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Canonical analysis of unimodular gravity

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This paper is devoted to the Hamiltonian analysis of the unimodular gravity. We treat the unimodular gravity as the general relativity action with the unimodular constraint imposed with the help of the Lagrange multiplier. We perform the canonical analysis of the given theory and determine its constraint structure.

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I. INTRODUCTION AND SUMMARY

Unimodular gravity is obtained from the Einstein-Hilbert action in which the unimodular condition

$$\sqrt{-\det \hat{g}_{\mu\nu}} = 1 \tag{1}$$

is imposed from the beginning [1,2]. The resulting field equations correspond to the traceless Einstein equations and can be shown that they are equivalent to the full Einstein equations with the cosmological constant term Λ, where Λ enters as an integration constant.[1] This fact that the cosmological constant arises as an integration constant is very attractive, and it is one of the motivations for the study of the unimodular gravity; for a recent analysis, see [3–14].

The fact that the determinant of the metric is fixed has clearly important consequences on the structure of the given theory. First of all, it reduces the full group of diffeomorphisms to invariance under the group of unimodular general coordinate transformations which are transformations that leave the determinant of the metric unchanged. Further, unimodular condition (1) could have important consequences for the Hamiltonian formulation of the given theory. Some aspects of the Hamiltonian treatment of unimodular gravity were analyzed in [15,16]. Then, a very important contribution to this analysis was presented in [17], where the condition (1) was fixed by hand from the beginning. On the other hand, we mean that it would be desirable to impose this condition using the Lagrange multiplier term that is added to the gravity action. In fact, a similar analysis was performed in [18] using a very elegant formalism of geometrodynamics [19,20] which is manifestly diffeomorphism invariant. However, this elegant formulation can be achieved with the help of introducing of the collection of the scalar fields, which, on the other hand, makes the analysis more complicated. Our goal is to perform the Hamiltonian analysis in a more straightforward manner when we consider the general relativity action where the constraint (1) is imposed with the help of the Lagrange multiplier. Clearly, this expression breaks the diffeomorphism invariance explicitly, and we would like to analyze the consequence of the presence of this term on the Hamiltonian structure of the given theory.[2] It turns out that the given structure is rather interesting. Explicitly, we consider the Lagrange multiplier as the dynamical variable. Because of the fact that there is no time derivative of the given multiplier, we find that its conjugate momentum is the primary constraint of the theory. We also find that the momentum conjugate to the lapse N is not the first class constraint but together with (1) forms the collection of the second class constraints. Then, we find another set of constraints that implies that the Lagrange multiplier corresponding to the unimodular constraint (1) has to depend on time only. Finally, we split the Hamiltonian constraints into a collection of $\infty^3 - 1$ constraints (in the terminology of [18]) and one constraint that together with the momentum conjugate to the zero mode part of the Lagrange multiplier forms the second class

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[1]It is important to stress that this is not exactly true in the presence of matter. In general relativity, we find that the stress energy tensor of matter is conserved when the Einstein equations are valid since the left side of these equations corresponds to the Einstein tensor that is covariantly constant. In the case of the unimodular gravity, the left side of the equation of motion does not correspond to the Einstein tensor, and, hence, the conservation of the stress energy tensor does not follow from Einstein equations but has to be imposed.

[2]There is a natural question why the condition that the metric tensor is symmetric should not be imposed to the action in the similar way. Explicitly, we could consider metric g_{ij} as a general tensor and impose the condition of its symmetry by the additional term in the action $\lambda^{ij}(g_{ij} - gji)$ with λ^{ij} the corresponding Lagrange multiplier. In principle, this can be done, and it can be shown that the presence of this constraint induces another constraint that together with the original one are the second class constraints. With the help of these constraints, we can determine the Dirac bracket between g_{ij} and π^{kl}, and we find that it takes the form $\{g_{ij}, \pi^{kl}\}_D = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k)$. On the other hand, it is common practice to use the given Dirac bracket as the definition of the Poisson bracket between the symmetric tensors g_{ij} and π^{ij} from the beginning, and we will proceed in the same way as well.

constraints. This is a subtle difference with respect to the case of general relativity that possesses $4\infty^3$ first class constraints. On the other hand, the presence of the global constraint that relates the dynamical gravity fields and embedding fields was mentioned in [18], and we mean that our result has close overlap with the conclusion derived there.

As the next step, we perform the Hamiltonian analysis of the unimodular theory proposed in [17]. Now, due to the fact that the given theory is manifestly covariant, the analysis is more straightforward than in the previous case, and we derive the $4\infty^3$ first class constraints. On the other hand, the structure of the Hamiltonian constraint is different from the Hamiltonian constraint of general relativity since now it contains the term corresponding to the momentum conjugate to the time component of the vector field $\mathcal{F}^\mu$. Now, due to the fact that the Hamiltonian does not depend on this field explicitly, we find that this momentum is constantly on shell, and, hence, its constant value can be considered as an effective cosmological constant.

Let us outline the results derived in the given paper. The main goal was to understand the meaning of the unimodular constraint from the Hamiltonian point of view in order to identify the number of physical degrees of freedom of the theory. For that reason, we were very careful and explicit in order to distinguish between the first class constraints, second class constraints, global, and local ones. We found that the constraint structure is almost the same as in the case of general relativity, and we can expect that the number of physical degrees of freedom is the same. However, there is a subtle difference due to the existence of the global constraint that could have significant impact on solutions of the given theory. This is similar situation as in case of global Hamiltonian constraint that exists in the projectable version of Hořava-Lifshitz gravity [21]. We hope to return to the analysis of the given constraint in the future.

This paper is organized as follows. In Sec. II, we perform the Hamiltonian analysis of the unimodular theory with constraint (1) included in the action using the Lagrange multiplier. Then in Sec. III, we perform the Hamiltonian analysis of the formulation of unimodular gravity proposed in [17].

II. HAMILTONIAN ANALYSIS OF UNIMODULAR GRAVITY

In this section, we perform the Hamiltonian analysis of the unimodular gravity where the condition (1) is imposed using the Lagrange multiplier term included in the action. Explicitly, we consider the action

$$S = \frac{1}{16\pi G}\int d^4x(\sqrt{-\hat{g}}{}^{(4)}R[\hat{g}] - \Lambda(\sqrt{-\hat{g}} - 1)), \quad (2)$$

where ${}^{(4)}R$ is a four-dimensional curvature and where $\Lambda(x)$ is the Lagrange multiplier.

To proceed to the canonical formulation, we use the well-known $3+1$ formalism, which is the fundamental ingredient of the Hamiltonian formalism of any theory of gravity.[3] We consider the $(3+1)$-dimensional manifold $\mathcal{M}$ with the coordinates x^μ, $\mu = 0, \ldots, 3$, and where $x^\mu = (t, \mathbf{x})$, $\mathbf{x} = (x^1, x^2, x^3)$. We presume that this spacetime is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-,+,+,+)$. Suppose that $\mathcal{M}$ can be foliated by a family of spacelike surfaces Σ defined by $t = x^0$. Let $g_{ij}, i, j = 1,2,3$ denote the metric on Σ with inverse g^{ij} so that $g_{ij}g^{jk} = \delta_i^k$. We further introduce the operator ∇_i that is a covariant derivative defined with the metric g_{ij}. We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables, we write the components of the metric $\hat{g}_{\mu\nu}$ as

$$\hat{g}_{00} = -N^2 + N_i g^{ij} N_j, \qquad \hat{g}_{0i} = N_i, \qquad \hat{g}_{ij} = g_{ij},$$
$$\hat{g}^{00} = -\frac{1}{N^2}, \qquad \hat{g}^{0i} = \frac{N^i}{N^2}, \qquad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2}. \quad (3)$$

Then, the standard canonical analysis leads to the bare Hamiltonian in the form

$$H = \int d^3\mathbf{x}(N\mathcal{H}_T + N^i\mathcal{H}_i + \Omega(\sqrt{g}N - 1) + v_N\pi_N + v_i\pi_i + v_\Omega p_\Omega), \quad (4)$$

where

$$\mathcal{H}_T = \frac{16\pi G}{\sqrt{g}}\pi^{ij}\mathcal{G}_{ijkl}\pi^{kl} - \frac{\sqrt{g}}{16\pi G}R, \qquad \mathcal{H}_i = -2g_{ik}\nabla_j\pi^{jk}, \quad (5)$$

where

$$\mathcal{G}_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) - \frac{1}{2}g_{ij}g_{kl}, \quad (6)$$

and where R is a three-dimensional curvature. Further, π^{ij} are momenta conjugate to g_{ij} with a nonzero Poisson bracket

$$\{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta(\mathbf{x} - \mathbf{y}). \quad (7)$$

Finally, $\pi_N \approx 0$, $\pi_i \approx 0$, $p_\Omega \approx 0$ are primary constraints where π_N, π_i, p_Ω are momenta conjugate to N, N^i, and Ω, respectively, with nonzero Poisson brackets

[3] For a recent review, see [22].

$$\{N(\mathbf{x}),\pi_N(\mathbf{y})\}=\delta(\mathbf{x}-\mathbf{y}),$$
$$\{N^i(\mathbf{x}),\pi_j(\mathbf{y})\}=\delta^i_j\delta(\mathbf{x}-\mathbf{y}),$$
$$\{\Omega(\mathbf{x}),p_\Omega(\mathbf{y})\}=\delta(\mathbf{x}-\mathbf{y}). \tag{8}$$

It is also useful to introduce the smeared form of the constraints $\mathcal{H}_T,\mathcal{H}_i$,

$$\mathbf{T}_T(X)=\int d^3\mathbf{x}X\mathcal{H}_T,\qquad \mathbf{T}_S(X^i)=\int d^3\mathbf{x}X^i\mathcal{H}_i, \tag{9}$$

where X, X^i are functions on Σ. For further purposes, we also introduce the well-known Poisson brackets

$$\{\mathbf{T}_T(X),\mathbf{T}_T(Y)\}=\mathbf{T}_S((X\partial_iY-Y\partial_iX)g^{ij}),$$
$$\{\mathbf{T}_S(X),\mathbf{T}_T(Y)\}=\mathbf{T}_T(X^i\partial_iY),$$
$$\{\mathbf{T}_S(X^i),\mathbf{T}_S(Y^j)\}=\mathbf{T}_S(X^j\partial_jY^i-Y^j\partial_jX^i). \tag{10}$$

Now we proceed to the analysis of the preservation of the primary constraints. Explicitly, from (4) we find

$$\partial_t\pi_N=\{\pi_N,H\}=-\mathcal{H}_T-\sqrt{g}\Omega\equiv-\mathcal{H}'_T\approx0,$$
$$\partial_tp_\Omega=\{p_\Omega,H\}=-(N\sqrt{g}-1)\equiv-\Gamma\approx0,$$
$$\partial_t\pi_i=\{\pi_i,H\}=-\mathcal{H}_i\approx0. \tag{11}$$

Then the total Hamiltonian with all constraints included has the form

$$H_T=\int d^3\mathbf{x}(N\mathcal{H}_T+v_T\mathcal{H}'_T+(v_\Gamma+\Omega)\Gamma$$
$$+N^i\mathcal{H}_i+v_\Omega p_\Omega+v_N\pi_N), \tag{12}$$

where N^i can now be considered as Lagrange multipliers corresponding to the constraints $\mathcal{H}_i$. On the other hand, we still keep N as a dynamical variable while v_T and v_Ω are the Lagrange multipliers corresponding to the constraints $\mathcal{H}'_T$ and Γ, respectively.

Now we proceed to the analysis of the stability of all constraints. Using (12), we find

$$\partial_tp_\Omega=\{p_\Omega,H_T\}=-\Gamma-v_T\sqrt{g}\approx-v_T\sqrt{g} \tag{13}$$

that implies $v_T=0$. In case of the constraint $\pi_N\approx0$, we have

$$\partial_t\pi_N=\{\pi_N,H_T\}=-\mathcal{H}_T-(v_\Gamma+\Omega)\sqrt{g}$$
$$=-\mathcal{H}'_T-v_\Gamma\sqrt{g}=0 \tag{14}$$

that again implies that $v_\Gamma=0$. Let us now consider the time evolution of the constraint Γ,

$$\partial_t\Gamma=\{\Gamma,H_T\}=\equiv-8\pi GN^2\pi^{ij}g_{ij}$$
$$+\partial_iN^iN\sqrt{g}+v_N\sqrt{g}=0 \tag{15}$$

that can be considered as the equation for the Lagrange multiplier v_N. In the case of the constraint $\mathcal{H}'_T$, we find

$$\partial_t\mathcal{H}'_T=\{\mathcal{H}'_T,H_T\}=\int d^3\mathbf{x}(\{\mathcal{H}'_T,N\mathcal{H}'_T\}$$
$$+\sqrt{g}v_\Omega\approx\sqrt{g}v_\Omega=0, \tag{16}$$

where in the first step we used (10). Then, (16) implies $v_\Omega=0$. Finally, we consider the time evolution of the constraint $\mathcal{H}_i$. Because of the fact that Ω and N are dynamical variables, it is natural to extend the constraint $\mathcal{H}_i$ with the appropriate combination of the primary constraints p_Ω and π_N so that

$$\tilde{\mathcal{H}}_i=\mathcal{H}_i+p_\Omega\partial_i\Omega+\pi_N\partial_iN,\qquad \mathbf{T}_S(N^i)=\int d^3\mathbf{x}N^i\tilde{\mathcal{H}}_i. \tag{17}$$

Now the time evolution of the smeared form of the constraint $\mathbf{T}_S(M^i)$ is equal to

$$\partial_t\mathbf{T}_S(M^i)=\{\mathbf{T}_S(M^i),H_T\}\approx\left\{\mathbf{T}_S(M^i),\int d^3\mathbf{x}N\mathcal{H}'_T\right\}$$
$$-\{\mathbf{T}_S(M^i),\Omega\}\approx M^i\partial_i\Omega=0, \tag{18}$$

where we again used (10). Since the equation above has to be valid for all M^i, we see that it corresponds to some form of the constraint on Ω. In order to explicitly identify the nature of the given constraint, we split Ω into the zero mode part and the remaining part as follows:

$$\Omega(\mathbf{x},t)=\Omega_0(t)+\bar{\Omega}(\mathbf{x},t),$$
$$\Omega(t)=\frac{1}{\int d^3\mathbf{x}\sqrt{g}}\int d^3\mathbf{x}\sqrt{g}\Omega(\mathbf{x},t), \tag{19}$$

where by definition, $\int d^3\mathbf{x}\sqrt{g}\bar{\Omega}(\mathbf{x},t)=0$. Then, Eq. (18) implies

$$\bar{\Omega}(\mathbf{x},t)=K(t), \tag{20}$$

where from the definition of $\bar{\Omega}$, we obtain

$$\int d^3\mathbf{x}\sqrt{g}\bar{\Omega}(\mathbf{x},t)=K(t)\int d^3\mathbf{x}\sqrt{g}=0, \tag{21}$$

and, hence, we find $K(t)=0$. In other words, we have following constraint,

$$\bar{\Omega}(\mathbf{x},t)=0, \tag{22}$$

while the zero mode $\Omega_0(t)$ is still nonspecified. It is useful to perform the similar separation of the zero mode part of p_Ω as well,

$$p_\Omega(\mathbf{x},t) = \frac{\sqrt{g}}{\int d^3\mathbf{x}\sqrt{g}} P_\Omega(t) + \bar{p}_\Omega(\mathbf{x},t),$$
$$p_\Omega(t) = \int d^3\mathbf{x}\, p_\Omega(\mathbf{x},t),$$
$$\int d^3\mathbf{x}\,\bar{p}_\Omega(\mathbf{x},t) = 0. \quad (23)$$

Note that we included the factor $\frac{\sqrt{g}}{\int d^3\mathbf{x}\sqrt{g}}$ in front of p_Ω in order to have the canonical Poisson bracket

$$\{\Omega_0, P_\Omega\} = 1 \quad (24)$$

and also in order to ensure that p_Ω transforms as the density since P_Ω is scalar. Then, by definition we also find

$$\{\bar{\Omega}(\mathbf{x}), P_\Omega\} = 0, \qquad \{\bar{p}_\Omega, \Omega_0\} = 0. \quad (25)$$

It turns out that it is useful to perform the similar separation in the case of the constraint $\mathcal{H}_T$,

$$\mathcal{H}_T = \frac{\sqrt{g}}{\int d^3\mathbf{x}\sqrt{g}}\mathcal{H}_0 + \bar{\mathcal{H}}_T,$$
$$\mathcal{H}_0 = \int d^3\mathbf{x}\,\mathcal{H}_T,$$
$$\int d^3\mathbf{x}\,\bar{\mathcal{H}}_T = 0 \quad (26)$$

and also in case of the Lagrange multiplier v_N,

$$v_N = v_0^N + \bar{v}_N,$$
$$v_0^N = \frac{1}{\int d^3\mathbf{x}\sqrt{g}}\int d^3\mathbf{x}\sqrt{g}\,v_N,$$
$$\int d^3\mathbf{x}\sqrt{g}\,\bar{v}_N = 0. \quad (27)$$

Note that the Poisson brackets between $\bar{\mathcal{H}}_T$ still have the form as (10). Explicitly, let us define the smeared form of this constraint,

$$\bar{\mathbf{T}}_T(N) = \int d^3\mathbf{x}\,N(\mathbf{x})\bar{\mathcal{H}}_T(\mathbf{x}) = \int d^3\mathbf{x}\,\bar{N}(\mathbf{x})\bar{\mathcal{H}}_T(\mathbf{x}), \quad (28)$$

where we performed the separation $N = N_0 + \bar{N}$, $\int d^3\mathbf{x}\bar{N}\sqrt{g} = 0$. Then we have

$$\{\bar{\mathbf{T}}_T(N), \bar{\mathbf{T}}(M)\} = \{\mathbf{T}_T(\bar{N}), \mathbf{T}_T(\bar{M})\}$$
$$= \int d^3\mathbf{x}((\bar{N}\partial_i\bar{M} - \partial_i\bar{N}\,\bar{M})g^{ij}\mathcal{H}_j), \quad (29)$$

and, hence, the right side vanishes on the constraint surface $\mathcal{H}_i \approx 0$. With the help of the separation (26) and (27), we find the total Hamiltonian in the form

$$H_T = \int d^3\mathbf{x}\Big(N\mathcal{H}_T + \bar{v}_N\bar{\mathcal{H}}_T + v_0^N\int d^3\mathbf{x}\sqrt{g}\Phi + N^i\tilde{\mathcal{H}}_i$$
$$+ (v_\Gamma + \Omega_0)\Gamma + v_\Omega P_\Omega + v_N\pi_N\Big), \quad (30)$$

where

$$\Phi \equiv \frac{1}{\int d^3\mathbf{x}\sqrt{g}}\mathcal{H}_0 + \Omega_0 \approx 0, \quad (31)$$

and where now we do not consider the canonically conjugate pairs $\bar{\Lambda} \approx 0, \bar{p}_\Omega \approx 0$ which are the second class constraints that decouple from the theory. Before we proceed further, we should also modify the constraint $\bar{\mathcal{H}}_T$ and $\tilde{\mathcal{H}}_i$ in such a way that they Poisson commute with Γ and Φ. In fact, let us consider the following modification of the constraint $\bar{\mathcal{H}}_T$,

$$\bar{\mathcal{H}}'_T = \bar{\mathcal{H}}_T + \frac{1}{32\pi Gg}g^{ij}\pi_{ij}\pi_N. \quad (32)$$

Now it is easy to see that

$$\{\bar{\mathcal{H}}'_T(\mathbf{x}), \Gamma(\mathbf{y})\} = 0. \quad (33)$$

Further, we have to ensure that the $\bar{\mathcal{H}}'_T$ Poisson commutes with the constraint Φ. Clearly, we have $\{\bar{\mathcal{H}}_T, H_0\} \approx 0$, while

$$\left\{\bar{\mathbf{T}}_T(N), \frac{1}{\int d^3\mathbf{x}\sqrt{g}}\right\} = 8\pi G\left(\frac{1}{\int d^3\mathbf{x}\sqrt{g}}\right)^2\int d^3\mathbf{x}\bar{N}\pi^{ij}g_{ij}, \quad (34)$$

so that in order to cancel this contribution, we extend the constraint $\bar{\mathcal{H}}'_T$ so that it has the form

$$\bar{\mathcal{H}}''_T = \bar{\mathcal{H}}'_T + 8\pi G\left(\frac{1}{\int d^3\mathbf{x}\sqrt{g}}\right)^2\overline{\pi^{ij}g_{ij}}P_\Omega, \quad (35)$$

where by definition $\int d^3\mathbf{x}\overline{\pi^{ij}g_{ij}} = 0$. In the similar way, we modify the diffeomorphism constraint $\tilde{\mathcal{H}}_i$ so that it Poisson commutes with Γ (note that it has a vanishing Poisson bracket with Φ on the constraint surface automatically)

$$\bar{\mathcal{H}}_i = \tilde{\mathcal{H}}_i + \partial_i \left[\frac{\pi_N}{\sqrt{g}}\right], \tag{36}$$

so that

$$\{\mathbf{T}_S(N^i), \Gamma(\mathbf{y})\} = -N^k \partial_k \Gamma - \partial_i N^i \Gamma \approx 0. \tag{37}$$

In summary, we have following total Hamiltonian,

$$H_T = \int d^3\mathbf{x} \left(N\mathcal{H}_T + \bar{v}_N \bar{\mathcal{H}}''_T + v_0^N \int d^3\mathbf{x} \sqrt{g}\Phi + N^i \bar{\mathcal{H}}_i + (v_\Gamma + \Omega_0)\Gamma + v_\Omega P_\Omega + v_N \pi_N \right), \tag{38}$$

and check the stability of all constraints,

$$\partial_t \pi_N = \{\pi_N, H_T\} = -\mathcal{H}_T - \Omega_0 \sqrt{g} - v_\Gamma \sqrt{g} \approx -v_\Gamma \sqrt{g} = 0, \tag{39}$$

which implies $v_\Gamma = 0$. For P_Ω, we obtain

$$\partial_t P_\Omega = \{P_\Omega, H_T\} = -\Gamma - v_0^N \int d^3\mathbf{x} \sqrt{g} = 0, \tag{40}$$

which determines v_0^N to be equal to zero. In case of the constraint Γ, we find

$$\partial_t \Gamma = \{\Gamma, H_T\} = v_N = 0 \tag{41}$$

and, hence, $v_N = 0$. Finally, in the case of the constraint Φ, we obtain

$$\partial_t \Phi = \{\Phi, H_T\} = v_\Omega = 0, \tag{42}$$

and we again find $v_\Omega = 0$. Now we should proceed to the analysis of the time evolution of the constraints $\bar{\mathcal{H}}_i$, $\bar{\mathcal{H}}''_T$. However, these constraints Poisson commute with the second class constraints by construction, and the Poisson brackets among themselves vanish on the constraint surface according to (10) and (29).

In summary, we found that $P_\Omega \approx 0, \pi_N \approx 0, \Gamma \approx 0, \Phi \approx 0$ are the second class constraints. Solving these constraints, we eliminate $N, \pi_N, \Omega_0, P_\Omega$ as functions of dynamical variables. Then, the remaining constraints $\bar{\mathcal{H}}''_T, \bar{\mathcal{H}}_i$ form the set of $4\infty^3 - 1$ first class constraints with agreement with [18].

III. UNIMODULAR GRAVITY IN HENNEAUX-TEITELBOIM FORM

In this section, we consider the Henneaux-Teitelboim formulation of unimodular gravity that is based on the existence of the space-time vector density $\mathcal{F}^\mu$. In this case, the action has the form [17]

$$S = \frac{1}{16\pi G} \int d^4x [\sqrt{-\hat{g}} ({}^{(4)}R - 2\Lambda) + 2\Lambda \partial_\mu \mathcal{F}^\mu], \tag{43}$$

where $\Lambda(\mathbf{x}, t)$ is the space-time-dependent Lagrange multiplier. Our goal is to perform the canonical analysis of the given theory. First, we find the following collection of primary constraints,

$$\pi_N \approx 0, \qquad \pi_i \approx 0, \qquad \Gamma \equiv p_t^{\mathcal{F}} - \frac{1}{8\pi G}\Lambda \approx 0, \qquad p_i^{\mathcal{F}} \approx 0, \qquad p_\Lambda \approx 0, \tag{44}$$

where $p_t^{\mathcal{F}}$, $p_i^{\mathcal{F}}$ are momenta conjugate to $\mathcal{F}^t$, $\mathcal{F}^i$, respectively, with the following canonical Poisson brackets,

$$\{\mathcal{F}^t(\mathbf{x}), p_t^{\mathcal{F}}(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \qquad \{\mathcal{F}^i(\mathbf{x}), p_j^{\mathcal{F}}(\mathbf{y})\} = \delta_i^j \delta(\mathbf{x} - \mathbf{y}). \tag{45}$$

Then, we again find that the bare Hamiltonian with primary constraints included has the form

$$H = \int d^3\mathbf{x} \left(N(\mathcal{H}_T + p_t^{\mathcal{F}} \sqrt{g}) + N^i \mathcal{H}_i - \frac{1}{8\pi G} \Lambda \partial_i \mathcal{F}^i + v_N \pi_N + v^i \pi_i + v_\Lambda p_\Lambda + u_i p^i_{\mathcal{F}} + u_\Gamma \Gamma \right), \tag{46}$$

where using the constraint Γ, we replaced $\frac{1}{8\pi G}\sqrt{g}\Lambda$ with $p_t^{\mathcal{F}} \sqrt{g}$. Now the requirement of the preservation of the primary constraints implies the following secondary constraints,

$$\begin{aligned}
\partial_t \pi_N &= \{\pi_N, H_T\} = -(\mathcal{H}_T + p_t^{\mathcal{F}} \sqrt{g}) \equiv -\mathcal{H}'_T \approx 0, \\
\partial_t \pi_i &= \{\pi_i, H_T\} = -\mathcal{H}_i \approx 0, \\
\partial_t \Gamma &= \{\Gamma, H_T\} = -\frac{v_\Lambda}{8\pi G} = 0, \\
\partial_t p_i^{\mathcal{F}} &= \{p_i^{\mathcal{F}}, H_T\} = \frac{1}{8\pi G} \partial_i \Lambda, \\
\partial_t p_\Lambda &= \{p_\Lambda, H_T\} = \frac{1}{8\pi G} \partial_i \mathcal{F}^i + \frac{1}{8\pi G} u_\Gamma = 0.
\end{aligned} \tag{47}$$

The third and fifth equation determine the Lagrange multipliers v_Λ and u_Γ. As in the previous section, we find that the fourth equation implies that $\bar{\Lambda}(t, \mathbf{x}) = 0$ while the zero mode part Λ_0 is not determined. In other words, we have the second class constraints $\bar{\Lambda} = 0$, $p_{\bar{\Lambda}} = 0$, so we will not consider these modes anywhere and restrict ourselves to the case of the zero mode of Λ. We also modify $\mathcal{H}_i$ in order to take into account the transformation rule for Λ,

$$\mathcal{H}'_i = \mathcal{H}_i + p_\Lambda \partial_i \Lambda, \tag{48}$$

so that the total Hamiltonian has the form

$$H_T = \int d^3\mathbf{x}(N\mathcal{H}'_T + N^i\mathcal{H}'_i + v_\Lambda p_\Lambda + u_\Gamma\Gamma), \qquad (49)$$

where we also used integration by parts that eliminates the term $\Lambda\partial_i\mathcal{F}^i$. Finally, we see that p_Λ and Γ are the second class constraints so that we can eliminate p_Λ and Λ from the theory. As a result, we find the theory with $4\infty^3$ the first class constraints $\mathcal{H}'_T, \mathcal{H}'_i$ for the dynamical variables $g_{ij}, \pi^{ij}, p_t^{\mathcal{F}}, \mathcal{F}^t$. Note that the Hamiltonian does not depend on $\mathcal{F}^t$ explicitly, and, hence, we see that $p_t^{\mathcal{F}}$ is constantly on shell. In other words $p_t^{\mathcal{F}}$ plays the role of the cosmological constant, which, however, is not included into the theory by hand but it arises as a consequence of the dynamics of the unimodular theory in Henneaux-Teitelboim formulation.

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