1\right)}{\det\left(\frac{\alpha_p R_{k_p} - \alpha_q R_{k_q}}{k_p - k_q} + 1\right)}, \tag{23}$$

where $p = 1, 3, \ldots, N-1$, $q = 2, 4, \ldots, N$, α_n is the solution to (21) corresponding to the integration constants (k_n, μ_n), and the functions R_{k_n} are given by

$$R_{k_n} = \sqrt{\rho^2 + (z - k_n)^2}. \tag{24}$$

Having obtained the Ernst potential for the new solution, the metric functions χ and ω can be extracted from it by considering its definition (17), and integrating the relations for the twist potential (16). In the case of a twofold Bäcklund transformation their explicit form is found to be

$$e^{2\chi} = e^{2\chi_0}\frac{W_1}{W_2}, \qquad \omega = e^{-2\chi_0}\frac{\hat\omega}{W_1} + C_\omega, \tag{25}$$

introducing the following notations:

$$W_1 = [(R_{k_1} + R_{k_2})^2 - (\Delta k)^2](1 + ab)^2 + [(R_{k_1} - R_{k_2})^2 - (\Delta k)^2](a - b)^2,$$
$$W_2 = [(R_{k_1} + R_{k_2} + \Delta k) + (R_{k_1} + R_{k_2} - \Delta k)ab]^2 + [(R_{k_1} - R_{k_2} - \Delta k)a - (R_{k_1} - R_{k_2} + \Delta k)b]^2, \tag{26}$$

$$\hat\omega = [(R_{k_1} + R_{k_2})^2 - (\Delta k)^2](1 + ab)[(R_{k_1} - R_{k_2} + \Delta k)b + (R_{k_1} - R_{k_2} - \Delta k)a]$$
$$- [(R_{k_1} - R_{k_2})^2 - (\Delta k)^2](b - a)[(R_{k_1} + R_{k_2} + \Delta k) - (R_{k_1} + R_{k_2} - \Delta k)ab],$$

where $\Delta k = k_2 - k_1$ and C_ω is a constant. The functions a and b are related to the functions Φ_1 and Φ_2, which represent the solutions of (22) with integration constants k_1 and k_2, respectively

$$a = \mu_1^{-1} e^{2\Phi_1}, \qquad b = -\mu_2 e^{-2\Phi_2}. \tag{27}$$

In this way if we consider a 5D Weyl solution as a seed, which is described by the metric

$$ds^2 = -e^{2\chi_0 - u_0}dt^2 + e^{-2\chi_0 - u_0}\rho^2 d\psi^2 + e^{-2\chi_0 - u_0}e^{2\Gamma_0}(d\rho^2 + dz^2) + e^{2u_0}d\phi^2,$$

we can construct a 5D stationary axisymmetric solution with single rotation in the form

$$ds^2 = -\frac{W_1}{W_2}e^{2\chi_0 - u_0}(dt - \omega d\psi)^2 + \frac{W_2}{W_1}e^{-2\chi_0 - u_0}\rho^2 d\psi^2 + e^{2u_0}d\phi^2 + \frac{W_2}{W_1}e^{2\Gamma}e^{-2\chi_0 - u_0}(d\rho^2 + dz^2), \tag{28}$$

by applying a twofold Bäcklund transformation on it. The functions W_1, W_2, and ω are given by the expressions (25) and (26), and the only analytical operation involved is solving Eq. (22) for the particular Ernst potential of the seed solution. The metric function u satisfies a Laplace equation in Weyl coordinates, as well as the corresponding metric function for the seed solution u_0. Therefore, it is convenient to preserve the metric function for the seed solution, and choose such a seed that u_0 coincides with the metric function u for the solution we want to construct. The remaining metric function Γ can be expressed in the form [47,48]

$$e^{2\Gamma} = C_1 \frac{W_1 e^{2\gamma}}{(R_{k_1} + R_{k_2})^2 - (\Delta k)^2}, \tag{29}$$

where C_1 is an integration constant and γ is a solution to the linear system

$$\rho^{-1}\partial_\rho\gamma = (\partial_\rho\tilde\chi_0)^2 - (\partial_z\tilde\chi_0)^2 + \frac{3}{4}[(\partial_\rho u_0)^2 - (\partial_z u_0)^2],$$
$$\rho^{-1}\partial_z\gamma = 2\partial_\rho\tilde\chi_0\partial_z\tilde\chi_0 + \frac{3}{2}\partial_\rho u_0\partial_z u_0. \tag{30}$$

The function $\tilde\chi_0$, which is involved in it, is related to the metric function of the seed solution χ_0 as

$$\tilde{\chi}_0 = \chi_0 + \frac{1}{2}\ln\frac{R_{k_1} + R_{k_2} - \Delta k}{R_{k_1} + R_{k_2} + \Delta k}. \qquad (31)$$

IV. CONSTRUCTION OF 5D DISTORTED MYERS-PERRY BLACK HOLE

In this section, we will construct a five-dimensional distorted Myers-Perry black hole rotating only in a single plane by applying a twofold Bäcklund transformation. The solution is an asymptotically nonflat generalization of the 5D Myers-Perry black hole presented in Sec. II, and can be interpreted as a local solution describing a black hole in the presence of an external distribution of matter fields. The twofold Bäcklund transformation involves the integration of the Riccati equation (22), and is parametrized by two couples of integration constants $(k_1,\ k_2)$ and $(\mu_1,\ \mu_2)$, introduced by the double integration of Eq. (19). The integration constants k_1 and k_2 are actually not independent. The Weyl coordinate z is defined only up to a translation, therefore we can always set $k_1 = \sigma$ and $k_2 = -\sigma$ for some real positive parameter σ. This identification corresponds to a shift in the z coordinate $z \to z + z_0$, where $z_0 = \frac{1}{2}(k_1 + k_2)$. Furthermore, we expect that the distorted Myers-Perry solution will be represented most conveniently in prolate spheroidal coordinates, in analogy with the distorted Kerr black hole [8]. The prolate spheroidal coordinates x and y are closely related to the Weyl coordinates ρ and z. They are defined by the transformation

$$\rho = \kappa\sqrt{(x^2-1)(1-y^2)}, \qquad z = \kappa xy, \qquad (32)$$

where κ is a real constant. As we already mentioned, the prolate spheroidal coordinates take the ranges $x \geq 1$ and $-1 \leq y \leq 1$, and the physical infinity corresponds to the limit $x \to \infty$. We can always set κ equal to the parameter of the Bäcklund transformation σ, which will simplify the form of the constructed solution. In order to obtain a solution in prolate spheroidal coordinates by means of the twofold Bäcklund transformation we described, we should transform the metric functions W_1, W_2, and ω in terms of x and y, and represent the differential equations (22) and (30), which we need to solve, in the same coordinates. Consequently, the general solution (28) acquires the form

$$\begin{aligned} ds^2 &= -\frac{W_1}{W_2}e^{2\chi_0 - u_0}(dt - \omega d\psi)^2 + \frac{W_2}{W_1}e^{-2\chi_0-u_0}\rho^2 d\psi^2 \\ &\quad + e^{2u_0}d\phi^2 + C_1\frac{W_2 e^{2\gamma}}{x^2-1} \\ &\quad \times e^{-2\chi_0-u_0}(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right), \\ \omega &= 2\sigma e^{-2\chi_0}\frac{\hat{\omega}}{W_1} + C_\omega, \end{aligned} \qquad (33)$$

where the zero indices refer to the metric functions of the seed solution, and the metric functions W_1, W_2, and $\hat{\omega}$ are determined by the expressions

$$\begin{aligned} W_1 &= (x^2-1)(1+ab)^2 - (1-y^2)(b-a)^2, \\ W_2 &= [(x+1) + (x-1)ab]^2 + [(1+y)a + (1-y)b]^2, \\ \hat{\omega} &= (x^2-1)(1+ab)[b-a-y(a+b)] \\ &\quad + (1-y^2)(b-a)[1+ab+x(1-ab)], \end{aligned} \qquad (34)$$

with C_1 and C_ω being real constants. Instead of representing the differential equation for Φ (22) in term of x and y, it is more convenient to obtain directly the equations for the related metric functions a and b in prolate spheroidal coordinates. They possess the form [49]

$$\begin{aligned} (x-y)\partial_x a &= 2a[(xy-1)\partial_x\chi_0 + (1-y^2)\partial_y\chi_0], \\ (x-y)\partial_y a &= 2a[-(x^2-1)\partial_x\chi_0 + (xy-1)\partial_y\chi_0], \\ (x+y)\partial_x b &= -2b[(xy+1)\partial_x\chi_0 + (1-y^2)\partial_y\chi_0], \\ (x+y)\partial_y b &= -2b[-(x^2-1)\partial_x\chi_0 + (xy+1)\partial_y\chi_0], \end{aligned} \qquad (35)$$

and depend as expected on the metric function of the seed solution χ_0. The equations for the remaining metric function γ are transformed into the following system:

$$\begin{aligned} \partial_x\gamma &= \frac{1-y^2}{(x^2-y^2)}[x(x^2-1)(\partial_x\chi')^2 - x(1-y^2)(\partial_y\chi')^2 - 2y(x^2-1)\partial_x\chi'\partial_y\chi'] \\ &\quad + \frac{3(1-y^2)}{4(x^2-y^2)}[x(x^2-1)(\partial_x u_0)^2 - x(1-y^2)(\partial_y u_0)^2 - 2y(x^2-1)\partial_x u_0\partial_y u_0], \\ \partial_y\gamma &= \frac{x^2-1}{(x^2-y^2)}[y(x^2-1)(\partial_x\chi')^2 - y(1-y^2)(\partial_y\chi')^2 + 2x(1-y^2)\partial_x\chi'\partial_y\chi'] \\ &\quad + \frac{3(x^2-1)}{4(x^2-y^2)}[y(x^2-1)(\partial_x u_0)^2 - y(1-y^2)(\partial_y u_0)^2 + 2x(1-y^2)\partial_x u_0\partial_y u_0], \end{aligned} \qquad (36)$$

where $\chi' = \frac{1}{2}\ln(\frac{x-1}{x+1}) + \chi_0$.

A. Seed solution

A nontrivial step in the construction of solutions by means of Bäcklund transformations is choosing the seed solution. As we mentioned, a convenient class of seed solutions is the static Weyl class, since they are comparatively simple and minimize the technical difficulties. A general 5D Weyl solution in prolate spheroidal coordinates possesses the form

$$ds^2 = -e^{2\chi_0-u_0}dt^2 + e^{-2\chi_0-u_0}\rho^2 d\psi^2 + e^{2u_0}d\phi^2 + e^{2\gamma}e^{-2\chi_0-u_0}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right), \quad (37)$$

where all the metric functions depend only on x and y, and the metric functions χ_0 and u_0 obey a Laplace equation in 3D flat space. The most general solution to the Laplace equation, which is regular on the symmetry axes, can be presented in the form [50,51]

$$\chi_0 = \sum_{n=0}^{\infty}\frac{c_n}{R^{n+1}}P_n\left(\frac{xy}{R}\right) + \sum_{n=0}^{\infty}b_n R^n P_n\left(\frac{xy}{R}\right), \quad (38)$$

and equivalently for u_0. The parameters c_n and b_n are real constants, n is a natural number, P_n are the Legendre polynomials, and the function R is defined by

$$R = \sqrt{x^2+y^2-1}. \quad (39)$$

The first part of the sum in the solution of the Laplace equation corresponds to an asymptotically flat solution of the static Einstein equations in vacuum. It describes a deformed mass source, and the parameters c_n are related to the mass multipole moments of the solution. The second part of the sum corresponds to an asymptotically nonflat solution, which is interpreted as a local solution in an external gravitational field, and the constants b_n are related to the characteristics of the external field. The twofold Bäcklund transformation preserves the asymptotic structure of the seed solution. Therefore, if we want to obtain a solution describing a black hole in an external gravitational field, we should choose a seed with nonzero constants b_n. A suitable seed solution is a regular Weyl solution describing a static external distribution of matter, which contains no horizons, and in the limit when the external gravitational fields vanish reduces to the 5D Minkowski spacetime. The 5D Minkowski spacetime can be represented in prolate spheroidal coordinates by the metric

$$ds^2 = -dt^2 + e^{-2W_0}d\psi^2 + e^{-2U_0}d\phi^2 + \frac{\sigma}{2}(x-y)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right),$$
$$e^{-2W_0} = \sigma(x-1)(1-y), \qquad e^{-2U_0} = \sigma(x+1)(1+y). \quad (40)$$

The functions U_0 and W_0 are solutions to the Laplace equation in 3D flat space, and in Weyl coordinates their sum satisfies $U_0 + W_0 = \ln\rho$, as required for a 5D Weyl solution [45]. If we replace the functions U_0 and W_0 by asymptotically nonflat solutions of the Laplace equation, and calculate the remaining metric function γ, we can obtain an asymptotically nonflat Weyl solution which contains no horizons.[3] It can be interpreted as describing locally a vacuum region in spacetime which is influenced by the presence of some static and axisymmetric distribution of matter situated in its exterior. Therefore, it can be called distorted Minkowski spacetime in analogy with the distorted black hole solution. For general static and axisymmetric external gravitational fields the metric of the distorted Minkowski spacetime acquires the form

$$ds^2 = -e^{2(\widehat{U}+\widehat{W})}dt^2 + e^{-2W}d\psi^2 + e^{-2U}d\phi^2 + \frac{\sigma}{2}(x-y)e^{2V}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right), \quad (41)$$

where the functions $U = U_0 + \widehat{U}$ and $W = W_0 + \widehat{W}$ include a contribution from the nondistorted Minkowski metric (40) denoted with zero index, and terms $\widehat{U}$ and $\widehat{W}$ characterizing the external sources. The most general form of $\widehat{U}$ and $\widehat{W}$ is given by the expressions

$$\hat{U} = \sum_{n=0}^{\infty}a_n R^n P_n\left(\frac{xy}{R}\right), \qquad \hat{W} = \sum_{n=0}^{\infty}b_n R^n P_n\left(\frac{xy}{R}\right), \quad (42)$$

according to the solution of the Laplace equation (38). The metric function V is obtained in the form

$$V = \sum_{n,k=1}^{\infty}\frac{nk}{n+k}(a_n a_k + a_n b_k + b_n b_k) \times R^{n+k}(P_n P_k - P_{n-1}P_{k-1}) + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - b_n)\sum_{k=0}^{n-1}(-1)^{n-k+1}(x+y)R^k P_k\left(\frac{xy}{R}\right) - \frac{1}{2}\sum_{n=0}^{\infty}(a_n+b_n)R^n P_n\left(\frac{xy}{R}\right). \quad (43)$$

[3]The metric function γ is a solution to (36) with $\chi' = \chi_0$, where χ_0 and u_0 are the metric functions appearing in the line element (37).

For general values of the parameters a_n and b_n characterizing the external gravitational field the solution is not completely regular, since it contains conical singularities. They can be removed by requiring that the solution is Lorenzian (elementary flat) in the vicinity of the axes of the spacelike Killing fields $\partial/\partial\phi$ and $\partial/\partial\psi$ located at $y=-1$ and $y=1$, respectively. The condition ensuring elementary flatness is given by the relation [46,52]

$$\frac{1}{4X}g^{\mu\nu}\partial_\mu X\partial_\nu X \longrightarrow 1, \qquad (44)$$

which should be satisfied by the norm X of each of the spacelike Killing fields in the neighborhood of their axes. It leads to the following constraints:

$$\begin{aligned}\exp\left[-\sum_{n=0}^{\infty}(-1)^n(a_n-b_n)\right]&=1, \qquad y=1,\\ \exp\left[\sum_{n=0}^{\infty}(-1)^n(a_n-b_n)\right]&=1, \qquad y=-1. \end{aligned} \qquad (45)$$

Therefore, the distorted Minkowski solution is free of conical singularities only if the parameters characterizing the external gravitational field satisfy the relation $\sum_{n=0}^{\infty}(-1)^n(a_n-b_n)=0$.

B. Distorted 5D Myers-Perry black hole

We will apply the twofold Bäcklund transformation on the distorted Minskowski spacetime (41) as a background. The transformation can be performed with respect to either of the spacelike Killing fields, leading to a solution rotating around its axis. Since we aim to construct a Myers-Perry black hole, which is axially symmetric, both approaches will result in equivalent solutions. Here we choose to perform the transformation with respect to the Killing field $\partial/\partial\psi$. Then, comparing the metric of the distorted Minkowski spacetime with the general expression (37) we conclude that the functions χ_0 and u_0 are given by

$$2\chi_0=-U_0+2\widehat{W}+\widehat{U}, \qquad u_0=-U_0-\hat{U}. \qquad (46)$$

Taking advantage of these expressions we solve equations (35) and obtain the functions a and b in a similar way as in [8]

$$\begin{aligned} a&=\alpha\sqrt{\frac{x+1}{y+1}}\exp\left[\sum_{n=1}^{\infty}(a_n+2b_n)\sum_{k=0}^{n-1}(x-y)R^kP_k\left(\frac{xy}{R}\right)\right],\\ b&=\beta\frac{\sqrt{(x+1)(y+1)}}{x+y}\exp\left[\sum_{n=1}^{\infty}(a_n+2b_n)(x+y)\right.\\ &\quad\left.\times\sum_{k=0}^{n-1}(-1)^{n-k}R^kP_k\left(\frac{xy}{R}\right)\right], \end{aligned} \qquad (47)$$

where α and β are real constants.[4] The equations for the metric function γ can be also solved in resemblance to [8], leading to the following result:

$$\begin{aligned} \gamma&=\gamma_0+\hat{\gamma},\\ \gamma_0&=\frac{1}{2}\ln(x^2-1)-\frac{1}{2}\ln(x^2-y^2)+\frac{1}{2}\ln(y+1),\\ \hat{\gamma}&=\sum_{n,k=1}^{\infty}\frac{nk}{n+k}(a_na_k+a_nb_k+b_nb_k)\\ &\quad\times R^{n+k}(P_nP_k-P_{n-1}P_{k-1})\\ &\quad+\frac{1}{2}\sum_{n=1}^{\infty}(2a_n+b_n)\sum_{k=0}^{n-1}(-1)^{n-k+1}(x+y)R^kP_k\left(\frac{xy}{R}\right)\\ &\quad-\frac{1}{2}\sum_{n=1}^{\infty}(a_n+2b_n)\sum_{k=0}^{n-1}(x-y)R^kP_k\left(\frac{xy}{R}\right)\\ &\quad-\frac{1}{2}\sum_{n=0}^{\infty}(a_n-b_n)R^nP_n\left(\frac{xy}{R}\right), \end{aligned} \qquad (48)$$

where all the Legendre polynomials have the same argument xy/R.

These functions determine completely a new stationary and axisymmetric solution to the 5D Einstein equations in vacuum with metric given by (1). The constructed solution possesses an event horizon with S^3 topology located at $x=1, -1\leq y\leq 1$, which rotates with respect to the axis of the Killing field $\partial/\partial\psi$. The axes of the spacelike Killing fields $\partial/\partial\phi$ and $\partial/\partial\psi$ correspond to $y=-1$ and $y=1$, respectively. The solution is not regular outside the black hole horizon for general values of the transformation parameters, since the metric function W_1 gets singular at $x=1$, $y=-1$. This pathological feature can be avoided if we set the parameter β equal to zero. Then, the function b vanishes and the explicit form of the solution simplifies considerably. It can be presented as

$$\begin{aligned} ds^2&=-\frac{x-1-\hat{a}^2(1-y)}{x+1+\hat{a}^2(1+y)}e^{2(\widehat{U}+\widehat{W})}(dt-\omega d\psi)^2\\ &\quad+\frac{x+1+\hat{a}^2(1+y)}{x-1-\hat{a}^2(1-y)}e^{-2W}d\psi^2+e^{-2U}d\phi^2\\ &\quad+C_1[x+1+\hat{a}^2(1+y)]e^{2(\hat{\gamma}-\widehat{W})}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right), \end{aligned} \qquad (49)$$

$$\omega=-2\sqrt{\sigma}\frac{(x-y)\hat{a}e^{-2\widehat{W}-\widehat{U}}}{(x-1)-(1-y)\hat{a}^2}+C_\omega, \qquad (50)$$

[4]The parameters α and β are the equivalent of the parameters of the Bäcklund transformation μ_1 and μ_2, which we introduced before. They are related as $\alpha=\mu_1^{-1}$ and $\beta=-\mu_2$.

$$\hat{a} = \alpha \exp\left[\sum_{n=1}^{\infty}(a_n + 2b_n)\sum_{k=0}^{n-1}(x-y)R^k P_k\left(\frac{xy}{R}\right)\right], \quad (51)$$

where the metric functions U and W correspond to the seed solution (41), and $\hat{\gamma}$ is given by the expression (48). The solution contains two integration constants C_1 and C_ω, which should be chosen appropriately in order to avoid pathological behavior. The value of C_ω should be determined by the requirement that the solution is regular on the axis of rotation, i.e. it should be satisfied that $\omega = 0$ on the axis $y = 1$. Examining the behavior of the function (50) at $y = 1$ we obtain

$$C_\omega = 2\sqrt{\sigma}\alpha \exp\left[-\sum_{n=0}^{\infty}(a_n + 2b_n)\right]. \quad (52)$$

The value of the constant C_1 is connected with the requirement that the solution is Lorentzian (elementary flat) in the vicinity of the axes of the spacelike Killing fields located at $y = -1$ and $y = 1$. Otherwise, the solution will contain conical singularities. The absence of conical singularities is ensured if the norm X of each of the spacelike Killing fields $\partial/\partial\phi$ and $\partial/\partial\psi$ satisfies the condition [46,52]

$$\frac{1}{4X}g^{\mu\nu}\partial_\mu X \partial_\nu X \longrightarrow 1, \quad (53)$$

in the vicinity of the corresponding rotational axis. It is equivalent to requiring that the orbits of the Killing fields are 2π periodic in the neighborhood of their axes. Evaluating (53) at $y = 1$ leads to the constraint

$$\frac{2C_1}{\sigma}\exp\left[\sum_{n=0}^{\infty}(b_{2n} - a_{2n}) + 3\sum_{n=0}^{\infty}(a_{2n+1} + b_{2n+1})\right] = 1, \quad (54)$$

while at $y = -1$ we obtain the relation

$$\frac{2C_1}{\sigma}\exp\left[-\sum_{n=0}^{\infty}(b_{2n} - a_{2n}) - 3\sum_{n=0}^{\infty}(a_{2n+1} + b_{2n+1})\right] = 1. \quad (55)$$

Both conditions are compatible only if the parameters a_n and b_n characterizing the external gravitational field satisfy

$$\sum_{n=0}^{\infty}(b_{2n} - a_{2n}) + 3\sum_{n=0}^{\infty}(a_{2n+1} + b_{2n+1}) = 0. \quad (56)$$

Then, the constant C_1 is determined to be $C_1 = \frac{\sigma}{2}$.

Assigning the constants C_ω and C_1 the values we obtained, leads to a black hole solution, which contains a nonsingular horizon, and is completely regular in the domain of outer communications. If we take the limit when $a_n = 0$, $b_n = 0$ for every n the metric functions describing the external sources vanish. Then, the solution reduces to the asymptotically flat Myers-Perry black hole with single rotation, which is represented in prolate spheroidal coordinates by (12). Another interesting limit is if we set the rotation parameter $\alpha = 0$. In this case the solution becomes static, and we recover the 5D distorted Schwarzschild-Tangherlini black hole, which is obtained in [15]. The coordinates (η, θ), in which the solution is represented, are related to the prolate spheroidal coordinates (x, y) as $x = \eta$, $y = \cos\theta$, and the solution parameter r_0 is connected with σ as $\sigma = r_0^2/4$. Furthermore, the parameters a_n and b_n characterizing the external gravitational field are interchanged in the two solutions $a_n \longleftrightarrow b_n$.

The 5D stationary and axisymmetric solutions are conveniently described by their interval structure [53]. It specifies the location of the horizons and the axes of the spacelike Killing fields in the factor space of the spacetime with respect to the isometry group. If we introduce the Weyl coordinates ρ and z on the two-dimensional surfaces orthogonal to the Killing fields, the factor space is represented by the upper (ρ, z) half-plane, and its boundary coincides with the z axis. The fixed point sets of the spacelike Killing fields and the horizons correspond to intervals on the z axis. In addition, a direction vector consisting of integer numbers is associated with each interval. It specifies the coefficients in the linear combination of Killing fields which vanishes on it.

The interval structure of the 5D distorted Minkowski spacetime and the distorted Myers-Perry black hole are presented in Fig. 1. The distorted solutions are interpreted as local solutions describing a certain vacuum neighborhood which is influenced by external matter sources. Therefore, the interval structure corresponding to them should also be interpreted as characterizing only the region where the distorted solution is valid, and not the whole spacetime. The direction vectors corresponding to each interval are specified above it, as the directions are given with respect to a basis of Killing vectors $\{\partial/\partial t, \partial/\partial\psi, \partial/\partial\phi\}$.

In Fig. 1 it is demonstrated that the axes of the spacelike Killing fields $\partial/\partial\phi$ and $\partial/\partial\psi$ for the distorted Myers-Perry black hole are located in Weyl coordinates at $z \le -\sigma$, and $z \ge \sigma$, respectively. The interval $-\sigma \le z \le \sigma$ corresponds to a Killing horizon for the Killing field $V = \partial/\partial t + \Omega\partial/\partial\psi$, i.e. V becomes null on the 3D hypersurface at constant t located at $-\sigma \le z \le \sigma$, $\rho = 0$. The constant coefficient in the linear combination of Killing fields Ω represents the angular velocity of the horizon rotating with respect to the axis of the Killing field $\partial/\partial\psi$. It is equal to the value of the metric function ω^{-1} on the hypersurface $-\sigma \le z \le \sigma$, $\rho = 0$, or equivalently at $x = 1$, $-1 \le y \le 1$ in prolate spheroidal coordinates.

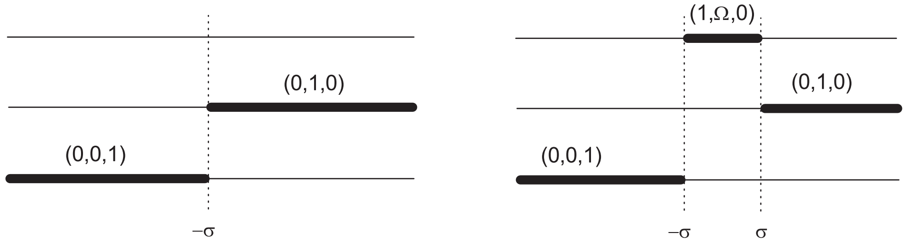

FIG. 1. Rod structure of the seed solution (left panel) and the distorted Myers-Perry black hole with single rotation (right panel).

Taking into account the relations (50) and (52), the angular velocity is calculated to be

$$\Omega = \frac{\alpha}{2\sqrt{\sigma}(1+\alpha^2)} \exp\left[\sum_{n=0}^{\infty}(a_n + 2b_n)\right]. \quad (57)$$

In the limit when the parameters a_n and b_n vanish, the expression reduces to the angular velocity of the asymptotically flat Myers-Perry black hole (13). Therefore, the exponential factor can be interpreted as describing the influence of the external matter fields on the horizon rotation. The interval structure contains also information about the topology of the horizon hypersurface. If we consider the directions of the intervals adjacent to the horizon interval, we can determine whether the horizon is topologically a sphere, a ring ($S^1 \times S^2$), or a general lens space $L(p,q)$ [53]. In our case the horizon possesses S^3 topology, as the nondistorted Myers-Perry black hole. In fact the interval structure for the distorted Myers-Perry black hole in Fig. 1 is equivalent in the vicinity of the horizon to the interval structure of the nondistorted Myers-Perry solution (12), differing only in the value of the angular velocity Ω. Consequently, the horizon location also coincides with that in the isolated Myers-Perry case.

V. MASS, ANGULAR MOMENTUM, AND SMARR-LIKE RELATIONS

The distorted Myers-Perry solution is stationary and axisymmetric, therefore we can define locally mass and angular momentum of a particular region of the spacetime by Komar integrals [29]. We denote by ξ the 1-form dual to the Killing field $\partial/\partial t$, and by ζ the 1-form dual to the Killing field $\partial/\partial\psi$. Then, the Komar mass M_H and angular momentum J_H associated with the black hole horizon are defined as

$$M_H = -\frac{3}{32\pi}\int_H \star d\xi,$$
$$J_H = \frac{1}{16\pi}\int_H \star d\zeta, \quad (58)$$

where the integration is performed over the horizon cross section. Calculating the Komar integrals we obtain the expressions

$$M_H = \frac{3\pi}{2}\sigma(1+\alpha^2), \quad (59)$$

$$J_H = 2\pi\sigma^{\frac{3}{2}}\alpha(1+\alpha^2)\exp\left[-\sum_{n=0}^{\infty}(a_n + 2b_n)\right], \quad (60)$$

which can be interpreted as the intrinsic mass and the angular momentum of the black hole. Comparing with the corresponding expressions for the asymptotically flat Myers-Perry black hole (13), we see that the mass possesses the same form, while the angular momentum differs by the exponential factor. We should also note that the mass expression in the asymptotically flat case refers both to the Komar mass at the horizon, and the ADM mass of the spacetime, since in that case they coincide. The mass of the distorted black hole (59) is only the local Komar mass on the horizon. Since the solution is not asymptotically flat, and an appropriate extension is required in order to construct an asymptotically flat solution, the ADM mass is not defined. Assuming that the solution is extended outside the horizon to some asymptotically flat solution containing matter fields, the ADM mass, which will correspond to it, will contain also terms characterizing the matter fields. The ADM mass is equivalent to the Komar integral ([29]), however evaluated on a 3D-sphere at infinity S_∞, instead of the horizon. Using Stokes's theorem, the integral can be represented as a sum of the Komar integral on the horizon, representing the local mass of the black hole, and a bulk integral of the Ricci 1-form[5]

$$M_{ADM} = -\frac{3}{32\pi}\int_{S_\infty}\star d\xi = -\frac{3}{32\pi}\int_H \star d\xi - \frac{3}{16\pi}\int_{\hat{M}}\star R(\xi). \quad (61)$$

The Ricci 1-form $R(\xi)$ is connected to the stress-energy tensor of the matter fields through the field equations. Therefore, the bulk integral will introduce corrections to the horizon mass M_H, which depend on the external matter fields, in the expression for the total ADM mass of the spacetime. A similar argument applies also for the angular

[5]We assume that the solution contains no other horizons or bolts for the spacelike Killing fields.

momentum on the horizon, which will not coincide with the total angular momentum of the spacetime if external matter is present.

The horizon mass and angular momentum satisfy a local Smarr-like relation defined on the black hole horizon which is equivalent to the Smarr relation for the nondistorted Myers-Perry black hole. We can define a surface gravity on the horizon by the standard relation

$$\kappa_H^2 = -\frac{1}{4\lambda} g^{\mu\nu}\partial_\mu\lambda\partial_\nu\lambda, \tag{62}$$

where $\lambda = g(V,V)$ is the norm of the Killing field $V = \partial/\partial t + \Omega\partial/\partial\psi$, which becomes null on the horizon. Performing the calculations we obtain the expression

$$\kappa_H = \frac{1}{2\sqrt{\sigma}(1+\alpha^2)} \times \exp\left[\frac{3}{2}\sum_{n=0}^{\infty}(a_{2n}+b_{2n}) + \frac{1}{2}\sum_{n=0}^{\infty}(b_{2n+1}-a_{2n+1})\right]. \tag{63}$$

The surface gravity is defined only up to a constant scale factor, since we do not have a normalization of the Killing field V at infinity. We have fixed the freedom by assuming that the distorted solution can be extended to an asymptotically flat one. We can calculate also the horizon area using the restriction of the metric on the horizon g_H defined by $x = 1$ and $\mathrm{t} = \mathrm{const}$:

$$\begin{aligned}\mathcal{A}_H &= \int_H \sqrt{\det g_H}\, d\phi d\psi dy \\ &= 16\pi^2\sigma^{\frac{3}{2}}(1+\alpha^2) \\ &\quad\times \exp\left[-\frac{3}{2}\sum_{n=0}^{\infty}(a_{2n}+b_{2n}) - \frac{1}{2}\sum_{n=0}^{\infty}(b_{2n+1}-a_{2n+1})\right].\end{aligned} \tag{64}$$

In the limit when no external matter is present, i.e. $a_n = b_n = 0$, the surface gravity and the horizon area reduce to the corresponding quantities for the nondistorted Myers-Perry black hole. The surface gravity is related to the temperature associated with the horizon $T = \kappa_H/2\pi$, and the horizon area is proportional to the entropy of the black hole $S = \mathcal{A}_H/4$.

Taking advantage of the expression for the angular velocity (57), we see that a local Smarr-like relation is satisfied for the physical quantities defined on the black hole horizon

$$M_H = \frac{3}{16\pi}\kappa_H\mathcal{A}_H + \frac{3}{2}\Omega J_H. \tag{65}$$

It coincides exactly with the local Smarr-like relation for the nondistorted Myers-Perry black hole, which in this case is also a global relation for the ADM mass and angular momentum, since they are equivalent to the corresponding quantities on the black hole horizon.

Finally, we will discuss the possible values that the horizon mass M_H and angular momentum J_H can attain. In the case of the asymptotically flat Myers-Perry black hole their values are restricted, since the following relation should be satisfied:

$$\frac{27\pi}{32}\frac{J_H^2}{M_H^3} < 1, \tag{66}$$

in order for the event horizon to exist [see (10)]. In the representation of the solution in prolate spheroidal coordinates the parameters σ and α are introduced in such a way that (66) is automatically satisfied,

$$\frac{27\pi}{32}\frac{J_H^2}{M_H^3} = \frac{\alpha^2}{1+\alpha^2} < 1, \tag{67}$$

i.e. the metric (12) always describes a black hole. If we examine the relation J_H^2/M_H^3 for the distorted Myers-Perry black hole we obtain the expression

$$\frac{27\pi}{32}\frac{J_H^2}{M_H^3} = \frac{\alpha^2}{1+\alpha^2}\exp\left[-2\sum_{n=0}^{\infty}(a_n+2b_n)\right]. \tag{68}$$

The parameters a_n and b_n characterize the external matter field, and their values are not connected with the existence of a horizon. Therefore, for some matter distributions they can certainly possess such values that the ratio $\frac{27\pi}{32}\frac{J_H^2}{M_H^3}$ exceeds one. Moreover, since it is not bounded, it is allowed to grow unlimitedly.

A similar effect was noticed in the investigation of a numerical solution representing the Kerr black hole surrounded by a perfect fluid ring [26,27]. The corresponding ratio of the angular momentum and the mass for an isolated Kerr black hole should satisfy the bound $|J|/M^2 \leq 1$. However, when a matter ring is present, it is demonstrated that the ratio can grow arbitrarily large. The effect can be easily foreseen from the analytical expressions for the horizon mass and the angular momentum of the Kerr black hole situated in an arbitrary stationary and axisymmetric external gravitational field, which were obtained in [8]. The horizon mass and angular momentum ratio is given by

$$\frac{|J|}{M^2} = \frac{2\alpha}{1+\alpha^2}\exp\left[-2\sum_{n=1}^{\infty}a_{2n}\right], \qquad 0 < \alpha < 1, \tag{69}$$

where α is a rotation parameter and the parameters a_n characterize the external field. Again, for certain values of the parameters a_n characterizing the external field the ratio can grow unlimitedly.

A. Particular case

We consider a particular case of the general distorted Myers-Perry solution (49) in which the parameters a_n and b_n characterizing the external matter field obey the relation $a_n + 2b_n = 0$ for every n. Then, the metric function $\hat{a}$ reduces to the rotation parameter α and the metric acquires the form

$$ds^2 = -\frac{x-1-\alpha^2(1-y)}{x+1+\alpha^2(1+y)}e^{-2\hat{W}}(dt-\omega d\psi)^2 + \frac{x+1+\alpha^2(1+y)}{x-1-\alpha^2(1-y)}e^{-2W}d\psi^2 + e^{-2U}d\phi^2 + \frac{\sigma}{2}[x+1+\alpha^2(1+y)]e^{2(\hat{\gamma}-\hat{W})}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right), \tag{70}$$

$$\omega = -2\sqrt{\sigma}\alpha\frac{(x-y)}{(x-1)-(1-y)\alpha^2} + 2\sqrt{\sigma}\alpha, \tag{71}$$

where $\hat{\gamma}$ is given by the expression (48) with $a_n = -2b_n$. The influence of the external matter fields appears only in the exponents containing the functions $\widehat{U}$, $\widehat{W}$, and $\hat{\gamma}$, while the remaining part of the line element coincides with the metric of the nondistorted solution. In particular, the metric function ω is equal in this case to the corresponding metric function for the asymptotically flat Myers-Perry black hole. Consequently, the angular velocity of the horizon is also equal in the two cases, and the ergoregion behaves as in the asymptotically flat case (see Sec. VI). It is confined to a compact region in spacetime encompassing the horizon, and the ergosurface intersects the axes of the spacelike Killing fields at the points $\{x=1, y=1\}$ and $\{x=1+2\alpha^2, y=-1\}$. Since the metric function ω is already asymptotically flat, it is in general possible to extend the distorted black hole solution to an asymptotically flat solution involving matter fields, which do not possess intrinsic angular velocity.

The condition for absence of conical singularities takes the form

$$\sum_{n=0}^{\infty}(-1)^n b_n = 0, \tag{72}$$

and examining the expressions for the horizon area, surface gravity, and angular momentum, we see that they coincide with the corresponding quantities for the asymptotically flat Myers-Perry black hole. Thus, the physical characteristics of the solution remain unaffected by the external distribution of matter.

VI. PROPERTIES OF THE HORIZON AND THE ERGOREGION

The three-dimensional surface of the horizon is defined by t = const and $x = 1$. The corresponding metric on the horizon surface is given by the expression

$$ds_H^2 = \frac{4\sigma(1+\alpha^2)^2}{2+\hat{a}^2(1+y)}(1-y)e^{-2\hat{W}}d\psi^2 + 2\sigma(1+y)e^{-2\hat{U}}d\phi^2 + \frac{\sigma}{2}[2+\hat{a}^2(1+y)]e^{2(\hat{\gamma}-\hat{W})}\frac{dy^2}{1-y^2}, \tag{73}$$

where the metric functions $\hat{a}$, $\widehat{W}$, $\widehat{U}$, and $\hat{\gamma}$ possess the following form:

$$\hat{U} = \sum_{n=0}^{\infty}a_n y^n, \qquad \hat{W} = \sum_{n=0}^{\infty}b_n y^n,$$

$$\hat{a} = \alpha\exp\left[-\sum_{n=0}^{\infty}(a_n+2b_n)(y^n-1)\right],$$

$$\hat{\gamma} = \sum_{n=0}^{\infty}(a_n+2b_n)y^n - \frac{3}{2}\sum_{n=0}^{\infty}(a_{2n}+b_{2n}) - \frac{1}{2}\sum_{n=0}^{\infty}(b_{2n+1}-a_{2n+1}).$$

The structure of the metric on the horizon cross section is more transparent if we introduce the angular coordinate $0 \le \theta \le \pi$, such that $y = \cos\theta$. Then, it acquires the form

$$ds_H^2 = \frac{4\sigma(1+\alpha^2)^2}{1+\hat{a}^2(\theta)\cos^2(\frac{\theta}{2})}\sin^2\left(\frac{\theta}{2}\right)e^{-2\hat{W}(\theta)}d\psi^2 + 4\sigma\cos^2\left(\frac{\theta}{2}\right)e^{-2\hat{U}(\theta)}d\phi^2 + \sigma\left[1+\hat{a}^2(\theta)\cos^2\left(\frac{\theta}{2}\right)\right]e^{2(\hat{\gamma}(\theta)-\hat{W}(\theta))}d\theta^2, \tag{74}$$

which can be written also as

$$ds_H^2 = 4\sigma\left[F(\lambda)e^{2(\hat{\gamma}(\lambda)-\widehat{W}(\lambda))}d\lambda^2 + \frac{(1+\alpha^2)^2 e^{-2\widehat{W}(\lambda)}}{F(\lambda)}\sin^2\lambda d\psi^2 + e^{-2\widehat{U}(\lambda)}\cos^2\lambda d\phi^2\right], \tag{75}$$

where $\lambda = \theta/2$ and $F(\lambda) = 1 + \hat{a}^2(\lambda)\cos^2\lambda$. In the limit when the rotation parameter α and the external matter parameters a_n and b_n vanish, we obtain the metric on the 3-sphere with radius $R = 2\sqrt{\sigma}$:

$$ds_H^2 = 4\sigma[d\lambda^2 + \sin^2\lambda d\psi^2 + \cos^2\lambda d\phi^2], \tag{76}$$

in the Hopf coordinates $\{0 \le \lambda \le \pi/2, 0 < \psi < 2\pi, 0 < \phi < 2\pi\}$. Comparing the metric on the 3-sphere with (75) we can see that the horizon geometry is deformed from the spherical, and all the analytic axisymmetric horizon geometries are possible depending on the type of the external matter fields. In the absence of external matter the horizon is also deformed:

$$ds_H^2 = 4\sigma\left[F(\lambda)d\lambda^2 + \frac{(1+\alpha^2)^2}{F(\lambda)}\sin^2\lambda d\psi^2 + \cos^2\lambda d\phi^2\right]. \tag{77}$$

However, the deformation is more restricted since it is determined only by the function $F(\lambda) = 1 + \alpha^2\cos^2\lambda$, where α is the rotation parameter.

The horizon is encompassed by an ergoregion defined as the region where the Killing field $\partial/\partial t$ is spacelike:

$$g_{tt} = -\frac{x - 1 - \hat{a}^2(1-y)}{x + 1 + \hat{a}^2(1+y)} e^{2(\hat{U}+\hat{W})} > 0. \tag{78}$$

Since the denominator is always positive, the ergoregion is determined by the sign of the function

$$G = x - 1 - (1-y)\hat{a}^2,$$
$$\hat{a} = \alpha \exp\left[\sum_{n=1}^{\infty}(a_n + 2b_n)\sum_{k=0}^{n-1}(x-y)R^k P_k\left(\frac{xy}{R}\right)\right], \tag{79}$$

and the boundary of the ergoregion $g_{tt} = 0$ defines the ergosurface. In the case of the nondistorted Myers-Perry black hole the ergosurface is always a compact 3D hypersurface which touches the horizon at the point $\{x = 1, y = 1\}$, and intersects the symmetry axis $y = -1$ at $x = 1 + \alpha^2$. Increasing the rotation parameter α the ergoregion extends further from the horizon (see Fig. 2). In the case of the Myers-Perry black hole in an external gravitational field the ergoregion has a more complicated behavior. We can see that there always exists a small neighborhood of the horizon $x \to 1$, $x > 1$, where (79) is positive. Since the ergoregion should be axially symmetric, we can estimate its qualitative behavior by investigating the points in which the ergosurface intersects the axes of the spacelike Killing fields. As for the nondistorted Myers-Perry black hole the ergosurface always touches the horizon $x = 1$ at the axis $y = 1$. The intersection point with the axis of the Killing field $\partial/\partial\phi$ depends on the values of the parameters a_n, b_n, and α. The restriction of the function $\hat{a}$ on the axis $y = -1$ is given by the expression

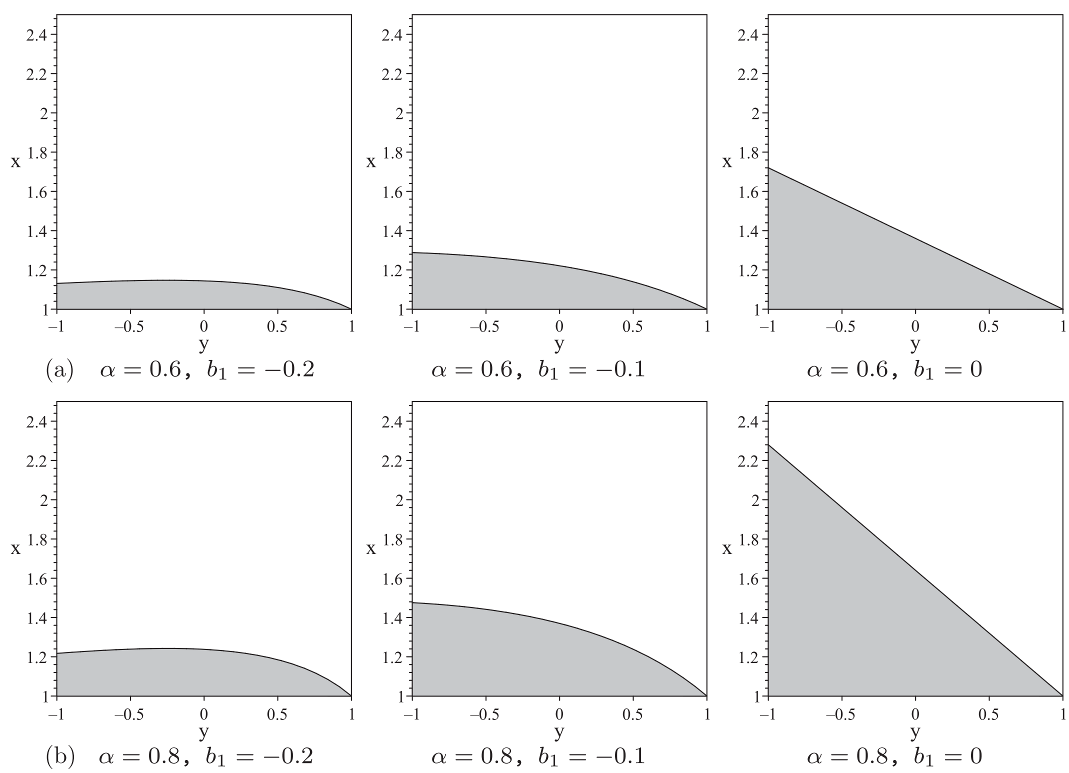

FIG. 2. Behavior of the ergoregion (grey area) for negative distortion parameter b_1 and different values of the rotation parameter : the dependence of the ergoregion on b_1 is investigated for fixed values of $\alpha = 0.6$ (a), and $\alpha = 0.8$ (b). The ergoregion for the non-distorted Myers-Perry black hole ($b_1 = 0$) is also presented.

$$\hat{a} = \alpha \exp\left[\sum_{n=0}^{\infty}(a_n + 2b_n)(1 - (-1)^n x^n)\right],$$

and examining Eq. (79) we can see that there exist values of the parameters a_n and b_n for which it is never satisfied. This behavior means that the ergoregion extends to infinity, if we consider the distorted Myers-Perry black hole as a global solution. Considering it as a local solution, which is valid only in some neighborhood of the horizon, it means that in some cases the ergoregion is not compact in the region of validity.

We demonstrate some of the possible configurations of the ergoregion by investigating some simple cases of distortion. The ergoregion is independent of the solution parameters σ, a_0, and b_0. Therefore, the most simple nontrivial type of distortion is described by setting $b_1 \neq 0$, and requiring the other external matter parameters to vanish. We will refer to this case as dipole distortion. Setting $a_1 \neq 0$ instead, and requiring the other distortion parameters to vanish leads to the same qualitative behavior of the ergoregion, and the two cases are related by the shift $a_1 \longleftrightarrow 2b_1$. Similarly, all the arguments for the case $b_1 \neq 0$ refer also for distortions of the type $a_1 \neq 0$, $b_1 \neq 0$, and the other distortion parameters vanishing.

Examining the behavior of (79) in the case of dipole distortion $b_1 \neq 0$ we obtain that for negative values of b_1 the ergoregion is always a compact region encompassing the horizon, similar to the nondistorted solution. If the distortion parameter b_1 is positive, however, the ergoregion always contains a noncompact part which extends to infinity if we consider our solution as a global one. Furthermore, the ergoregion can be either simply connected, or include two parts separated in spacetime, one of which is a compact region in the vicinity of the horizon.

In the case of the dipole distortion the number of the separated parts of the ergoregion is determined by the number of the crossing points of the ergosurface with the axis $y = -1$, or equivalently by the number of the real roots of the equation

$$G = x - 1 - (1 - y)\hat{a}^2 = 0, \qquad (80)$$

evaluated at $y = -1$. For negative values of the external matter parameter b_1, Eq. (80) has always a single real root, and consequently a single intersection point of the ergosurface with the $y = -1$ axis. This corresponds to a compact ergoregion encompassing the horizon for all values of $b_1 < 0$, and the rotation parameter α. If we keep α fixed and vary b_1, we obtain that the ergoregion gets smaller as $|b_1|$ grows. Keeping b_1 fixed and varying α, we observe that the ergoregion gets larger with the increase of the rotation parameter, and extends further from the horizon. The behavior of the ergoregion for negative b_1 is presented in Fig. 2. For comparison we have also illustrated the ergoregion for the nondistorted Myers-Perry black hole for the same values of the rotation parameter α.

If we consider positive values of the distortion parameter $b_1 > 0$, Eq. (80) can possess at most two real roots. Provided we keep the rotation parameter α constant and vary b_1, two roots occur for small b_1, and there exists a critical value $b_{1_{\rm crit}}$, depending on the particular value of α, when Eq. (80) possesses only a single real root. For $b_1 > b_{1_{\rm crit}}$ no real roots exist. Consequently, for $b_1 < b_{1_{\rm crit}}$ we observe two separated ergoregions, one of which extends to infinity. When b_1 approaches $b_{1_{\rm crit}}$ the two regions get closer to each other, and for $b_1 = b_{1_{\rm crit}}$ they touch at the common crossing point of their boundaries with the axis $y = -1$. When b_1 exceeds $b_{1_{\rm crit}}$ the two parts of the ergoregion merge into a single noncompact ergoregion which extends to infinity. This is in agreement with the fact that for $b_1 > b_{1_{\rm crit}}$ the ergosurface does not intersect the $y = -1$ axis.

If we keep the parameter b_1 fixed and vary the rotation parameter α a similar behavior is observed. Again, there exists a critical value of the rotation parameter $\alpha_{\rm crit}$, when Eq. (80) has a single real root, two real roots for $0 < \alpha < \alpha_{\rm crit}$, and none for $\alpha > \alpha_{\rm crit}$. Thus, for $0 < \alpha < \alpha_{\rm crit}$ two separated parts of the ergoregion are observed, which touch at $\alpha = \alpha_{\rm crit}$, and merge for $\alpha > \alpha_{\rm crit}$ into a single ergoregion extending to infinity. The behavior of the ergoregion for positive b_1 is demonstrated in Fig. 3. In Fig. 3(a) we investigate the ergoregion for a constant value of the rotation parameter $0 < \alpha < 1$, and different values of $b_1 > 0$. In Fig. 3(b) we keep the distortion parameter b_1 fixed and study the influence of the rotation on the ergoregion by varying α.

We will obtain a richer structure of the ergoregion if we consider more complicated types of distortion by keeping other external matter parameters a_n and b_n different from zero. We illustrate some possible configurations in the case when the distortion parameter $b_2 \neq 0$, and the others are vanishing, which we call quadrupole distortion (see Figs. 4 and 5). In contrast to the case of dipole distortion, for all values of b_2 the ergoregion is not compact. For negative b_2 the ergoregion behaves qualitatively as in the case of positive dipole distortion $b_1 > 0$. For low values of $|b_2|$ two separated parts of the ergoregion exist, intersecting the axis $y = -1$, which merge into a single one as $|b_2|$ grows. The same structure is observed if we keep $b_2 < 0$ fixed and vary the rotation parameter α (Fig. 4). For positive values of the distortion parameter b_2 another type of behavior of the ergoregion is demonstrated. For low values of b_2 the ergoregion consists of two parts—a compact one in the vicinity of the horizon and a noncompact one separated from it in spacetime. However, the noncompact part is situated in the region $0 < y < 1$, and does not intersect the axis $y = -1$. When b_2 grows, the two parts merge in a single ergoregion,

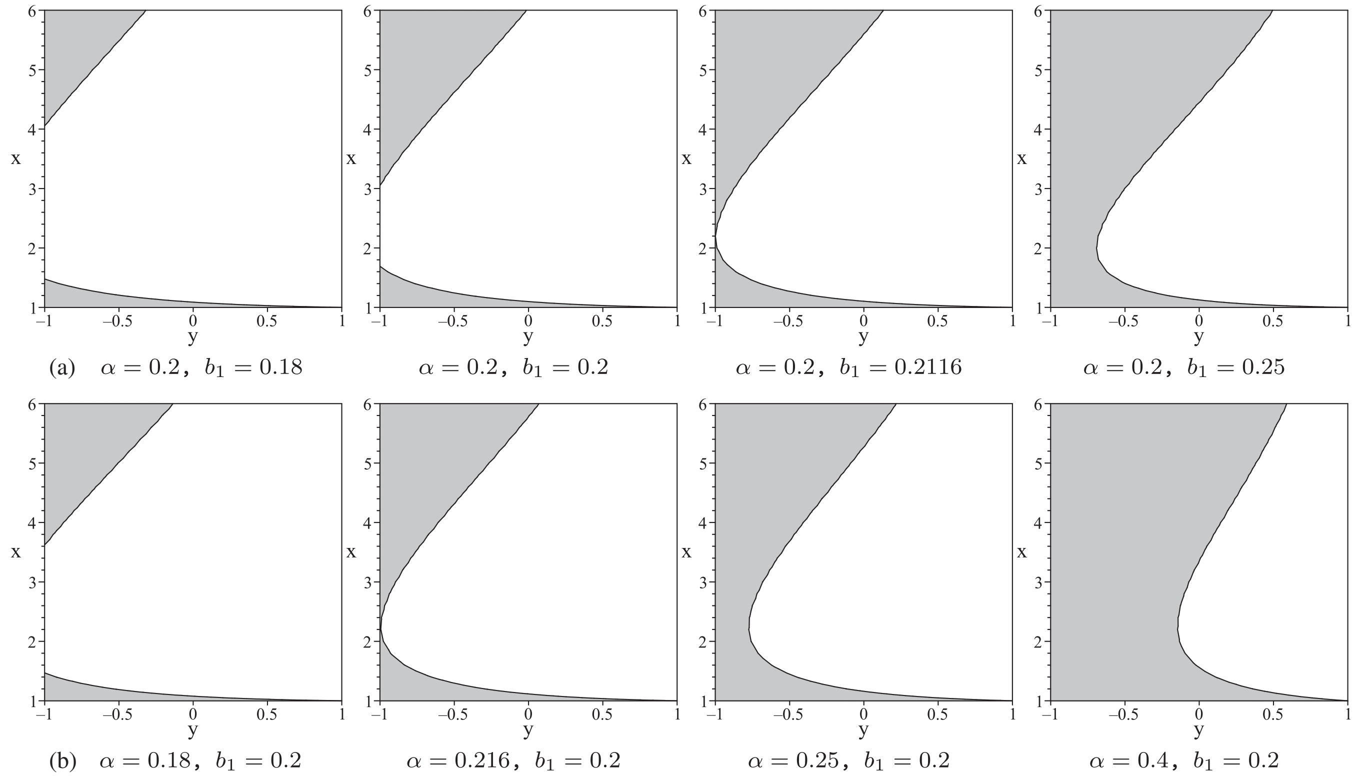

FIG. 3. Behavior of the ergoregion (grey area) for positive distortion parameter b_1: (a) the dependence of the ergoregion on b_1 is investigated for fixed value of the rotation parameter $\alpha = 0.2$; (b) the dependence of the ergoregion on the rotation is investigated for fixed value of $b_1 = 0.2$.

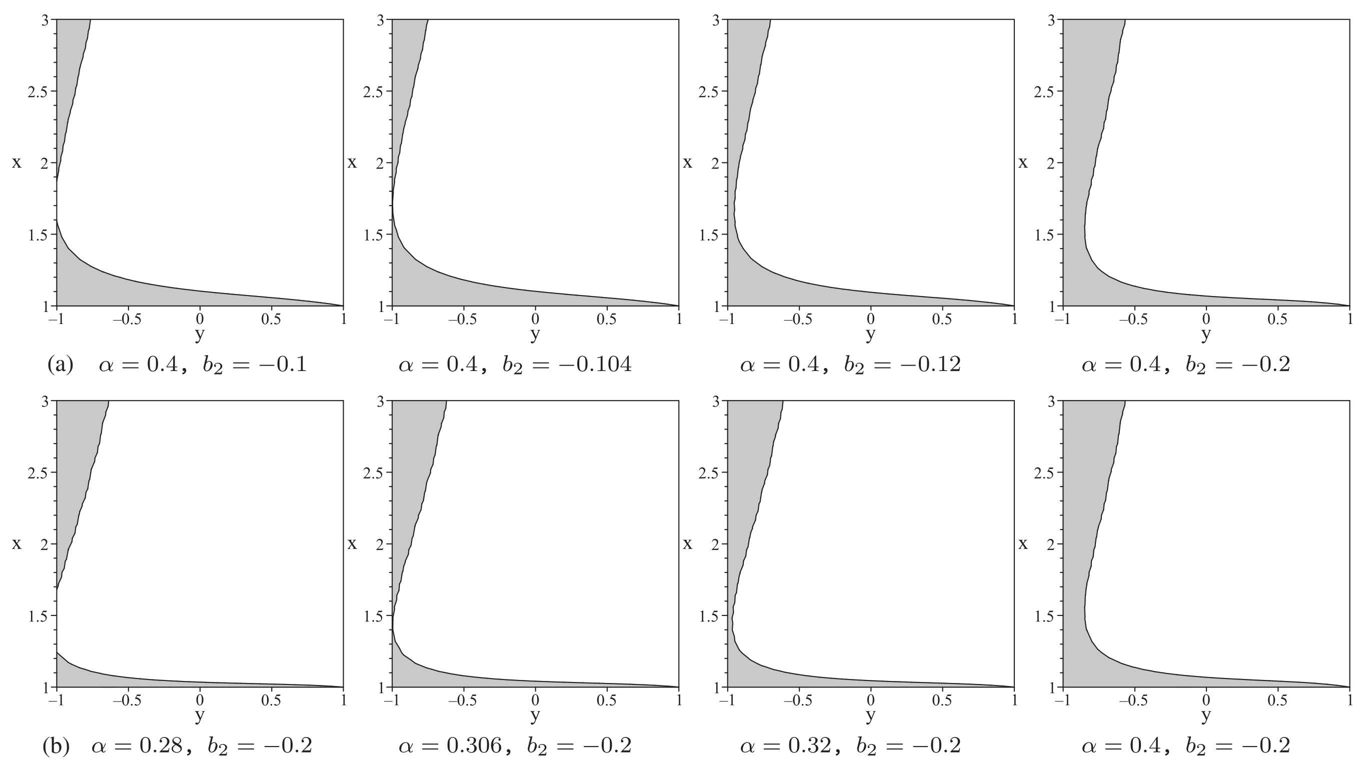

FIG. 4. Behavior of the ergoregion (grey area) for quadrupole distortion $b_2 < 0$: (a) the dependence of the ergoregion on b_2 is investigated for fixed value of the rotation parameter $\alpha = 0.4$; (b) the dependence of the ergoregion on the rotation is investigated for fixed value of $b_2 = -0.2$.

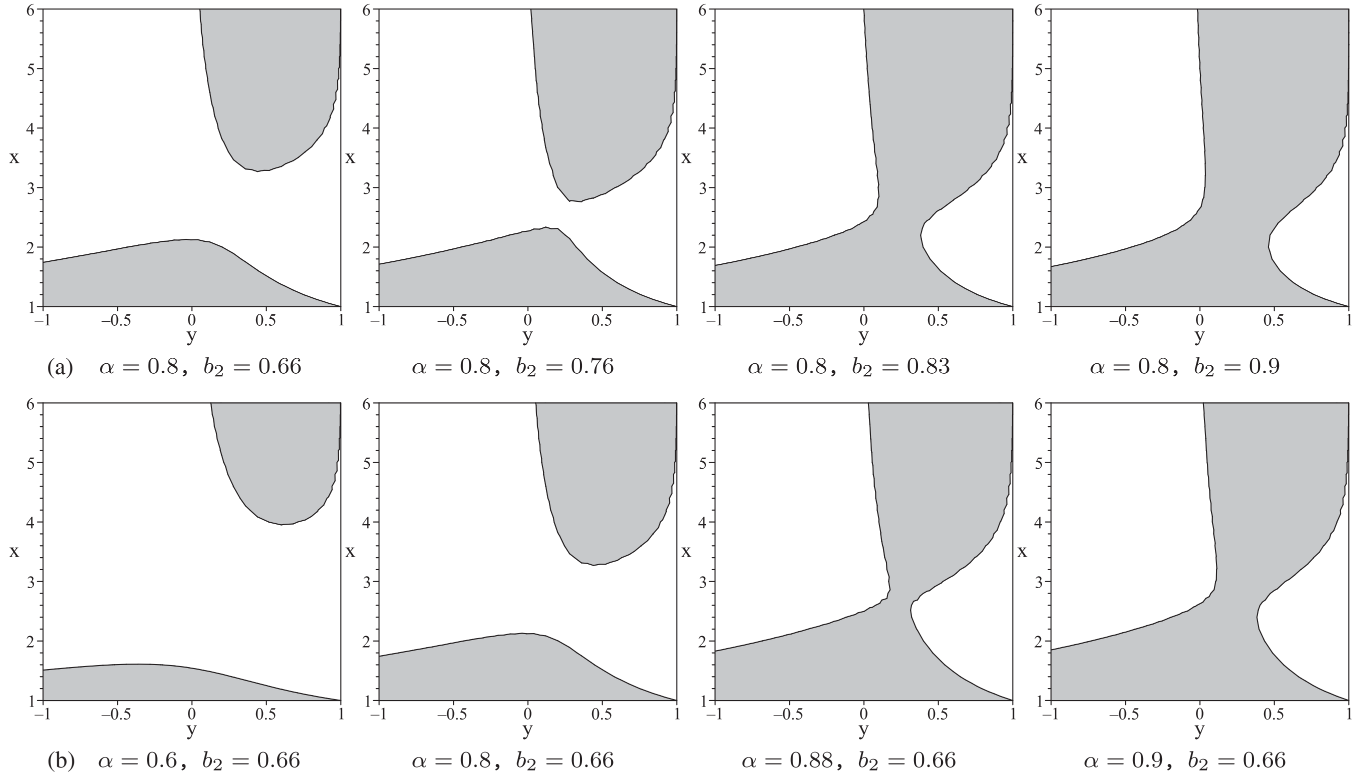

FIG. 5. Behavior of the ergoregion (grey area) for quadrupole distortion $b_2 > 0$: (a) the dependence of the ergoregion on b_2 is investigated for fixed value of the rotation parameter $\alpha = 0.8$; (b) the dependence of the ergoregion on the rotation is investigated for fixed value of $b_2 = 0.66$.

and the same behavior is observed if we keep $b_2 > 0$ fixed, and vary the rotation parameter α (Fig. 5).

In this work we consider the distorted Myers-Perry black hole (49) as a local solution, which is valid only in a certain neighborhood of the horizon. In general, a global solution can be constructed if (49) is extended to an asymptotically flat solution by some sewing technique. This can be realized by cutting the spacetime manifold in the region where the metric (49) is valid and attaching to it another spacetime manifold where the solution is not vacuum anymore, but the sources of the distorting matter are also included. In the cases where the ergoregion consists of a compact region close to the horizon and another noncompact region, one can always choose to cut the manifold at such a value of the x and y coordinates that the noncompact region is not included. In this way one may be able to construct an extension to an asymptotically flat solution with a compact ergoregion.

VII. CONCLUSION

We have obtained a new exact solution of the 5D Einstein equations in vacuum representing a Myers-Perry black hole with a single angular momentum in an external gravitational field. Locally, the solution is interpreted as a black hole distorted by a stationary $U(1) \times U(1)$ symmetric distribution of external matter. We have constructed the solution by applying a twofold Bäcklund transformation on a 5D distorted Minkowski spacetime as a seed. The physical quantities of the solution have been calculated, and a local Smarr-like relation on the black hole horizon has been derived. The solution satisfies the same Smarr-like relation as the asymptotically flat Myers-Perry black hole. However, in contrast to the nondistorted Myers-Perry black hole the ratio of the horizon mass and angular momentum, J_H^2/M_H^3, can become arbitrarily large. We have then considered a particular case of the general distorted Myers-Perry solution (49), in which the parameters a_n and b_n characterizing the external matter field obey the relation $a_n + 2b_n = 0$ for every n. Consequently, the angular velocity of the horizon is equal to the one of the asymptotically flat Myers-Perry case, and the ergoregion behaves also as in the asymptotically flat Myers-Perry case.

We have further considered the effect of dipole distortions and quadrupole distortions on the ergoregions. There exists a small neighborhood of the horizon $x \to 1$, $x > 1$ belonging to the ergoregion. In some cases, the ergoregion appears to be noncompact in the region of validity of the solution. In the cases when the ergoregion consists of a compact region close to the horizon and another noncompact region, one can always choose to cut the manifold at a value of the x and y coordinates such that the noncompact region in not included, in order to construct an asymptotically flat solution with a compact ergoregion. However, we have not discussed the particular extension of the solution to the asymptotically flat cases.

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