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Stationary axisymmetric spacetimes with a conformally coupled scalar field

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Solution-generating techniques for general relativity with a conformally (and minimally) coupled scalar field are pushed forward to build a wide class of asymptotically flat, axisymmetric, and stationary spacetimes continuously connected to Kerr spacetime. This family contains, amongst other things, rotating extensions of the Bocharova-Bronnikov-Melnikov-Bekenstein black hole and also its angular and mass multipolar generalizations. The addition of Newman-Unti-Tamburino charge is also discussed.

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I. INTRODUCTION

Fundamental scalar fields have been studied for a long time in gravity and high-energy theoretical physics, with various aims ranging from cosmology to the standard model (of particles), scalar-tensor theories, and strings. But lately they are enjoying renewed attention after the experimental confirmation of the Higgs scalar field at CERN. Historically, the interest in the scalar matter field coupled to general relativity (GR) in a conformal-invariant way (such as standard Maxwell electromagnetism in four dimensions) began in the 1970s, when Bekenstein showed that coupling could admit a black hole solution [1,2]. At the time this constituted the first counterexample to the black hole no-hair theorem, which states that in the gravitational collapse forming a black hole all degrees of freedom vanish, apart from the mass and the angular momentum (and electric charge, if we are considering also the electromagnetic coupling). This black hole—found by Bocharova, Bronnikov, Melnikov, and Bekenstein [1,2] (henceforth BBMB)—presents certain issues, which were summarized in Ref. [3]. The main ones are the fact that the spacetime is not stable under linear perturbations [4], and the fact that the scalar field is divergent on the event horizon.[1] Note that in the presence of a cosmological constant the scalar-field infinities are hidden behind the event horizon [5], and therefore the solution becomes more regular.

Nevertheless, lately there has been some interest in solution-generating techniques for general relativity with a conformally coupled scalar field [6,7] and in its main application, i.e., the rotating generalization of the BBMB black hole, which is still missing. Some stationary generalizations of BBMB spacetime were produced that included acceleration [8], an external magnetic field [3,6], or Newman-Unti-Tamburino (NUT) charge [7,9]. The main inconvenience shared by these constructions is that they are not asymptotically flat and they do not have a proper limit to the Kerr black hole. Recently the possibility of having a slowly rotating generalization of the BBMB metric was discussed in Refs. [9,10].

The aim of this paper is to fill this gap, i.e., to exploit and enhance the techniques developed in Ref. [6] to find a general asymptotically flat, axisymmetric, and stationary rotating family of metrics for the conformally coupled scalar matter, which include as a static limit the BBMB black hole. This is done in Sec. II. For this purpose we have to integrate the methods of Ref. [6] (based on the Ernst formalism [11]) with the Hoenselaers-Kinnersley-Xanthopoulos (HKX) transformation [12], which was originally developed to add rotation to a static axisymmetric spacetime in general relativity while preserving asymptotic flatness. For example, these are the best transformations for generating the Kerr black hole from the Schwarzschild one. Basically we want to generalize some of the results presented in Refs. [13,14] in the presence of a conformally coupled scalar field, and hence multipolar metrics are considered in Sec. III.

As explained in Ref. [6], when a scalar field is conformally coupled with general relativity[2] the most generic axisymmetric and stationary spacetime is not modeled by the Lewis-Weyl-Papapetrou metric. Therefore, in order to take advantage of the Weyl coordinates and of the integrability of the system, we shift from the conformally coupled theory (CC) to the minimally coupled one (MC), thanks to a conformal transformation of the metric. Then we make use of the explicit symmetries of the minimally coupled theory, which allow us to perform an $H\hat{K}X$ transformation that is able to generate rotation, and finally we come back to the conformally coupled theory, thanks to a conformal transformation (which is the inverse of the first one). With this procedure we can also generate an HKX

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[1] In Ref. [2] it is explained (as suggested by de Witt) that this divergency does not cause any pathological behavior on physical observables; for example, while crossing the horizon there is no potential barrier and tidal forces remain finite.

[2] We are not considering the cosmological constant here because a solution-generating technique in that case is not available at the moment [15].

transformation in the conformally coupled theory. Pictorially this is illustrated in the following figure:

$$\begin{array}{ccc} MC & \xleftarrow{\Omega^{-1}} & CC \\ {\scriptstyle H\hat{K}X}\downarrow & & \downarrow{\scriptstyle HKX=\Omega\circ H\hat{K}X\circ\Omega^{-1}} \\ MC & \xrightarrow[\Omega]{} & CC \end{array}$$

To be more precise, let us consider the action for general relativity with a conformally coupled scalar field Ψ:

$$I[g_{\mu\nu},\Psi] = \int d^4x\sqrt{-g}\left[\frac{\mathrm{R}}{16\pi G} - \frac{1}{2}\partial_\mu\Psi\partial^\mu\Psi - \frac{\mathrm{R}}{12}\Psi^2\right]. \tag{1.1}$$

Extremizing the action with respect to the metric $g_{\mu\nu}$ yields the Einstein field equations, while extremizing with respect to the scalar field Ψ gives the scalar field equation:

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}g_{\mu\nu} = 8\pi G\left[\partial_\mu\Psi\partial_\nu\Psi - \frac{1}{2}g_{\mu\nu}\partial_\sigma\Psi\partial^\sigma\Psi + \frac{1}{6}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\Psi^2\right], \tag{1.2}$$

$$\Box\Psi - \frac{1}{6}\mathrm{R}\Psi = 0. \tag{1.3}$$

We now focus on a subclass of stationary axisymmetric spacetimes that contain the BBMB black hole in the static case, which can be generally written as

$$ds^2 = \Omega\{-f(dt-\omega d\varphi)^2 + f^{-1}[\rho^2 d\varphi^2 + e^{2\gamma}(d\rho^2+dz^2)]\}, \tag{1.4}$$

where all the functions f, γ, ω,and Ω depend on the (ρ,z) coordinates only. Ω is the conformal factor that relates the minimally coupled theory to the conformally coupled one (1.1):

$$\Omega(\rho,z) \coloneqq \left[1 - \frac{4\pi G}{3}\Psi^2(\rho,z)\right]^{-1}. \tag{1.5}$$

Actually, any solution of general relativity with a minimally coupled scalar field $(\hat{g},\hat{\Psi})$, whose action is

$$\hat{I}[\hat{g}_{\mu\nu},\hat{\Psi}] = \int d^4x\sqrt{-\hat{g}}\left[\frac{\hat{R}}{16\pi G} - \frac{1}{2}\nabla_\mu\hat{\Psi}\nabla^\mu\hat{\Psi}\right] \tag{1.6}$$

and whose field equations are

$$\hat{\mathrm{R}}_{\mu\nu} - \frac{\hat{\mathrm{R}}}{2}\hat{g}_{\mu\nu} = 8\pi G\left(\partial_\mu\hat{\Psi}\partial_\nu\hat{\Psi} - \frac{1}{2}\hat{g}_{\mu\nu}\partial_\sigma\hat{\Psi}\partial^\sigma\hat{\Psi}\right), \tag{1.7}$$

$$\Box\hat{\Psi} = 0, \tag{1.8}$$

can be mapped into a solution (g,Ψ) of the conformally coupled theory (1.1) by the following conformal transformation:

$$\hat{\Psi} \longrightarrow \Psi = \sqrt{\frac{6}{8\pi G}}\tanh\left(\sqrt{\frac{8\pi G}{6}}\hat{\Psi}\right), \tag{1.9}$$

$$\hat{g}_{\mu\nu} \longrightarrow g_{\mu\nu} = \Omega\hat{g}_{\mu\nu}. \tag{1.10}$$

At this point it is possible to use the solution-generating technique developed in Ref. [6] for the theory (1.1). It consists of building Ernst potentials for the minimally coupled theory (1.6) and then uplifting it to the conformally coupled theory by the conformal transformation (1.9)–(1.10). For generating purposes, usually the best coordinates are the prolate spherical ones (x,y), which are related to (ρ,z) by the following transformations:

$$\rho \coloneqq \kappa\sqrt{(x^2-1)(1-y^2)}, \qquad z \coloneqq \kappa xy, \tag{1.11}$$

where κ is a constant. In Ref. [6] we have learned that the symmetries of axisymmetric and stationary solutions of standard general relativity are inherited by the conformally coupled theory, so we can also use the improvements of the Ernst technique [11] developed by Hoenselaers, Kinnersley, and Xanthopoulos in Ref. [12] (see also Ref. [14]) to generate a stationary version of the BBMB metric from the static one.

As a starting point we consider the Fisher-Janis-Robinson-Winnicour metric (FJRW), which is a static solution for the minimally coupled theory; in prolate spherical coordinates it can be written as

$$d\hat{s}^2 = -\left(\frac{x-1}{x+1}\right)^\delta dt^2 + \left(\frac{x+1}{x-1}\right)^\delta \kappa^2\left[dx^2 + \frac{x^2-1}{1-y^2}dy^2 + (x^2-1)(1-y^2)d\varphi^2\right]. \tag{1.12}$$

It is supported by the following scalar field:

$$\hat{\Psi}_0(x) = \sqrt{\frac{1-\delta^2}{16\pi G}}\log\left(\frac{x-1}{x+1}\right). \tag{1.13}$$

From this seed metric we can extract its Ernst potential (since the metric is static and electromagnetically uncharged, $\mathcal{E} = f$),

$$\mathcal{E}_0 = \left(\frac{x-1}{x+1}\right)^\delta, \tag{1.14}$$

where the distortion (or Zipoy-Voorhees) parameter $\delta \in \mathbb{R}$. We recall that for $\delta = 1$ we have the Schwarzschild spacetime (note that in this case the scalar field vanishes), while for $\delta = 1/2$ we have the BBMB black hole up to the conformal transformation (1.5) (as explicitly shown in the next section). Note that the scalar field (1.13) is not the most general solution of Eq. (1.8), but rather just the one that gives the BBMB metric; this is our motivation for picking it. Other possible generalizations of the scalar field (1.13) are considered in Appendix D. Note also that the HKX transformations do not affect the scalar field, like all transformations inherited by the vacuum symmetries.

II. ADDING ROTATION TO THE BBMB BLACK HOLE

In this section we want to find a stationary generalization of the BBMB black hole. Thus we apply two rank-zero HKX transformations to the static (and therefore real) seed Ernst potential (1.14), as done in Refs. [13,14] for general relativity. The presentation of the HKX formalism is rather involved and beyond the scope of the present paper; for a detailed introduction to HKX transformations and their applications to vacuum Weyl metrics see Refs. [16,17]. However, we can present the action of N rank-zero HKX transformations on a static seed Ernst potential $\mathcal{E}_0$ to get a new stationary potential $\mathcal{E}$,

$$\mathcal{E}_0 \longrightarrow \mathcal{E} = \mathcal{E}_0 \frac{D_-}{D_+}, \tag{2.1}$$

where

$$D_\pm = \det\left\{\delta_{ij} + i\frac{\alpha_k U_k}{2S(U_k)}\exp[2B(U_k)] \times \left[\frac{U_j + U_k - 4U_jU_kz}{U_jS(U_k) + U_kS(U_j)} \pm 1\right]\right\}. \tag{2.2}$$

This transformation adds $2N$ parameters α_k and U_k because $j,k = 1,2,\ldots,N$. The function $B(U_k)$ satisfies the differential equation[3]

$$S(U_k)\vec{\nabla}B(U_k)(1-2U_kz)\vec{\nabla}\left(\frac{1}{2}\log\mathcal{E}_0\right) + 2U_k\rho\vec{e}_\varphi \times \vec{\nabla}\left(\frac{1}{2}\log\mathcal{E}_0\right), \tag{2.3}$$

with

$$S^2(U_k) = (1-2U_kz)^2 + (2U_k\rho)^2. \tag{2.4}$$

For the two rank-zero HKX transformations $k \in \{1,2\}$, so they add four new constants α_1, α_2, U_1, and U_2, two of which can be reabsorbed in a coordinate transformation,

$$U_1 = -U_2 = \frac{1}{2\kappa} = U; \tag{2.5}$$

then,

$$S(\pm U) = x \mp y.$$

By inserting this latter into Eq. (2.2) and redefining the constants $\alpha_1 := \alpha$ and $\alpha_2 := \beta$, we get a new rotating (and therefore complex) Ernst potential for the stationary version of the FJRW metric:

$$\mathcal{E} = \frac{d_-}{d_+} = \frac{\xi - 1}{\xi + 1}, \quad \text{with } \xi := \frac{d_+ + d_-}{d_+ - d_-}, \tag{2.6}$$

where

$$d_\pm(x,y) := (x\pm 1)^{\delta-1}[x(1-\lambda\mu) + iy(\lambda+\mu) \pm (1+\lambda\mu) \mp i(\lambda-\mu)], \tag{2.7}$$

$$\lambda(x,y) := \alpha(x^2-1)^{1-\delta}(x+y)^{2\delta-2}, \tag{2.8}$$

$$\mu(x,y) := \beta(x^2-1)^{1-\delta}(x-y)^{2\delta-2}. \tag{2.9}$$

The two rank-zero HKX transformations add two new independent parameters α and β, which are usually called the rotation and reflection parameters. In general, for $\delta \neq 1$, the presence of α and β (with $\alpha \neq \beta$) may break the equatorial symmetry with respect to the plane $y = 0$, while the axisymmetry is always granted by construction through Eq. (1.4). The HKX-transformed potential generally may have a NUT charge, which can spoil the asymptotic flatness of the seed metric. Therefore we perform an additional Ehlers transformation to add another NUT charge, parametrized by τ, which can elide the possible preexisting one. The Ehlers transformation in terms of ξ consists in adding a multiplying phase, $\xi \longrightarrow \bar{\xi} = \xi e^{i\tau}$; therefore, the final Ernst potential $\bar{\mathcal{E}}$ reads

$$\bar{\mathcal{E}} = \frac{\bar{\xi} - 1}{\bar{\xi} + 1} = \frac{(d_+ + d_-)e^{i\tau} - (d_+ - d_-)}{(d_+ + d_-)e^{i\tau} + (d_+ - d_-)}. \tag{2.10}$$

The Ernst potential (2.10) represents the stationary rotating version of the FJRW metric, describing a mass monopole, which additionally is asymptotically flat, or at most NUT. Mass multipolar solutions can also be constructed with the help of the solution-generating techniques; this will be done in Sec. III. We remember that a spacetime can have both mass multipoles and angular momentum multipoles, but generally these latter vanish in the Newtonian limit.

Moreover, we note that the δ parameter remains a real number in the stationary case as well, and it is not limited to integers, as is the case for the standard Tomimatsu-Sato family.

[3] The differential operator $\vec{\nabla}$ refers to the flat cylindrical gradient in (ρ, z, φ) coordinates.

A. $\alpha \neq 0$ and $\beta = 0$

For the sake of simplicity let us restrict to the case $\beta = 0$ in Eq. (2.9), because this is the simplest case containing the Kerr metric. In Sec. II B and Appendix B some more general cases are considered.

First of all, we want to check that the case $\delta = 1$ contains the Kerr Black hole. For $\delta = 1$ the Ernst potential becomes

$$\mathcal{E}_{(1)} = \frac{(x+iy\alpha)(\cos\tau + i\sin\tau) - (1-i\alpha)}{(x+iy\alpha)(\cos\tau + i\sin\tau) + (1-i\alpha)}. \tag{2.11}$$

Then we can cancel the NUT charge by demanding asymptotic flatness. In practice this means we have to impose the following constraints on the parameters:

$$\cos\tau = \frac{\kappa}{m}, \qquad \sin\tau = -\frac{a}{m}, \qquad \alpha = \frac{a}{\kappa}, \qquad \kappa^2 = m^2 - a^2. \tag{2.12}$$

Hence the Ernst potential for the pure Kerr metric is found:

$$\mathcal{E}_{(1)} = \frac{x\frac{\kappa}{m} + iy\frac{a}{m} - 1}{x\frac{\kappa}{m} + iy\frac{a}{m} + 1}. \tag{2.13}$$

In this case the parameters a and m represent, respectively, the mass and the angular momentum of the Kerr black hole. Note that $\delta = 1$ implies the vanishing of the scalar field and, as a consequence, the trivialization of the conformal factor (1.5), which becomes $\Omega = 1$. It means that the Ernst potential (2.11) (if it is properly cleaned) describes the Kerr metric in both the Einstein and Jordan frames.

Since we want to build a stationary version of the BBMB black hole we have to consider $\delta = 1/2$. In fact, for this value of δ, the static BBMB black hole can be obtained by a conformal transformation [Eq. (1.5)] of the FJRW spacetime. So for $\delta = 1/2$ the Ernst potential becomes

$$\mathcal{E}_{(\frac{1}{2})} = \frac{\sqrt{x+1}\sin\frac{\tau}{2}[-\alpha(x-1)(y-1) + i\sqrt{x^2-1}(x+y)] + \sqrt{x-1}\cos\frac{\tau}{2}[\sqrt{x^2-1}(x+y) + i\alpha(x+1)(y+1)]}{\sqrt{x-1}\sin\frac{\tau}{2}[-\alpha(x+1)(y+1) + i\sqrt{x^2-1}(x+y)] + \sqrt{x+1}\cos\frac{\tau}{2}[\sqrt{x^2-1}(x+y) + i\alpha(x-1)(y-1)]}. \tag{2.14}$$

From the definition of the Ernst potential,

$$\mathcal{E} \coloneqq f + ih, \tag{2.15}$$

we can directly infer that the f field of the metric (1.4) is the real part of Eq. (2.14), while ω can be obtained from the definition of h:

$$\vec{\nabla}h \coloneqq -\frac{f^2}{\rho}\vec{e}_\varphi \times \vec{\nabla}\omega. \tag{2.16}$$

The differential operators in spheroidal coordinates can be written as[4]

$$\vec{\nabla}f(x,y) \propto \frac{\vec{e}_x}{\kappa}\sqrt{\frac{x^2-1}{x^2-y^2}}\partial_x f(x,y) + \frac{\vec{e}_y}{\kappa}\sqrt{\frac{1-y^2}{x^2-y^2}}\partial_y f(x,y), \tag{2.17}$$

while the two-dimensional line element in spheroidal coordinates is

$$d\rho^2 + dz^2 = \kappa^2(x^2-y^2)\left[\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right]. \tag{2.18}$$

Up to this point the effects of the minimally coupled scalar field have not been taken into account, because at the level of the Ernst formalism the minimally coupled scalar field is actually decoupled from the Ernst potentials. But to find γ the contributions of the scalar stress-energy tensor are relevant. To obtain γ a quadrature is usually sufficient, once the other fields are known. In this case, from the $EE^\rho{}_\rho$ and $EE^\rho{}_z$ components of the Einstein equations in the minimally coupled theory (1.7), we have, respectively,

$$\partial_\rho\gamma = -\frac{1}{4}\frac{f^2}{\rho}[(\partial_\rho\omega)^2 - (\partial_z\omega)^2] + \frac{1}{4}\frac{\rho}{f^2}[(\partial_\rho f)^2 - (\partial_z f)^2] + 4\pi G\rho[(\partial_\rho\hat{\Psi})^2 - (\partial_z\hat{\Psi})^2], \tag{2.19}$$

$$\partial_z\gamma = \frac{\rho}{2f^2}(\partial_z f)(\partial_\rho f) - \frac{f^2}{2\rho}(\partial_z\omega)(\partial_\rho\omega) + 8\pi G\rho(\partial_z\hat{\Psi})(\partial_\rho\hat{\Psi}). \tag{2.20}$$

Note that by defining $\gamma = \gamma_0 + \gamma_\Psi$, where γ_0 is a solution for general relativity (when $\Psi = 0$), the previous system of partial differential equations (2.19)–(2.20) (thanks to its linearity) reduces to

[4] The orthonormal frame is defined by the ordered triad $(\vec{e}_x, \vec{e}_y, \vec{e}_\varphi)$.

$$\partial_\rho \gamma_\Psi = 4\pi G \rho [(\partial_\rho \hat{\Psi})^2 - (\partial_z \hat{\Psi})^2], \tag{2.21}$$

$$\partial_z \gamma_\Psi = 8\pi G \rho (\partial_z \hat{\Psi})(\partial_\rho \hat{\Psi}). \tag{2.22}$$

This means that from any axisymmetric and stationary solution of general relativity we can generate a new solution for the same theory with the addition of a minimally [or conformally, depending on whether it is properly conformally transformed according to Eqs. (1.9)–(1.10)] coupled scalar field. This can be done by adding the γ_Ψ contribution given by a harmonic scalar field satisfying Eqs. (2.21)–(2.22). The harmonicity is required by the scalar field equation (1.8).

The most general solution of Eq. (1.8) achievable by separation of variables can be expressed, in prolate spherical coordinates, as an expansion in terms of the Legendre polynomials of the first and second kind (more details in Appendix A), denoted by $P_n(x)$ and $Q_n(x)$, respectively,

$$\hat{\Psi} = \sum_{n=0}^{\infty} [a_n Q_n(x) + b_n P_n(x)][c_n Q_n(y) + d_n P_n(y)]. \tag{2.23}$$

By imposing some regularity properties on the scalar field, it is possible to constrain the coefficients a_n, b_n, c_n, and d_n; for instance, imposing regularity along the symmetry axis ($y = \pm 1$) fixes $c_n = 0$. In Appendix D the first orders of the scalar field expansion (2.23) and their contributions to γ are considered, for some suitable boundary conditions.

The scalar field (1.13) that we are focusing on in this paper (i.e., the one that gives the BBMB black hole) can be obtained from the general solution (2.23) by keeping only the a_0 and d_0 coefficients non-null, such that $a_0 d_0 = \sqrt{(1-\delta^2)/(16\pi G)}$. In this case it is easy to evaluate the scalar field contribution γ_Ψ to the total γ; integrating Eqs. (2.21)–(2.22), we have

$$\gamma_\Psi = \kappa_2 - \frac{1}{2}(\delta^2 - 1)\log\left(\frac{x^2-1}{x^2-y^2}\right), \tag{2.24}$$

where κ_2 is an integrating constant, which can be fixed to fulfill the desired boundary conditions or guarantee the regularity of the metric, such as elementary asymptotic flatness. To sum up, the resulting fields for the conformally coupled theory and $\delta = 1/2$ are

$$f(x,y) = \frac{\sqrt{x^2-1}[(x+y)^2 - \alpha^2(1-y^2)]}{\cos\tau[(x+y)^2 - \alpha^2(1+2xy+y^2)] + \alpha^2(xy^2+x+2y) + (x+2\alpha\sin\tau)(x+y)^2}, \tag{2.25}$$

$$\omega(x,y) = \kappa \frac{\sin\tau[y(x+y)^2 + \alpha^2(1-y^2)(2x+y)] - 2\alpha y\cos\tau(x+y)^2 + 2\alpha^3(1-y^2)}{(x+y)^2 + \alpha^2(y^2-1)}, \tag{2.26}$$

$$\gamma(x,y) = \frac{1}{2}\log\left[\frac{x^2-1}{x^2-y^2} - \frac{\alpha^2(x^2-1)(1-y^2)}{(x+y)^2(x^2-y^2)}\right], \tag{2.27}$$

$$\Psi(x) = \sqrt{\frac{3}{4\pi G}}\tanh\left[\frac{1}{4}\log\left(\frac{x-1}{x+1}\right)\right]. \tag{2.28}$$

γ is independent of the NUT parameter τ, but not of ω. When $\alpha = 0$ we recover the NUT-BBMB metric recently found in Refs. [7,9]. In order for the metric to be free from the NUT charge we have to ask that $\omega(x,y) \to 0$ at spatial infinity, that is, for large x. Therefore, we have properly fixed the arbitrary integration constant of ω, and furthermore we have to constrain the τ parameter as follows:

$$\tau = \mathrm{ArcTan}\left(-\frac{\alpha}{\delta}\right). \tag{2.29}$$

Under these flat boundary conditions the functions f and ω simplify to

$$\omega = \frac{2\alpha^3\kappa(1-y^2)(\sqrt{1+4\alpha^2} + 2x + y)}{\sqrt{1+4\alpha^2}[(x+y)^2 - \alpha^2(1-y^2)]}, \tag{2.30}$$

$$f = \frac{\sqrt{1+4\alpha^2}\sqrt{x^2-1}[(x+y)^2 - \alpha^2(1-y^2)]}{\sqrt{1+4\alpha^2}[\alpha^2(xy^2+x+2y) + x(x+y)^2] + (1+4\alpha^2)(x+y)^2 - \alpha^2(2xy+y^2+1)}. \tag{2.31}$$

The metric is free from conical singularities on the axes of symmetry, since $\lim_{y\to\pm1}\gamma = 0$ and asymptotically it approaches the Minkowski spacetime. When the parameter $\alpha = 0$, one recovers the BBMB static black hole,

$$ds^2|_{\alpha=0} = -\left(1-\frac{m}{R}\right)^2 d\tau^2 + \frac{dR^2}{(1-\frac{m}{R})^2} + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.32}$$

$$\Psi(R) = \pm\sqrt{\frac{3}{4\pi G}}\left(1-\frac{R}{m}\right)^{-1}, \tag{2.33}$$

where the following relation between the coordinate x and the radial coordinate[5] R is used:

$$x := \frac{R^2}{2m(R-m)} - 1. \tag{2.34}$$

The double-degenerate horizon is located at $R = m$. Therefore, given that $R(x) = m(x+1\mp\sqrt{x^2-1})$, in terms of the x coordinate the horizon can be approached in the limit $x \to \infty$ by taking the minus branch, and the radial coordinate $R(x)$ points towards spatial infinity for $x \to \infty$ when taking the plus branch.

In the stationary case we do not have a unique criterion to define a radial coordinate, as can be done in the static case requiring, for instance, a spherically symmetric base manifold. Therefore, several possibilities for the radial coordinate can be considered in the rotating case, which physically may not be equivalent everywhere because of the nondifferentiability of the change of coordinates. The fact that the two charts are not diffeomorphic everywhere stems from the only constraint we have to impose, namely, that the radial coordinate converges to the static one (2.34) in the nonrotating limit ($\alpha = 0$). The easiest radial coordinate in the rotating case we can define is[6]

$$x := \frac{2R^2}{\kappa(2R-\kappa)} - 1 \xrightarrow[\alpha\to 0]{} \frac{R^2}{2m(R-m)} - 1. \tag{2.35}$$

The mass and angular momentum can be read from the asymptotic behavior of the metric, because the scalar field does not contribute to the charges. This is because the scalar field depends only on the radial coordinate and it quickly decays to zero at spatial infinity, and in the Hamiltonian formalism one can see that it does not contribute. For large values of the radial coordinate R the metric approaches spatial infinity as

$$ds^2 \sim -\left(1-\frac{2m}{R}\right)dt^2 + \left(1+\frac{2m}{R}\right)dR^2 + \frac{8\kappa^2\alpha^3\sin^2\theta}{R\sqrt{1+4\alpha^2}}dtd\varphi + R^2(d\theta^2+\sin^2\theta d\varphi^2) + O\left(\frac{1}{R^2}\right). \tag{2.36}$$

We now try to adapt the definition of the constant parameters κ and α, as in the Kerr case, while also taking into account the extra constant δ:

$$\kappa := \frac{m}{\sqrt{\delta^2+\alpha^2}}, \qquad \alpha := \frac{a\delta}{\kappa}. \tag{2.37}$$

This value we have chosen for κ coincides (setting $\beta = 0$) with the more general one given in Ref. [18],

$$\kappa = \frac{m(1-\alpha\beta)}{\sqrt{[\delta(\alpha\beta-1)-2\alpha\beta]^2 + (\alpha-\beta)^2}}. \tag{2.38}$$

With these definitions the mass M and angular momentum J become, respectively,

$$M = m, \qquad J = -\frac{8\alpha^3 m^2}{(1+4\alpha^2)^{3/2}}. \tag{2.39}$$

With the help of Appendix C we can compute the mass and angular multipole moments up to the octupole for the scalar generalization of the FJRW metric (with $\delta = 1/2$) defined by Eqs. (2.31), (2.30), and (2.27) in the Einstein frame,

$$\begin{aligned} M_0 &= m, \qquad J_0 = 0,\\ M_1 &= -\frac{4\alpha^2 m^2}{(1+4\alpha^2)^{3/2}}, \qquad J_1 = -\frac{8\alpha^3 m^2}{(1+4\alpha^2)^{3/2}},\\ M_2 &= \frac{(1+8\alpha^2-16\alpha^4)m^3}{(1+4\alpha^2)^2}, \qquad J_2 = \frac{16\alpha^3 m^3}{(1+4\alpha^2)^2},\\ M_3 &= -\frac{4\alpha^2(4\alpha^2+3)m^4}{(1+4\alpha^2)^{5/2}}, \qquad J_3 = \frac{8\alpha^3(1-4\alpha^2)m^4}{(1+4\alpha^2)^{5/2}}. \end{aligned} \tag{2.40}$$

A spacetime that is symmetric with respect to the equatorial plane $y = 0$ has a multipolar expansion characterized by even (power of 2) mass poles (monopole, quadrupole, …) and odd angular poles (dipole, octupole, …), such as the Kerr spacetime (see Appendix C). The fact that both even and odd multipole moments are present means that the metric is asymmetric with respect to the equatorial plane. In fact, odd powers of y are present in the metric functions (2.27)–(2.31).

[5] In order to minimize the confusion between the radial coordinate R and the scalar curvature invariants (such as the Ricci scalar R), a different font is used for the latter.

[6] Note that in the Kerr case this difficulty is not present because the rotating metric we want to recover is already a known solution, and therefore the change of coordinate can be easily established. For instance, an alternative radial coordinate [which recovers Eq. (2.34) in the static limit] is $x(R) := \frac{R^2}{\kappa(R-m)} - \frac{2m}{\kappa}$, but other choices are possible.

Moreover, the spacetime (2.27)–(2.31) presents divergences of the scalar curvature invariants, such as the Riemann squared $R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$, which are not covered by a horizon.

B. $\alpha = \beta \neq 0$

Interestingly enough the Kerr spacetime can be obtained using the general potential (2.10) in ways other than that used in Sec. II A. We will see that, although for $\delta = 1$ the two constructions coincide, whenever $\delta \neq 1$ they give rise to inequivalent Ernst potentials. Therefore, we can have different stationary solutions with the same δ that also have the same static limit to the BBMB black hole. This occurs even without adding mass multipoles, which produce extra degeneracy; we will further consider these multipolar generalizations of the FJRW metric in Sec. III.

In this section let us also consider a non-null $\mu(x, y)$, but for simplicity we set $\beta = \alpha$ in Eq. (2.9), and thus we will again keep only one rotation/reflection-independent parameter. With these settings, fixing $\delta = 1$ in Eq. (2.10) gives us the usual Ernst potential for the Kerr-NUT spacetime [19]:

$$\mathcal{E} = \frac{\xi e^{i\tau} - 1}{\xi e^{i\tau} + 1}, \quad \text{with} \quad \xi = px + iqy, \tag{2.41}$$

where

$$p = \frac{1 - \alpha^2}{1 + \alpha^2}, \qquad q = \frac{2\alpha}{1 + \alpha^2}. \tag{2.42}$$

Note that $p^2 + q^2 = 1$, as is expected for the Kerr solution. In order to neutralize the NUT charge in this case an Ehlers transformation is not necessary; we can achieve the same result by simply imposing $\tau = 0$. In this way we remain with the Ernst potential for the Kerr black hole, as in Eq. (2.13), and the $\mathcal{E}$ simplifies to d_-/d_+.

Now we will play the same game as in the previous section (where $\beta = 0$) for the FJRW metric with $\delta = 1/2$, but under the assumption $\alpha = \beta \neq 0$. In the same way we can derive ω through Eq. (2.16) and then analyze its asymptotic behavior for large x:

$$\omega \approx \frac{-4\alpha^3\kappa + \alpha^2\omega_0 + (3\alpha^2 + 1)\kappa y \sin(\tau) - \omega_0}{\alpha^2 - 1} - \frac{8(\alpha^3\kappa(y^2 - 1)\cos(\tau))}{(\alpha^2 - 1)^2 x} + O\left(\frac{1}{x^2}\right). \tag{2.43}$$

In order to have a good falloff behavior we require that $\omega \to 0$ at spatial infinity, so we impose

$$\omega_o = \frac{4\alpha^3\kappa}{\alpha^2 - 1} \quad \text{and} \quad \tau = 0.$$

Therefore, as in the $\delta = 1$ case, when $\alpha = \beta$ the vanishing of the NUT charge is achieved for $\tau = 0$. A general expression for τ in the case $\alpha \neq 0 \neq \beta$ is given in Ref. [18],

$$\tau = \frac{\alpha - \beta}{\delta(\alpha\beta - 1) - \alpha\beta}. \tag{2.44}$$

Thus, when $\alpha = \beta$, τ is independent from δ, in contrast with what happened in Sec. II A. Hence, for these values of the parameters, the asymptotically flat Ernst potential $\mathcal{E}$ is just d_-/d_+,

$$\mathcal{E} = \frac{\sqrt{x^2 - 1}[\alpha^2(x+1)^2 + y^2 - x^2] - 2i\alpha x^2 y + 2i\alpha y}{(x+1)\{\alpha^2 - 2iy\alpha\sqrt{x^2 - 1} + x[\alpha^2(x-2) - x] + y^2\}}. \tag{2.45}$$

In order to avoid a conical singularity on the axis of symmetry, when integrating γ one has to set the arbitrary integration constant to fulfill

$$\lim_{y \to \pm 1} \gamma = 0. \tag{2.46}$$

Finally, after having imposed the elementary flat boundary conditions, we have

$$f = \frac{\sqrt{x^2 - 1}[\alpha^4(x^2 - 1)^2 + (x^2 - y^2)^2 - 2\alpha^2[x^4 + x^2(1 - 3y^2) + y^2]]}{-2\alpha^2(x^2 - 1)[(x-1)x^2 - (3x+1)y^2] + (x+1)(x^2 - y^2)^2 + \alpha^4(x-1)^4(x+1)}, \tag{2.47}$$

$$\omega = \frac{4\alpha^3\kappa(y^2 - 1)[2x^3 + x^2 - \alpha^2(x-1)^2(2x+1) + y^2]}{(\alpha^2 - 1)[\alpha^4(x^2 - 1)^2 + (x^2 - y^2)^2 - 2\alpha^2[x^4 + x^2(1 - 3y^2) + y^2]]}, \tag{2.48}$$

$$e^{2\gamma} = \frac{1}{(\alpha^2 - 1)^2}\left[\frac{x^2 - 1}{x^2 - y^2} - \frac{2\alpha^2(x^2 - 1)(x^4 - 3x^2y^2 + x^2 + y^2)}{(x^2 - y^2)^3} + \frac{\alpha^4(x^2 - 1)^3}{(x^2 - y^2)^3}\right]. \tag{2.49}$$

Note that for $\delta = 1$ the metrics built here and in the previous section coincide with the Kerr spacetime. But for $\delta = 1/2$ (and possibly $\forall\, \delta \neq 1$) the two constructions give rise to inequivalent Ernst potentials. Since the coordinate system (x, y) used for both constructions is the same (i.e., prolate spherical), the two spacetimes are different, as scalar curvature invariants show. Another difference between the two rotating

BBMB spacetimes presented here and in Sec. II A lies in the multipolar expansion. In fact, with the help of Appendix C and Eq. (2.38), we can compute the mass and angular multipole moments up to the octupole, for the metric (2.47)–(2.49) in the Einstein frame:

$$
\begin{aligned}
&M_0 = m, \qquad J_0 = 0,\\
&M_1 = 0, \qquad J_1 = -\frac{16\alpha^3 m^2}{(1+3\alpha^2)^2},\\
&M_2 = \frac{(-5\alpha^6 - 69\alpha^4 + 9\alpha^2 + 1)m^3}{(1+3\alpha^2)^3}, \qquad J_2 = 0,\\
&M_3 = 0, \qquad J_3 = \frac{8\kappa^4\alpha^3(\alpha^2+1)}{(\alpha^2-1)^4}.
\end{aligned} \tag{2.50}
$$

This multipolar moment expansion differs both qualitatively and quantitatively with respect to the one in Eq. (2.40). Since the multipole moments considered here are not coordinate dependent, for $\delta = 1/2$ the metrics constructed in here and in Sec. II A are not diffeomorphic, and thus they describe different spacetimes, in contrast with the case $\delta = 1$. In particular, the multipole expansion (2.50) is typical of metrics that are symmetric with respect to the equatorial plane $y = 0$, as can be directly checked in Eqs. (2.47)–(2.49).

For large values of the radial coordinate R [as defined in Eq. (2.35) and taking into account the relation (2.38)] the metric approaches spatial infinity as

$$
\begin{aligned}
ds^2 \sim &-\left(1-\frac{2m}{R}\right)dt^2 + \left(1+\frac{2m}{R}\right)dR^2\\
&+\frac{64m^2\alpha^3\sin^2\theta}{R(1-\alpha^2)^2}dtd\varphi + R^2(d\theta^2+\sin^2\theta d\varphi^2)\\
&+O\left(\frac{1}{R^2}\right).
\end{aligned}
$$

As explained in Ref. [20], for this class of stationary and axisymmetric spacetimes, the null horizons can be found from the relation

$$
g_{tt}g_{\varphi\varphi} = g_{\varphi t}^2 \Rightarrow \frac{\rho^2}{(1-\frac{8\pi G}{6}\Psi^2)^2} = 0. \tag{2.51}
$$

So [using the radial coordinate (2.35)] the $R_H = \kappa/2$ hypersurface is null, $g^{RR}(R_H) = 0$, and in the no-rotation limit it coincides with the BBMB event horizon,

$$
R_H = m\frac{1-\alpha^2}{1+3\alpha^2}\underset{\alpha\to 0}{\longrightarrow} m. \tag{2.52}
$$

Actually, the hypersurface $R_H = \kappa/2$ is double degenerate, as it occurs in the static case, where the geometry is extremal even though the mass parameter is free (and the addition of electromagnetic charges to the BBMB black hole cannot alter its extremality). For a reasonable range of the mass and rotation parameters and radial coordinate R, the scalar curvature invariants (such as the Riemann squared $\mathrm{R}_{\mu\nu\sigma\lambda}\mathrm{R}^{\mu\nu\sigma\lambda}$) diverge only at $R = 0$, as can be seen

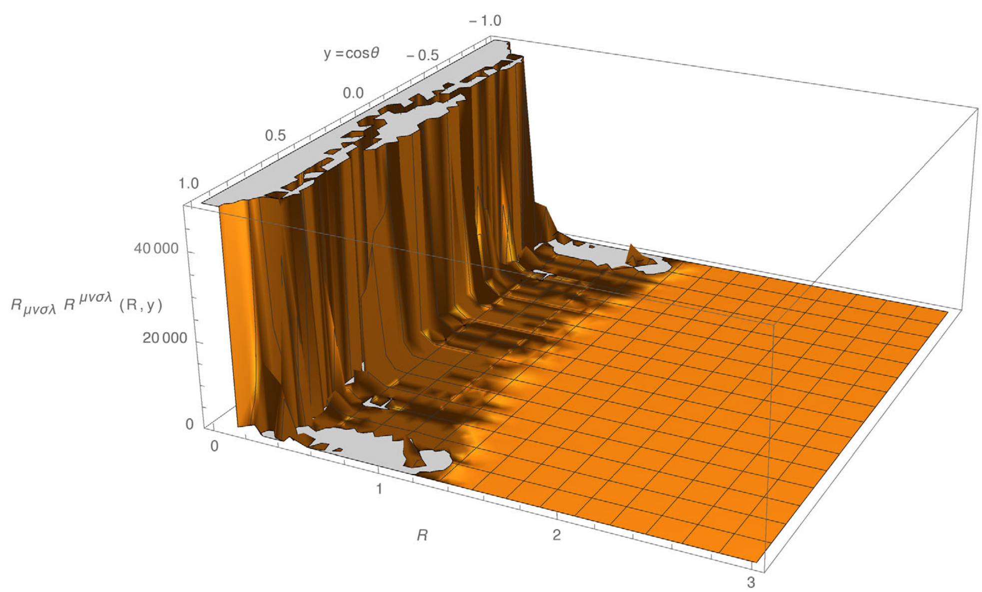

FIG. 1 (color online). Plot of the $\mathrm{R}_{\mu\nu\sigma\lambda}\mathrm{R}^{\mu\nu\sigma\lambda}(R, y)$ curvature invariant for particular values of the parameters $\delta = 1/2$, $\alpha = 1/2$, and $\kappa = 1$. It diverges when the radial coordinate $R \to 0$. This behavior remains qualitatively the same for other values of α, κ in the range $0 \le \alpha \le 1$ and $\kappa \ge 0$.

in Fig. 1. The symmetry axis is located at $y = \pm 1$, as can be checked by the fact that $g_{t\varphi}$ and $g_{\varphi\varphi}$ vanish there.

The surface horizon area, defined by $R = \kappa/2$, is given by

$$S_H = \int_0^{2\pi} d\varphi \int_{-1}^{1} dy \sqrt{g_{yy} g_{\varphi\varphi}}|_{R=m} = \pi\kappa^2. \tag{2.53}$$

Therefore, similarly to the standard GR case where the Kerr black hole's event horizon area is given by $8\pi m(m + \sqrt{m^2 - a^2})$, the presence of the rotation shrinks the size of the horizon for a given value of the mass. Nevertheless its geometry remains spherical as in the static case; this can be understood by looking at the equatorial and polar circumferences, which are, respectively,

$$C_e = \int_0^{2\pi} \sqrt{g_{\varphi\varphi}} d\varphi = \pi\kappa, \tag{2.54}$$

$$C_p = \int_{-1}^{1} \sqrt{g_{yy}} dy = \pi\kappa. \tag{2.55}$$

The topology of the $\mathcal{S}_h$ surface can be checked with the help of the Gauss-Bonnet theorem. The Euler characteristic is given by

$$\chi(\mathcal{S}_h) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_{-1}^{1} dy \sqrt{\bar{g}} \bar{\mathrm{R}} = 2, \tag{2.56}$$

where $\bar{g}$ and $\bar{\mathrm{R}}$ are the determinant and Ricci scalar curvature of the metric defined on the surface's horizon $\mathcal{S}_h$ at constant time. Therefore the genus $\mathfrak{g} = \chi(\mathcal{S}_h)/2 - 1$ of the surface $\mathcal{S}_h$ is null, so the horizon topology is spherical.

The radial coordinate (2.35) was chosen because it is the simplest one that contains the static radial coordinates (2.34) in the limit of null rotation. But a better-suited coordinate transformation $x(R)$ might exist for describing the stationary spacetime, in particular for a black hole interpretation.

Thanks to the Yamakazi potentials [18] it is possible to write the spacetime defined by the Ernst potential (2.6) in a closed metric form with the parameters δ, α, κ free. This is useful to recognize the limits to some notable spacetimes, such as Schwarzschild, Kerr or BBMB. Thus, when the scalar field is conformally coupled and for $\alpha = \beta$ [and consequently, according to Eq. (2.44), $\tau = 0$], the structure functions in the metric (1.4) become[7]

$$f = \frac{R_+}{L_+} \left(\frac{x-1}{x+1}\right)^{\delta - 1}, \tag{2.57}$$

$$\omega = \kappa_1 - 2\kappa \frac{M_+}{R_+} \left(\frac{x-1}{x+1}\right)^{1-\delta}, \tag{2.58}$$

$$\gamma = \frac{1}{2} \log\left[\frac{\kappa_2 R_+}{x^2 - y^2}\right], \tag{2.59}$$

$$\Psi = \sqrt{\frac{6}{8\pi G}} \tanh\left[\sqrt{\frac{1-\delta^2}{12}} \log\left(\frac{x-1}{x+1}\right)\right], \tag{2.60}$$

where

$$R_+(x,y) = (x^2-1)(1-\lambda\mu)^2 - (1-y^2)(\lambda+\mu)^2, \tag{2.61}$$

$$L_+(x,y) = (1-\lambda\mu)[(x+1)^2 - \lambda\mu(x-1)^2] + (\lambda+\mu)[\lambda(1-y)^2 + \mu(1+y)^2], \tag{2.62}$$

$$M_+(x,y) = (x^2-1)(1-\lambda\mu)[\lambda+\mu-y(\lambda-\mu)] + (1-y^2)(\lambda+\mu)[1-\lambda\mu+x(1+\lambda\mu)], \tag{2.63}$$

while the conformal factor Ω is given by Eq. (1.5), and $\lambda(x,y)$ and $\mu(x,y)$ are the same as in Eqs. (2.8) and (2.9), respectively. The integration constants κ_1 and κ_2 are fixed by requiring elementary asymptotic flatness of the metric (2.57)–(2.60) as follows:

$$\lim_{x\to\infty} \omega = 0 \Rightarrow \kappa_1 = \frac{4\kappa\alpha}{\alpha^2 - 1}, \tag{2.64}$$

$$\lim_{y\to\pm 1} \gamma = 0 \Rightarrow \kappa_2 = (\alpha^2 - 1)^{-2}, \tag{2.65}$$

while κ remains the same as in Eq. (2.38). The constraint (2.64) for κ_1 also arise demanding the regularity of the metric on the rotation axis. In fact, according to Ref. [20] $g_{\varphi\varphi}$ and $g_{t\varphi}$ have to vanish where the killing vector $\partial_\varphi = 0$. The main difference with respect to standard general relativity [14] appears in $\gamma(x,y)$, which in our case [according to Eqs. (2.19)–(2.24)] assumes the simple expression (2.59). Actually, when $\delta = 1$ the scalar field vanishes, and we recover the rotating black hole of Einstein theory: the Kerr spacetime.

Some limits to notable spacetimes are shown in Table I.

In order to have the Kerr spacetime in the standard Boyer-Lindquist coordinate representation, we define

$$x = \frac{r-m}{\kappa}, \qquad y = \cos\theta, \tag{2.66}$$

while to recover the static BBMB black hole (2.32) one has to use the coordinate transformation (2.34).

Even though the distortion parameter δ continuously connects the Kerr black hole with the rotating version of the BBMB black hole we do not expect to have a physical process that actually connects these two black holes. This is because even in the static limit (when $1/2 < \delta < 1$) one has naked singularities.

[7] A MATHEMATICA notebook with this metric can be found at https://sites.google.com/site/marcoastorino/papers/1412-3539.

TABLE I. Some specializations of the metric (1.4) and (2.57)–(2.63), for some values of its parameters. m and a denote the standard mass and angular momentum (in units of mass) of the Kerr spacetime.

Spacetime	$\alpha = \beta$	κ	δ
Kerr black hole	$\pm\sqrt{\frac{m-\sqrt{m^2-a^2}}{m+\sqrt{m^2-a^2}}}$	$\sqrt{m^2-a^2}$	1
Schwarzschild black hole	0	m	1
BBMB black hole	0	$2m$	1/2
Rotating BBMB	α	$2m\frac{(1-\alpha^2)}{1+3\alpha^2}$	1/2

Note that these spacetimes are naturally NUT free (because $\alpha = \beta$), but is possible to add NUT charge with an extra Ehlers transformation, as we have done to obtain the more general case (2.10).

III. MULTIPOLAR FJRW METRICS

It is possible to push the solution-generating mechanism with the minimally and conformally coupled scalar field further to construct mass and angular multipolar generalizations of the FJRW solutions with an infinite number of independent parameters. We recall that the mass multipole solutions have the peculiar property that they do not vanish in the Newtonian limit, unlike the angular multipoles (i.e., the ones carried by the Tomimatsu-Sato solution). On the other hand, the angular multipoles are produced by the mass deformation of the body due to the rotation. The simplest example—the case of a null scalar field—is given by the Erez-Rosen metric which is a static spacetime endowed with a quadrupole moment. Of course these solutions in general have a curvature singularity that is not covered by an event horizon, and therefore they are not suitable to describe black holes (but they can describe other astrophysical objects). By applying the HKX transformation it is possible to build new exact stationary and axisymmetric vacuum solutions possessing an arbitrary large number of independent parameters [14].

These results can be directly generalized to the case of a minimally or conformally coupled scalar field, as we have done in the monopolar solutions of Secs. II A and II B. To do so one has to generalize Eqs. (2.8)–(2.10) to

$$\bar{\lambda} = \alpha(x^2-1)^{1-\delta}(x+y)^{2\delta-2}\exp\left[2\delta\sum_{n=1}^{\infty}(-1)^n q_n B_{n-}\right], \tag{3.1}$$

$$\bar{\mu} = \beta(x^2-1)^{1-\delta}(x-y)^{2\delta-2}\exp\left[2\delta\sum_{n=1}^{\infty}(-1)^n q_n B_{n+}\right], \tag{3.2}$$

$$\bar{\mathcal{E}} = \frac{(d_+ + d_- e^{2\delta\psi})e^{i\tau} - (d_+ - d_- e^{2\delta\psi})}{(d_+ + d_- e^{2\delta\psi})e^{i\tau} + (d_+ - d_- e^{2\delta\psi})}, \tag{3.3}$$

where, for $n \geq 0$,

$$B_{n\pm} = \frac{(\pm 1)^n}{2}\log\left[\frac{(x\mp y)^2}{x^2-1}\right] - (\pm 1)^n Q_1(x) + P_n(y)Q_{n-1}(x) - \sum_{k=1}^{n-1}(\pm 1)^k P_{n-k}(y)[Q_{n-k+1}(x) - Q_{n-k-1}(x)],$$

$$\psi(x,y) = \sum_{n=1}^{\infty}(-1)^{n+1}q_n P_n(y)Q_n(x), \tag{3.4}$$

where $P_n(y)$ are the Legendre polynomials and $Q_n(x)$ are the Legendre functions of the second kind[8]; $d_\pm$ follows the definition (2.7). $\{q_n\}_{n=0,1,2,\ldots}$ are independent constants related to the metric multipolar expansion, for both angular or mass multipole moments. To be more precise, the q_n term gives contributions to the 2^n multipole; further details can be found in Appendix C or in Ref. [14]. Here, integration constants are set to zero according to

$$\lim_{x\to\infty} B_{n\pm} = 0.$$

In Secs. II A and II B we considered the simplest case, where $q_0 = 1$ and $q_j = 0\ \forall\ j > 0$; in that case Eqs. (3.1)–(3.3) trivially reduced to Eqs. (2.8)–(2.10).

Up to this point the Ernst potential worked well for both the vacuum case (describing stationary rotating multipolar Zipoy-Woorhees metrics) and the scalar coupling (describing stationary rotating FJRW metrics). From the Ernst potential we can extract the $f(x,y)$ and $\omega(x,y)$ fields. But the main difference in the two theories consists of the remaining $\gamma(x,y)$ structure function of the Lewis-Weyl-Papapetrou metric, and a further possible conformal transformation if we want to work in the conformally coupled theory. To obtain $\gamma(x,y)$ one has to integrate Eqs. (2.19)–(2.20), where the presence of a nontrivial scalar field becomes relevant. For the scalar field (1.13) considered in this paper, the correction with respect to standard general relativity is given in Eq. (2.24).

As a significant example we will now build the Erez-Rosen metric with a minimally coupled scalar field. The standard Erez-Rosen metric can be built from Eqs. (3.1)–(3.4) by fixing the parameters as follows:

$$q_0 = 1, \quad q_1 = 0, \quad q_2 \neq 0, \quad q_j = 0 \quad (j > 2),$$
$$\kappa = m, \qquad \alpha = \beta = \tau = 0, \qquad \delta = 1.$$

Analogously, if we want to have a Erez-Rosen metric in the presence of a minimally (or conformally) coupled scalar field (1.13) [or Eq. (2.28)] we have to choose the same values for the parameters q_j, κ, α, β as in the vacuum case,

[8] See Appendix A for more information about the Legendre functions of the second kind.

so that asymptotically and in the weak-field limit, for small m, the scalar coupled cases have a similar multipolar behavior with respect to the vacuum case. Obviously in this case $\delta = 1/2$ because the metric has to reduce to FJRW (or BBMB) spacetime when the quadrupole moment of the source vanishes (i.e., $q_2 = 0$), in the same way the Erez-Rosen metric reduces to the Schwarzschild black hole. With this parametric imposition the Ernst potential (3.3) becomes

$$\mathcal{E} = f = \exp\left\{q_2(3y^2-1)\right.$$
$$\left.\times\left[\frac{1}{4}(3x^2-1)\log\left(\frac{x-1}{x+1}\right)+\frac{3}{2}x\right]\right\}\sqrt{\frac{x-1}{x+1}}. \quad (3.5)$$

Since the spacetime is static, the Ernst potential is not complex and $\omega = 0$; therefore, the remaining unknown function can be obtained by integrating Eqs. (2.19) and (2.20), which gives

$$\gamma = \frac{1}{2}\left(1+\frac{q_2}{2}+\frac{q_2^2}{4}\right)\log\left(\frac{x^2-1}{x^2-y^2}\right)$$
$$+\frac{9q_2^2}{256}(x^2-1)(y^2-1)$$
$$\times[x^2(9y^2-1)-y^2+1]\log^2\left(\frac{x-1}{x+1}\right)$$
$$+\frac{3q_2}{64}(y^2-1)\left\{\left[1+x\log\left(\frac{x-1}{x+1}\right)\right]\right.$$
$$\times[8+q_2[3(9x^2-7)y^2-3x^2+5]]$$
$$\left.+[8+q_2(9y^2-1)]\right\}.$$

Here the arbitrary integration constant was set to fulfill Eq. (2.46) to avoid conical singularities on the symmetry axis. The scalar field remains as in Eq. (1.13) or Eq. (2.28) depending on if we are considering the Einstein or Jordan frame, respectively. Let us compute the first mass and angular multipole moments for the above spacetime. Using the general results of Appendix C we have, for the minimally coupled system,

$$M_0 = m, \qquad M_1 = 0, \qquad M_2 = m^3\left(1+\frac{8}{15}q_2\right), \quad (3.6)$$

$$J_j = 0, \quad \forall\ j \geq 0. \quad (3.7)$$

There is a difference with respect to the Erez-Rosen mass multipole moments that is basically due to the different value of the Zipoy parameter δ, which (for instance) can be seen by looking at the mass quadrupole moment (the Erez-Rosen value is $M_2^{\mathrm{ER}} = 2q_2m^3/15$).

IV. COMMENTS AND CONCLUSIONS

In this paper, the Ernst solution-generating technique, in the context of standard Einstein gravity with a (minimally or) conformally coupled scalar field, was enhanced to include the HKX transformations. These transformations are able to add rotation while preserving asymptotic and elementary flatness. Applying these methods, we were able to generate a large family of asymptotically flat, axisymmetric, and stationary solutions for both the minimally and conformally coupled theories, containing (apart from the Zipoy-Woorhees distortion parameter δ and the mass m) two independent parameters: the rotation and reflection parameters α and β. We explained how to remove the possible NUT charge emerging from the HKX transformation. As significant examples, we analyzed some special cases that are continuously connected to the Kerr black hole by the distortion parameter, where only one independent extra parameter was left: the rotation (i.e., $\beta = 0$ and $\alpha = \beta$). In the minimal frame they can be considered as the stationary extension of the Janis-Winnicour-Robinson-Fisher solution, while in the conformally coupled theory they include a rotating generalization of the BBMB black hole. Although both cases have a clear limit to the BBMB black hole when the rotation parameter is turned off, the case with $\alpha = \beta$ is the most similar to the rotating black hole in GR, that is, an angular and mass multipolar expansion and a geometry similar to the extremal Kerr spacetime. Depending on the relative values of the α and β parameters introduced by the HKX transformation, these axisymmetric spacetimes may or may not be symmetric with respect to the equatorial plane. The more general case where both the rotation and reflection parameters are not null and independent remains to be studied.

This family has been further generalized to contain an arbitrary number of independent parameters related to additional mass multipoles. As an example, we provided an Erez-Rosen-like spacetime in the presence of a scalar field.

Note that the static seed metric of the BBMB black hole coincides with that of the extremal Reissner-Nordström black hole. Therefore, if one wants to apply the Janis-Newman (JN) algorithm for adding rotation, the extremal Kerr-Newman metric would be obtained, which is not a solution for the theory we are dealing with. This occurs because the JN algorithm was discovered *a posteriori* to work within Einstein-Maxwell general relativity, and it is just a (complex) coordinate transformation and thus not dependent of the specific theory one is actually considering. On the other hand, the resulting stationary metrics we have built (after the HKX transformation in the Ernst formalism) are different from the Kerr-Newman metric, and they are proper solutions of the field equations.

It may also be interesting for future work to add the cosmological constant term, because it turned out to be useful in regularizing the behavior of the scalar field on the horizon. This is because the cosmological constant (of the appropriate positivity) shifts the position of the horizon so that the divergence of the scalar field is protected by the event horizon [5]. Of course, this is not a trivial task since a solution-generating technique that includes the cosmological term is not known at the moment [15].

HKX transformations can be adapted in other gravity theories connected to general relativity with a minimally coupled scalar field by a conformal transformation [such as Brans-Dicke or some $f(R)$ gravity theories], in basically the same way as described in this paper for general relativity with a conformally coupled scalar field.

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APPENDIX A: LEGENDRE POLYNOMIALS AND FUNCTIONS OF THE SECOND KIND

Legendre polynomials $P_n(x)$ can be obtained by the Rodrigues formula,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]. \tag{A1}$$

Legendre functions of the second kind $Q_n(x)$ can be built by means of $P_n(x)$ with the following prescription:

$$Q_n(x) = \frac{1}{2}P_n(x)\log\left(\frac{x+1}{x-1}\right) - W_{n-1}(x), \tag{A2}$$

where

$$W_{n+1} = \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(x)P_{n-1}(x). \tag{A3}$$

Thus, the first few are

$$Q_0(x) = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right), \tag{A4}$$

$$Q_1(x) = \frac{1}{2}x\log\left(\frac{x+1}{x-1}\right) - 1, \tag{A5}$$

$$Q_2(x) = \frac{1}{4}(3x^2-1)\log\left(\frac{x+1}{x-1}\right) - \frac{3}{2}x, \tag{A6}$$

$$Q_3(x) = \frac{1}{4}(5x^3-3x)\log\left(\frac{x+1}{x-1}\right) - \frac{5}{2}x^2 + \frac{2}{3}. \tag{A7}$$

APPENDIX B: COSGROVE'S METRICS WITH A SCALAR FIELD

For sake of completeness we also present the extension of another solution-generating technique (based on Ernst equations and complex potentials) that is able to achieve stationarity without spoiling the asymptotic flatness, given by Cosgrove in Refs. [21,22]. It provides the rotating generalisation of the Zipoy-Woorhess metric and the generalization of the Tomimatsu-Sato metric for noninteger δ (which is not equivalent to the generalizations studied in Secs. II A, II B, and III, which are based on the HKX transformation). For a generic δ, it is concise enough to work directly in the metric formalism, and not only in the Ernst picture. Let us begin by considering an example containing both the Kerr and Zipoy-Woorhess metrics. We will present the standard separable Cosgrove solution of Ref. [22] and we will show how to adapt it to the presence of the scalar field according to Eqs. (2.21)–(2.22). It can be most compactly expressed when the NUT charge is not null; further on we will show how to remove it, if so desired. When the scalar field is null the axisymmetric stationary metric is given by the following Ernst potential:

$$\mathcal{E} = \frac{d_+ - d_-}{d_+ + d_-}, \tag{B1}$$

with

$$\begin{aligned} d_\pm = {}& \frac{p}{2}(x^2-1)^{\bar\delta}[(x+1)^{\bar\delta+1}(1-y)^{\bar\delta} \\ & \pm (x-1)^{\bar\delta+1}(1+y)^{\bar\delta}] + \frac{iq}{2}(1-y^2)^{\bar\delta} \\ & \times [(x+1)^{\bar\delta}(1-y)^{\bar\delta+1} \mp (x-1)^{\bar\delta}(1+y)^{\bar\delta+1}]. \end{aligned} \tag{B2}$$

p and q are two dependent parameters related to the mass and angular momentum: when $q=0$ the Ernst potential remains real, so the metric is static; in this case the parameters are related by the usual constraint $p^2+q^2=1$. $\bar\delta$ is chosen to fit the notation of Ref. [22] and is related to ours by $\delta = \bar\delta + 1$; hence, the Kerr spacetime is now given for $\bar\delta = 0$. Note that for $-1 \le \bar\delta \le 0$ (or $0 \le \delta \le 1$) the scalar field is real; otherwise, it is imaginary.

Explicitly, the Ernst potential (B1) has the form

$$\begin{aligned} \mathcal{E} = {}& \left[\frac{(x-1)(1+y)}{(x+1)(1-y)}\right]^{\bar\delta} \\ & \times \frac{p(x^2-1)^{\bar\delta}(x-1) - iq(1-y^2)^{\bar\delta}(1+y)}{p(x^2-1)^{\bar\delta}(x+1) - iq(1-y^2)^{\bar\delta}(y-1)}. \end{aligned} \tag{B3}$$

Note that this potential does not contain the static BBMB spacetime, and therefore it cannot considered a good seed for a stationary BBMB.

The structure functions of the Lewis-Weyl-Papapetrou metric descending from the potential (B1) are

$$f = \left[\frac{(x-1)(1+y)}{(x+1)(1-y)}\right]^{\bar{\delta}} \times \frac{p^2(x^2-1)^{2\bar{\delta}+1} - q^2(1-y^2)^{2\bar{\delta}+1}}{p^2(1+x)^2(x^2-1)^{2\bar{\delta}} + q^2(1-y)^2(1-y^2)^{2\bar{\delta}}}, \tag{B4}$$

$$\omega = -\kappa \frac{2pq[(x+1)(1-y)]^{2\bar{\delta}+1}(x+y)}{p^2(x^2-1)^{2\bar{\delta}+1} - q^2(1-y^2)^{2\bar{\delta}+1}}, \tag{B5}$$

$$e^{2\gamma_0} = b\frac{(x^2-1)^{\bar{\delta}^2}(1-y^2)^{\bar{\delta}^2}}{(x-y)^{(2\bar{\delta}+1)^2}(x+y)} \times [p^2(x^2-1)^{2\bar{\delta}+1} - q^2(1-y^2)^{2\bar{\delta}+1}], \tag{B6}$$

where b is an arbitrary integration constant. When the scalar field (1.13) is present the only structure function of the Lewis-Weyl-Papapetrou metric that changes is γ. It can be found thanks to Eqs. (2.19)–(2.20):

$$e^{2\gamma_\Psi} = b\frac{(x+y)^{\bar{\delta}^2+2\bar{\delta}-1}(1-y^2)^{\bar{\delta}^2}}{(x-y)^{3\bar{\delta}^2+2\bar{\delta}+1}(x^2-1)^{2\bar{\delta}}} \times [p^2(x^2-1)^{2\bar{\delta}+1} - q^2(1-y^2)^{2\bar{\delta}+1}] = e^{2\gamma_0}\left(\frac{x^2-y^2}{x^2-1}\right)^{\bar{\delta}^2+2\bar{\delta}}. \tag{B7}$$

For $\bar{\delta} = 0$ the scalar field is null, $\gamma_\Psi \to \gamma$, and the spacetime becomes a Kerr-NUT black hole. We can remove the NUT charge by applying an Ehlers transformation to the Ernst potential of Ref. [21] and requiring the appropriate falloff boundary conditions. So we add an extra NUT charge, parametrized by τ, as done in Sec. II. The Ehlers-transformed Ernst potential (B1) is

$$\mathcal{E} = \frac{d_+ e^{i\tau} - d_-}{d_+ e^{i\tau} + d_-}. \tag{B8}$$

When $\bar{\delta} = 0$ the ω function coming from this potential is given by

$$\omega = \omega_0 + \frac{2\kappa}{q}(p\cos\tau + q\sin\tau) - \frac{2\kappa p(x^2-1)[(p^2x - q^2y)\cos\tau + p^2 + pq(x+y)\sin\tau + q^2]}{p^2q(x^2-1) + q^3(y^2-1)}, \tag{B9}$$

whose asymptotic behavior for large x is given by

$$\omega \approx \left(-\frac{2\kappa q}{p} - \frac{2\kappa p}{q} + \frac{2\kappa qy\cos\tau}{p} - 2\kappa y\sin\tau + \omega_0\right) + \frac{2\kappa q(y^2-1)[p\cos\tau + q\sin\tau]}{p^2x} + O\left(\frac{1}{x^2}\right). \tag{B10}$$

Requiring the usual falloff at spatial infinity $[O(1/x)]$, we impose

$$\cos\tau = p \quad \text{and} \quad \omega_0 = \frac{2\kappa}{pq}. \tag{B11}$$

Note that Eq. (B11) with $p^2 + q^2 = 1$ implies that $\sin\tau = q$. By fine-tuning the NUT charge we have erased the previous existing one. Therefore we remain with a pure Kerr spacetime. To convince oneself of this, it is sufficient to check the constrained Ernst potential, which is exactly that of Kerr spacetime:

$$\mathcal{E}|_{\bar{\delta}=0} = 1 - \frac{2(p+iq)}{p+iq+e^{i\tau}(px-iqy)} \to \frac{px-iqy-1}{px-iqy+1}. \tag{B12}$$

For $\delta > 0$ the spacetimes (B4)–(B7) are NUT free, so we do not need an additional Ehlers transformation (but Ψ becomes imaginary). On the other hand, γ and γ_Ψ remain the same as before [Eqs. (B6) and (B7), respectively], because the Ehlers transformations do not affect Eqs. (2.19) and (2.20) [19].

APPENDIX C: MULTIPOLAR MOMENTS

It is possible to compute the angular and mass multipole moments from the Ernst potential in prolate spheroidal coordinate [14,23]. This can clarify the role of the independent constants that appear in the general multipolar metric presented in Sec. III. There are several definitions of multipole moments for axisymmetric fields; we consider here those given by Geroch-Hansen ones [24]. These have the advantages of being coordinate independent and they coincide with the Newtonian moments (in the case of flat spacetime).

According to the notation used in Eqs. (3.1)–(3.3), we list the first mass M_j and angular J_j multipole moments (for more details see Ref. [14][9]):

[9]After the completion of this paper, Ref. [25] was published where the contribution of the scalar field is also taken into account.

$$M_0 = \kappa\left(\delta q_0 + \frac{2\alpha\beta}{1-\alpha\beta}\right), \quad \text{(C1)}$$

$$\begin{aligned}
M_1 &= \kappa^2\left[-\frac{\delta q_1}{3} + \frac{\beta^2-\alpha^2}{(1-\alpha\beta)^2}\right],\\
M_2 &= \kappa^3\left\{\frac{2\delta q_2}{5} - \frac{\delta^3}{3} - \frac{2\delta^2\alpha\beta}{1-\alpha\beta} + \delta\left[\frac{1}{3} + \frac{\alpha\beta(-2-2\alpha\beta+3\alpha^2+3\beta^2+4\alpha^2\beta^2)-3(\alpha^2+\beta^2)}{(1-\alpha\beta)^3}\right]\right.\\
&\quad\left. + 2\frac{(\alpha+\beta)^2 - \alpha\beta(1+2\alpha^2+2\beta^2+2\alpha\beta+\alpha^2\beta^2)}{(1-\alpha\beta)^3}\right\},\\
M_3 &= -\frac{k^4(\alpha^2-\beta^2)\{\alpha^2[3\beta^2(\delta-1)^2-1] - 2\alpha\beta[3(\delta-2)\delta+4] - \beta^2 + 3(\delta-1)^2\}}{(\alpha\beta-1)^4},\\
J_0 &= -\kappa\frac{\alpha-\beta}{1-\alpha\beta},\\
J_1 &= -\kappa^2\frac{\alpha+\beta}{(1-\alpha\beta)^2}[3\alpha\beta + 2\delta(1-\alpha\beta) - 1],\\
J_2 &= -\frac{\kappa^3}{1-\alpha\beta}\left[-\frac{2}{3}\delta q_1(\alpha+\beta) + (\alpha-\beta)(1-\delta)^2\right] - \frac{\kappa^3}{(1-\alpha\beta)^3}(\beta^3-\alpha^3+\alpha\beta^2-\alpha^2\beta),\\
J_3 &= \frac{\kappa^4(\alpha+\beta)[5\alpha^3(\beta^3+\beta) + \alpha^2(\beta^2-3) + \alpha\beta(5\beta^2-3) - 3\beta^2+1]}{(1-\alpha\beta)^4},\\
&\quad + \frac{\delta\kappa^4(\alpha+\beta)\{\alpha^2[\beta^2(2\delta^2-15\delta+28)+12] - 4\alpha\beta(\delta^2-3\delta-1) + 12\beta^2 + 2\delta^2 + 3\delta - 8\}}{3(1-\alpha\beta)^3}.
\end{aligned} \quad \text{(C2)}$$

For simplicity, we put $q_i = 0 \ \forall\, i$ in M_3 and J_3. When the NUT parameter $\tau \neq 0$ the angular and mass multipole moments, for $n \leq 3$, are modified as follows:

$$\begin{aligned}
M'_n &= M_n \cos\tau - J_n \sin\tau,\\
J'_n &= M_n \sin\tau + J_n \cos\tau.
\end{aligned} \quad \text{(C3)}$$

The presence of odd mass multipole and even angular multipole moments means that the metric is not symmetric with respect to the equatorial plane, $y = 0$. Using Eqs. (C1)–(C3), it is easy to obtain the first multipole moments for the Kerr black hole of Secs. II A and II B:

$$M_0 = m, \quad M_1 = 0, \quad M_2 = -ma^2, \quad M_3 = 0, \quad \text{(C4)}$$

$$J_0 = 0, \quad J_1 = am, \quad J_2 = 0, \quad J_3 = -ma^3. \quad \text{(C5)}$$

The monopole term is the mass of the black hole, while the angular dipole moment coincides with the angular momentum. The higher multipoles are due to the rotation and reflect the fact that the stationary Kerr black hole loses the spherical symmetry typical of the static Schwarzschild one.

APPENDIX D: MORE GENERAL SCALAR FIELDS

The most general form for the scalar field in the minimal frame that can be obtained by variable separation is given by Eq. (2.23),

$$\hat{\Psi} = \sum_{n=0}^{\infty} [a_n Q_n(x) + b_n P_n(x)][c_n Q_n(y) + d_n P_n(y)]. \quad \text{(D1)}$$

Applying the condition of asymptotic flatness, we set to zero the coefficients b_n and c_n, and considering $\delta = 1/2$, the scalar field becomes

$$\hat{\Psi} = \sum_{n=0}^{\infty} a_n Q_n(x) P_n(y) = \frac{a_0}{2}\log\left(\frac{x-1}{x+1}\right) + a_1\left[\frac{x}{2}\log\left(\frac{x-1}{x+1}\right) + 1\right]y + a_2\left[\frac{3x^2-1}{4}\log\left(\frac{x-1}{x+1}\right) + \frac{3}{2}x\right]\left(\frac{3y^2-1}{2}\right) + \ldots. \quad \text{(D2)}$$

We can evaluate the contribution of the scalar's first-term expansion to the $\gamma = \gamma_0 + \sum_{n=0}^{\infty}\gamma_{\Psi_n}$ field. According to Eqs. (2.21)–(2.22) the first contributions are given by

$$\gamma_{\Psi_0} = c_0 + \frac{a_0^2}{4} 8\pi G \log\left(\frac{x^2-1}{x^2-y^2}\right), \tag{D3}$$

$$\gamma_{\Psi_1} = \frac{a_1^2}{16} 8\pi G \left\{4\log\left(\frac{x^2-1}{x^2-y^2}\right) + (y^2-1)\log\left(\frac{x-1}{x+1}\right)\left[4x + (x^2-1)\log\left(\frac{x-1}{x+1}\right)\right]\right\}, \tag{D4}$$

$$\begin{aligned}\gamma_{\Psi_2} = \frac{a_2^2}{32} 8\pi G \Big\{ & 8\log\left(\frac{x^2-1}{x^2-y^2}\right) + 9x^2 + 6y^2(8-15x^2) + 9y^4(9x^2-4) + \frac{3}{4}(y^2-1)\log\left(\frac{x-1}{x+1}\right) \\ & \cdot\left[4x[5-3x^2+3(9x^2-7)y^2] + 3(x^2-1)[1-x^2+(9x^2-1)y^2]\log\left(\frac{x-1}{x+1}\right)\right]\Big\},\end{aligned} \tag{D5}$$

where c_0 is an integration constant that can be fixed by physical requirements, such as the absence of conical singularities.

If we relax the boundary conditions a little by allowing a constant falloff of the scalar field, and the coefficient b_0 can be turned on. The effect of a non-null b_0 is simply a constant shift of the scalar field in the minimally coupled theory [which is a symmetry in the action (1.6)], but it reflects nontrivially in the conformally coupled theory. In fact, by starting from any seed solution (ds_0^2, Ψ_0) of the conformally coupled theory it is possible to obtain a nonequivalent new solution in this way:

$$ds_0^2 \mapsto ds^2 = \frac{1-\frac{8\pi G}{6}\Psi_0^2}{1-\frac{8\pi G}{6}\Psi^2} ds_0^2, \tag{D6}$$

$$\Psi_0 \mapsto \Psi = \sqrt{\frac{6}{8\pi G}}\tanh \times \left\{\sqrt{\frac{8\pi G}{6}}\left[b_0 + \sqrt{\frac{6}{8\pi G}}\operatorname{arctanh}\left(\sqrt{\frac{8\pi G}{6}}\Psi_0\right)\right]\right\}. \tag{D7}$$

These transformations, parametrized by the real number b_0, map solutions of the theory of general relativity with a conformally coupled scalar field onto itself. In particular, when the seed metric is the BBMB black hole (2.32) we obtain after the transformation (D6)–(D7),

$$ds^2 = \frac{[\rho(s+1)-2ms]^2}{4s[\rho-m]^2}\left[-\left(1-\frac{m}{\rho}\right)^2 dt^2 + \frac{d\rho^2}{(1-\frac{m}{\rho})^2} + \rho^2 d\theta^2 + \rho^2\sin^2\theta d\varphi^2\right], \tag{D8}$$

$$\Psi = -\sqrt{\frac{8\pi G}{6}}\frac{\rho(s-1)-2ms}{\rho(s+1)-2ms}, \tag{D9}$$

where for simplicity we have defined the parameter $b_0 = \sqrt{\frac{8\pi G}{6}}\frac{1}{2}\log s$. Of course when the parameter b_0 vanishes (and thus $s=1$) the transformation (D6)–(D7) becomes the identity and we recover the standard BBMB black hole (2.32). On the other hand, for non-null b_0 the transformation is not trivial, as can be seen (for instance) by looking at the contribution of the s parameter in the scalar curvature invariants.

This solution was first found, by direct integration, in Ref. [26] and interpreted as a traversable wormhole. In the case where the cosmological constant is not null, the constant shift in the scalar field have the effect to map, in the action, the conformal scalar potential from a quartic power[10] to a quartic polynomial; for further details see Ref. [27]. Recently a solution to this system was found in Ref. [28]. It admits a black hole interpretation and generalizes Ref. [26] to the presence of the cosmological constant.

[10]Generically an additional scalar potential proportional to Ψ^4 can be considered in the action (1.1) without spoiling the conformal invariance. It is not compatible with HKX transformations and usually it becomes relevant in the presence of the cosmological constant. For these reasons it is not taken into account in the present work.

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